INVOLUTIONS ON MODULI SPACES AND REFINEMENTS OF THE VERLINDE FORMULA

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Abstract. The moduli space M of semi-stable rank 2 bundles with trivial determinant over a complex curve Σ carries involutions naturally associated to 2-torsion points on the Jacobian of the curve. For every lift of a 2-torsion point to a 4-torsion point, we define a lift of the involution to the determinant line bundle $\mathcal{L}$. We obtain an explicit presentation of the group generated by these lifts in terms of the order 4 Weil pairing. This is related to the triple intersections of the components of the fixed point sets in M, which we also determine completely using the order 4 Weil pairing. The lifted involutions act on the spaces of holomorphic sections of powers of $\mathcal{L}$, whose dimensions are given by the Verlinde formula. We compute the characters of these vector spaces as representations of the group generated by our lifts, and we obtain an explicit isomorphism (as group representations) with the combinatorial-topological TQFT-vector spaces of [BHMV]. As an application, we describe a 'brick decomposition', with explicit dimension formulas, of the Verlinde vector spaces. We also obtain similar results in the twisted (*i.e.*, degree one) case.

Contents

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Introduction and Motivation.

The celebrated Verlinde formula [Ve] gives the dimension of certain vector spaces of so-called 'conformal blocks' appearing in conformal field theory. In this paper, we will take the point of view of algebraic geometry and think of the conformal blocks as holomorphic sections of powers of the determinant line bundle over moduli spaces of semi-stable bundles with fixed determinant over a complete non-singular complex curve Σ. According to Atiyah [At2] and Witten [Wi], the spaces of holomorphic sections should also fit into $2+1$-dimensional 'Topological Quantum Field Theories' (TQFT). This geometric construction has been studied quite a lot (see *e.g.* [ADW, MS, RSW, Hi, CLM, Th1, Th2, Sz, BSz, BL, KNR, Fa]).

In [BHMV], a combinatorial-topological construction of TQFT-functors was described, based on a particularly simple construction of the Witten-Reshetikhin-Turaev 3-manifold invariant [RT] in the $SU(2)$-case through the Kauffman bracket. In that paper one also constructed certain involutions on the TQFT-vector spaces which were then used to decompose the vector spaces into direct summands. These involutions are associated to simple closed curves on the underlying smooth surface of Σ, and they generate a kind of Heisenberg group presented in terms of the *mod* 4 intersection form. These ideas were developed further in [BM] to construct spin-refined TQFT's.

The starting point for the present paper was the idea that the involutions of [BHMV] should correspond on the algebraic-geometric side to the involutions on moduli space naturally associated to 2-torsion points on the Jacobian of Σ. More precisely, the 2-torsion points should correspond to the *mod* 2 homology classes of the simple closed curves. Note that the involutions on the spaces of holomorphic sections require a choice of lift to the determinant line bundle $\mathcal{L}$. It is easy to see that these lifts generate a central extension of the group of 2-torsion points, whose cocycle is given by the order 2 Weil pairing. It follows that this extension is indeed abstractly isomorphic to the one that appeared in [BHMV]. The ambiguity in the choice of lift is reflected on the topological side by the choice of a simple closed curve within its *mod* 2 homology class.

One of our motivations in this paper is to establish a more precise correspondance between the two theories. The key idea is to define, for every lift of a 2-torsion point α to a 4-torsion point a, an involution ρ_a on $\mathcal{L}$ which covers the action of α on the moduli space M. The sign of this lift ρ_a is fixed by requiring it to act as the identity over a certain component of the fixed point set of the involution acting on M; this component is simply the one containing the class of the semi-stable bundle $L_a \oplus L_a^{-1}$ (see section 1 for more details). We will see in section 3 that the action of this lift ρ_a on holomorphic sections corresponds precisely to the involution τ_γ which is associated in the [BHMV]-theory to a simple closed curve γ whose homology class is Poincaré dual to a. In this way, we obtain an explicit isomorphism (as group representations) with the TQFT-vector spaces of [BHMV]. We believe this constitutes a nice confirmation that there should be a natural correspondence between the two theories.

The main work in this paper is, however, algebraic-geometric in nature and consists of a detailed study of our lifts ρ_a and their action on the Verlinde vector spaces. Our main results are as follows (see section 2 for more complete statements). We show in theorem 2.1 that our lifts satisfy

$$\rho_a \, \rho_b \;=\; \lambda_4(a,b)\, \rho_{a+b},$$

where λ_4 is the order 4 Weil pairing (which is the algebraic-geometric analogue of the *mod* 4 intersection form). Along the way, we also determine in theorem 2.2 the triple intersections of the components of the fixed point sets of the action on the various moduli spaces. The *rôle* played by the order 4 Weil pairing in this context doesn't seem to have been observed before. We then compute in theorem 2.3 the trace of the induced involutions $\rho_a^{\otimes k}$ on the spaces of holomorphic sections of the k-th tensor power of $\mathcal{L}$. This determines the characters of these vector spaces as representations of the group generated by our lifts. We also obtain similar results in the twisted (*i.e.*, degree one) case, where the situation is somewhat simpler, as only the order 2 Weil pairing is needed.

A corollary of our results is a 'brick decomposition' of the spaces of holomorphic sections, the structure of which depends on the value of the level k modulo 4. This is, of course, analogous to the decomposition in [BHMV], but we will derive it, as well as explicit dimension formulas, directly from the character of the representation. At low levels, similar decompositions have appeared previously in the work of Beauville [Be1] (see also Laszlo [L]) for $k = 2$, and of van Geemen and Previato [vGP1, vGP2], Oxbury and Pauly [OP], and Pauly [P] for $k = 4$ (in our notation). Their approach is based on the relationship with abelian theta-functions and seems quite different from ours.

This paper is organized as follows. After giving the basic definitions in section 1, we state our main results in section 2. The relationship with the [BHMV]-theory is discussed in more detail in section 3, and the 'brick decompositions' are described in section 4. The remainder of the paper is devoted to the proofs. The reader interested only in the algebraic geometry may skip section 3, and no familiarity with [BHMV] is necessary to understand the results and their proofs.

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1. Basic definitions and notation.

Let Σ be a complete, non-singular curve over the complex numbers of genus $g \geq 2$. Let M_d be the moduli space of semi-stable bundles of rank 2 and degree $d \in \mathbb{Z}$ on Σ. There is a natural algebraic action of the degree zero Picard group $\mathrm{Pic}_0(\Sigma)$ on M_d gotten by tensoring. In this paper, we will use the standard identification of $\mathrm{Pic}_0(\Sigma)$ with the Jacobian $J(\Sigma)$ and speak of an action of $J(\Sigma)$ on M_d.

Fix a point $p \in \Sigma$ and let $[p]$ be the associated line bundle. We have the determinant morphism $\det : M_d \to \mathrm{Pic}_d(\Sigma)$. We put $M = \det^{-1}(\mathcal{O}_\Sigma)$ and $M' = \det^{-1}([p])$.

As it is explained in [DN] there is a natural construction of a determinant line bundle $\mathcal{L}_d$ over M_d. We will denote the restriction of $\mathcal{L}_0$ to M by $\mathcal{L}$ and the restriction of $\mathcal{L}_1$ to M' by $\mathcal{L}'$. These two determinant bundles are generators respectively of the Picard group of M and of M' (see [DN]).

Let $k \geq 1$ be an integer called the *level*. Put

$$Z_k(\Sigma) = H^0(M, \mathcal{L}^k)$$

$$Z'_k(\Sigma) = \begin{cases} H^0(M', \mathcal{L}'^{k/2}) & \text{if } k \equiv 0 \bmod 2 \\ 0 & \text{if } k \equiv 1 \bmod 2 \end{cases}$$

It is by now well-known that the dimensions of these vector spaces are given by the celebrated Verlinde formulas (to be recalled in section 4). Following Thaddeus [Th1], we will refer to $Z'_k(\Sigma)$ as the *twisted case*, and to $Z_k(\Sigma)$ as the *untwisted case.*

Let $J^{(r)}$ be the subgroup of order r points on $J(\Sigma)$. We denote by L_α (resp. L_a) the line bundle on Σ corresponding to $\alpha \in J^{(2)}$ (resp. $a \in J^{(4)}$). Note that the group $J^{(r)}$ is identified with $H^1(\Sigma; \mu_r)$, where $\mu_r \subset \mathbb{C}$ is the group of r-th roots of unity.

The action of $\alpha \in J^{(2)}$ on M_d is given by tensoring with L_α. Since $L_\alpha^{\otimes 2} \cong \mathcal{O}_\Sigma$, this preserves M and M'. By abuse of notation, the automorphisms of M and M' induced by $\alpha \in J^{(2)}$ will again be denoted by α.

The lifts ρ_a and ρ'_α. By a lift of α to $\mathcal{L}$ we mean an invertible bundle map from $\mathcal{L}$ to itself covering α. For a lift to exist it suffices that $\alpha^*\mathcal{L} \cong \mathcal{L}$. This is the case for every $\alpha \in J^{(2)}$, since the Picard group of M is isomorphic to $\mathbb{Z}$, and $\mathcal{L}$ is ample, so that α must act trivially on the Picard group. Therefore the action of any $\alpha \in J^{(2)}$ can be lifted to $\mathcal{L}$, and also to $\mathcal{L}'$, for the same reason. Since the only algebraic functions on M and M' are the constant ones, it follows that we can actually choose involutive lifts of each element of $J^{(2)}$. Any two involutive lifts of α agree up to sign.

To fix the signs of the lifts of α to $\mathcal{L}$ and $\mathcal{L}'$, we use the fact that the sign can be read off in the fiber over a fixed point of α. Given $0 \neq \alpha \in J^{(2)}$, we use the notation $|X|_\alpha$ for the fixed point variety of the automorphism induced by α on the various moduli spaces X.

It is well-known that $|M'|_\alpha$ is isomorphic to the *Prym variety* P_α associated to α, and that $|M|_\alpha$ is isomorphic to two copies of the *Kummer variety* $P_\alpha/\langle \pm 1\rangle$. This result is due to Narasimhan and Ramanan [NR]. In particular, $|M'|_\alpha$ is connected and non-empty, and $|M|_\alpha$ has two components. For any $a \in J^{(4)}$ such that $2a = \alpha$, we denote by $|M|_a^+$ the component of $|M|_\alpha$ containing the S-equivalence class of the semistable bundle $L_a \oplus L_a^{-1}$. (Note that this is indeed fixed under tensoring with L_α, since $L_a \otimes L_\alpha \cong L_a^{-1}$.) The other component of $|M|_\alpha$ is denoted by $|M|_a^-$.

Definition 1.1. *For $0 \neq \alpha \in J^{(2)}$, we define ρ'_α to be the involutive lift to $\mathcal{L}'$ of α acting on M' such that ρ'_α acts by* minus *the identity on the restriction of $\mathcal{L}'$ to $|M'|_\alpha$.*

Definition 1.2. *For $a \in J^{(4)}$ such that $2a = \alpha \neq 0$, we define ρ_a to be the involutive lift to $\mathcal{L}$ of α acting on M such that ρ_a acts by the identity on the restriction of $\mathcal{L}$ to the component $|M|_a^+$ of $|M|_\alpha$ specified by a.*

It will be convenient to extend this definition by letting ρ'_α (resp. ρ_a) be the identity if $\alpha = 0$ (resp. $2a = 0$).

Note. According to Theorem F in [DN], we have that $\mathcal{L}'^{-2} \cong K$, where K is the canonical bundle of M'. The natural action of α on K coincides with the one induced by both possible involutive lifts $\pm\rho'_\alpha$. This is because α acts as the identity on the fiber of K over the fixed point set $|M'|_\alpha$, since $|M'|_\alpha$ has even codimension (see section 5). Similar comments apply in the untwisted case.

The Weil pairing. (See *e.g.* [Ho].[1]) Let $\mathcal{M}(\Sigma)$ be the field of meromorphic functions on Σ. The divisor of $f \in \mathcal{M}(\Sigma)$ is denoted by (f). As already mentioned, we consider $J^{(r)}$ to be the r-torsion points on the group $\mathrm{Pic}_0(\Sigma)$ which is naturally identified with $\mathrm{Div}_0(\Sigma)/\mathrm{Div}^{pr}(\Sigma)$. (Here, $\mathrm{Div}_d(\Sigma)$ is the group of degree d divisors, and $\mathrm{Div}^{pr}(\Sigma)$ are the principal divisors.) If $D = \sum n_j x_j$ is a divisor and $f \in \mathcal{M}(\Sigma)$ a meromorphic function, we put $f(D) = \prod f(x_j)^{n_j}$. The Weil pairing

$$\lambda_r : J^{(r)} \times J^{(r)} \to \mu_r$$

is defined as follows. Given $a, b \in J^{(r)}$, represent them by divisors D_a, D_b with disjoint support, and pick $f, g \in \mathcal{M}(\Sigma)$ such that $(f) = rD_a$ and $(g) = rD_b$. Then

$$\lambda_r(a,b) = \frac{g(D_a)}{f(D_b)}.$$

The Weil pairing is antisymmetric and non-degenerate. The fact that it takes values in μ_r follows from Weil reciprocity (see [GH], p. 242).

2. Statement of the main results.

Let $\mathcal{E}$ be the group generated by the involutions ρ_a acting on $\mathcal{L}$, and let $\mathcal{E}'$ the group generated by the involutions ρ'_α acting on $\mathcal{L}'$. By general theory, these groups are central extensions of $J^{(2)}$. Our first main result gives a presentation of these groups in terms of our lifts ρ_a and ρ'_α.

Theorem 2.1. *The involutions ρ_a and ρ'_α satisfy the following relations:*

(1) $$\rho_a \, \rho_b \;=\; \lambda_4(a,b)\, \rho_{a+b}$$

(2) $$\rho'_\alpha \, \rho'_\beta \;=\; \lambda_2(\alpha,\beta)\, \rho'_{\alpha+\beta}$$

[1] We thank A. Beauville for pointing out this reference.

It follows that the group $\mathcal{E}$ is a non-trivial extension of $J^{(2)}$ by μ_4. The cocycle $c(\alpha,\beta)$ of this extension is the order 2 Weil pairing λ_2; indeed:

$$c(\alpha,\beta) = \rho_a\rho_b\rho_a^{-1}\rho_b^{-1} = \lambda_4(a,b)^2 = \lambda_2(\alpha,\beta).$$

The group $\mathcal{E}'$ is a trivial extension of $J^{(2)}$, *i.e.*, it is isomorphic to $\mu_2 \times J^{(2)}$.

Note. The cocycle of the extensions $\mathcal{E}$ was known before; *e.g.* it is deduced in [BLS] from the isomorphism of $Z_1(\Sigma)$ with the space of abelian theta-functions due to Beauville, Narasimhan, and Ramanan [BNR]. Given the cocycle, it follows immediately that for *any* choice of involutive lifts $\tilde\rho_\alpha$, $\tilde\rho_\beta$, and $\tilde\rho_{\alpha+\beta}$, say, of $\alpha, \beta, \alpha+\beta$ to the determinant line bundle $\mathcal{L}$, one has $\tilde\rho_\alpha\tilde\rho_\beta = \pm\tilde\rho_{\alpha+\beta}$ if $\lambda_2(\alpha,\beta)=1$, and $\tilde\rho_\alpha\tilde\rho_\beta = \pm i\,\tilde\rho_{\alpha+\beta}$ if $\lambda_2(\alpha,\beta) = -1$. The contribution of theorem 2.1 is to show that with our precise choice of lifts, the prefactors ± 1 and $\pm i$ are given by the order 4 Weil pairing. Similar remarks apply in the twisted case.

The above result is related to the triple intersections of the components of the fixed point set on M. In fact, a key step in the proof is the following result which does not seem to have been observed before.

Theorem 2.2. *Assume that α and β are distinct non-zero elements of $J^{(2)}$ such that $\lambda_2(\alpha,\beta) = 1 \in \mu_2$. Let $a, b \in J^{(4)}$ such that $2a = \alpha$ and $2b = \beta$. Then*

$$|M|_a^+ \cap |M|_b^+ \cap |M|_{a+b}^+ \neq \emptyset \;\Leftrightarrow\; \lambda_4(a,b) = 1 \in \mu_4$$

$$|M|_a^+ \cap |M|_b^+ \cap |M|_{a+b}^- \neq \emptyset \;\Leftrightarrow\; \lambda_4(a,b) = -1 \in \mu_4$$

Note. Given theorem 2.2, theorem 2.1 in the case $\lambda_4(a,b) = 1$ follows immediately. Indeed, if the triple intersection $|M|_a^+ \cap |M|_b^+ \cap |M|_{a+b}^+$ is non-empty, it follows from the definition of the lifts that $\rho_{a+b} = \rho_a\rho_b$ (since $\rho_a\rho_b\rho_{a+b}$ must be a constant, and this constant can be computed in the fiber of $\mathcal{L}$ over a triple intersection point.) The proof in the general case, however, requires some further arguments. The most interesting case is when $\lambda_4(a,b) = \pm i$. In this situation, the fixed point sets $|M|_\alpha$, $|M|_\beta$, $|M|_{\alpha+\beta}$ don't intersect, but there is a $\mathbb{P}^1$ intersecting each one of the six components $|M|_a^+$, $|M|_a^-$, $|M|_b^+$, $|M|_b^-$, $|M|_{a+b}^+$, $|M|_{a+b}^-$, in a point. See section 8.

Our next result describes $Z_k(\Sigma)$ (resp. $Z'_k(\Sigma)$) as representations of the group $\mathcal{E}$ (resp. $\mathcal{E}'$). (Here, $\mathcal{E}$ acts on $Z_k(\Sigma)$ *via* the natural action of $\rho_a^{\otimes k}$ on $\mathcal{L}^{\otimes k}$, and similarly for $\mathcal{E}'$.) This is based on the following result obtained by applying the Lefschetz-Riemann-Roch formula. We assume $\alpha \in J^{(2)}$ is non-zero, and $2a = \alpha$.

Theorem 2.3. *One has*

$$\mathrm{Tr}(\rho_a^{\otimes k}) = \frac{1+(-1)^k}{2}\left(\frac{k+2}{2}\right)^{g-1}$$

and (for even k)

$$\mathrm{Tr}({\rho'_\alpha}^{\otimes k/2}) = (-1)^{k/2}\left(\frac{k+2}{2}\right)^{g-1}$$

Note. In the twisted case and for levels divisible by 4, this result is due to Pantev [Pa]. In the untwisted case and for levels divisible by 4, it is also contained in Beauville's recent paper [Be2]. Our computation was done independently of his.

The characters of $Z_k(\Sigma)$ and $Z'_k(\Sigma)$ (as representations of the groups $\mathcal{E}$ and $\mathcal{E}'$, respectively) are determined by the formulas in theorem 2.3 together with the trace of the identity, given by the Verlinde formulas. Therefore the above result determines these representations up to isomorphism.

A remarkable consequence of this is the following theorem, which was actually the main motivation for the present paper.

Theorem 2.4. *The spaces $Z_k(\Sigma)$ and $Z'_k(\Sigma)$ are isomorphic, as representations of the groups $\mathcal{E}$ and $\mathcal{E}'$, respectively, to the TQFT-vector spaces constructed in* [BHMV].

This will be discussed in more detail in section 3.

Corollary 2.5. *If the level k is even, the spaces $Z_k(\Sigma)$ and $Z'_k(\Sigma)$ are decomposed as direct sums of isotypic components (called 'bricks' in what follows) for the action of $\mathcal{E}$ and $\mathcal{E}'$. If $k \equiv 0$ mod 4, the bricks are indexed by characters of $J^{(2)}$. If $k \equiv 2$ mod 4, the bricks are indexed by θ-characteristics on the curve Σ (or, equivalently, by spin structures on Σ). If the level is odd, $Z_k(\Sigma)$ is isomorphic, as representation of the group $\mathcal{E}$, to a direct sum of copies of $Z_1(\Sigma)$ or to a direct sum of copies of the conjugate representation, $\overline{Z_1(\Sigma)}$, according to the parity of $(k-1)/2$.*

Remarks 2.6. (i) Note that formula (1) implies

$$\rho_a^{\otimes k}\,\rho_b^{\otimes k} \;=\; \lambda_4(a,b)^k\,\rho_{a+b}^{\otimes k}. \tag{3}$$

Hence the action of $\mathcal{E}$ on $Z_k(\Sigma)$ factors through an action of $\mathcal{E}'$ if k is even, and furthermore through an action of $J^{(2)}$ if $k \equiv 0$ mod 4. This explains why the bricks are indexed by characters of $J^{(2)}$ if $k \equiv 0$ mod 4. If $k \equiv 2$ mod 4, the index set are the characters of $\mathcal{E}'$ which do not factor through $J^{(2)}$; such characters can be identified with θ-characteristics.

(ii) The dimensions of the bricks are the same for all non-trivial characters of $J^{(2)}$ in the case $k \equiv 0$ mod 4, and depend only on the parity of the θ-characteristic in the case $k \equiv 2$ mod 4. In section 4, we will give explicit formulas for their dimensions.

Note. Our brick decomposition can be viewed as a generalisation of an old result of Beauville's [Be1] (see also Laszlo [L]) at level 2. Beauville constructed bases of $Z_2(\Sigma)$ (resp. $Z'_2(\Sigma)$) whose basis elements are indexed by even (resp. odd) θ-characteristics. This corresponds from our point of view to the fact that the bricks in level 2 are zero, if the θ-characteristic has the 'wrong' parity, and one-dimensional otherwise. Note, however, that the situation in level 2 is very special; in general, the bricks are non-zero for both parities. We would like to mention also the work of van Geemen and Previato [vGP1, vGP2], Oxbury and Pauly [OP], and Pauly [P]. We believe their work contains a description of the bricks in level 4, but a detailed comparison hasn't been done yet.

3. Relationship with the [BHMV]-theory.

In [BHMV], the Kauffman bracket (at a $2p$-th root of unity called A) was used to give a combinatorial-topological construction of TQFT-functors V_p on a certain $2+1$-dimensional cobordism category. It is expected that these functors for $p = 2k+4$ correspond (in some natural sense) to Witten's ones for $SU(2)$ and level k. For example, the dimensions of the vector space $V_p(\Sigma) \otimes \mathbb{C}$ [2] associated to a closed oriented surface Σ is also given by the Verlinde formula ([BHMV], cor. 1.16).

The aim of the present section is to explain the following more precise statement of theorem 2.4. (As already mentioned in the introduction, the reader interested only in the algebraic geometry may proceed directly to section 4.)

Theorem 3.1. *Let $p = 2k+4$. For the 'right' choice of $2p$-th root of unity A, the vector spaces $V_p(\Sigma) \otimes \mathbb{C}$ and $Z_k(\Sigma)$ are isomorphic as representations of $\mathcal{E}$ (this group is denoted by $\Gamma(\Sigma)$ in* [BHMV]*). This isomorphism sends the involution $\rho_a^{\otimes k}$ acting on $Z_k(\Sigma)$ to the involution τ_γ acting on $V_p(\Sigma) \otimes \mathbb{C}$, where γ is a simple closed curve on Σ representing the Poincaré dual of a. A similar statement holds in the twisted case.*

Note. For simplicity of exposition, a few technical details related to framing issues will be suppressed from the discussion here. The reader is also referred to the survey article [MV].

By definition, elements of $V_p(\Sigma)$ are represented by linear combinations of compact oriented 3-manifolds M with boundary equal to Σ. (No complex structure on Σ is needed here.) The 3-manifolds may also contain colored links or, more generally, colored trivalent graphs. In the following, we assume p is even and put $p = 2k+4$. Then a color is just an element of $\{0, 1, 2, \dots, k\}$.

In the [BHMV]-theory, one has involutions τ_γ of $V_p(\Sigma)$ associated to unoriented simple closed curves γ on Σ. They are defined as follows. Consider a vector represented by some (M, L), where M is a 3-manifold with boundary Σ, and L stands for some colored link inside M. Then the action of τ_γ consists of adding to the link L already present in M, an additional component consisting of the curve γ pushed

[2] By definition, $V_p(\Sigma)$ is a module over a certain abstract cyclotomic ring k_p. By $V_p(\Sigma) \otimes \mathbb{C}$, we mean the vector space obtained by extending coefficients from k_p to $\mathbb{C}$; this depends on a choice of a $2p$-th root of unity A in $\mathbb{C}$.

slightly inside M in a neighborhood of $\Sigma = \partial M$, where γ is colored by k (the last color). It was shown in section 7 of [BHMV] that (because of the relations which hold in $V_p(\Sigma)$) this endomorphism τ_γ is an involution.

Following [BHMV], p. 917, the groups generated by these involutions acting on $V_p(\Sigma)$ can be described as follows. Let $\Gamma(\Sigma)$ be the group with one generator $[a]$ for each $a \in H_1(\Sigma;\mathbb{Z}/4)$ plus one additional generator u, and the following relations: the element u is central, $u^4 = 1$, $[a]^2 = 1$ for all $a \in H_1(\Sigma;\mathbb{Z}/4)$, and

$$[a]\,[b] \;=\; u^{a\cdot b}\,[a+b] \tag{4}$$

for all $a, b \in H_1(\Sigma;\mathbb{Z}/4)$. Here, $a \cdot b \in \mathbb{Z}/4$ denotes the *mod* 4 intersection form determined by the orientation of Σ.

If a *mod* 2 class $\alpha \in H_1(\Sigma;\mathbb{Z}/2)$ is given, every lift of α to a *mod* 4 class $a \in H_1(\Sigma;\mathbb{Z}/4)$ determines an element $[a] \in \Gamma(\Sigma)$. However, up to multiplication by u^2, this element $[a]$ depends only on α. (Indeed, if a_1, a_2 are two such lifts, then $[a_1] = u^{2x}[a_2]$ where $a_1 - a_2 = 2x$.) From this it follows easily that the group $\Gamma(\Sigma)$ is a central extension of $H_1(\Sigma;\mathbb{Z}/2)$ by $\mathbb{Z}/4$. (It can be viewed as some kind of 'reduced' Heisenberg group associated to twice the intersection form.)

Here is the relationship of the group $\Gamma(\Sigma)$ with the involutions τ_γ. Given $\alpha \in H_1(\Sigma;\mathbb{Z}/2)$, a lift of α to an element of $\Gamma(\Sigma)$ can also be specified by an unoriented simple closed curve γ representing α, as follows. The curve γ determines an element $a \in H_1(\Sigma;\mathbb{Z}/4)$ up to sign, and the induced element $[a] \in \Gamma(\Sigma)$ is well-defined, since $a \cdot a = 0$ and hence $[-a] = [a]$.

It is clear from the geometric description of the τ_γ's given above that the commutation properties of these involutions are related to the intersection properties of the associated curves; for example two such involutions obviously commute if the corresponding simple closed curves on Σ don't intersect. The fact that the τ_γ's satisfy precisely the relations in $\Gamma(\Sigma)$ is shown in prop. 7.5 of [BHMV]. Moreover, it is shown there that the central element u acts on $V_{2k+4}(\Sigma)$ as multiplication by $(-1)^{k+1}A^{(k+2)^2}$.

Now let us identify $\mathbb{Z}/4 = \mu_4$ such that $1 \in \mathbb{Z}/4$ corresponds to $i \in \mu_4$. Also, let us use Poincaré duality to identify $J^{(4)}(\Sigma) = H^1(\Sigma;\mu_4)$ with $H_1(\Sigma;\mathbb{Z}/4)$. By a well-known folk theorem, there exists a sign $\varepsilon = \pm 1$ such that the Weil pairing is related to the intersection form as follows: [3]

$$\lambda_4(a,b) = (\varepsilon i)^{a\cdot b}.$$

We now choose the primitive root of unity A of order $2p = 4k+8$ such that $(-1)^{k+1}A^{(k+2)^2} = (\varepsilon i)^k$. For instance, we may choose $A = -\varepsilon e^{2i\pi/(4k+8)}$. Comparing (1) and (4), we see that the assignment

$$\tau_\gamma \mapsto \rho_a^{\otimes k}$$

[3] The value of ε should be known, but we have been unable to locate it in the literature.

(where the curve γ represents the Poincaré dual of a) defines an isomorphism from the image of $\Gamma(\Sigma)$ in the general linear group of $V_{2k+4}(\Sigma) \otimes \mathbb{C}$ to the image of the group $\mathcal{E}$ in the general linear group of $Z_k(\Sigma)$. (In particular, for $k = 1$ we have an isomorphism $\Gamma(\Sigma) \cong \mathcal{E}$.)

Using the dimension formulas given in section 7 of [BHMV], one can check that the traces of the involutions τ_γ acting on $V_p(\Sigma)$ coincide precisely with the traces computed in our theorem 2.3. This verification will be omitted.

Since representations of finite groups are determined by their characters, this proves theorem 3.1 in the untwisted case.

The twisted case, where $Z_k(\Sigma)$ is replaced with $Z'_k(\Sigma)$, is similar. Here we must replace $V_p(\Sigma)$ with $V_p(\Sigma')$, where Σ' is Σ with one puncture colored by k (the last color). (We maintain the convention $p = 2k+4$.) Then we have again an isomorphism of representations (the group corresponding to $\mathcal{E}'$ is called $\Gamma'(\Sigma)$ in [BHMV]). The dimension of $V_p(\Sigma')$ is given by the 'twisted Verlinde formula', see [BHMV], Remark 5.11. This space is zero if k is odd, and this is why we define $Z'_k(\Sigma)$ to be zero for odd k. (See also Thaddeus [Th1].) (N.b., the vector spaces V_p are also defined for surfaces with colored punctures; they are called 'surfaces with colored structure' in [BHMV]. The theory of the involutions τ_γ works the same for these, with the *caveat* that the curves γ must avoid the punctures. We expect analogues of our results to hold in the case of general colored punctures, using moduli spaces of parabolic bundles.)

Remark. Although it has been known for some time that $V_p(\Sigma)$ and $Z_k(\Sigma)$ have the same dimensions, it seems, to the best of our knowledge, that a *natural* isomorphism between the two theories is still missing. Of course, the fact that these two spaces are isomorphic also as representations (of canonically isomorphic groups) gives further evidence that there should be a natural isomorphism between the two theories.

4. Brick decomposition and dimension formulas.

In this section, we describe $Z_k(\Sigma)$ and $Z'_k(\Sigma)$ as representations of the groups $\mathcal{E}$ and $\mathcal{E}'$, respectively. We also give explicit dimension formulas. The results of this section are immediate consequences of the isomorphism of theorem 3.1 and the computations in [BHMV] and [BM]. In order to make this paper self-contained, we will, however, derive them directly from theorems 2.1 and 2.3.

We first recall the Verlinde formula and its twisted counterpart. Put $d_g(k) = \dim\ Z_k(\Sigma_g)$ and $d'_g(k) = \dim\ Z'_k(\Sigma_g)$, where the subscript g indicates the genus of the curve Σ_g. [4] Then one has

$$d_g(k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}$$

[4]These numbers are denoted by $d_g(p)$ and $\widehat{d}_g(p)$ in [BHMV, MV, BM], where $p = 2k+4$.

$$d'_g(k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} (-1)^{j+1} \left(\sin\frac{\pi j}{k+2}\right)^{2-2g}$$

The case $k \equiv 0$ mod 4.

In view of formula (3) in section 2, the action of the group $\mathcal{E}$ on $Z_k(\Sigma_g)$ factors through an action of $J^{(2)}(\Sigma_g) = H^1(\Sigma_g; \mu_2)$. We have a direct sum decomposition

$$Z_k(\Sigma_g) = \bigoplus_h Z_k(\Sigma_g; h)$$

where h runs through the characters of $J^{(2)}$, and $Z_k(\Sigma_g; h)$ is the subspace of $Z_k(\Sigma_g)$ where α acts as multiplication by $h(\alpha) \in \mu_2 = \pm 1$, for all $\alpha \in J^{(2)}$. We will refer to $Z_k(\Sigma_g; h)$ as the *brick* associated to h.

By theorem 2.3, the character of the representation $Z_k(\Sigma_g)$ takes the same value on all non-trivial elements of $J^{(2)}$, and is therefore invariant under the automorphism group of $J^{(2)}$. It follows that the dimension of the brick $Z_k(\Sigma_g; h)$ is the same for all non-trivial characters h of the group, since the automorphism group acts transitively on the set of non-trivial characters. We denote this dimension by $d_g^{(1)}(k)$, and we put $d_g^{(0)}(k) = dim\ Z_k(\Sigma_g; 0)$, where 0 denotes the trivial character. (Thus, the space $Z_k(\Sigma_g; 0)$ is the fixed point set of the action of the group $J^{(2)}$ on $Z_k(\Sigma_g)$.) These numbers can be computed from the following two formulas:

$$d_g(k) = d_g^{(0)}(k) + (2^{2g} - 1)\, d_g^{(1)}(k)$$

$$d_g^{(0)}(k) - d_g^{(1)}(k) = tr(\alpha) = \left(\frac{k+2}{2}\right)^{g-1}, \quad \alpha \neq 0$$

For example, one has

$$d_g^{(0)}(k) = \frac{1}{2^{2g}} \left(d_g(k) + (2^{2g} - 1) \left(\frac{k+2}{2}\right)^{g-1} \right)$$

Similarly, $Z'_k(\Sigma_g)$ is the direct sum of the bricks $Z'_k(\Sigma_g; h)$, and the dimensions ${d'_g}^{(0)}(k) = dim\ Z'_k(\Sigma_g; 0)$ and ${d'_g}^{(1)}(k) = dim\ Z'_k(\Sigma_g; h)$ for $h \neq 0$, can be computed as before (just replace $d_g(k)$ with $d'_g(k)$ and $d_g^{(\varepsilon)}(k)$ with ${d'_g}^{(\varepsilon)}(k)$ in the above).

Example: The case $k = 4$. The numbers ${d_g}^{(1)}(4)$ and ${d'_g}^{(1)}(4)$ are equal to $(3^{g-1} + 1)/2$ and $(3^{g-1} - 1)/2$, respectively, and the numbers ${d_g}^{(0)}(4)$ and ${d'_g}^{(0)}(4)$ are obtained by adding 3^{g-1}. These numbers have appeared in Oxbury and Pauly [OP] and Pauly [P].

The case $k \equiv 2$ mod 4.

In this case, the action of $\mathcal{E}$ factors through an action of the group $\mathcal{E}'$ but not through an action of $J^{(2)}$. Indeed, the involutions $\rho_a^{\otimes 2}$ acting on $\mathcal{L}^2$ depend only on α, and satisfy the same relations as the ρ'_α's (see formulas (2) and (3) in section 2). Therefore one has a direct sum decomposition

$$Z_k(\Sigma_g) = \bigoplus_q Z_k(\Sigma_g; q)$$

where q runs through the characters of the group $\mathcal{E}'$ which do not factor through $J^{(2)}$, *i.e.*, such that q takes the value -1 on the central element $-1 \in \mu_2 \subset \mathcal{E}'$. Such characters are in 1-to-1 correspondence with functions $q\colon J^{(2)} \to \mu_2$ such that

$$q(\alpha+\beta) \;=\; q(\alpha)\, q(\beta)\, \lambda_2(\alpha,\beta).$$

In other words, q runs through the set of quadratic forms on $J^{(2)} \cong H^1(\Sigma_g;\mathbb{Z}/2)$ inducing the Weil pairing, *i.e.*, the *mod* 2 intersection form. It is well-known that such quadratic forms correspond to spin structures, or equivalently, θ-characteristics, on Σ. (See Atiyah [At1], Johnson [Jo].)

Let $\mathcal{G} \cong Sp(2g;\mathbb{F}_2)$ be the group of automorphisms of $J^{(2)}$ preserving the order 2 Weil pairing λ_2. The group $\mathcal{G}$ acts on $\mathcal{E}'$. As in the case $k \equiv 0$ mod 4, it follows from theorem 2.3 that the character of the representation $Z_k(\Sigma_g)$ is invariant under the action of $\mathcal{G}$. It is well-known that the induced action of $\mathcal{G}$ on quadratic forms has two orbits which are characterized by the *Arf invariant*, *i.e.*, the action is transitive on forms q with the same Arf invariant $\mathrm{Arf}(q) \in \mathbb{Z}/2$. (The Arf invariant of the quadratic form corresponds to the parity of the θ-characteristic.) This shows that the dimension of the brick $Z_k(\Sigma_g; q)$ depends only on $\mathrm{Arf}(q)$. Put $d_g^{(\varepsilon)}(k) = dim\ Z_k(\Sigma_g; q_\varepsilon)$, where q_ε has Arf invariant $\varepsilon \in \mathbb{Z}/2$. These dimensions can be computed from the following two formulas:

$$d_g(k) = 2^{g-1}(2^g+1)\, d_g^{(0)}(k) + 2^{g-1}(2^g-1)\, d_g^{(1)}(k)$$

$$2^{g-1}(d_g^{(0)}(k) - d_g^{(1)}(k)) = tr(\rho_a^{\otimes k}) = \left(\frac{k+2}{2}\right)^{g-1}, \quad \alpha \neq 0$$

The first formula follows from the fact that the number of quadratic forms with zero Arf invariant is equal to $2^{g-1}(2^g+1)$. The second formula follows from the fact[5] that for $\alpha \neq 0$, one has

$$\sharp\{q \mid q(\alpha) = -1,\ \mathrm{Arf}(q) = 0\} = 2^{2g-2} = \sharp\{q \mid q(\alpha) = -1,\ \mathrm{Arf}(q) = 1\}$$

Similarly, $Z'_k(\Sigma_g)$ is the direct sum of the bricks $Z'_k(\Sigma_g; q)$, and the dimensions $d'^{(\varepsilon)}_g(k) = dim\ Z'_k(\Sigma_g; q_\varepsilon)$ can be computed as before (just replace $d_g(k)$ with $d'_g(k)$

[5] A nice way to think about this is to observe that there is a natural bijection between quadratic forms q with fixed Arf invariant and such that $q(\alpha) = -1$, and quadratic forms with arbitrary Arf invariant on the $2g-2$-dimensional space $\langle\alpha\rangle^\perp/\langle\alpha\rangle$.

and $d_g^{(\varepsilon)}(k)$ with ${d'_g}^{(\varepsilon)}(k)$ in the above, but notice that $tr({\rho'_\alpha}^{\otimes k/2})$ is now equal to $-\left((k+2)/2\right)^{g-1}$.)

Here are explicit formulas for the dimensions. They are equivalent to the formulas on p. 264 of [BM] [6].

$$\begin{aligned} d_g^{(\varepsilon)}(k) &= \frac{1}{2^{2g}}\left(d_g(k) + \left(\frac{k+2}{2}\right)^{g-1}((-1)^\varepsilon 2^g - 1)\right) \\ {d'_g}^{(\varepsilon)}(k) &= \frac{1}{2^{2g}}\left(d'_g(k) + \left(\frac{k+2}{2}\right)^{g-1}(1-(-1)^\varepsilon 2^g)\right) \end{aligned}$$

Example: The case $k=2$. The numbers ${d_g}^{(0)}(2)$ and ${d'_g}^{(1)}(2)$ are equal to 1, and the numbers ${d_g}^{(1)}(2)$ and ${d'_g}^{(0)}(2)$ are zero. Therefore $d_g(2) = 2^{g-1}(2^g+1)$ and $d'_g(2) = 2^{g-1}(2^g-1)$ (see Beauville [Be1]).

Note. In [BM], a $\mathbb{Z}/2$-graded TQFT-functor is constructed on a cobordism category of surfaces equipped with spin structures (and other things). Given a connected surface Σ with a spin structure σ, the even (resp. odd) part of this functor is isomorphic to $Z_k(\Sigma_g; q_\sigma)$ (resp. $Z'_k(\Sigma_g; q_\sigma)$), where q_σ is the quadratic form corresponding to σ.

Remark 4.1. The action of the symplectic group $\mathcal{G} \cong Sp(2g;\mathbb{F}_2)$ permuting the bricks has the following geometric interpretation. Recall that all elements of $\mathcal{G}$ can be represented by diffeomorphisms of Σ. In the [BHMV]-theory, one has, more or less by definition, a natural action of a certain extended mapping class group on $V_{2k+4}(\Sigma)$. On the geometric side, there is also a (projective-linear) action of the mapping class group of Σ on $Z_k(\Sigma)$; this action is constructed using Hitchin's projectively-flat connection [Hi]. It is, of course, expected that $V_{2k+4}(\Sigma)\otimes\mathbb{C}$ and $Z_k(\Sigma)$ are isomorphic as representations of the extended mapping class group. In any case, it is easy to see in both theories that the action of a diffeomorphism f takes the brick associated to a character h (resp. a quadratic form q) to the brick associated to $f^*(h)$ (resp. $f^*(q)$). On the geometric side, the main reason for this is that Hitchin's connection is (projectively) invariant under the actions of both the mapping class group and the group $\mathcal{E}$.

The odd-level case.

Proposition 4.2. *If $k \equiv 1$ mod 2, $Z_k(\Sigma)$ is isomorphic, as representation of the group $\mathcal{E}$, to a direct sum of copies of $Z_1(\Sigma)$, if $k\equiv 1$ mod 4, and to a direct sum of copies of the conjugate representation, $\overline{Z_1(\Sigma)}$, if $k \equiv 3$ mod 4.*

[6] Warning: Note that k does not denote the level in [BM]; one has $p=8k$ in [BM] while in the present paper $p=2k+4$.

Proof. Note that the character of the representation $Z_1(\Sigma)$ is the function $\chi\colon \mathcal{E} \to \mathbb{C}$ which is zero on all group elements not in the central subgroup $\mu_4 \subset \mathcal{E}$, while the trace of a central element $\lambda \in \mu_4$ is $\chi(\lambda) = 2^g \lambda$. The character of the conjugate representation, $\overline{Z_1(\Sigma)}$, is of course the conjugate character $\overline{\chi}$. Now it follows from theorems 2.1 and 2.3 that the character of $Z_k(\Sigma)$ is a multiple of χ if $k \equiv 1 \bmod 4$, and a multiple of $\overline{\chi}$, if $k \equiv 3 \bmod 4$. It is however well-known that χ is an irreducible character, and this proves the proposition.

Note. This result corresponds, *via* the isomorphism $Z_k(\Sigma) \cong V_{2k+4}(\Sigma) \otimes \mathbb{C}$, to theorems 1.5 and 1.6 of [BHMV]. It is shown there that (for odd k)

$$V_{2k+4}(\Sigma) \cong V_2'(\Sigma) \otimes V_{k+2}(\Sigma)$$

as representations of the group $\mathcal{E}$, where $V_2'(\Sigma)$ and $V_{k+2}(\Sigma)$ are defined in [BHMV]. Moreover, the group $\mathcal{E}$ acts trivially on $V_{k+2}(\Sigma)$, and $V_2'(\Sigma_g)$ is isomorphic to $V_6(\Sigma)$, and hence to $Z_1(\Sigma)$ or to $\overline{Z_1(\Sigma)}$, after a change of coefficients. It would be interesting to have an algebro-geometric interpretation of the space $V_{k+2}(\Sigma)$, which in the [BHMV]-theory can be interpreted as a $SO(3)$-TQFT vector space. Here, the name $SO(3)$-TQFT just means that the allowed colors in this TQFT are even, or in other words, are $SU(2)$-representations which lift to $SO(3)$.

5. The fixed point varieties.

We need to analyze the action of $J^{(2)}$ on M and M' in order to obtain information about our lifts. In this section, we describe the various fixed point varieties, mainly following Narasimhan and Ramanan [NR]. We also discuss when two lifts of $\alpha \in J^{(2)}$ to $J^{(4)}$ determine the same component of the fixed point set $|M|_\alpha$.

Notation. Throughout this paper, we denote by L_α (resp. L_a) the line bundle on Σ corresponding to $\alpha \in J^{(2)}$ (resp. $a \in J^{(4)}$).

Let $\alpha \in J^{(2)}$ be nonzero. Let $\pi_\alpha : \Sigma^\alpha \to \Sigma$ be the 2-sheeted unramified covering of Σ corresponding to α, and let ϕ_α be the covering transformation of π_α. Using the line bundle L_α, we can explicitly construct Σ^α as

$$\Sigma^\alpha = \{\xi \in L_\alpha | \xi \otimes \xi = 1\}$$

using an isomorphism $L_\alpha^2 \cong \mathcal{O}_\Sigma$. The involution ϕ_α is then induced by multiplication by -1 on L_α.

Given a line bundle L over Σ^α, the push-down $E = \pi_{\alpha *}(L)$ can be obtained by descending $L \oplus \phi_\alpha^*(L)$ (which is naturally an equivariant bundle) to Σ:

$$E = (L \oplus \phi_\alpha^*(L))/\langle \phi_\alpha \rangle.$$

Here, the natural involution of $L \oplus \phi_\alpha^*(L)$ covering ϕ_α is again denoted by ϕ_α. The fundamental observation is that $E \otimes L_\alpha$ is isomorphic to E. This follows formally from the pull-push formula $\pi_{\alpha *}(L)\otimes L' \cong \pi_{\alpha *}(L\otimes \pi_\alpha^* L')$ and the fact that L_α pulls back to the trivial bundle on Σ^α.

Remark 5.1. We will later need the following explicit isomorphism from $E=\pi_{\alpha *}(L)$ to $E\otimes L_\alpha$. Note that $\pi_\alpha^* L_\alpha$, as an equivariant bundle, is isomorphic to $\mathcal{O}^-_{\Sigma^\alpha}$, that is, the trivial line bundle $\Sigma^\alpha\times\mathbb{C}$, but with non-trivial action, given by $(x,z)\mapsto(\phi_\alpha(x),-z)$. Therefore

$$E\otimes L_\alpha = (L\oplus\phi_\alpha^*(L))/\langle -\phi_\alpha\rangle.$$

It follows that the diagonal automorphism $1\oplus(-1)$ of $L\oplus\phi_\alpha^*(L)$ descends to an isomorphism from E to $E\otimes L_\alpha$.

Let $\mathrm{Nm}_\alpha\colon\ \mathrm{Pic}(\Sigma^\alpha)\to\mathrm{Pic}(\Sigma)$ be the classical albanese or norm map induced by the following map on divisors. If $D=\sum n_j x_j\in\mathrm{Div}(\Sigma^\alpha)$ then $\mathrm{Nm}_\alpha(D)=\sum n_j\pi_\alpha(x_j)\in\mathrm{Div}(\Sigma)$.

The *Prym variety* P_α associated to α is by definition the connected component of $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$ containing $\mathcal{O}_{\Sigma^\alpha}$. It is a principally polarized abelian variety of dimension $g-1$. The quotient variety $P_\alpha/\langle\pm 1\rangle$ is called the *Kummer variety.*

We define $\theta_\alpha:\mathrm{Pic}(\Sigma^\alpha)\to\mathrm{Pic}(\Sigma)$ by

$$\theta_\alpha(L)=\det(\pi_{\alpha *}(L))=\mathrm{Nm}_\alpha(L)\otimes L_\alpha$$

(see [NR] for the second equality). Note that ϕ_α acts on $\mathrm{Pic}(\Sigma^\alpha)$ by sending L to $\phi_\alpha^*(L)$.

Proposition 5.2 (Narasimhan and Ramanan [NR])**.** *(i) The map $L\mapsto(\pi_\alpha)_*(L)$ induces isomorphisms*

$$\theta_\alpha^{-1}(\mathcal{O}_\Sigma)/\langle\phi_\alpha\rangle \xrightarrow{\sim} |M|_\alpha \tag{5}$$

$$\theta_\alpha^{-1}([p])/\langle\phi_\alpha\rangle \xrightarrow{\sim} |M'|_\alpha \tag{6}$$

(ii) Moreover, $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)/\langle\phi_\alpha\rangle$ is isomorphic to two copies of $P_\alpha/\langle\pm1\rangle$, while $\theta_\alpha^{-1}([p])/\langle\phi_\alpha\rangle$ is isomorphic to P_α.

Remark 5.3. In the twisted case, $\pi_{\alpha *}\colon\theta_\alpha^{-1}([p])\to|M'|_\alpha$ is a double covering, with covering transformation ϕ_α. In the untwisted case, the same holds on the open subvariety of $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$ where ϕ_α acts freely, while the fixed points of ϕ_α are sent by the map $\pi_{\alpha *}$ bijectively to the points in $|M|_\alpha$ represented by semi-stable but not stable bundles. (See remark 5.6 below.)

For later use, we need the following more explicit description of $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$ and $\theta_\alpha^{-1}([p])$. For $d=0,1$, define $\Phi_\alpha^d:\mathrm{Pic}_d(\Sigma^\alpha)\to\mathrm{Pic}_0(\Sigma^\alpha)=J(\Sigma^\alpha)$ by

$$\Phi_\alpha^d(L)=L\otimes\phi_\alpha^* L^{-1}.$$

The following two properties (7) and (8) of the maps Φ_α^d are elementary facts from the classical theory of line bundles on curves, see *e.g.* App. B. in [ACGH].

First, one has the disjoint union

$$\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma) = \mathrm{Im}\,\Phi_\alpha^0 \cup \mathrm{Im}\,\Phi_\alpha^1 \tag{7}$$

and the Prym variety P_α is equal to the component $\mathrm{Im}\,\Phi_\alpha^0$.

Second, note that $\mathrm{Nm}_\alpha(\pi_\alpha^*(L_\beta)) = L_\beta^2 = \mathcal{O}_\Sigma$ for all $\beta \in J^{(2)}$. Moreover, one has

$$\pi_\alpha^* L_\beta \in \mathrm{Im}\,\Phi_\alpha^d \ \Leftrightarrow\ \lambda_2(\alpha,\beta) = (-1)^d. \tag{8}$$

Here, $\lambda_2 : J^{(2)} \times J^{(2)} \to \mu_2$ is the order 2 Weil pairing.

Now pick $a \in J^{(4)}$ such that $2a = \alpha$, and $\beta \in J^{(2)}$ such that $\lambda_2(\alpha,\beta) = -1$. Note that $a' = a + \beta$ is another element of $J^{(4)}$ such that $2a' = \alpha$. Also, pick a point $p_\alpha \in \pi_\alpha^{-1}(p) \subset \Sigma^\alpha$. Note that $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$ and $\theta_\alpha^{-1}([p])$ are both isomorphic to $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$. From (7) and (8), we have the following description of $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$ and $\theta_\alpha^{-1}([p])$ as disjoint unions:

$$\theta_\alpha^{-1}(\mathcal{O}_\Sigma) = \pi_\alpha^* L_a \otimes \mathrm{Im}\,\Phi_\alpha^0 \ \cup\ \pi_\alpha^* L_{a'} \otimes \mathrm{Im}\,\Phi_\alpha^0 \tag{9}$$

$$\theta_\alpha^{-1}(\mathcal{O}_\Sigma) = \pi_\alpha^* L_a \otimes \mathrm{Im}\,\Phi_\alpha^0 \ \cup\ \pi_\alpha^* L_a \otimes \mathrm{Im}\,\Phi_\alpha^1 \tag{10}$$

$$\theta_\alpha^{-1}([p]) = \pi_\alpha^* L_a \otimes [p_\alpha] \otimes \mathrm{Im}\,\Phi_\alpha^0 \ \cup\ \pi_\alpha^* L_a \otimes [\phi_\alpha(p_\alpha)] \otimes \mathrm{Im}\,\Phi_\alpha^0 \tag{11}$$

Remark 5.4. It follows from (10) and (11) that the action of ϕ_α preserves the two components of $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$, while it exchanges the two components of $\theta_\alpha^{-1}([p])$. By (7), the action of ϕ_α on $\mathrm{Pic}(\Sigma^\alpha)$ restricts to multiplication by -1 (that is, the map $L \mapsto L^{-1}$) on $\mathrm{Nm}_\alpha^{-1}(\mathcal{O}_\Sigma)$. This proves 5.2(ii).

We can now describe when two lifts of $\alpha \in J^{(2)}$ to $J^{(4)}$ determine the same component of the fixed point set $|M|_\alpha$. Recall that $|M|_a^+$ was defined to be the component of $|M|_\alpha$ containing the S-equivalence class of the semi-stable bundle $L_a \oplus L_a^{-1}$.

Proposition 5.5. *(i) Let $a_1, a_2 \in J^{(4)}$ such that $2a_1 = 2a_2 = \alpha$. Then*

$$|M|_{a_1}^+ = |M|_{a_2}^+ \ \Leftrightarrow\ \lambda_2(a_1 - a_2, \alpha) = 1 \in \mu_2.$$

(ii) For $\beta \in J^{(2)}$, the action of β on $|M|_\alpha$ interchanges the two components of $|M|_\alpha$ if and only if $\lambda_2(\alpha,\beta) = -1$.

Proof. Note that $\pi_{\alpha *}(\pi_\alpha^*(L_{a_i})) \cong L_{a_i} \oplus L_{a_i} \otimes L_\alpha \cong L_{a_i} \oplus L_{a_i}^{-1}$. Therefore $|M|_{a_i}^+$ is the component $\pi_{\alpha *}(\pi_\alpha^* L_{a_i} \otimes \mathrm{Im}\,\Phi_\alpha^0)$, and (i) follows from formula (9). Now part (ii) follows from (8), since the action of β on $|M|_\alpha$ lifts to tensoring with $\pi_\alpha^* L_\beta$ on $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$. This completes the proof.

Note. Translating the 'multiplicative' notation of the Weil pairing into the 'additive' notation in section 3, the condition $\lambda_2(a_1 - a_2, \alpha) = 1 \in \mu_2$ becomes the condition $((a_1 - a_2)/2) \cdot \alpha = 0 \in \mathbb{Z}/2$. Thus, prop. 5.5(i) implies that $|M|_{a_1}^+ = |M|_{a_2}^+$ if and only if $[a_1] = [a_2]$, where $[a_i]$ is the lift of α to the group $\Gamma(\Sigma)$ defined in section 3.

Remark 5.6. Let M^{sing} denote the set of points of M represented by semistable, but not stable, bundles. A semi-stable bundle E with $\mathrm{Gr}(E) \cong L_1 \oplus L_2$ represents a point in M^{sing} if and only if $L_2 \cong L_1^{-1}$, and this point lies in $|M|_\alpha$ if and only if $L_1 \otimes L_\alpha \cong L_2$. This shows that $M^{sing} \cap |M|_\alpha$ is precisely the set of points represented by bundles of the form $L_a \oplus L_a^{-1}$ with $a \in J^{(4)}$ and $2a = \alpha$. Since $L_a \oplus L_a^{-1} \cong \pi_{\alpha *}(\pi_\alpha^*(L_a))$, and $\pi_\alpha^*(\{L \in \mathrm{Pic}_0(\Sigma) | L^2 \cong L_\alpha\})$ is precisely the fixed point set of ϕ_α on $\theta_\alpha^{-1}(\mathcal{O}_\Sigma)$, we see that $\pi_{\alpha *}$ induces a bijection from that fixed point set to $M^{sing} \cap |M|_\alpha$, as asserted above.

6. Intersections and triple intersections.

In this section, we describe how the various fixed point varieties intersect. The triple intersections will be used in the proof of theorem 2.1.

Note. The intersection properties of the fixed point sets in relation to the order 2 Weil pairing are well-known; they are used for example in van Geemen and Previato [vGP1]. On the other hand, the relationship of the triple intersections of their individual *components* (in the untwisted case) with the order 4 Weil pairing seems to be new.

Proposition 6.1. *For $\alpha, \beta \in J^{(2)}$ non-zero distinct elements, we have that*

$$|M|_\alpha \cap |M|_\beta \neq \emptyset \;\Leftrightarrow\; \lambda_2(\alpha, \beta) = 1 \in \mu_2$$

$$|M'|_\alpha \cap |M'|_\beta \neq \emptyset \;\Leftrightarrow\; \lambda_2(\alpha, \beta) = -1 \in \mu_2$$

Proof. The morphism $\pi_{\alpha *} \colon \mathrm{Pic}_d(\Sigma^\alpha) \to |M_d|_\alpha$ is surjective and $J^{(2)}$-equivariant. (Here, $\beta \in J^{(2)}$ acts on $\mathrm{Pic}_d(\Sigma^\alpha)$ by $L \mapsto L \otimes \pi_\alpha^* L_\beta$.) Hence we get the following description of the intersection

$$|M_d|_\alpha \cap |M_d|_\beta = \pi_{\alpha *}(\{L \in \mathrm{Pic}_d(\Sigma^\alpha) | L \otimes \pi_\alpha^* L_\beta \cong \phi_\alpha^* L \text{ or } L\}).$$

Hence we see that

$$|M_d|_\alpha \cap |M_d|_\beta \neq \emptyset$$

if and only if

$$\pi_\alpha^* L_\beta \in \mathrm{Im}\, \Phi_\alpha^d.$$

From (8), this is the case if and only if $\lambda_2(\alpha, \beta) = (-1)^d$. Using the action of $J(\Sigma)$ on $|M_d|_\alpha \cap |M_d|_\beta$, the results for $|M|_\alpha \cap |M|_\beta$ and $|M'|_\alpha \cap |M'|_\beta$ follow from this.

Proposition 6.2. *If $\lambda_2(\alpha, \beta) = 1$, the quotient group $J^{(2)}/\langle \alpha, \beta \rangle$ acts simply transitively on $|M|_\alpha \cap |M|_\beta$. In particular, this intersection has 2^{2g-2} elements. If $\lambda_2(\alpha, \beta) = -1$, the same holds for $|M'|_\alpha \cap |M'|_\beta$.*

Proof. Put $I_{\alpha,\beta} = \{L \in \theta_\alpha^{-1}(\mathcal{O}_\Sigma) \mid \pi_\alpha^*(L_\beta) \cong L \otimes \phi_\alpha^* L^{-1}\}$. Then $\pi_{\alpha *}\colon I_{\alpha,\beta} \to |M|_\alpha \cap |M|_\beta$ is a double covering. Note that $J^{(2)}/\langle\alpha\rangle$ acts simply transitively on $I_{\alpha,\beta}$. The involution ϕ_α is on $I_{\alpha,\beta}$ the same as tensoring with $\pi_\alpha^* L_\beta$, in other words, the action of β. This proves the result in the untwisted case. The twisted case is proved similarly.

Note. In view of remark 5.6, this description shows that $|M|_\alpha \cap |M|_\beta$ is contained in the stable part of M.

Proposition 6.3. *$|M|_\alpha \cap |M|_\beta$ is the disjoint union of the sets $|M|_a^\varepsilon \cap |M|_b^\mu$, where $\varepsilon = \pm$ and $\mu = \pm$, each of which sets has 2^{2g-4} elements.*

Proof. Recall from proposition 5.5(ii) that the action of $\gamma \in J^{(2)}$ exchanges the components of $|M|_\alpha$ if and only if $\lambda_2(\alpha,\gamma) = -1$. Thus, the result follows by exploiting the fact that for every choice of signs $\varepsilon = \pm 1$ and $\mu = \pm 1$, there exists γ such that $\lambda_2(\alpha,\gamma) = \varepsilon$ and $\lambda_2(\beta,\gamma) = \mu$.

We now turn to the triple intersections. The first observation is the following easy lemma.

Lemma 6.4. *Let α, β, γ be distinct non-zero elements of $J^{(2)}$ such that the triple intersection $|M|_\alpha \cap |M|_\beta \cap |M|_\gamma$ is non-empty. Then $\gamma = \alpha + \beta$. The same holds for the triple intersections in the twisted case.*

Proof. Indeed, it follows from the description in prop. 6.2 that the triple intersection can only be non-empty if $\pi_\alpha^*(L_\beta) = \pi_\alpha^*(L_\gamma)$ which implies $\gamma = \alpha + \beta$.

Note. Since the group $J^{(2)}$ is commutative, we have

$$|M|_\alpha \cap |M|_\beta = |M|_\alpha \cap |M|_\beta \cap |M|_{\alpha+\beta},$$

and similarly in the twisted case.

In the untwisted case, the fixed point sets have two components each, and we may ask about the triple intersections of the individual components. Our answer was already stated in theorem 2.2. We will prove it in the following equivalent form.

Theorem 6.5. *Assume that α and β are distinct non-zero elements of $J^{(2)}$ such that $\lambda_2(\alpha,\beta) = 1 \in \mu_2$. Let $a, b \in J^{(4)}$ such that $2a = \alpha$ and $2b = \beta$. Let $\varepsilon, \mu, \nu = \pm 1$ be three signs. Then*

$$|M|_a^\varepsilon \cap |M|_b^\mu \cap |M|_{a+b}^\nu \neq \emptyset \;\Leftrightarrow\; \lambda_4(a,b) = \varepsilon\mu\nu$$

Note. It follows that if $|M|_a^\varepsilon \cap |M|_b^\mu \cap |M|_{a+b}^\nu$ is non-empty, then it is equal to all of $|M|_a^\varepsilon \cap |M|_b^\mu$. This fact can of course be seen directly using prop. 6.3. The important information in the theorem is that it tells us when $|M|_a^\varepsilon \cap |M|_b^\mu$ intersects $|M|_{a+b}^+$ and when $|M|_{a+b}^-$.

Proof. To simplify notation, we put $\gamma = \alpha + \beta$ and $c = a + b$. Let E represent a point in $|M|_\alpha \cap |M|_\beta \cap |M|_\gamma$. The description of $|M|_\alpha$ in section 5 tells us that there is a unique $\mathcal{L}_a \in \pi_\alpha^*(L_a) \otimes \operatorname{Im} \Phi_\alpha^d \subset \operatorname{Pic}_0(\Sigma^\alpha)$ such that $E \cong \pi_{\alpha*}\mathcal{L}_a$; moreover E lies in $|M|_a^+$ if and only d is even. (See formula (10).) Similarly we have $E \cong \pi_{\beta*}\mathcal{L}_b \cong \pi_{\gamma*}\mathcal{L}_c$, where $\mathcal{L}_b \in \pi_\beta^*(L_b) \otimes \operatorname{Im} \Phi_\beta^{d'} \subset \operatorname{Pic}_0(\Sigma^\beta)$ and $\mathcal{L}_c \in \pi_\gamma^*(L_c) \otimes \operatorname{Im} \Phi_\gamma^{d''} \subset \operatorname{Pic}_0(\Sigma^\gamma)$. Thus, the theorem is equivalent to the following lemma.

Lemma 6.6. *One has* $\lambda_4(a, b) = (-1)^{d+d'+d''}$.

The proof of this lemma will occupy the remainder of this section.

There is a curve $\widetilde{\Sigma}$ naturally double covering Σ^α, Σ^β and Σ^γ:

$$\widetilde{\Sigma} = \left\{ (\xi, \eta) \in L_\alpha \oplus L_\beta \,|\, \xi^2 = 1 = \eta^2 \right\}.$$

The projections onto the two factors induce projections $\pi^\alpha \colon \widetilde{\Sigma} \to \Sigma^\alpha$ and $\pi^\beta \colon \widetilde{\Sigma} \to \Sigma^\beta$. The bilinear map $L_\alpha \oplus L_\beta \to L_\alpha \otimes L_\beta$ induces the projection $\pi^\gamma \colon \widetilde{\Sigma} \to \Sigma^\gamma$.

$$\begin{tikzcd}
 & \widetilde{\Sigma} \arrow[dl, "\pi^\alpha"'] \arrow[d, "\pi^\beta"] \arrow[dr, "\pi^\gamma"] & \\
\Sigma^\alpha \arrow[dr, "\pi_\alpha"'] & \Sigma^\beta \arrow[d, "\pi_\beta"] & \Sigma^\gamma \arrow[dl, "\pi_\gamma"] \\
 & \Sigma &
\end{tikzcd}$$

The deck-transformations of the coverings $\pi^\alpha, \pi^\beta, \pi^\gamma$ will be denoted respectively by $\phi^\alpha, \phi^\beta, \phi^\gamma$. Note that ϕ^α (resp. ϕ^β) is induced by multiplication by -1 in the fibers of L_β (resp. L_α), and that $\phi^\gamma = \phi^\alpha \circ \phi^\beta$. We denote the projection $\widetilde{\Sigma} \to \Sigma$ by $\tilde{\pi}$, so that

$$\tilde{\pi} = \pi_\alpha \circ \pi^\alpha = \pi_\beta \circ \pi^\beta = \pi_\gamma \circ \pi^\gamma.$$

Notice also that the involution ϕ_α of Σ^α is covered by both ϕ^β and ϕ^γ (but of course not by ϕ^α, since $\Sigma^\alpha = \widetilde{\Sigma}/\langle \phi^\alpha \rangle$). Similar comments apply to ϕ_β and ϕ_γ.

Lemma 6.7. *One has* $\pi^{\alpha*}(\mathcal{L}_a) \cong \pi^{\beta*}(\mathcal{L}_b) \cong \pi^{\gamma*}(\mathcal{L}_c)$.

Proof. Since $E \cong \pi_{\alpha*}(\mathcal{L}_a)$ lies in $|M|_\alpha \cap |M|_\beta$, we have $\phi_\alpha^*(\mathcal{L}_a) \cong \mathcal{L}_a \otimes \pi_\alpha^*(L_\beta)$ (see the proof of prop. 6.2). Since $\pi^{\alpha*}\pi_\alpha^*(L_\beta) = \tilde{\pi}^*(L_\beta)$ is trivial, it follows that

$$\tilde{\pi}^*(E) \cong \tilde{\pi}^*\pi_{\alpha*}(\mathcal{L}_a) \cong \pi^{\alpha*}(\mathcal{L}_a \oplus \phi_\alpha^*(\mathcal{L}_a)) \cong \pi^{\alpha*}(\mathcal{L}_a) \oplus \pi^{\alpha*}(\mathcal{L}_a).$$

Similarly $\tilde{\pi}^*(E) \cong \pi^{\beta*}(\mathcal{L}_b) \oplus \pi^{\beta*}(\mathcal{L}_b) \cong \pi^{\gamma*}(\mathcal{L}_c) \oplus \pi^{\gamma*}(\mathcal{L}_c)$. Since line bundles are simple, the lemma follows.

We now turn to the computation of the Weil pairing $\lambda_4(a, b)$. Represent $a, b \in J^{(4)}$ by divisors $D_a, D_b \in \operatorname{Div}_0(\Sigma)$ with disjoint support, and put $D_c = D_a + D_b$. Pick $D \in Div_d(\Sigma^\alpha)$ (resp. $D' \in Div_{d'}(\Sigma^\beta)$, resp. $D'' \in Div_{d''}(\Sigma^\gamma)$) such that

$\pi_\alpha^*(D_a) + (1-\phi_\alpha^*)(D)$ (resp. $\pi_\beta^*(D_b) + (1-\phi_\beta^*)(D')$, resp. $\pi_\gamma^*(D_c) + (1-\phi_\gamma^*)(D''))$ represents $\mathcal{L}_a$ (resp. $\mathcal{L}_b$, resp. $\mathcal{L}_c$). Pulling everything up to $\widetilde{\Sigma}$, we get divisors

$$F_a = \tilde{\pi}^*(D_a) + \pi^{\alpha *}(1-\phi_\alpha^*)(D)$$

$$F_b = \tilde{\pi}^*(D_b) + \pi^{\beta *}(1-\phi_\beta^*)(D')$$

$$F_c = \tilde{\pi}^*(D_c) + \pi^{\gamma *}(1-\phi_\gamma^*)(D'')$$

such that F_a represents $\pi^{\alpha *}(\mathcal{L}_a)$, F_b represents $\pi^{\beta *}(\mathcal{L}_b)$, and F_c represents $\pi^{\gamma *}(\mathcal{L}_c)$. Since these three bundles are isomorphic by lemma 6.7, there exist meromorphic functions $h_1, h_2 \in \mathcal{M}(\widetilde{\Sigma})$ such that

$$(h_2) + F_a \;=\; F_c \;=\; (h_1) + F_b.$$

Let $\mathrm{Nm}^\alpha \colon \mathcal{M}(\widetilde{\Sigma}) \to \mathcal{M}(\Sigma^\alpha)$ be the norm map on meromorphic functions associated to the covering π^α. The norm maps associated to the various other coverings will similarly be denoted by $\mathrm{Nm}^\beta, \mathrm{Nm}^\gamma, \mathrm{Nm}_\alpha, \mathrm{Nm}_\beta, \mathrm{Nm}_\gamma$, and $\widetilde{\mathrm{Nm}}$.

Lemma 6.8. *(i) Define $f, g \in \mathcal{M}(\Sigma)$ by $f = \widetilde{\mathrm{Nm}}(h_1)$, $g = \widetilde{\mathrm{Nm}}(h_2)$. Then*

$$(f) = 4\,D_a, \quad (g) = 4\,D_b.$$

(ii) Define $f_\alpha = \mathrm{Nm}^\alpha(h_1) \in \mathcal{M}(\Sigma^\alpha)$, $f_\beta = \mathrm{Nm}^\beta(h_2) \in \mathcal{M}(\Sigma^\beta)$, $f_\gamma = \mathrm{Nm}^\gamma(h_1/h_2) \in \mathcal{M}(\Sigma^\gamma)$. Then

$$f_\alpha \circ \phi_\alpha = -f_\alpha, \;\; f_\beta \circ \phi_\beta = -f_\beta, \quad f_\gamma \circ \phi_\gamma = -f_\gamma.$$

Proof. Using that ϕ^α covers ϕ_β and ϕ_γ, one computes that

$$\pi^{\alpha *}((f_\alpha)) = (h_1) + \phi^{\alpha *}(h_1) = 2\,\tilde{\pi}^*(D_a).$$

It follows that $(f_\alpha) = 2\pi_\alpha^*(D_a)$ and hence $(f) = (\mathrm{Nm}_\alpha(f_\alpha)) = 4D_a$, as asserted. This also shows that the divisor (f_α) is ϕ_α-invariant. Therefore one has $f_\alpha \circ \phi_\alpha = \pm f_\alpha$. But f_α itself cannot be ϕ_α-invariant, because if it were, it would descend to a function $h \in \mathcal{M}(\Sigma)$ such that $(h) = 2D_a$, which is impossible since a has order 4. Therefore $f_\alpha \circ \phi_\alpha = -f_\alpha$. The other assertions of the lemma are proved similarly.

By lemma 6.8(i), we can compute the Weil pairing $\lambda_4(a,b)$ using the functions f and g (see the definition in section 1). Note that by lemma 6.8(ii), we have that

$$h_1(\pi^{\alpha *}(1-\phi_\alpha^*)(D)) = f_\alpha((1-\phi_\alpha^*)(D)) = \frac{f_\alpha(D)}{(f_\alpha \circ \phi_\alpha)(D)} = (-1)^{\deg D}.$$

Thus

$$\begin{aligned}
\lambda_4(a,b) &= \frac{g(D_a)}{f(D_b)} = \frac{f(-D_b)}{g(-D_a)} = \frac{h_1(-\tilde{\pi}^*(D_b))}{h_2(-\tilde{\pi}^*(D_a))} \\
&= \frac{h_1(-\tilde{\pi}^*(D_b) + (h_2))}{h_2(-\tilde{\pi}^*(D_a) + (h_1))} \\
&= \frac{h_1(\pi^{\gamma *}(1-\phi_\gamma^*)(D'') - \pi^{\alpha *}(1-\phi_\alpha^*)(D))}{h_2(\pi^{\gamma *}(1-\phi_\gamma^*)(D'') - \pi^{\beta *}(1-\phi_\beta^*)(D'))} \\
&= f_\gamma((1-\phi_\gamma^*)(D''))\frac{f_\beta((1-\phi_\beta^*)(D'))}{f_\alpha((1-\phi_\alpha^*)(D))} \\
&= (-1)^{\deg(D)+\deg(D')+\deg(D'')} = (-1)^{d+d'+d''}
\end{aligned}$$

where we have used Weil reciprocity in the fourth equality. This proves lemma 6.6 and hence theorem 6.5.

7. The action of $J^{(2)}$ on the Hecke correspondence.

We will make use of the Hecke correspondence in our analysis of the involutions in section 8. This is a pair of morphisms

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "q"'] \arrow[dr, "q'"] & \\ M & & M' \end{tikzcd}$$

which allows one to 'transfer' information from M to M'. In this section, we describe the fixed point varieties $|\mathcal{P}|_\alpha$ of the action of the various $\alpha \in J^{(2)}$ on $\mathcal{P}$.

Notation. Given a bundle E over Σ, we denote by E_x the fiber of E at the point $x \in \Sigma$. Also, for a bundle E representing a point in the moduli spaces M or M', we use the notation $[E]$ for that point.

We briefly review the construction of $\mathcal{P}$. (See *e.g.* Bertram and Szenes [BSz].) Let $\mathcal{U}$ be a Poincaré bundle over $\Sigma \times M'$. Thus, if $[E'] \in M'$, the restriction of $\mathcal{U}$ to $\Sigma \times \{[E']\}$ is isomorphic to E'. We can uniquely fix $\mathcal{U}$ by requiring that $\det(\mathcal{U}|_{\{p\}\times M'})$ is an ample generator of $\mathrm{Pic}(M')$.

We put $\mathcal{P} = \mathbb{P}(\mathcal{U}|_{\{p\}\times M'})$ and let $q' : \mathcal{P} \to M'$ be the projection. Note that q' is a $\mathbb{P}^1$-fibration and for $[E'] \in M'$, the fiber $(q')^{-1}([E'])$ is isomorphic to the projective space $\mathbb{P}(E'_p)$. In fact, $\mathcal{P}$ can be viewed as the moduli space of pairs $(E', \mathcal{F})$ where E' is a stable rank 2 bundle with $\det(E') = [p]$, and $\mathcal{F} \subset E'_p$ is a one-dimensional subspace, *i.e.*, $\mathcal{P}$ is a moduli space of semi-stable parabolic bundles. We will refer to $\mathcal{F}$ as a *line* in E'_p. Points in $\mathcal{P}$ will be denoted as $[(E',\mathcal{F})]$, and we have $q'([(E',\mathcal{F})]) = [E']$.

The map q is obtained by the operation of elementary modification at p. This means that we have $q([(E',\mathcal{F})]) = [E]$ if and only if there is a short exact sequence (of sheaves)

$$0 \to E \to E' \xrightarrow{\lambda} \mathbb{C}_p \to 0$$

such that $\ker_p(\lambda) = \mathcal{F} \subset E'_p$. Here, $\mathbb{C}_p$ is the skyscraper sheaf at p.

The group $J^{(2)}$ acts naturally on $\mathcal{P}$. The action of $\alpha \in J^{(2)}$ on $\mathcal{P}$ sends $[(E',\mathcal{F})]$ to $[(E' \otimes L_\alpha, \mathcal{F} \otimes L_\alpha)]$. The morphisms q and q' are $J^{(2)}$-equivariant.

Let $\alpha \in J^{(2)}$ be non-zero. We now describe the fixed point variety $|\mathcal{P}|_\alpha$. Recall that $\pi_{\alpha *}\colon \theta_\alpha^{-1}([p]) \to |M'|_\alpha$ is a double covering, where $\theta_\alpha^{-1}([p]) \subset \mathrm{Pic}_1(\Sigma^\alpha)$ consists of two 'translates' of the Prym variety P_α. Let $p_\alpha \in \Sigma^\alpha$ be such that $\pi_\alpha(p_\alpha) = p$. If $L \in \mathrm{Pic}_1(\Sigma^\alpha)$, the projection gives a canonical isomorphism

$$(L \oplus \phi_\alpha^*(L))_{p_\alpha} \xrightarrow{\sim} (\pi_{\alpha *}(L))_p.$$

Proposition 7.1. *We have an isomorphism*

$$j_{p_\alpha}\colon \theta_\alpha^{-1}([p]) \xrightarrow{\sim} |\mathcal{P}|_\alpha$$

defined by $j_{p_\alpha}(L) = [(\pi_{\alpha *}(L), (L \oplus 0)_{p_\alpha})]$.

Proof. Put $E' = \pi_{\alpha *}(L)$. It is clear that $|\mathcal{P}|_\alpha \subset (q')^{-1}(|M'|_\alpha)$. Therefore the only question is which lines in E'_p correspond to fixed points of α acting on $\mathbb{P}(E'_p) \cong (q')^{-1}([E'])$. Let $\psi\colon E' \xrightarrow{\sim} E' \otimes L_\alpha$ be the isomorphism described in remark 5.1. It is covered by the diagonal automorphism $\tilde{\psi} = 1 \oplus (-1)$ of $L \oplus \phi_\alpha^*(L)$. Since E' is stable, it is simple, hence any other isomorphism is a non-zero multiple of ψ. Therefore $(E',\mathcal{F})$ represents a point in $|\mathcal{P}|_\alpha$ if and only if $\psi(\mathcal{F}) = \mathcal{F} \otimes L_\alpha$. Letting $\widetilde{\mathcal{F}}$ denote the line in $(L \oplus \phi_\alpha^*(L))_{p_\alpha}$ projecting down to $\mathcal{F} \subset E'_p$, this condition is equivalent to $\tilde{\psi}(\widetilde{\mathcal{F}}) = \widetilde{\mathcal{F}}$. The only lines in $(L \oplus \phi_\alpha^*(L))_{p_\alpha}$ that $\tilde{\psi}$ preserves are $(L \oplus 0)_{p_\alpha}$ and $(0 \oplus \phi_\alpha^*(L))_{p_\alpha}$. The first line defines the point $j_{p_\alpha}(L)$ in $\mathcal{P}$, and the second line defines the point $j_{p_\alpha}(\phi_\alpha^*(L))$. This shows that j_{p_α} is bijective. It is clearly an algebraic morphism, and since its domain is smooth, this shows j_{p_α} is an isomorphism.

Remark 7.2. One has $q' \circ j_{p_\alpha} = \pi_{\alpha *}$. In other words, j_{p_α} is an isomorphism of coverings over the identity of $|M'|_\alpha$. Note also that $j_{\phi_\alpha(p_\alpha)} = j_{p_\alpha} \circ \phi_\alpha^*$.

Notation. Given $a \in J^{(4)}$ such that $2a = \alpha$, we denote by $|\mathcal{P}|_a^+$ the component of $|\mathcal{P}|_\alpha$ containing the point $j_{p_\alpha}(\pi_\alpha^*(L_a) \otimes [p_\alpha])$. (See formula (11).) Note that this point, and hence the definition of the component $|\mathcal{P}|_a^+$, depends only on a, not on the choice of p_α.

Proposition 7.3. *One has* $q(|\mathcal{P}|_a^+) = |M|_a^-$.

Proof. Put $L = \pi_\alpha^*(L_a)$ and $L' = L \otimes [p_\alpha]$. Then $E = \pi_{\alpha *}(L)$ represents a point in $|M|_a^+$, and $E' = \pi_{\alpha *}(L')$ represents a point in $|M'|_\alpha$. The short exact sequence of sheaves

$$0 \to L \to L' \to \mathbb{C}_{p_\alpha} \to 0$$

induces the short exact sequence

$$0 \to E \to E' \xrightarrow{\lambda} \mathbb{C}_p \to 0. \tag{12}$$

We need to determine the line $\mathcal{F} = \ker_p(\lambda) \subset E'_p$. Pulling (12) back to Σ^α and restricting to the fiber at p_α, the map λ becomes

$$(L' \oplus \phi_\alpha^*(L'))_{p_\alpha} \to (\mathbb{C}_{p_\alpha} \oplus \phi_\alpha^*(\mathbb{C}_{p_\alpha}))_{p_\alpha} = \mathbb{C}_{p_\alpha} \oplus 0.$$

This shows that $\mathcal{F} = \ker_p(\lambda)$ is the projection of the line $(0 \oplus \phi_\alpha^*(L'))_{p_\alpha}$. Hence

$$[(E', \mathcal{F})] = j_{p_\alpha}(\phi_\alpha^*(L')) = j_{\phi_\alpha(p_\alpha)}(L').$$

Since $j_{p_\alpha}(L') \in |\mathcal{P}|_a^+$, this shows that $[(E', \mathcal{F})]$ lies in $|\mathcal{P}|_a^-$ (see remark 7.2). Recalling that $q([(E', \mathcal{F})] = [E]$, it follows that $q(|\mathcal{P}|_a^-) = |M|_a^+$, and also that $q(|\mathcal{P}|_a^+) = |M|_a^-$.

The following observation will be used in section 9.

Proposition 7.4. *Let ν be the relative cotangent sheaf of $q' \colon \mathcal{P} \to M'$. Then the restriction of ν to $|\mathcal{P}|_\alpha$ is numerically trivial.*

Proof. Note that $\nu|_{|\mathcal{P}|_\alpha}$ is the dual of the normal bundle, N, say, of the inclusion $|\mathcal{P}|_\alpha \subset (q')^{-1}(|M'|_\alpha)$. By prop. 7.1, it suffices to show that $j_{p_\alpha}^*(N)$ is numerically trivial. Let Λ be a Poincaré bundle over $\mathrm{Pic}_1(\Sigma^\alpha) \times \Sigma^\alpha$. For $x \in \Sigma^\alpha$, let Λ_x denote its restriction to $\theta_\alpha^{-1}([p]) \times \{x\}$. We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(\Lambda_{p_\alpha} \oplus \Lambda_{\phi_\alpha(p_\alpha)}) & \xrightarrow{\Pi_{\alpha *}} & (q')^{-1}(|M'|_\alpha) & \subset & \mathcal{P} \\ \downarrow & & \downarrow q' & & \\ \theta_\alpha^{-1}([p]) & \xrightarrow{\pi_{\alpha *}} & |M'|_\alpha & & \end{array}$$

Here, $\Pi_{\alpha *}$ is the obvious map covering $\pi_{\alpha *}$. (A point in $\mathbb{P}(\Lambda_{p_\alpha} \oplus \Lambda_{\phi_\alpha(p_\alpha)})$ is a point $[L] \in \theta_\alpha^{-1}([p])$ together with a line in $L_{p_\alpha} \oplus L_{\phi_\alpha(p_\alpha)} = (L \oplus \phi_\alpha^*(L))_{p_\alpha}$. This is sent by $\Pi_{\alpha *}$ to the point represented by $\pi_{\alpha *}(L)$ and the induced line in $(\pi_{\alpha *}(L))_p$.)

Let s_{p_α} be the section of the fibration on the left defined by $s_{p_\alpha}(L) = L_{p_\alpha} \oplus 0$, for $[L] \in \theta_\alpha^{-1}([p])$. Then $j_{p_\alpha} = \Pi_{\alpha *} \circ s_{p_\alpha}$, and hence the inclusion $|\mathcal{P}|_\alpha \subset (q')^{-1}(|M'|_\alpha)$ corresponds to the inclusion of the image of s_{p_α} in $\mathbb{P}(\Lambda_{p_\alpha} \oplus \Lambda_{\phi_\alpha(p_\alpha)})$. This shows

$$j_{p_\alpha}^*(N) \cong \Lambda_{p_\alpha}^* \otimes \Lambda_{\phi_\alpha(p_\alpha)}.$$

Tensoring Λ by the pull-back of a bundle over $\mathrm{Pic}_1(\Sigma^\alpha)$ if necessary, we may assume Λ_{p_α} is trivial. Hence $j_{p_\alpha}^*(N)$ is numerically trivial, proving the proposition.

8. Investigation of the involutions ρ_a and ρ'_α.

Let $a \in J^{(4)}$ such that $2a = \alpha \neq 0$. Recall that the involution ρ_a is the lift of α to $\mathcal{L}$ which acts as the identity over the fixed point component $|M|_a^+$.

Proposition 8.1. *The involution ρ_a acts as minus the identity over the fixed point component $|M|_a^-$.*

Proof. Let E' represent a point $[E']$ in $|M'|_\alpha$. Now α acts on the fiber of q' over $[E']$ and from our description of $|\mathcal{P}|_\alpha$, we have that α has exactly two fixed points on $(q')^{-1}([E'])$. Consider now $q^*\mathcal{L}|_{(q')^{-1}([E'])}$ with its lift of α induced by ρ_a. Let s_1, s_2 be the signs by which ρ_a acts over the two fixed points. By lemma 2.1 in [BSz] we have that $q^*\mathcal{L}|_{(q')^{-1}([E'])}$ is isomorphic to $\mathcal{O}(1)$ over $(q')^{-1}([E']) \cong \mathbb{P}(E'_p)$. From this we conclude that $s_1 s_2 = -1$, proving the proposition.

Remark 8.2. At this point, it follows easily that the cocycle of the extension $\mathcal{E}$ generated by the involutions ρ_a is equal to the order 2 Weil pairing λ_2. Indeed, recall that the cocycle is defined by $c(\alpha, \beta) = \rho_a \rho_b \rho_a^{-1} \rho_b^{-1}$ where a is a lift of α and b is a lift of β. Assume first that $\lambda_2(\alpha, \beta) = 1$. Let us then evaluate $\rho_a \rho_b \rho_a^{-1} \rho_b^{-1}$ in a point in $|M|_a^+$. Recalling from 5.5(ii) that β preserves this component of $|M|_\alpha$, we get that

$$c(\alpha, \beta) = 1\, \rho_b\, (1)^{-1}\, \rho_b^{-1} = 1.$$

If however $\lambda_2(\alpha, \beta) = -1$, then β exchanges the two components, and

$$c(\alpha, \beta) = 1\, \rho_b\, (-1)^{-1}\, \rho_b^{-1} = -1.$$

Thus $c = \lambda_2$, as asserted.

Theorem 8.3. *We have $\rho'_\alpha \rho'_\beta = \lambda_2(\alpha, \beta) \rho'_{\alpha+\beta}$.*

Proof. We may assume none of the classes α, β, $\alpha+\beta$, is zero, the result being obvious otherwise. Consider first the case $\lambda_2(\alpha, \beta) = -1$. Then by prop. 6.1 we have that the triple intersection $|M'|_\alpha \cap |M'|_\beta \cap |M'|_{\alpha+\beta}$ is non-empty. Since by definition ρ'_α acts as minus the identity on the fiber over $|M'|_\alpha$, it follows that

$$\rho'_\alpha \rho'_\beta = -\rho'_{\alpha+\beta},$$

proving the result in this case.

Now consider the case $\lambda_2(\alpha, \beta) = 1$. Pick $a, b \in J^{(4)}$ such that $2a = \alpha$ and $2b = \beta$, and consider the involutions $\rho_a^{\otimes 2}$, $\rho_b^{\otimes 2}$, and $\rho_{a+b}^{\otimes 2}$, acting on $\mathcal{L}^2$. In fact, those involutions depend only on α and β, and not on the choice of a and b. By prop. 6.1 we have that the triple intersection $|M|_\alpha \cap |M|_\beta \cap |M|_{\alpha+\beta}$ is non-empty. Note that $\rho_a^{\otimes 2}$ acts as the identity over both components of $|M|_\alpha$. Hence

$$\rho_a^{\otimes 2} \rho_b^{\otimes 2} = \rho_{a+b}^{\otimes 2}. \tag{13}$$

Now consider the Hecke correspondence. From Corollary 2.2 in [BSz] (see also Lemma 10.3 in [BLS]) we have that

$$K_{\mathcal{P}} \cong (q')^*(\mathcal{L}'^{-1}) \otimes q^*(\mathcal{L}^{-2}). \tag{14}$$

From Proposition 7.1 we see that $|\mathcal{P}|_\alpha$ has odd codimension, hence α acts by -1 on the restriction of $K_{\mathcal{P}}$ to $|\mathcal{P}|_\alpha$. Our lifts $\rho_a^{\otimes 2}$ and ρ'_α thus make the isomorphism (14) a $J^{(2)}$-equivariant isomorphism. The action of $J^{(2)}$ on $K_{\mathcal{P}}$ is obviously a group action. This enables us to compare the lift ρ'_α acting on $\mathcal{L}'$ and the lift $\rho_a^{\otimes 2}$ acting on $\mathcal{L}^2$. Thus (13) implies

$$\rho'_\alpha \rho'_\beta = \rho'_{\alpha+\beta},$$

proving the result in the case $\lambda_2(\alpha, \beta) = 1$. This completes the proof.

Note. The equivariance of the isomorphism (14) is the reason why we defined ρ'_α to be the lift which acts as *minus* the identity over the fixed point set.

Theorem 8.4. *We have* $\rho_a \rho_b = \lambda_4(a, b)\rho_{a+b}$.

Proof. We first deal with the case where $\lambda_4(a, b) = \pm 1$, or, equivalently, $\lambda_2(\alpha, \beta) = 1$. If $\alpha = \beta = 0$, there is nothing to show. If $\alpha = \beta \neq 0$, then ρ_a and ρ_b are lifts of the same class, hence $\rho_a = \pm\rho_b$. By prop. 8.1, we have $\rho_a = \rho_b$ if and only if a and b define the same component of $|M|_\alpha$, which in turn is equivalent, by prop. 5.5(i), to $\lambda_2(b - a, \alpha) = 1$. But if $\alpha = \beta$ then $\lambda_2(b - a, \alpha) = \lambda_4(a, b)$ and ρ_{a+b} is the identity (by definition). This proves the result in the case $\alpha = \beta$. Finally, if α, β, and $\alpha + \beta$ are all three non-zero, the triple intersection $|M|_\alpha \cap |M|_\beta \cap |M|_{\alpha+\beta}$ is non-empty, and we can compute $\rho_a \rho_b \rho_{a+b}$ in the fiber over an intersection point. By theorem 2.2 and prop. 8.1, it follows that $\rho_a \rho_b \rho_{a+b} = \lambda_4(a, b)$, completing the proof in the case $\lambda_4(a, b) = \pm 1$.

The remainder of this section is devoted to the proof in the case where $\lambda_4(a, b) = \pm i$. We will again use the notations $\gamma = \alpha + \beta$ and $c = a + b$.

Let the bundle E' represent a point $[E']$ in $|M'|_\alpha \cap |M'|_\beta \cap |M'|_\gamma$. The three involutions α, β, γ induce involutions on $(q')^{-1}([E']) \subset \mathcal{P}$; recall that $(q')^{-1}([E'])$ is identified with the projective space $\mathbb{P}(E'_p)$. Each of these involutions has two fixed points on $\mathbb{P}(E'_p)$; these are precisely the intersection points of $\mathbb{P}(E'_p)$ with the fixed point varieties $|\mathcal{P}|_\alpha, |\mathcal{P}|_\beta$, and $|\mathcal{P}|_\gamma$. Note that ρ_a acts as ∓ 1 on the fiber of $q^*\mathcal{L}$ at the intersection point of $\mathbb{P}(E'_p)$ with the component $|\mathcal{P}|_a^\pm$, since $q(|\mathcal{P}|_a^+) = |M|_a^-$ by prop. 7.3. Let $\mathcal{F}_a^\pm$, $\mathcal{F}_b^\pm$, $\mathcal{F}_c^\pm$ be the lines in E'_p corresponding to the intersection points of $\mathbb{P}(E'_p)$ with the components $|\mathcal{P}|_a^\pm, |\mathcal{P}|_b^\pm$, and $|\mathcal{P}|_c^\pm$. As already used in the proof of prop. 8.1, the restriction of $q^*\mathcal{L}$ to $\mathbb{P}(E'_p)$ is the bundle $\mathcal{O}(1)$. It will be convenient to transfer the calculation to the tautological bundle $\mathcal{O}(-1)$, whose fiber over a point represented by a line $\mathcal{F}$ is that line. Of course, $\mathcal{O}(-1)$ is the restriction of $q^*\mathcal{L}^{-1}$ to $\mathbb{P}(E'_p)$. For $a \in J^{(4)}$, let us denote by $\hat{\rho}_a$ the involution $\rho_a^{\otimes(-1)}$ acting on $\mathcal{L}^{-1}$. Then $\hat{\rho}_a$ acts as ∓ 1 on the line $\mathcal{F}_a^\pm$, and similarly for $\hat{\rho}_b$ and $\hat{\rho}_c$ on the lines $\mathcal{F}_b^\pm$ and $\mathcal{F}_c^\pm$.

It follows easily from this description (or from the computation of the cocycle in remark 8.2) that one has $\hat\rho_a\hat\rho_b = \varepsilon\hat\rho_c$ where $\varepsilon \in \{\pm i\}$. (In fact, the involutions $\hat\rho_a, \hat\rho_b$ generate a quaternion subgroup $Q_8 \subset Sl_2(\mathbb{C})$, covering the commutative subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset PSl_2(\mathbb{C})$ generated by α, β.) Of course, the sign of ε is determined by the relative position of the six lines. The following lemma computes this relative position in terms of the Weil pairing $\lambda_4(a,b)$.

Lemma 8.5. *Let $\lambda = \lambda_4(b,a) \in \{\pm i\}$. There is an isomorphism of E'_p with $\mathbb{C}^2$ sending the six lines $\mathcal{F}_a^+$, $\mathcal{F}_a^-$, $\mathcal{F}_b^+$, $\mathcal{F}_b^-$, $\mathcal{F}_c^+$, $\mathcal{F}_c^-$, to the lines generated by the vectors*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

The proof will be given later. Assuming lemma 8.5 for the moment, we see that $\hat\rho_a, \hat\rho_b$ and $\hat\rho_c$ correspond to the matrices

$$T_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_b = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad T_c = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}.$$

Note that $T_aT_b = \lambda T_c$, and hence $\hat\rho_a\hat\rho_b = \lambda\hat\rho_c$ and $\rho_a\rho_b = \lambda^{-1}\rho_c$. Thus the remaining case of theorem 8.4 follows directly from lemma 8.5.

Now let us prove lemma 8.5. The proof uses again the coverings Σ^α, Σ^β, Σ^γ, and their common covering $\widetilde\Sigma$ (see section 6).

Choose a point $\tilde p \in \tilde\pi^{-1}(p) \subset \widetilde\Sigma$ and put

$$p_\alpha = \pi^\alpha(\tilde p), \; p_\beta = \pi^\beta(\tilde p), \; p_\gamma = \pi^\gamma(\tilde p). \tag{15}$$

Since $[E'] \in |M'|_\alpha \cap |M'|_\beta \cap |M'|_\gamma$, there exist line bundles $\mathcal{L}_a$ over Σ^α, $\mathcal{L}_b$ over Σ^β, and $\mathcal{L}_c$ over Σ^γ, such that $E' \cong \pi_{\alpha*}(\mathcal{L}_a) \cong \pi_{\beta*}(\mathcal{L}_b) \cong \pi_{\gamma*}(\mathcal{L}_c)$. We can fix $\mathcal{L}_a$ (resp. $\mathcal{L}_b$, resp. $\mathcal{L}_c$) uniquely up to isomorphism by requiring that $\mathcal{L}_a \in \pi_\alpha^*(L_a) \otimes [p_\alpha] \otimes \operatorname{Im}\Phi^0_\alpha$ (resp. $\mathcal{L}_b \in \pi_\beta^*(L_b) \otimes [p_\beta] \otimes \operatorname{Im}\Phi^0_\beta$, resp. $\mathcal{L}_c \in \pi_\gamma^*(L_c) \otimes [p_\gamma] \otimes \operatorname{Im}\Phi^0_\gamma$) (see formula (11)).

Let us denote the bundle $\pi^{\gamma*}(\mathcal{L}_c)$ on $\widetilde\Sigma$ by L. Being a pull-back bundle, L has a canonical involution, C, say, covering the involution ϕ^γ on $\widetilde\Sigma$, and such that $L/\langle C\rangle$ is the bundle $\mathcal{L}_c$ on $\Sigma^\gamma = \widetilde\Sigma/\langle\phi^\gamma\rangle$. Proceeding as in lemma 6.7, it is easy to check that the bundles $\pi^{\alpha*}(\mathcal{L}_a)$ and $\pi^{\beta*}(\mathcal{L}_b)$ are isomorphic to $L = \pi^{\gamma*}(\mathcal{L}_c)$. Therefore L also has canonical involutions A and B, covering ϕ^α and ϕ^β, respectively, such that $L/\langle A\rangle \cong \mathcal{L}_a$ and $L/\langle B\rangle \cong \mathcal{L}_b$.

Lemma 8.6. *One has $AB = \lambda_4(b,a)C$.*

Proof. As in the proof of lemma 6.6, represent $\pi^{\alpha*}(\mathcal{L}_a)$, $\pi^{\beta*}(\mathcal{L}_b)$, and $L = \pi^{\gamma*}(\mathcal{L}_c)$, by divisors F_a, F_b, F_c, respectively, such that

$$F_a = \tilde\pi^*(D_a) + \pi^{\alpha*}(p_\alpha + (1-\phi_\alpha^*)(D))$$

$$F_b = \tilde{\pi}^*(D_b) + \pi^{\beta *}(p_\beta + (1-\phi_\beta^*)(D'))$$
$$F_c = \tilde{\pi}^*(D_c) + \pi^{\gamma *}(p_\gamma + (1-\phi_\gamma^*)(D''))$$

where $D_a, D_b \in \mathrm{Div}_0(\Sigma)$ represent $a, b \in J^{(4)}$, $D_c = D_a + D_b$, $D \in Div_0(\Sigma^\alpha)$, $D' \in Div_0(\Sigma^\beta)$, and $D'' \in Div_0(\Sigma^\gamma)$. As before, since the three bundles are isomorphic, there exist meromorphic functions $h_1, h_2 \in \mathcal{M}(\widetilde{\Sigma})$ such that

$$(h_2) + F_a \;=\; F_c \;=\; (h_1) + F_b.$$

The action of our involutions A, B and C on L can be described on local sections as follows. Since $L = \mathcal{O}(F_c)$, a local section over some open set $U \subset \widetilde{\Sigma}$ is just a meromorphic function s on U such that $(s) + F_c|_U \geq 0$. The action of C is simply given by

$$C \;: s \mapsto s \circ \phi^\gamma,$$

since the divisor F_c was pulled back from Σ^γ, and C is the canonical involution of the pull-back bundle.

The involution A is nothing but the canonical involution of the pull-back bundle $\pi^{\alpha *}(\mathcal{L}_a) = \mathcal{O}(F_a)$, conjugated by an isomorphism with $L = \mathcal{O}(F_c)$. Since $(h_2) = F_c - F_a$, multiplication by h_2 gives such an isomorphism $\mathcal{O}(F_c) \xrightarrow{\sim} \mathcal{O}(F_a)$. Therefore the action of A on local sections of $L = \mathcal{O}(F_c)$ is

$$A \;: s \mapsto ((sh_2) \circ \phi^\alpha)h_2^{-1} = (s \circ \phi^\alpha)\, k_A = (s\; k_A^{-1}) \circ \phi^\alpha$$

where we have put $k_A = (h_2 \circ \phi^\alpha)/h_2$. (N.b., one may think about this as follows: $s \circ \phi^\alpha$ is a local section of $\phi^{\alpha *}L = \mathcal{O}(\phi^{\alpha *}F_c)$, and multiplication by k_A describes an isomorphism $\mathcal{O}(\phi^{\alpha *}F_c) \xrightarrow{\sim} \mathcal{O}(F_c)$.)

Similarly, B acts on local sections of $\mathcal{O}(F_c)$ as

$$B \;: s \mapsto ((sh_1) \circ \phi^\beta)h_1^{-1} = (s \circ \phi^\beta)\, k_B = (s\; k_B^{-1}) \circ \phi^\beta$$

where $k_B = (h_1 \circ \phi^\beta)/h_1$.

Put

$$\lambda = \frac{k_B}{k_A} = \frac{h_2}{h_1}\, \frac{h_1 \circ \phi^\beta}{h_2 \circ \phi^\alpha} \tag{16}$$

Note that λ is a constant, since $(k_B) = (k_A)$. Since AB acts on local sections by

$$AB \;: s \mapsto (((s \circ \phi^\beta)\, k_B) \circ \phi^\alpha)\, k_A = ((s \circ \phi^\beta)\, k_B\, k_A^{-1}) \circ \phi^\alpha = \lambda\, s \circ \phi^\gamma,$$

we have $AB = \lambda C$.

Now let us show that $\lambda = \lambda_4(b, a)$. As in section 6, we use the functions $f, g \in \mathcal{M}(\Sigma)$ defined by $f = \widetilde{\mathrm{Nm}}(h_1)$ and $g = \widetilde{\mathrm{Nm}}(h_2)$. A computation shows that the statements of lemma 6.8 hold word for word. We can therefore compute $\lambda_4(a, b) = g(D_a)/f(D_b)$ exactly as before. Note that the divisors D, D', D'' have degree zero, so that the terms involving the functions $f_\alpha, f_\beta, f_\gamma$ are now equal to 1. We thus obtain

$$\begin{aligned}\lambda_4(a,b) &= \frac{h_1(\pi^{\gamma *}(p_\gamma)-\pi^{\alpha *}(p_\alpha))}{h_2(\pi^{\gamma *}(p_\gamma)-\pi^{\beta *}(p_\beta))} = \frac{h_1(\phi^\gamma(\tilde p)-\phi^\alpha(\tilde p))}{h_2(\phi^\gamma(\tilde p)-\phi^\beta(\tilde p))}\\ &= \frac{h_1}{h_2}\,\frac{h_2\circ\phi^\alpha}{h_1\circ\phi^\beta}\,(\phi^\gamma(\tilde p))\end{aligned}$$

where we have used (15) in the last but one step. Comparing this with formula (16), we have $\lambda = \lambda_4(a,b)^{-1} = \lambda_4(b,a)$, proving lemma 8.6.

We return to the proof of lemma 8.5. Recall that

$$E' \cong \pi_{\alpha *}(\mathcal{L}_a) = (\mathcal{L}_a \oplus \phi_\alpha^* \mathcal{L}_a)/\langle \phi_\alpha \rangle. \tag{17}$$

Since $\mathcal{L}_a \in \pi_\alpha^*(L_a)\otimes[p_\alpha]\otimes \operatorname{Im}\Phi_\alpha^0$, we have

$$j_{p_\alpha}(\mathcal{L}_a) \in |\mathcal{P}|_a^+,$$

where j_{p_α} is the isomorphism of prop. 7.1. Thus, the line $\mathcal{F}_a^+ \subset E'_p$ (representing the unique intersection point in $\mathbb{P}(E'_p)\cap|\mathcal{P}|_a^+$) is the image of the line $(\mathcal{L}_a\oplus 0)_{p_\alpha} \subset (\mathcal{L}_a \oplus \phi_\alpha^*\mathcal{L}_a)_{p_\alpha}$ under the natural projection. Also, the line $\mathcal{F}_a^-$ is the image of $(0\oplus\phi_\alpha^*(\mathcal{L}_a))_{p_\alpha}$. Note that since E' is stable, the isomorphism (17) is unique up to scalar multiples, hence the lines $\mathcal{F}_a^\pm$ are well-determined by this description.

Similarly, using the isomorphism of E' with $\pi_{\beta *}(\mathcal{L}_b)$ and with $\pi_{\gamma *}(\mathcal{L}_c)$, the lines $\mathcal{F}_b^+$ and $\mathcal{F}_b^-$ correspond to the projections of the lines $(\mathcal{L}_b\oplus 0)_{p_\beta}$ and $(0\oplus\phi_\beta^*(\mathcal{L}_b))_{p_\beta}$, and the lines $\mathcal{F}_c^+$ and $\mathcal{F}_c^-$ correspond to the projections of the lines $(\mathcal{L}_c\oplus 0)_{p_\gamma}$ and $(0\oplus\phi_\gamma^*(\mathcal{L}_c))_{p_\gamma}$.

Let us now understand the relative position of the six lines in E'_p.

Note that since $AB=-BA$, the involution B of L covering the involution ϕ^β on $\widetilde\Sigma$ induces a map

$$L/\langle -A\rangle \longrightarrow L/\langle A\rangle$$

covering the involution ϕ_α on Σ^α. We may choose isomorphisms $\mathcal{L}_a \cong L/\langle A\rangle$ and $\phi_\alpha^*\mathcal{L}_a \cong L/\langle -A\rangle$ such that this map becomes the canonical map $\phi_\alpha^*\mathcal{L}_a \to \mathcal{L}_a$ covering ϕ_α. Therefore $\pi_{\alpha *}(\mathcal{L}_a)$ is isomorphic to the bundle E_a defined by

$$E_a \;=\; L\oplus L\Big/\Big\langle \begin{pmatrix} A & 0\\ 0 & -A\end{pmatrix}, \begin{pmatrix} 0 & B\\ B & 0\end{pmatrix}\Big\rangle.$$

(Notice that the two matrices commute, so that E_a is indeed a well-defined bundle on $\Sigma = \widetilde\Sigma/\langle\phi^\alpha,\phi^\beta\rangle$.) Moreover, the lines $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$ in E'_p correspond, *via* an isomorphism $E'\cong E_a$ (which is unique up to scalar multiples), to the images of the lines $(L\oplus 0)_{\tilde p}$ and $(0\oplus L)_{\tilde p}$ under the projection from $L\oplus L$ to E_a. Here, we have used that $\pi^\alpha(\tilde p) = p_\alpha$ by our choice of p_α in (15).

Similarly, $\pi_{\beta *}(\mathcal{L}_b)$ is isomorphic to the bundle E_b defined by

$$E_b = L \oplus L \Big/ \left\langle \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \right\rangle,$$

and since $\pi^\beta(\tilde p) = p_\beta$, the lines $\mathcal{F}_b^+$ and $\mathcal{F}_b^-$ in E'_p correspond to the images of the same lines $(L \oplus 0)_{\tilde p}$ and $(0 \oplus L)_{\tilde p}$, but now projected from $L \oplus L$ to E_b.

Lastly, $\pi_{\gamma *}(\mathcal{L}_c)$ is isomorphic to the bundle E_c defined by

$$E_c = L \oplus L \Big/ \left\langle \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \right\rangle,$$

and as before, since $\pi^\gamma(\tilde p) = p_\gamma$, the lines $\mathcal{F}_c^+$ and $\mathcal{F}_c^-$ in E'_p correspond to the images of the lines $(L \oplus 0)_{\tilde p}$ and $(0 \oplus L)_{\tilde p}$ under the projection from $L \oplus L$ to E_c.

The three bundles E_a,E_b, E_c are all isomorphic to E'. In fact, we have isomorphisms $\psi_X \colon E_b \xrightarrow{\sim} E_a$ and $\psi_Y \colon E_c \xrightarrow{\sim} E_a$ induced by the endomorphisms of $L \oplus L$ defined by the matrices

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 1 \\ -\lambda & \lambda \end{pmatrix}.$$

The verification, which uses that $AB = \lambda C$ by lemma 8.6, is left to the reader.

Let us identify the fiber $(L \oplus L)_{\tilde p}$ in the obvious way with $\mathbb{C} \oplus \mathbb{C}$ and consider the isomorphism

$$E'_p \xrightarrow{\sim} (E_a)_p \xleftarrow{\sim} (L \oplus L)_{\tilde p} = \mathbb{C} \oplus \mathbb{C},$$

where the first map is induced by an isomorphism $E' \cong E_a$ and the second map is the projection. Then the six lines $\mathcal{F}_a^+$, $\mathcal{F}_a^-$, $\mathcal{F}_b^+$, $\mathcal{F}_b^-$, $\mathcal{F}_c^+$, $\mathcal{F}_c^-$, correspond to the lines in $\mathbb{C} \oplus \mathbb{C}$ generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X\begin{pmatrix} 1 \\ 0 \end{pmatrix}, X\begin{pmatrix} 0 \\ 1 \end{pmatrix}, Y\begin{pmatrix} 1 \\ 0 \end{pmatrix}, Y\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These vectors are precisely the ones in the statement of lemma 8.5. This proves lemma 8.5, and completes the proof of theorem 8.4.

9. The trace computation.

In this section, we prove theorem 2.3 by computing the trace of $\rho_\alpha'^{\otimes k/2}$ and $\rho_a^{\otimes k}$ using the Lefschetz-Riemann-Roch fixed point formula. In the twisted case, the computation is rather straightforward, since M' is smooth, and the relevant cohomology classes on the fixed point set are given in [NR]. This computation has been done in a different context by Pantev [Pa]. We repeat the calculation here and in the process we correct a misprint in his formula. In the untwisted case, the moduli space and the fixed point sets are not smooth. We circumvent this problem by transferring the computation to $\mathcal{P}$, using some results of [BSz].

Note. Beauville [Be2] has recently computed the traces in the untwisted case in a different way by transferring the calculation to M'. [7] Beauville considers, more generally, rank r bundles, and his formula agrees with ours in the case $r = 2$. He does not, however, choose lifts to the line bundle $\mathcal{L}$, and his result (for $r = 2$) concerns only the case $k \equiv 0 \bmod 4$, where one has a group action of $J^{(2)}$.

Proposition 9.1. *The trace of the involution* ${\rho'_\alpha}^{\otimes k/2}$ *is given by*

$$\mathrm{Tr}({\rho'_\alpha}^{\otimes k/2}) = (-1)^{k/2} \left(\frac{k+2}{2}\right)^{g-1}.$$

Proof. The Lefschetz-Riemann-Roch fixed point formula [AS] states that

$$\mathrm{Tr}({\rho'_\alpha}^{\otimes k/2}) = \widetilde{Ch}(\mathcal{L}'|_{|M'|_\alpha})^{k/2}\, \widetilde{Ch}(\lambda_{-1}N_\alpha)^{-1}\ \mathrm{Td}(|M'|_\alpha) \cap [|M'|_\alpha].$$

Here $\mathrm{Td}(|M'|_\alpha)$ is the Todd class, N_α is the conormal bundle of $|M'|_\alpha$, λ_t is the operation defined by $\lambda_t E = \sum t^i \Lambda^i E$, and

$$\widetilde{Ch}(E) = Ch(E_+ - E_-),$$

where Ch is the Chern character, and, for any $\mathbb{Z}/2$-equivariant bundle E over the fixed point set $|M'|_\alpha$, E_+ and E_- are the ± 1-eigenbundles.

Since the fixed point set $|M'|_\alpha$ is isomorphic to the Prym variety P_α, its Todd class is 1. The cohomology class $\widetilde{Ch}(\lambda_{-1}N_\alpha)$ was computed in Proposition 4.2 in [NR]. Let Θ be the restriction to P_α of the principal polarization of $J_0(\Sigma^\alpha)$. The result of Narasimhan and Ramanan then states that

$$\widetilde{Ch}(\lambda_{-1}N_\alpha) = 2^{2(g-1)} e^{-2\Theta}. \tag{18}$$

By the construction of $\mathcal{L}'$ in [DN], we have the following relation between Θ and $Ch(\mathcal{L}'|_{|M'|_\alpha})$.

$$Ch(\mathcal{L}'|_{|M'|_\alpha}) = e^{2\Theta}. \tag{19}$$

Note that $\widetilde{Ch}(\mathcal{L}'|_{|M'|_\alpha}) = -Ch(\mathcal{L}'|_{|M'|_\alpha})$, by our definition of ρ'_α. Hence we get that

$$\mathrm{Tr}({\rho'_\alpha}^{\otimes k/2}) = (-1)^{k/2} 2^{-2(g-1)} e^{(k+2)\Theta} \cap [P_\alpha].$$

Using Corollary 4.16 in [NR], which states that

$$\Theta^{g-1} \cap [P_\alpha] = (g-1)! 2^{g-1}, \tag{20}$$

the result follows.

Proposition 9.2. *The trace of the involution* $\rho_a^{\otimes k}$ *is given by*

$$\mathrm{Tr}(\rho_a^{\otimes k}) = \frac{1+(-1)^k}{2} \left(\frac{k+2}{2}\right)^{g-1}.$$

[7] Our computation was done independently of his.

Proof. The morphism q induces a $J^{(2)}$-equivariant morphism

$$q^* : H^0(M, \mathcal{L}^k) \to H^0(\mathcal{P}, q^*\mathcal{L}^k).$$

According to [BSz] we have that this morphism is an isomorphism and that

$$H^i(\mathcal{P}, q^*\mathcal{L}^k) = 0,$$

for $i > 0$. Hence we just need to apply the Lefschetz-Riemann-Roch fixed point theorem to $(\mathcal{P}, q^*\mathcal{L}^k)$. The fixed point set $|\mathcal{P}|_\alpha$ has two components, $|\mathcal{P}|_a^+$ and $|\mathcal{P}|_a^-$, each of which is isomorphic to the Prym variety P_α, and hence has trivial Todd class. In order to understand $q^*\mathcal{L}|_{|\mathcal{P}|_\alpha}$ and the conormal bundle $N(|\mathcal{P}|_\alpha)$, consider the following exact sequence

$$0 \to (q')^* T^*_{M'} \to T^*_{\mathcal{P}} \to \nu \to 0,$$

where ν is the relative cotangent sheaf of q'. We conclude that

$$K_{\mathcal{P}} \cong \nu \otimes (q')^* K_{M'} \cong \nu \otimes (q')^*(\mathcal{L}')^{-2},$$

since $K_{M'} \cong (\mathcal{L}')^{-2}$ [DN]. By cor. 2.2 in [BSz] (see equation (14) in the proof of theorem 8.3), we get that

$$q^*\mathcal{L}^2 \cong (q')^*\mathcal{L}' \otimes \nu^{-1}.$$

Now recall from prop. 7.4 that the line bundle $\nu|_{|\mathcal{P}|_\alpha}$ is numerically trivial. This shows

$$Ch(q^*\mathcal{L}|_{|\mathcal{P}|_a^\pm}) = \big(Ch((q')^*\mathcal{L}'|_{|\mathcal{P}|_a^\pm})\big)^{1/2} = e^\Theta$$

where in the last equality we have used formula (19), after identifying both $|\mathcal{P}|_a^+$ and $|\mathcal{P}|_a^-$ with the Prym variety P_α. Next, observe that α acts as -1 on $\nu|_{|\mathcal{P}|_\alpha}$, since one has an exact sequence (of bundles over $|\mathcal{P}|_\alpha$)

$$0 \to (q')^* N_\alpha \to N(|\mathcal{P}|_\alpha) \to \nu|_{|\mathcal{P}|_\alpha} \to 0. \tag{21}$$

where $N_\alpha = N(|M'|_\alpha)$ as before. Therefore

$$\widetilde{Ch}(\lambda_{-1}\nu|_{|\mathcal{P}|_a^\pm}) = \widetilde{Ch}(1 - \nu|_{|\mathcal{P}|_a^\pm}) = Ch(1+1) = 2,$$

and the exact sequence (21) gives us that

$$\widetilde{Ch}(\lambda_{-1}(N(|\mathcal{P}|_a^\pm))) = 2\,\widetilde{Ch}(\lambda_{-1}(N_\alpha)) = 2 \cdot 2^{2(g-1)} e^{-2\Theta},$$

where we have used formula (18) in the last equality, after again identifying $|\mathcal{P}|_a^+$ and $|\mathcal{P}|_a^-$ with P_α.

Now recall that ρ_a acts with opposite signs on the restriction of $q^*\mathcal{L}$ to the two components $|\mathcal{P}|_a^+$ and $|\mathcal{P}|_a^-$. In fact, it acts as ∓ 1 over $|\mathcal{P}|_a^\pm$, by prop. 7.3. Therefore

$$\widetilde{Ch}(q^*\mathcal{L}|_{|\mathcal{P}|_a^\pm}) = \mp Ch(q^*\mathcal{L}|_{|\mathcal{P}|_a^\pm}) = \mp e^\Theta.$$

Putting everything together, the fixed point formula gives

$$\mathrm{Tr}(\rho_a^{\otimes k}) = (1 + (-1)^k) e^{k\Theta}\, 2^{-1} 2^{-2(g-1)} e^{2\Theta} \cap [P_\alpha].$$

The proposition now follows as in the twisted case from formula (20).

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths & J. Harris. Geometry of Algebraic Curves. Vol. I. Springer-Verlag New York 1985.

[At1] M. F. Atiyah. Riemann surfaces and spin structures. *Ann. Scien. Ec. Norm. Sup.* 4^e série **4** (1971), 47-62.

[At2] M. F. Atiyah. The geometry and physics of knots. Cambridge University Press, Cambridge 1990.

[AS] M. F. Atiyah, I. M. Singer. The index of elliptic operators III. *Ann. of Math.* **87**, 546-604 (1968).

[ADW] S. Axelrod, S. Della Pietra & E. Witten. Geometric quantization of Chern Simons gauge theory. *J. Diff. Geom.* **33** (1991) 787-902.

[Be1] A. Beauville. Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta II. *Bull. Soc. Math. France* **119** (1991) 259-291.

[Be2] A. Beauville. The Verlinde formula for PGL_p. Preprint alg-geom/9609017.

[BL] A. Beauville, Y. Laszlo. Conformal blocks and generalized theta functions. *Comm. Math. Phys.* **164** (1994) 385-419.

[BLS] A. Beauville, Y. Laszlo, Ch. Sorger. The Picard Group of the Moduli of G-bundles on a Curve. Preprint alg-geom/9608002

[BNR] A. Beauville, M. S. Narasimhan, S. Ramanan. Spectral curves and the generalised theta divisor. *J. Reine angew. Math.* **398** (1989) 169-179.

[BSz] A. Bertram & A. Szenes. Hilbert polynomials of moduli spaces of rank 2 vector bundles II. *Topology* **32** (1993) 599-609.

[BM] C. Blanchet, G. Masbaum. Topological quantum field theories for surfaces with spin structure. *Duke Math. J.* **82** (1996) 229-267.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel. Topological Quantum Field Theories derived from the Kauffman bracket. *Topology* **34** (1995), 883-927.

[CLM] S.E. Cappell, R. Lee & E.Y. Miller. Invariants of 3-manifolds from conformal field theory, *Topology 90, Ohio state University Mathematical Research Institute Publications 1.*

[DN] J.-M. Drezet et M.S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.* **97** (1989), 53-94.

[Fa] G. Faltings. Stable G-bundles and projective connections. *J.Alg.Geom.* **2** (1993) 507-568.

[vGP1] B. van Geemen, E. Previato. Prym varieties and the Verlinde formula. *Math. Ann.* **294** (1992) 741-754.

[vGP2] B. van Geemen, E. Previato. Heisenberg action and Verlinde Formulas. In *Verdier Memorial Conf.* (O. Babelon et al., Ed.) Birkhäuser 1994.

[GH] P. Griffiths, J. Harris. Principles of algebraic gometry. Wiley-Interscience (1978).

[Hi] N. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.* **131** (1990) 347-380.

[Ho] E. W. Howe. The Weil pairing and the Hilbert symbol. *Math. Ann.* **305** (1996) 387-392.

[Jo] D. Johnson. Spin structures and quadratic forms on surfaces. *J. London Math. Soc.* (2) **22** (1980) 365-377.

[KNR] S. Kumar, M.S. Narasimhan, A. Ramanathan. Infinite Grassmannians and moduli spaces of G-bundles *Math. Ann.* **300** (1994) 41-75.

[L] Y. Laszlo. Dimension de l'espace des sections du diviseur thêta généralisé, *Bull. Soc. Math. France* **119** (1991) 293-306.

[MS] G. Moore & N. Seiberg. Classical and quantum Conformal Field Theory. *Comm. Math. Phys.* **123** (1989) 177-254.

[MV] G. Masbaum, P. Vogel. Verlinde Formulae for Surfaces with Spin Structure. *Contemp. Math.* **164** (1994) 119-137.

[NR] M. S. Narasimhan, S. Ramanan. Generalised Prym varieties as fixed points. *J. Indian Math. Soc. (N.S.)* **39** (1975) 1-19.

[OP] W. Oxbury, Ch. Pauly. $SU(2)$-Verlinde spaces as Theta-spaces on Pryms. *Int. J. Math.* **7** (1996) 393-410.

[Pa] T. Pantev. Comparison of generalized theta functions. *Duke Math. J.* **76** (1994), 509-539.

[P] Ch. Pauly. On Pryms, rank 2 bundles and nonabelian theta functions. *Int. J. Math.* **8** (1997) 267-287.

[RSW] T.R. Ramadas, I.M. Singer and J. Weitsman. Some Comments on Chern - Simons Gauge Theory. *Commun. Math. Phys.* **126** (1989) 409-420.

[RT] N. Yu. Reshetikhin, V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547-597.

[S] G. Segal. The Definition of Conformal Field Theory. Preprint.

[Sz] A. Szenes. Hilbert polynomial of moduli spaces of rank 2 vector bundles. *Topology* **32** (1993) 587-597.

[Th1] M. Thaddeus. Conformal field theory and the cohomology of the moduli spaces of stable bundles. *J. of Diff. Geom.* **35** (1992), 131-150.

[Th2] M. Thaddeus. Stable pairs, linear systems and the Verlinde formula. *Invent. Math.* **117** (1994) 317-353.

[Ve] E. Verlinde. Fusion rules and modular transformations in $2d$ conformal field theory. *Nuclear Phys.* **B 300** (1988) 360-376.

[Wi] E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys* **121** (1989) 351-98.

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