BELLMAN FUNCTIONS AND TWO WEIGHT INEQUALITIES FOR HAAR MULTIPLIERS

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ABSTRACT. We are going to give necessary and sufficient conditions for two weight norm inequalities for Haar multipliers operators and for square functions. We also give sufficient conditions for two weight norm inequalities for the Hilbert transform.

0. INTRODUCTION

Weighted norm inequalities for singular integral operators appear naturally in many areas of analysis, probability, operator theory ect.

The one-weight case is now pretty well understood, and the answers are given by the famous Helson–Szegö theorem and the Hunt–Muckenhoupt–Wheden Theorem. The fist one state that the Hilbert Transform H is bounded in the weighted space $L^2(w)$ if and only if w can be represented as $w = \exp\{u + Hv\}$, where $u,\ v \in L^\infty$, $\|u\|_\infty < \pi/2$.

The Hunt–Muckenhoupt–Wheden Theorem states that the Hilbert transform H is bounded in $L^p(w)$ if and only if the weight w satisfies the so-called Muckenhoupt A_p condition

$$(A_p) \qquad \sup_I \Big(\frac{1}{|I|}\int_I w\Big)\cdot\Big(\frac{1}{|I|}\int_I w^{-1/(p-1)}\Big)^{p-1} < \infty,$$

where the *supremum* is taken over all intervals I. This condition is also necessary and sufficient fo boundedness for a wide class of singular integral operators, as well as for the boundedness of the maximal operator M,

$$Mf(x) = \sup_{I\ni x}\frac{1}{|I|}\int_I |f|;$$

here supremum is taken over all intervals I containing x.

It is worth mentioning, that there in no direct proof of equivalence the Helson–Szegö condition and the Muckenhoupt condition A_2.

Two weight inequalities, i. e. the problem when an operator acts from $L^2(w)$ to $L^2(v)$ (one can also consider L^p case, even with different exponents p, but the L^2 case is complicated enough, so we restrict our attention on it) appears naturally in many areas like the theory of Hankel and Toeplitz operators, perturbation theory, etc.

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Things look much more complicated in the two-weight case, and it is probably an agreement now that there is no simple (Muckenhoupt type) necessary and sufficient condition of boundedness of the Hilbert Transform.

It was a big surprise when Eric Sawyer [S1] found necessary and sufficient condition for a maximal operator M to be a bounded operator from $L^2(w)$ to $L^2(v)$: his theorem states that it is enough to test the boundedness on a very special class of test functions, namely only on functions $\chi_I w^{-1}$ (and we should do the same for the adjoint operator).[1] We will call such type of conditions *Sawyer type* conditions.

There is also a two weight analog of Helson–Szegö theorem due to M. Cotlar and C. Sadosky, see [CS1]. Their approach (which can be referred as Generalized Bochner Theorem) provides both integral representation and extension of forms and kernels invariant under the shift operator. Being applied to a special bilinear form built with the help of the Hilbert transform and two measures, this approach gives a necessary and sufficient condition for the Hilbert transform to be bounded between L^2-spaces with respect to these measures (see [CS1]). The approach of M.Cotlar, C.Sadosky is very intersting because it provides a direct link between the lifting theory of Sz.-Nagy and Foias (and thus the scattering theory) and the continuity of the Hilbert transform in weighted spaces (see [S]).

But there is no analog of the Muckenhoupt A_2 condition for two weights, which is necessary and sufficiend for the boundedness of the Hilbert transform. There are quite a few sufficient conditons, let us mention a very nice and simple one due to Dechao Zheng [Zh].

In this paper we are going to consider an operator (more precisely, a family of operators, the so called Haar multipliers) which can serve as a good "model" for singular integral operators. For such operator we give necessary and sufficient conditions (of Sawyer type) of the boundedness.

Our operators appear to be simpler than the Hilbert transform and we believe that our approach splits the difficulties of two weight singular integral estimates and allows to treat these difficulties separately.

So let us explain what is our "model" operator. Let $\mathcal{D}$ denote the set of dyadic subarcs of the real line $\mathbb{R}$. Let $\sigma = (\sigma_I)_{I\in\mathcal{D}}$ be a sequence of signs $\pm$. We will be dealing actually with the following family of operators. Let I_-, I_+ denote the left and the right halves of a dyadic interval I and let

$$h_I = \begin{cases} +|I|^{-1/2}\chi_{I_-}, \\ -|I|^{-1/2}\chi_{I_+}. \end{cases}$$

denote a Haar function normalized in $L^2 = L^2(R)$. Let $(\cdot,\cdot)$ denote the scalar product in L^2. We are interested in the following question: How to describe the pairs (μ,ν) of weights on $\mathbb{R}$ such that all operators T_σ,

[1] Sawyer's theorem states more, and treats the L^p-case as well, but we are not going into details here

$$T_\sigma f = \sum_{I\in D} \sigma_I (f, h_I) h_I \tag{0.1}$$

are uniformly bounded from $L^2(\mu)$ to $L^2(\nu)$ with respect to all possible choice of σ?

It is easy to see that the measure ν has to be absolutely continuous. It is also not difficult to see that singular part of μ does not help: if operators T_σ are uniformly bounded from $L^2(\mu)$ to $L^2(\nu)$, then the same holds if we replace μ by its absolutely continous part. So, without loss of generality one can assume that the measures μ and ν are absolutely continuous, $d\mu = udt$, $d\nu = vdt$.

If the operators T_σ are uniformly bounded, then the operators

$$f \mapsto (f, h_I)_{L^2} h_I, \qquad I \in \mathcal{D}$$

are uniformly bounded as well. For a fixed I the norm of the above operator can be easily computed — it is just a rank one operator — and it is equal to $\langle v\rangle_I^{1/2} \langle u^{-1}\rangle_I^{1/2}$. So we get a simple necessary condition

$$\sup_{I\in\mathcal{D}} \langle v\rangle_I \langle u^{-1}\rangle_I < \infty,$$

which can be considered as a two-weight analog of the Muckenhoupt A_2 condition. Unfortunately this condition is not sufficient.

It is convenient to denote $w := u^{-1}$. In this notation the unifrom boundedness of operator $T_\sigma L^2(u) \to L^2(v)$ is equivalent to the uniform boundedness of the operators $M_v^{1/2} T_\sigma M_w^{1/2}$ in usual (non-weighted) L^2; here M_v and M_w denote operators of multiplication by v and w respectively.

And as we have shown above, the following condition

$$\sup_{I\in\mathcal{D}} \langle v\rangle_I \langle w\rangle_I \le C$$

is necessary for the uniform boundedness of all $M_v^{1/2} T_\sigma M_w^{1/2}$ in L^2 or equivalently for the uniform boundedness of all operators $T_\sigma : L^2(w^{-1}) \to L^2(v)$.

The above uniform boundedness of T_σ admits a simple geometric interpretation in terms of so-called *multipliers*. Consider the family $\mathcal{M}$ of bounded operators $A : L^2(w^{-1}) \to L^2(v)$ such that they commute with all operators $f \mapsto (f, h_I)h_I$, $I \in \mathcal{D}$. We call this family the family of Haar multipliers.

These are operators given by a simple formula $Ah_I = a_I h_I$. If $a = \{a_I\}_{I\in\mathcal{D}}$ is given, let us call this operator A_a,

Now one can note that uniform boundedness of T_σ is equivalent to the inclusion

$$\ell^\infty \subset \mathcal{M} \tag{0.2}$$

in the sense that $A_a \in \mathcal{M}$ for all $a \in \ell^\infty$.

We are going to investigate the question when (0.2) holds, that is when the family of Haar multipliers contains ℓ^∞.

We are going to formulate three main results now. Notice that Theorems 0.2 and 0.3 together give the necessary and sufficient conditions for

$$\sup_\sigma \|M_v^{1/2} T_\sigma M_w^{1/2}\| < \infty \tag{0.3}$$

and Theorem 0.1 also gives the necessary and sufficient conditions in a completely diferent form.

Theorem 0.1. *The family of singular integrals* $M_v^{1/2} T_\sigma M_w^{1/2}$ *is uniformly bounded in* L^2 *if and only if*

1. $\forall\, J \in \mathcal{D} \quad \sup_\sigma \frac{1}{|J|}\int_J |T_\sigma(\chi_J w)|^2 v\,dx \le C\langle w\rangle_J\,;$
2. $\forall\, J \in \mathcal{D} \quad \sup_\sigma \frac{1}{|J|}\int_J |T_\sigma(\chi_J v)|^2 w\,dx \le C\langle v\rangle_J\,.$

Note, that any of the above conditions 1 or 2 immediately implies taht

$$\forall\, J \in \mathcal{D} \quad \langle v\rangle_J \langle w\rangle_J \le C < \infty,$$

which is the necessary condition we discussed above.

To formulate the next theorem, let us introduce some notation. Let us denote

$$\alpha_I = \left|\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right| \left|\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right| |I|,$$

where I_-, I_+ are left and right halves of I. Consider an integral operator T_0 given by the formula

$$T_0 f = \sum_{I\in\mathcal{D}} \frac{1}{|I|}\langle f\rangle_I \chi_I \alpha_I$$

whose kernel $k(x,y) = \sum_{I\in\mathcal{D}} |I|^{-2}\chi_I(x)\chi_I(y)\alpha_I$ is evidently positive.

Theorem 0.2. *The family of operators* $M_v^{1/2} T_\sigma M_w^{1/2}$ *is uniformly bounded in* L^2 *if and only if the following four assertions hold simultaneously:*

1. $\forall\, J \in \mathcal{D} \quad \langle v\rangle_J \langle w\rangle_J \le C < \infty;$
2. $\forall\, J \in \mathcal{D} \quad \frac{1}{|J|}\sum_{I\subseteq J} |\langle v\rangle_{I_-} - \langle v\rangle_{I_+}|^2 \langle w\rangle_I |I| \le C\langle v\rangle_J;$
3. $\forall\, J \in \mathcal{D} \quad \frac{1}{|J|}\sum_{I\subseteq J} |\langle w\rangle_{I_-} - \langle w\rangle_{I_+}|^2 \langle v\rangle_I |I| \le C\langle w\rangle_J;$
4. *The operator* T_0 *is bounded from* $L^2(w^{-1})$ *to* $L^2(v)$*, or, equivalently, the operator* $M_v^{1/2} T_0 M_w^{1/2}$ *is bounded in* L^2.

Theorem 0.3. *The operator* T_0 *is bounded from* $L^2(w^{-1})$ *to* $L^2(v)$ *if and only if*

1. $\forall\, J \in \mathcal{D} \quad \frac{1}{|J|}\int_J \Big(\sum_{I\subset J} \frac{1}{|I|}\chi_I \langle w\rangle_I \alpha_I\Big)^2 v\,dx \le C\langle w\rangle_J\,;$
2. $\forall\, J \in \mathcal{D} \quad \frac{1}{|J|}\int_J \Big(\sum_{I\subset J} \frac{1}{|I|}\chi_I \langle v\rangle_I \alpha_I\Big)^2 w\,dx \le C\langle v\rangle_J\,.$

Theorem 0.1 looks surprising. If it would concern an operator with positive kernel it would be in the vein of Sawyer's weighted theorems from [S1], [S2]. In fact, exactly as in [S1], [S2], Theorem 0.1 claims that (the family of) integral operators are bounded if and only if (the family of) integral operators are bounded on test functions χ_J and the adjoint operators are bounded on test functions χ_J. However, the family T_ε models a singular integral operator rather than a positive kernel integral operator. Unlike the case of positive kernel integral operators it now seems surprising that boundedness on χ_J implies boundedness on smaller positive functions.

On the other hand, the classical $T1$ theorem says exactly the same: if an operator with Calderón-Zygmund kernel is uniformly bounded on χ_J then it is bounded. Recently, (see [NTV] and [T]) the $T1$ theorem was extended to nonhomogeneous spaces (spaces with non-doubling measure). It turned out that this generalization plays an important role in the treatment of analytic capacity problems including a famous problem of Ahlfors-Vitushkin.

The method we use to prove these theorems consists of constructing the Bellman function of the problems we consider. Roughly speaking, we try to solve an extremal problem associated to a given problem (simply speaking,we try to consider the worst possible case). This leads to Bellman function of the problem. This approach resembles the approach of Burkholder ([Bu]). But there is a difference. We are not solving the extremal problem mentioned above (we would only wish). Instead, we are looking for a kind of subsolution of an associated system of Partial Differential (In)Equalities.

Theorem 0.3 can be most probably proved using the combination of ideas frim the papers of Kalton, Verbitsky [KV] and Sawyer, Wheeden [SW]. Following [KV] we can introduce the new metric $d(x,y) := \frac{1}{k(x,y)}$, where k is the kernel of T_0 and was written above. It is clearly a metric, and balls in this metric are just all dyadic intervals. Then kernel k satisfies the regularity condition from [SW]. Using the main result of [SW] we could have given an alternative proof of the theorem. Unfortunately it is not clear why the new metric space has certain regularity properties. For example, in [KV] one requires the property that all annuli be nonempty. This is false in our new metric space. However, it is not clear to us how essential are those regularity properties of the metric space for the application of the technique of [SW].

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1. Necessary conditions

As it was shown above in the Introduction, the uniform boundedness of $M_v^{1/2} T_\sigma M_w^{1/2}$ implies that the operators $f \mapsto (f, h_I) h_I$ are uniformly bounded as operators from $L^2(w^{-1})$ to $L^2(v)$, and the later condition is equivalent to

$$\langle v \rangle_I \langle w \rangle_I \leq C < \infty. \tag{1.1}$$

It is well known that for the case of one weight ($w^{-1} = v$ in our notation) this caondition is just the famous Muckenhoupt A_2 condition, and it is a sufficient condition for the unoform boundedness of $M_v^{1/2} T_\sigma M_w^{1/2}$.

But in general case we have other simple necessary conditions which are independent of (1.1). To get on of the condition let us apply the operator $M_v^{1/2}T_\sigma M_w^{1/2}$ to the test function $w^{1/2}\chi_J$. We get

$$\int_{\mathbb{R}}\Big|\sum_{I\in\mathcal{D}}\sigma_I(w\cdot\chi_J,h_I)h_I(x)\Big|^2 v(x)\,dx$$

Let us now take the average over all possible choices of signs σ_I.

Fix a function g in $L^2(w^{-1})$ and let $\{\varepsilon_I(\omega)\}$ be the sequence of independent random variables assuming values ± 1 with probabilities $1/2$, $1/2$. Then

$$\int_\Omega d\mathbb{P}\int_{\mathbb{T}}|\sum_{I\in\mathcal{D}}\varepsilon_I(\omega)(g,h_I)h_I(x)|^2 v\,dx\le C\|g\|^2_{L^2(w^{-1})}$$

and

$$\int_{\mathbb{T}}\sum_{x\in I}|(g,h_I)|^2\frac{1}{|I|}v(x)dx\le C\int_{\mathbb{T}}|g|^2w^{-1}$$

which is

$$\sum_{I\in\mathcal{D}}|\langle g\rangle_{I_-}-\langle g\rangle_{I_+}|^2\langle v\rangle_I\cdot|I|\le C\int_{\mathbb{T}}|g|^2w^{-1}$$

Choose $g=\chi_J w$. Then we come to the following necessary condition (notice that we keep only the summation over $I\subset J$)

$$\forall\, J\in\mathcal{D},\ \frac{1}{|J|}\sum_{I\subset J}|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|^2 v_I\cdot|I|\le C\langle w\rangle_J. \tag{1.2}$$

Symmetrically (we can interchange w and v)

$$\forall\, J\in\mathcal{D},\ \frac{1}{|J|}\sum_{I\subset J}|\langle v\rangle_{I_-}-\langle v\rangle_{I_+}|^2 w_I\cdot|I|\le C\langle v\rangle_J \tag{1.3}$$

If $w^{-1}=u=v$, then, say, (1.2) becomes

$$\frac{1}{|J|}\sum_{I\subset J}\langle w^{-1}\rangle_I\langle w\rangle_I^2\left(\frac{\langle w\rangle_{I_-}-\langle w\rangle_{I_+}}{\langle w\rangle_I}\right)^2|I|\le C\langle w\rangle_J \tag{1.4}$$

which follows from (A_2) condition (1.1) on w (nontrivially, see e.g. [FKP] or [B]).

But in general neither (1.2) nor (1.3) follows from (1.1). This is shown in [N]. In what follows we will try to establish to what extent (1.1), (1.2), (1.3) is the full list of necessary and sufficient conditions.

In general this is not the case, but first we separate some cases when (1.1)-(1.3) or their simple modifications are sufficient for

$$\sup_\varepsilon\|T_\varepsilon\|_{L^2(w^{-1})\to L^2(v)}\le C<\infty. \tag{1.5}$$

2. 2. Reduction of Theorem 0.1 to Therems 0.2, 0.3 and the proof of Theorem 0.2.

Let us use the notation $(\cdot,\cdot)$, $(\cdot,\cdot)_{w^{-1}}$, and $(\cdot,\cdot)_v$ for the dualities in L^2, $L^2(w^{-1})$, and $L^2(v)$. Let T be an arbitrary operator from $L^2(w^{-1})$ to $L^2(v)$ with the norm $\|T\|_{L^2(w^{-1})\to L^2(v)} = \sup_{\|f\|_2\le 1, \|g\|_2\le 1} \ |(Tw^{1/2}g, v^{1/2}f)|$, where $\|\cdot\|_2$ denotes the usual L^2-norm.

We are interested in estimating from above $\|T\|$. Thus we can assume for the time being that f and g are bounded as long as the final estimates of $\|T\|$ will not depend on these bounds.

Adopting this convention let us write for arbitrary $f,g\in L^\infty$ the following decompositions: the decomposition of $fw^{1/2}$ and $gv^{1/2}$ with respect to the usual Haar basis:

$$
\begin{aligned}
fw^{1/2} &= \sum (f, w^{1/2}h_J)h_J,\\
gv^{1/2} &= \sum (g, v^{1/2}h_I)h_I.
\end{aligned}
$$

The reader can notice that we are using the usual Haar system $\{h_I\}$ and its biorthogonal system $\{\mathrm{const}_I \frac{h_I}{v}\}$ in $L^2(v)$ to set up the decomposition in the output space $L^2(v)$ and we proceed similarly in the input space $L^2(w^{-1})$.

Now the expression $(Tw^{1/2}g, v^{1/2}f)$ involved in the formula for the norm $\|T\|$ becomes

$$
(Tw^{1/2}f, v^{1/2}g) = \sum (g, v^{1/2}h_I)(f, w^{1/2}h_J)(Th_J, h_I). \tag{2.1}
$$

For the operators T_ε we have

$$
(T_\varepsilon w^{1/2}f, v^{1/2}g) = \sum_{I\in\mathcal{D}} \varepsilon_I (g, v^{1/2}h_I)(f, w^{1/2}h_I). \tag{2.2}
$$

To estimate the latter expression we are going to use

2.1. **Disbalanced Haar functions.** This is the system of functions $\{h_I\}_{I\in\mathcal{D}}$ having the following properties:

1) h_I^w vanishes outside of I and equals to two different constants on the left and on the right halves of I,
2) $\int h_I^w w\,dx = 0$,
3) $\|h_I^w\|_{L^2(w)} = 1$.

Then this is an orthonormal system in $L^2(w)$. Such kind of system with nonpositive weight $z'(s)ds$ has been used in [CJS] to give a simple solution of a problem of Calderón. The following identity plays an important part below:

$$
x_I \cdot h_I = h_I^w + A_I \cdot \chi_I, \tag{2.3}
$$

where the constants x_I and A_I are uniquelty defined by the properties of $\{h_I^w\}$ listed above. Let us compute them. Using 2) and 3) we get

$$x_I^2 \cdot \langle w\rangle_I = \|h_I^w\|^2_{L^2(w)} + A_I^2|I|\langle w\rangle_I = 1 + A_I^2|I|\langle w\rangle_I$$

Considering the scalar product of both parts of (1.8) with the constant function in $L^2(w)$ we get $x_I(1, h_I)_w = A_I|I|\langle w\rangle_I$.

Thus

$$\begin{aligned}
x_I &= \sqrt{\frac{\langle w\rangle_I}{\langle w\rangle_{I_-}\langle w\rangle_{I_+}}};\\
A_I &= \frac{x_I}{2\sqrt{|I|}}\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I};\\
\frac{A_I}{x_I} &= \frac{1}{2\sqrt{|I|}}\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}.
\end{aligned}$$

And having this in mind let us plug (2.3) into (2.2) to get 4 sums:

$$\begin{aligned}
(T_\varepsilon w^{1/2}f, v^{1/2}g) = \sum_{I\in\mathcal{D}} \varepsilon_I \left(\frac{g}{v^{1/2}}, h_I^v\right)_v (fw^{-1/2}, h_I^w)_w \frac{1}{x_I^v x_I^w}\\
+\sum_{I\in\mathcal{D}} \varepsilon_I \left(\frac{g}{v^{1/2}}, h_I^v\right)_v (fw^{1/2})_I \frac{1}{x_I^v} A_I^w \frac{1}{x_I^w}|I|\\
+\sum_{I\in\mathcal{D}} \varepsilon_I (gv^{1/2})_I (fw^{-1/2}, h_I^w)_w \cdot \frac{1}{x_I^v} A_I^v \frac{1}{x_I^w}|I|\\
+\sum_{I\in\mathcal{D}} \varepsilon_I (gv^{1/2})_I \cdot (fw^{1/2})_I \frac{1}{x_I^v} A_I^v|I| \cdot \frac{1}{x_I^w} A_I^w|I|\\
= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.
\end{aligned}$$

To reduce Theorem 0.1 to Theorems 0.2, 0.3 we will prove that the assumptions of Theorem 0.1 imply the assumptions of Theorems 0.2, 0.3. We already saw that the averaging over signs of the second assumption of Theorem 0.1 implies (1.2) and the averaging over signs of the third assumption of Theorem 0.1 implies (1.3). So in our reduction we can already use (1.2), (1.3) freely.

Four sums above define four linear operators whose bilinear forms are given by the corresponding sums. Let us call them $D_\varepsilon, \Pi_\varepsilon, \Pi'_\varepsilon, T_{0,\varepsilon}$ correspondingly.

A very important remark now is the following. Suppose that all assumptions of Theorem 0.1 are satisfied. Then operators $v^{1/2}T_\varepsilon w^{1/2}$ are uniformly (in ε) bounded on f, g if and only if $T_{0,\varepsilon}$ are uniformly bounded on f, g.

This is true because of the following analysis of operators $D_\varepsilon, \Pi_\varepsilon, \Pi'_\varepsilon$, which says that these operators are uniformly (in ε) bounded if the assumptions of Theorem 0.1

are satisfied. To analyse these operators is the same as to analyse their bilinear forms given by $\sum_i, i = 1, 2, 3$.

2.2. **Estimates of $\sum_i, i = 1, 2, 3$.** There is nothing to estimate in $\sum_1$. Notice that $\frac{1}{x_I^v} \le \sqrt{\langle v\rangle_I}$ so we can use the bound $\frac{1}{x_I^v}\frac{1}{x_I^w} \le \sqrt{\langle v\rangle_I}\sqrt{\langle w\rangle_I} \le C$ to write

$$\Sigma_1 \le C\|\frac{g}{v^{1/2}}\|_{L^2(v)} \cdot \|fw^{-1/2}\|_{L^2(w)} = C\|g\|_2 \cdot \|f\|_2. \tag{2.4}$$

To estimate $\sum_2$ let us notice that

$$\left|\frac{1}{x_I^v}A_I^w\frac{1}{x_I^w}\right||I| \le \sqrt{\langle v\rangle_I}\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\sqrt{|I|}.$$

Then

$$\begin{aligned}\Sigma_2 &\le \left\|\frac{g}{v^{1/2}}\right\|_{L^2(v)}\left(\sum_{I\in\mathcal{D}}\langle fw^{1/2}\rangle_I^2\langle v\rangle_I\left(\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right)^2|I|\right)^{1/2}\\ &= \|g\|_2\cdot\left(\sum_{I\in\mathcal{D}}\langle fw^{1/2}\rangle_I^2\langle v\rangle_I\left(\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right)^2|I|\right)^{1/2}.\end{aligned}$$

To estimate the last expression let us use the following lemma.

Lemma 2.1. *Let $\{\alpha_I\}_{I\in\mathcal{D}}$ be a sequence of nonnegative numbers. Then*

$$\sum_{I\in\mathcal{D}}\langle fw^{1/2}\rangle_I^2\alpha_I \le C\|f\|_2^2$$

if and only if for all J, $\frac{1}{|J|}\sum_{I\subset J}\langle w\rangle_I^2\alpha_I \le C\langle w\rangle_J$.

We postpone the proof till Section 6. If we use this lemma and (1.2) we can finish the estimate of $\sum_2$:

$$\Sigma_2 \le C_2\|f\|_2\cdot\|g\|_2. \tag{2.5}$$

Similarly, (1.3) gives

$$\Sigma_3 \le C_3\|f\|_2\cdot\|g\|_2. \tag{2.6}$$

We conclude that if the assumptions of Theorem 0.1 hold then, for any given pair f, g of L^2-functions $\sup_\varepsilon|(v^{1/2}T_\varepsilon w^{1/2}f, g)| \le A\|f\|_2\|g\|_2$ if and only if $\sup_\varepsilon|(T_{0,\varepsilon}f, g)| \le B\|f\|_2\|g\|_2$. But this supremum can be computed as follows

$$\sup_\varepsilon\Sigma_4 = \sup_\varepsilon\sum\varepsilon_I\langle gv^{1/2}\rangle_I\langle fw^{1/2}\rangle_I\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}|I| =$$

$$\sum_{I\in\mathcal{D}}|(gv^{1/2})_I|\cdot|(fw^{1/2})_I|\frac{|\langle v\rangle_{I_-} - \langle v\rangle_{I_+}|}{\langle v\rangle_I}\cdot\frac{|\langle w\rangle_{I_-} - \langle w\rangle_{I_+}|}{\langle w\rangle_I}|I|$$

Clearly the estimate

$$\sum_{I\in\mathcal{D}} |\langle gv^{1/2}\rangle_I| \cdot |\langle fw^{1/2}\rangle_I| \frac{|\langle v\rangle_{I_-} - \langle v\rangle_{I_+}|}{\langle v\rangle_I} \cdot \frac{|\langle w\rangle_{I_-} - \langle w\rangle_{I_+}|}{\langle w\rangle_I} |I| \le C\|f\|_2\|g\|_2$$

is equivalent to

$$|(T_0 f, g)| \le C\|f\|_2\|g\|_2.$$

Now we can finish the reduction. Second and third assumptions of Theorem 0.1 say exactly that for pairs $f = \chi_J w^{1/2}, g = \chi_J g$ and $f = \chi_J f, g = \chi_J v^{1/2}$ the supremum of bilinear forms $\sup_\varepsilon |(v^{1/2}T_\varepsilon w^{1/2} f, g)|$ has the desired estimate. The consideration written above then implies that on such pairs the bilinear form $|(T_0 f, g)|$ has the desired estimate. But this means exactly that the second and third assumptions of Theorem 0.3 are satisfied. The reduction is finished. We saw that the assumptions of Theorem 0.1 imply the first three assumptions of Theorem 0.2 and all assumptions of Theorem 0.3, but they, in their turn, imply the fourth assumption of Theorem 0.2. So we are left to prove theorems 0.2 and 0.3.

Notice that Theorem 0.2 is already proved. In fact, we established that the estimate $\sup_\varepsilon |(T_\varepsilon f, g)| \le C\|f\|_2\|g\|_2$ implies (1.1)–(1.3). We also know that (1.1)–(1.3) guarantee the equivalence of the inequality $\sup_\varepsilon |(T_\varepsilon f, g)| \le C\|f\|_2\|g\|_2$ with the inequality $|(T_0 f, g)| \le C\|f\|_2\|g\|_2$. So Theorem 0.2 is completely proved.

3. 3. A_∞ conditions and sufficient conditions.

In this section we show that relatively simple conditions (1.1)–(1.3) are already sufficient for uniform boundedness of $\|T_\varepsilon\|_{L^2(w^{-1}\to L^2(v)}$ if weights have certain A_∞ properties. In the first theorem below we assume that either v or w belong to A_∞.

Theorem 3.1 *If necessary conditions (1.2), (1.3) hold and one of the functions v or w is in A_∞ then (2.7) holds too.*

Proof. Let $v \in A_\infty$. Then (see [FKP]) we have

$$\forall\ J, \quad \frac{1}{|J|}\sum_{I\subset J} \langle v\rangle_I \left(\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right)^2 |I| \le C\langle v\rangle_J. \tag{3.1}$$

Application of Lemma 2.1 now proves that

$$\sum_{I\in\mathcal{D}} |\langle gv^{1/2}\rangle_I|^2 \cdot \frac{1}{\langle v\rangle_I}\left(\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right)^2 ||I| \le C\|g\|_2. \tag{3.2}$$

Then the left part of (2.7) can be estimated as

$$\left(\sum |\langle gv^{1/2}\rangle_I|^2 \frac{1}{\langle v\rangle_I}\left(\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right)^2 I\right)^{1/2}$$

$$\left(\sum |\langle f w^{1/2}\rangle_I|^2 \left(\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right)^2 \langle v\rangle_I I\right)^{1/2}$$

$$\leq C\|g\|_2 \cdot \left(\sum |\langle f w^{1/2}\rangle_I|^2 \left(\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right)^2 \langle v\rangle_I I\right)^{1/2}.$$

Now we apply Lemma 2.1 to the second factor. Notice that the condition of Lemma 2.1 is satisfied (it is just (1.2)) we conclude that the second factor is at most $C\|f\|_2$, and the theorem is proved for the case $v \in A_\infty$. The case $w \in A_\infty$ is completely symmetric.

In the next theorem we require that

$$w \in A_\infty(v). \tag{3.3}$$

We remind the reader that there are many equivalent wordings of this assertion, which can be found in [St]. For us here it will be important that this property means the following

$$\forall \varepsilon > 0 \exists \delta > 0,\, \forall I, E, E \subset I \ \frac{v(E)}{v(I)} \leq \delta \Rightarrow \frac{w(E)}{w(I)} \leq \varepsilon.$$

Remind also that this property is symmetric.

Theorem 3.2 *If w is in $A_\infty(v)$ and if w, v have the doubling property then operator T_0 is bounded if and only if (1.1) holds. In particular, in this situation $\sup_\varepsilon \|T_\varepsilon\|_{L^2(w^{-1})\to L^2(v)} < \infty$ if and only if (1.1)–(1.3) hold.*

Boundedness of T_0 is equivalent to (2.7). We have to be able to prove (2.7) starting with our assumptions. An important particular case of (2.7) appears when one writes (2.7) for $g = \chi_J v^{1/2}$, $f = \chi_J w^{1/2}$ where J is an arbitrary dyadic interval. So we have to be able to prove the following "Carleson measure type inequality":

$$\forall J \in \mathcal{D},\ \frac{1}{|J|}\sum_{I\subset J} \langle v\rangle_I \cdot \langle w\rangle_I \frac{|\langle v\rangle_{I_-} - \langle v\rangle_{I_+}|}{\langle v\rangle_I} \cdot \frac{|\langle w\rangle_{I_-} - \langle w\rangle_{I_+}|}{\langle w\rangle_I}|I|$$
$$\leq C\sqrt{\langle v\rangle_J \langle w\rangle_J}$$

.

It turns out that the proof of this inequality plays an important role in the proof of Theorem 3.2.

Lemma 3.3 *If (1.1) holds then the above inequality holds. Moreover, for any $\alpha \in (0,1]$, $\alpha \neq \frac{1}{2}$, the following more general inequality holds*

$$\forall J \in \mathcal{D},\ \frac{1}{|J|}\sum_{I\subset J}(\langle v\rangle_I\cdot\langle w\rangle_I)^\alpha\frac{|\langle v\rangle_{I_-}-\langle v\rangle_{I_+}|}{\langle v\rangle_I}\cdot\frac{|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|}{\langle w\rangle_I}|I|$$
$$\le C(\alpha)(\langle v\rangle_J\langle w\rangle_J)^{\min(\alpha,\frac{1}{2})}$$

.

We postpone the proof of Lemma 3.3 till the next section. We will see there that the constant $C(\alpha)$ blows up when α approaches $\frac{1}{2}$. Now we use it to give the proof of Theorem 3.2.

Proof of Theorem 3.2 Let J be in $\mathcal{D}$ and let $\mathcal{I}$ be a disjoint family of its dyadic subintervals having the following property

$$\frac{\langle F\rangle_I\langle G\rangle_I}{\langle w\rangle_I\langle v\rangle_I}\ge B\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}.$$

Then $\frac{\cup_{I\in\mathcal{I}}v(I)}{v(J)}\le\varepsilon$ and $\frac{\cup_{I\in\mathcal{I}}w(I)}{w(J)}\le\varepsilon$, where ε depends on B and is small if B is large.

In fact, intervals I are either of the type that $\frac{\langle F\rangle_I}{\langle w\rangle_I}\ge\sqrt{B}\frac{\langle F\rangle_J}{\langle w\rangle_J}$ or of the type that $\frac{\langle G\rangle_I}{\langle v\rangle_I}\ge\sqrt{B}\frac{\langle G\rangle_J}{\langle v\rangle_J}$. Let Ω_1 be the union of the first type intervals, and let Ω_2 denote the union of the second type intervals, which are not in Ω_1. Then obviously

$$w(\Omega_1)/w(J)\le\delta,\ v(\Omega_1)/v(J)\le\delta.$$

We are ready to use that $w\in A_\infty(v)$ (which also means $v\in A_\infty(w)$) to conclude that

$$\frac{\cup_{I\in\mathcal{I}}v(I)}{v(J)}\le\varepsilon,\ \frac{\cup_{I\in\mathcal{I}}w(I)}{w(J)}\le\varepsilon. \tag{3.4}$$

To apply this remark let us fix f,g from $L^2(\mathbb{T})$ and denote $F=fw^{1/2}, G=gv^{1/2}$. Let $\langle F\rangle,\langle G\rangle$ are averages over $\mathbb{T}$. Fix a very large number C to be chosen later and let $\mathcal{J}_k$ denote maximal dyadic intervals such that

$$\frac{\langle F\rangle_I\langle G\rangle_I}{\langle w\rangle_I\langle v\rangle_I}\ge C^k\frac{\langle F\rangle\langle G\rangle}{\langle w\rangle\langle v\rangle}.$$

Also let $\mathcal{G}_k$ denote the collection of dyadic intervals, which are contained in some J from $\mathcal{J}_k$ and are not contained in any J from $\mathcal{J}_{k+1}$.

Now we use Lemma 3.3 and (1.1) to conclude the following

$$\sum_{I\in\mathcal{G}_k}\langle F\rangle_I\langle G\rangle_I\frac{|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|}{\langle w\rangle_I}\cdot\frac{|\langle v\rangle_{I_-}-\langle v\rangle_{I_+}|}{\langle v\rangle_I}|I|\le$$

$$\frac{\langle F\rangle\langle G\rangle}{\langle w\rangle\langle v\rangle}C^{k+1}\sum_{I\in\mathcal{G}_k}\langle w\rangle_I\langle v\rangle_I\frac{|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|}{\langle w\rangle_I}\cdot\frac{|\langle v\rangle_{I_-}-\langle v\rangle_{I_+}|}{\langle v\rangle_I}|I|\le$$

$$\frac{\langle F\rangle\langle G\rangle}{\langle w\rangle\langle v\rangle}C^{k+1}\sqrt{A}\sum_{J\in\mathcal{J}_k}\sqrt{\langle w\rangle_J\langle v\rangle_J}|J|\le C\sqrt{A}\sum_{J\in\mathcal{J}_k}\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}\sqrt{\langle w\rangle_J\langle v\rangle_J}|J|=$$

$$C\sqrt{A}\sum_{J\in\mathcal{J}_k}\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}\sqrt{w(J)v(J)}$$

Here $A\stackrel{\text{def}}{=}\sup_{J\in\mathcal{D}}\langle w\rangle_J\langle v\rangle_J$.

For an interval J from $\mathcal{J}_k$ let us denote by E_J the set $J\setminus\cup_{I\in\mathcal{J}_{k+1}}I$. We would like to replace $w(J),v(J)$ in the last sum by $w(E_J),v(E_J)$. Suppose for a moment that we can do that. Let us denote by M_v the dyadic maximal function with respect to measure vdm. Do similarly for w to obtain M_w. Notice that $\langle G\rangle_J/\langle v\rangle_J\le M_v(gv^{-1/2})(x)$ for any $x\in J$. Similarly, $\langle F\rangle_J/\langle w\rangle_J\le M_w(fw^{-1/2})(x)$ for any $x\in J$. Notice also that all E_J are disjoint, and we assume temporarily that, say, $w(J)\le 2w(E_J)$ and $v(J)\le 2v(E_J)$.

Then we finish our estimate as follows:

$$\sum_{I\in\mathcal{D}}\langle F\rangle_I\langle G\rangle_I\frac{|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|}{\langle w\rangle_I}\cdot\frac{|\langle v\rangle_{I_-}-\langle v\rangle_{I_+}|}{\langle v\rangle_I}|I|\le$$

$$\sum_k\sum_{I\in\mathcal{G}_k}...\le 2C\sqrt{A}\sum_k\sum_{J\in\mathcal{J}_k}\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}\sqrt{w(E_J)v(E_J)}\le$$

$$2C\sqrt{A}(\int(M_w(fw^{-1/2})(x))^2w(x)dx)^{1/2}(\int(M_v(gv^{-1/2})(x))^2v(x)dx)^{1/2}\le$$

$$2C\sqrt{A}\|M_w(fw^{-1/2}\|_{L^2(w)}\|M_v(gv^{-1/2}\|_{L^2(v)}\le C'\|f\|_2\|g\|_2$$

We are left to choose C so large that $w(J)\le 2w(E_J)$ and $v(J)\le 2v(E_J)$. To do that let us notice that the maximality of J in $\mathcal{J}_k$ and the doubling properties imply that

$$C^k\le\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}\le DC^k.$$

So if $I\subset J$ belongs to $\mathcal{J}_{k+1}$ we have that

$$\frac{\langle F\rangle_I\langle G\rangle_I}{\langle w\rangle_I\langle v\rangle_I}\ge\frac{C}{D}\frac{\langle F\rangle_J\langle G\rangle_J}{\langle w\rangle_J\langle v\rangle_J}.$$

The remark at the beginning of the proof of the theorem shows that if C is much larger than D then the v-measure of $J \cap (\cup_{I\in\mathcal{J}_{k+1}} I)$ is smaller than any given ε (say, $\frac{1}{2}$). And the same is true about w-measure of this set.

Thus $w(J) \le 2w(E_J)$ and $v(J) \le 2v(E_J)$, and the theorem is completely proved.

4. 4. Bellman function and Carleson measures

Here we are going to prove Lemma 3.3. We also apply the lemma to give another sufficient condition for uniform boundedness of our T_ε.

In the proof of Lemma 3.3 the Bellman functions approach appears for the first time. It will play the key role in the rest of the paper.

Proof of Lemma 3.3. Without loss of generality we assume that

$$\langle v\rangle_I \langle w\rangle_I \le 1 \tag{4.1}$$

Let us first prove the case $\alpha \in (0,1/2)$. Let us introduce the function

$$\Phi(x,y) = \sup \frac{1}{|J|}\sum_{I\subseteq J} (\langle v\rangle_I \langle w\rangle_I)^\alpha \left|\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right| \left|\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right| |I|$$

where supremum is taken over all nonnegative finite linear combinations v, w of characteristic functions of dyadic intervals such that $v_J = x, w_J = y$. Notice that Φ does not depend on J. Notice also that by the definition Φ is concave. Moreover if $x = \frac{x_+ + x_-}{2}$, $y = \frac{y_+ + y_-}{2}$ then

$$\Phi(x,y) - \frac{(\Phi(x_+,y_+) + \Phi(x_-,y_-))}{2} \ge (xy)^\alpha \left|\frac{x_+ - x_-}{x}\right| \left|\frac{y_+ - y_-}{y}\right| \tag{4.2}$$

If only one could prove that $\Phi(x,y) \le C(\alpha)(xy)^\alpha \dots$. Let us reverse the argument. Suppose we can construct a concave B in the domain $D = \{x \ge 0, y \ge 0, xy \le 1\}$ such that

$$0 \le B \le (xy)^\alpha, \quad (x,y) \in \mathcal{D} \tag{4.3}$$

and such that

$$B(x,y) - \frac{(B(x_+,y_+) + B(x_-,y_-))}{2} \ge c_\alpha (xy)^\alpha \left|\frac{x_+ - x_-}{x}\right| \left|\frac{y_+ - y_-}{y}\right| \tag{4.4}$$

if $(x,y), (x_+,y_+), (x_-,y_-) \in D$ and if

$$x = \frac{x_+ + x_-}{2}, \quad y = \frac{y_+ + y_-}{2}.$$

Then we are done.

And for $\alpha \in (0,1/2)$ it is easy to guess such a function:

$$B(x,y) = (xy)^\alpha.$$

Let us consider it in a larger domain $D_0 = \{x \ge 0, y \ge 0\}$ and prove (4.4) there. Consider the function

$$b(t) = B(x + t\xi, y + t\eta), \quad t \in [-1, 1]$$

where $\xi = \frac{x_+ - x_-}{2}$, $\eta = \frac{y_+ - y_-}{2}$. Clearly b is concave, because B is concave in $\{x \geq 0, y \geq 0\}$. We want to estimate $b''(t)$ and to prove that

$$b''(t) \leq -c_\alpha \frac{|\xi\eta|}{x^{1-\alpha}y^{1-\alpha}}, \quad t \in [-1/2, 1/2]. \tag{4.5}$$

If (4.5) were proved, then we could write

$$b(0) - \frac{1}{2}(b(-1) + b(1)) = -\int_{-1}^{1}(1 - |t|)b''(t)dt \geq c_\alpha \frac{|\xi\eta|}{x^{1-\alpha}y^{1-\alpha}}$$

which is exactly (4.4).

Let $\mathcal{H}_{x,y}(\xi, \eta)$ be the Hessian form of B at point (x, y) on the vector (ξ, η). Denote $x_t = x + t\xi$, $y_t = y + t\xi$. Then

$$b''(t) = \mathcal{H}_{x_t,y_t}(\xi, \eta).$$

Here is the computation of $\mathcal{H}_{x,y}(\xi, \eta)$:

$$\mathcal{H}_{x_t,y_t}(\xi, \eta) = -\alpha x_t^\alpha y_t^\alpha \left[(1 - \alpha)(\frac{\xi}{x_t})^2 + (1 - \alpha)(\frac{\eta}{y_t})^2 - 2\alpha\frac{\xi\eta}{x_t y_t}\right] =$$

$$-\alpha x_t^\alpha y_t^\alpha \left[(1 - 2\alpha)((\frac{\xi}{x_t})^2 + (\frac{\eta}{y_t})^2) + \alpha(\frac{\xi}{x_t} + \frac{\eta}{y_t})^2\right] \leq$$

$$-(1 - 2\alpha)\alpha\frac{|\xi\eta|}{x_t^{1-\alpha}y_t^{1-\alpha}}$$

Noticing that for $t \in [-1/2, 1/2]$, $x_t \geq \frac{x}{2}$, $y_t \geq \frac{y}{2}$ we get (4.5), which gives the proof of Lemma 3.3 for the case $\alpha < 1/2$.

Our key inequality (4.5) could be proved differently. Denote $t = xy$, $x_\pm = (1 \pm \lambda)x$, $y_\pm = (1 \pm \mu)y$, $\lambda, \mu \in [-1, 1]$ we come to proving that for $\lambda, \mu \in [-1, 1]$ the following holds

$$1 - \frac{1}{2}\left\{[(1 - \lambda)(1 - \mu)]^\alpha + [(1 + \lambda)(1 + \mu)]^\alpha\right\} \geq c_\alpha|\lambda\mu| \tag{4.6}$$

This elementary inequality is true when $\alpha \in (0, 1/2)$, which can be checked by direct computations.

Now let us consider the case $\alpha \in (1/2, 1]$. Again we should notice that it is enough to build a function B such that

$$0 \leq B(x, y) \leq C(xy)^{1/2}, \quad (x, y) \in D, \tag{4.7}$$

and

$$B(x, y) - \frac{B(x_+, y_+) + B(x_-, y_-)}{2} \geq c_\alpha(xy)^\alpha \left|\frac{x_+ - x_-}{x}\right| \left|\frac{y_+ - y_-}{y}\right| \tag{4.8}$$

where both inequalities hold in $D = \{(x,y) : x \geq 0, y \geq 0, xy \leq 1\}$ and $x = \frac{x_+ + x_-}{2}$, $y = \frac{y_+ + y_-}{2}$.

Unfortunately, $B = (xy)^{1/2}$ does not work. It is not concave enough near the diagonal $x = y$. Here is the function which satisfies (4.7), (4.8):

$$B(x,y) = (xy)^{1/2} - \frac{1}{4}(xy)^{\alpha}.$$

Again there is nothing to prove in (4.7). It is a nonneggative function because $xy \leq 1$ in D.

Consider the function

$$b(t) = B(x + t\xi, y + t\eta), \quad t \in [-1,1]$$

where $\xi = \frac{x_+ - x_-}{2}$, $\eta = \frac{y_+ - y_-}{2}$. Clearly b is concave, because B is concave in $\{x \geq 0, y \geq 0\}$. We want to estimate $b''(t)$ and to prove that

$$b''(t) \leq -c_\alpha \frac{|\xi\eta|}{x^{1-}y^{1-\alpha}}, \quad t \in [-1/2, 1/2]. \tag{4.9}$$

If (4.9) were proved, then we could write

$$b(0) - \frac{1}{2}(b(-1) + b(1)) = -\int_{-1}^{1}(1 - |t|)b''(t)dt \geq C(\alpha)\frac{|\xi\eta|}{x^{1-\alpha}y^{1-\alpha}}$$

which is exactly (4.8).

Let $\mathcal{H}_{x,y}(\xi,\eta)$ be the Hessian form of B at point (x,y) on the vector (ξ,η). Denote $x_t = x + t\xi$, $y_t = y + t\xi$. Then

$$b''(t) = \mathcal{H}_{x_t,y_t}(\xi,\eta).$$

Here is the computation of $\mathcal{H}_{x,y}(\xi,\eta)$:

$$\mathcal{H}_{x,y}(\xi,\eta)=\frac{1}{4}(xy)^{1/2}[2(1-\alpha^2(xy)^{\alpha-1/2})\frac{\xi\eta}{xy}-(1-\alpha(1-\alpha)(xy)^{\alpha-1/2})(\frac{\xi}{x})^2-$$
$$(1-\alpha(1-\alpha)(xy)^{\alpha-1/2})(\frac{\eta}{y})^2]$$

Let us denote by ρ the expression $\alpha^2(xy)^{\alpha-1/2}$ and by r the expression $\alpha(1-\alpha)(xy)^{\alpha-1/2}$. Then

$$\mathcal{H}_{x,y}(\xi,\eta)=\frac{1}{4}(xy)^{1/2}\Bigg\{-(1-\rho)\left[\frac{\xi}{x}+\frac{\eta}{y}\right]^2$$
$$-(\rho-r)\left[\left(\frac{\xi}{x}\right)^2+\left(\frac{\eta}{y}\right)^2\right]\Bigg\}\leq-\frac{1}{4}(\rho-r)(xy)^{1/2}\frac{|\xi\eta|}{xy}.$$

Let us look at $\rho-r=(\alpha^2-\alpha(1-\alpha))(xy)^{\alpha-1/2}=+\alpha(2\alpha-1)(xy)^{\alpha-1/2}$. It is positive, when $\alpha>1/2$. In particular,

$$\mathcal{H}_{x_t,y_t}(\xi,\eta)\leq-c_\alpha\frac{|\xi\eta|}{x^{1-\alpha}y^{1-\alpha}} \tag{4.10}$$

if $t\in[-1/2,1/2]$ because $x_t\geq\frac{x}{2}$, $y_t\geq\frac{y}{2}$ for such t. Now (4.10) implies (4.9) immediately. The proof of Lemma 3.3 is completed.

Let us show some corollaries of Lemma 3.3. Fix a number $q\in(0,1)$. We denote by $\mathcal{F}_k$, $k=0,1,2,\dots$ the family of all dyadic intervals I such that

$$q^{k+1}\leq v_Iw_I\leq q^k.$$

Consider the following measure in the disk

$$\sigma_k=\sum_{I\in\mathcal{F}_k}\left|\frac{v_{I_-}-v_{I_+}}{v_I}\right|\left|\frac{w_{I_-}-w_{I_+}}{w_I}\right||I|\delta_{c(I)},$$

where $c(I)$ denotes the center of the box $Q(I)$ built over the dyadic arc I.

Let us remind that measure σ in $\mathbb{D}$ is called Carleson if $\sigma(Q(I))\leq C|I|$ for all arcs I. The best constant in this inequality is called the Carleson norm of σ and is denoted by $\|\sigma\|_C$.

Lemma 4.1. *Measures σ_k are Carleson measures and $\|\sigma_k\|_C\leq B,\infty$.*

The proof follows immediately from Lemma 3.3 with $\alpha\in(0,1/2)$. Let us notice that in the case when $\langle v\rangle_I\langle w\rangle_I$ is bounded also from below (so in the case when we can assume $w=u^{-1}=v$, and we consider a one weight problem) we get that $\sum_{k=0}^\infty\sigma_k$ is a Carleson measure.

We will see now how a slightly strengthen version of (1.1) becomes sufficient for the boundedness of $\|T_0\|_{L^2(w^{-1})\to L^2(v)}$ (but it stops to be necessary of course). We use the Carleson measure result that has been just proved.

Theorem 4.2. *Suppose that*

$$\forall\, I \in \mathcal{D}, \quad \langle v^{1+\eta}\rangle_I \langle w^{1+\eta}\rangle_I \le A < \infty \tag{4.11}$$

Then $\|T_0\|_{L^2(w^{-1})\to L^2(v)} \le C(A) < \infty$.

Remark. There is a whole stream of results of this kind starting with Fefferman-Phong theorem from [F] and continuing in [ChWW], [Z].

Proof. Let us remind that the boundedness of T_0 is equivalent to the following estimate for all L^2-functions f, g

$$\sum_{I\in\mathcal{D}} \langle g v^{1/2}\rangle_I \langle f w^{1/2}\rangle_I \alpha_I \le C\|g\|_2\|f\|_2. \tag{4.12}$$

where

$$\alpha_I = \left|\frac{\langle v\rangle_{I_-} - \langle v\rangle_{I_+}}{\langle v\rangle_I}\right| \left|\frac{\langle w\rangle_{I_-} - \langle w\rangle_{I_+}}{\langle w\rangle_I}\right| |I|.$$

Clearly for $\gamma + \rho = 1/2, \gamma, \rho > 0$, we have

$$\langle g v^{1/2}\rangle_I \le \langle g^p\rangle_I^{1/p} \langle v^{\gamma q_1}\rangle_I^{1/q_1} \langle v^{\rho q_2}\rangle_I^{1/q_2}$$

where $1/p + 1/q_1 + 1/q_2 = 1$. Choose $\gamma = \rho = 1/4,\ q_1 = 4,\ q_2 = 4(1+\eta)$ to have $p = 4\frac{1+\eta}{2+3\eta} < 2$. Then

$$\begin{aligned} \langle g v^{1/2}\rangle_I &\le \langle g^p\rangle_I^{1/p} \langle v^{1+\eta}\rangle_I^{1/q_2} \langle v\rangle_I^{1/4}, \\ \langle f w^{1/2}\rangle_I &\le \langle f^p\rangle_I^{1/p} \langle w^{1+\eta}\rangle_I^{1/q_2} \langle w\rangle_I^{1/4}. \end{aligned}$$

Remind that measures from Lemma 4.1

$$\sigma_k = \sum_{I\in\mathcal{F}_k} \alpha_I \delta_{c(I)}$$

are proved to be uniformly Carleson. Thus the measure

$$\sigma \stackrel{def}{=} \sum_{I\in\mathcal{D}} (v_I w_I)^{1/4} \alpha_I \delta_{c(I)} \le \sum_k (q^k)^{1/4} \sigma_k$$

is a Carleson measure. Let us notice that $2/p > 1$, and let us use the Carleson embedding theorem in $L^{2/p}$ (see [G]) to estimate the sum in (4.12).

$$\sum_{I\in\mathcal{D}}\langle gv^{1/2}\rangle_I\langle fw^{1/2}\rangle_I\alpha_I \le \sup_I(\langle v^{1+\eta}\rangle_I\langle w^{1+\eta}\rangle_I)^{1/q_2}$$

$$\cdot\sum_{I\in\mathcal{D}}\langle g^p\rangle_I^{1/p}\langle f^p\rangle_I^{1/p}(\langle v\rangle_I\langle w\rangle_I)^{1/4}\alpha_I$$

$$\le\left(\sum_{I\in\mathcal{D}}\langle f^p\rangle_I^{2/p}(\langle v\rangle_I\langle w\rangle_I)^{1/4}\alpha_I\right)^{1/2}\left(\sum_{I\in\mathcal{D}}\langle f^p\rangle_I^{2/p}(\langle v\rangle_I\langle w\rangle_I)^{1/4}\alpha_I\right)^{1/2}$$

$$\le C(p)\|\sigma\|_C\|g^p\|_{L^{2/p}}^{1/p}\|f^p\|_{L^{2/p}}^{1/p}\le C\|f\|_{L^2}\|g\|_{L^2}.$$

We are done.

5. 5. The necessary and sufficient conditions: Bilinear weighted imbedding theorem

We saw in Section 2 that given necessary conditions (1.1), (1.2), (1.3) the uniform boundedness of $\|T_\varepsilon\|_{L^2(w^{-1})}\to L^2(v)$ is equivalent to the following estimate for all L^2-functions f, g

$$\sum\langle gv^{1/2}\rangle_I\langle fw^{1/2}\rangle_I\alpha_I\le C\|g\|_2\|f\|_2 \tag{5.1}$$

where $\alpha_I=\left|\frac{\langle v\rangle_{I_-}-\langle v\rangle_{I_+}}{\langle v\rangle_I}\right|\left|\frac{\langle w\rangle_{I_-}-\langle w\rangle_{I_+}}{\langle w\rangle_I}\right||I|$. We may (and should) consider (5.1) as a bilinear weighted imbedding estimate.

Consider the kernel function

$$k(x,y)=\sum_{I\in\mathcal{D}}\frac{1}{|I|^2}\chi_I(x)\chi_I(y)\alpha_I$$

It is obvious that (5.1) means the estimate for any nonnegative measurable G:

$$\int\left(\int k(x,y)G(y)w(y)dy\right)^2v(x)dx\le C\int G^2w\,dy \tag{5.2}$$

The inequalities (5.2) have been extensively studied in the works of Kalton, Sawyer, Verbitsky, and Wheeden. The next result is in the vein of those works. Inequalities (5.2) can be most probably proved using the combination of ideas frim the papers of Kalton, Verbitsky [KV] and Sawyer, Wheeden [SW]. Following [KV] we can introduce the new metric $d(x,y):=\frac{1}{k(x,y)}$, where k is the kernel of T_0 and was written above. It is clearly a metric, and balls in this metric are just all dyadic intervals. Then kernel k satisfies the regularity condition from [SW]. Using the main result of [SW] we could have given an alternative proof of inequalities (5.2). Unfortunately it is not clear why the new metric space has certain regularity properties. For example, in [KV] one requires the property that all annuli be nonempty. This is false in our new metric space. However, it is not clear to us how essential are those regularity properties of the metric space for the application of the technique of [SW].

So we will give another proof.

Plug into (5.2) the charecteristic functions of the intervals. Then we get a condition necessary for (5.2). Do the same with the dual inequality. Then we get another necessary condition. These are are the conditions of the Sawyer type:

$$\forall\, J\in\mathcal{D},\qquad \frac{1}{|J|}\int_J\left(\int_J k(x,y)w(y)dy\right)^2 v(x)dx\le C\langle w\rangle_J, \tag{5.3}$$

$$\forall\, J\in\mathcal{D},\qquad \frac{1}{|J|}\int_J\left(\int_J k(x,y)v(x)dx\right)^2 w(y)dy\le C\langle v\rangle_J. \tag{5.4}$$

In other words, for all dyadic arcs J

$$\frac{1}{|J|}\int_J\left(\sum_{I\subset J}\frac{1}{I}\chi_I\langle w\rangle_I\alpha_I\right)^2 v\,dx\le C\langle w\rangle_J, \tag{5.5}$$

$$\frac{1}{|J|}\int_J\left(\sum_{I\subset J}\frac{1}{I}\chi_I\langle v\rangle_I\alpha_I\right)^2 w\,dx\le C\langle v\rangle_J. \tag{5.6}$$

Let us denote by T_0 the operator with kernel k.

Theorem 5.1. *Let $\{\alpha_I\}$ be a nonnegative sequence. The following assertions are equivalent*

1) *Operator T_0 is bounded from $L^2(w^{-1})$ to $L^2(v)$;*
2) *Estimate (5.1) holds;*
3) *Estimate (5.2) holds;*
4) *For any $J\in\mathcal{D}$ (5.5) and (5.6) hold with the same constant C independent of J.*

Remark: Clearly (5.5) (or (5.6) as well) implies

$$\frac{1}{|J|}\sum_{I\subset J}\langle v\rangle_I\langle w\rangle_I\alpha_I\le C\sqrt{\langle v\rangle_J\langle w\rangle_J}, \tag{5.7}$$

which is the conclusion of Lemma 3.3 for the case $\alpha=1$, and so (5.7) follows from just (1.1). Unfortunately, (5.5), (5.6) themselves do not follow from (1.1).

On the other hand (5.7) is a particular case of our key estimate (5.1). In fact, (5.1) becomes (5.7) if $g=\chi_J v^{1/2}$, $f=\chi_J w^{1/2}$. The conclusion is that the particular case of (5.1) for test functions $\chi_J v^{1/2}$, $\chi_J w^{1/2}$ follows from (1.1). But this is not the case for the full of (5.1). However, (5.5) and (5.6) are in fact the conditions obtained by testing (5.1) by certain test functions. We are going to repeat what has been already said above. Namely, it is easy to see that (5.5) is the testing of (5.1) by $f=\chi_J w^{1/2}$

and $g = g^J$, where g^J is any L^2-function supported on J. Similarly (5.6) is the testing of (5.1) by $g = \chi_J v^{1/2}$ and $f = f^J$, where f^J is any L^2-function suppoprted on J.

Let us conclude that in a particular case $v = w$ the bilinear weighted imbedding estimate (5.1) becomes a usual weighted L^2-imbedding estimate for linear operator $f \to \{(fw^{1/2})_I\}_{I\in\mathcal{D}}$:

$$\sum \langle fw^{1/2}\rangle_I^2 \alpha_I \le C\|f\|_2^2. \tag{5.8}$$

Test conditions (5.5), (5.6) become

$$\forall\, J \in \mathcal{D},\ \frac{1}{|J|}\sum_{I\subset J} \langle w\rangle_I^2 \alpha_I \le C\langle w\rangle_J. \tag{5.9}$$

Let us start our proof of Theorem 5.1 with a special proof of the classical equivalence (5.8) $\Leftrightarrow$ (5.9) (see [S1], [S2], [TV]).

6. 6. The necessary and sufficient conditions: the proof of Theorem 5.1 for the case $w = v$

Theorem 6.1 (5.8) $\Leftrightarrow$ (5.9).

Proof. One needs to prove only the implication (5.9) $\Rightarrow$ (5.8). Let us consider the function

$$\Phi_J(X, x, w) = \sup \frac{1}{|J|}\sum_{I\subseteq J} \langle fw^{1/2}\rangle_I^2 \alpha_I$$

where supremum is taken over $f \ge 0$, $\langle fw^{1/2}\rangle_J = x$, $\langle f\rangle_J^2 = X$, $\langle w\rangle_J = w$. Clearly, this restricts us to

$$D = \{(X, x, w) : x^2 \le Xw,\ X, x, w \ge 0\}.$$

The family of functions Φ_J is concave in the following sense. Let $X = \frac{X_- + X_+}{2}$, $x = \frac{x_- + x_+}{2}$, $w = \frac{w_- + w_+}{2}$. Then

$$\Phi_J(X, x, w) - \frac{1}{2}(\Phi_{J_-}(X_-, x_-, w_-) + \Phi_{J_+}(X_+, x_+, w_+) \ge \frac{1}{|J|}x^2\alpha_J \tag{6.1}$$

In fact, fixing averages on J_-, J_+ separately leads to a smaller set of functions when if fixing the averages only on J (we keep the compatability by saying that the averages over J are arithmetic means of averages over J_-, J_+.

One wishes to prove that $\Phi_J \le CX$. This certainly would give the result. But this is certainly false because we did not use the condition (5.9) in the definition of Φ_J.

Let us try to correct this by introducing

$$\Psi_J(X, x, w) = \sup \frac{1}{|J|}\sum_{I\subseteq J} \langle fw^{1/2}\rangle_I^2 \alpha_I$$

where supremum is taken over the same set of conditions including $\langle w\rangle_J = w$ plus the condition that $\{\langle w\rangle_I\}_{I\subset J}$ are so distributed that

$$\frac{1}{|J'|}\sum_{I\subseteq J'}\langle w\rangle_I^2\alpha_I\le C\langle w\rangle_{J'},\quad \forall\ J'\subset J \tag{6.2}$$

Unfortunately, it is now not clear why (6.1) would hold with Ψ_J replacing Φ_J. Conditioning as (6.1) makes unclear why $\{\Phi_J\}$ has even simple concavity property. In fact, by passing to J_-, J_+ from J we did not deminished the set over which the supremum is taking. Unlike the case of Φ above we actually relaxed our requirements. Notice that inequality (6.2) for $J^{'}=J$ will not be required anymore after passing to J_-, J_+.

But one can overcome this difficulty by fixing the "main" sum in (6.1) and making its value M the new variable. To do that let us consider

$$\Phi_J(X,x,w,M)\stackrel{def}{=}\sup\frac{1}{|J|}\sum_{I\subseteq J}\langle fw^{1/2}\rangle_I^2\alpha_I$$

where supremum is taken over such functions that $f\ge 0$, $\langle fw^{1/2}\rangle_J=x$, $\langle f\rangle_J^2=X$, $\langle w\rangle_J=w$, and over sequences $\{\alpha_I\}$ and functions such that

$$\frac{1}{|J'|}\sum_{I\subseteq J'}\langle w\rangle_I^2\alpha_I\le C\langle w\rangle_{J'},\quad \forall\ J'\subset J, \tag{6.3}$$

and such that

$$M=\frac{1}{|J|}\sum_{I\subseteq J}\langle w\rangle_I^2\alpha_I. \tag{6.4}$$

Notice that thus defined B does not depend on J, and that it is cleary defined and concave in the domain

$$D_B\ =\ \{(X,x,w,M): x^2\le Xw,\ (X,x,w,M)\ge 0,\ M\le Cw\}.$$

On the top of concavity we know the following. Let $h=M-\frac{M_-+M_+}{2}$. Clearly,

$$B(X,x,w,M)-\frac{B(X,x,w,M_-)+B(X,x,w,M_+)}{2}\ge\frac{x^2}{w^2}h. \tag{6.5}$$

Thus we conclude that B satisfies the following properties

$$0\ \le\ B\le X,\quad\text{on}\quad\mathcal{D}_B, \tag{6.6}$$

$$\frac{\partial B}{\partial M}\ \ge\ \frac{x^2}{w^2},\quad\text{on}\quad\mathcal{D}_B, \tag{6.7}$$

$$d^2B\ \le\ 0,\quad\text{on}\quad\mathcal{D}_B, \tag{6.8}$$

Instead of looking for the exact formula for B let us consider

$$B(X,x,w,M)=CX-C\frac{x^2}{w+M},$$

which satisfies all the properties (6.8)-(6.10). To finish the proof of (5.8) it is enough to consider triples $x=\langle fw^{1/2}\rangle_J,\ X=\langle f\rangle_J^2,\ w=\langle w\rangle_J, x_\pm=\langle fw^{1/2}\rangle_{J\pm},\ X_\pm=\langle f\rangle^2_{J_\pm},\ w_\pm=\langle w\rangle_{J\pm}$ and $M, M_\pm$ such that $M-\frac{M_-+M_+}{2}=\frac{1}{|J|}w^2\alpha_J$, and to prove that

$$B(X,x,w,M)-\frac{B(X_-,x_-,w_-,M_-)+B(X_+,x_+,w_+,M_+)}{2}\geq\frac{1}{|J|}x^2\alpha_J.$$

But the last inequality follows immediately from (6.7) and the concavity of B.

Remark. It is interesting to notice that one can give another euristic explanation of appearance of new variable M. Let us think that all averages are variables and all sums are functions defined on the set of all averages involved. Then M is a pretty concave function of w in the following sense:

$$M(w_J)-\frac{1}{2}[M(w_{J_+})+M(w_{J_-})]\geq\frac{1}{|J|}w_J^2\alpha_J. \tag{6.9}$$

Thus it is resonable to look for $Q(X,x,w)$ in the form $Q=B(X,x,w,M(w))$, where B has a large derivative with respect to M. Then we can use the following formula for the Hessian of the composition

$$d^2Q=d^2B+\frac{\partial B}{\partial M}\cdot d^2M. \tag{6.10}$$

7. 7. Necessary and sufficient conditions. The proof of the bilinear weighted imbedding theorem

We try to repeat the argument of the previous section. First of all it would be nice to discern the variables of the future Bellman function. Some of them are ready right away. These are $X=\langle f^2\rangle_J, x=\langle fw^{1/2}\rangle_J, w=w_J, Y=\langle g^2\rangle_J, y=\langle gv^{1/2}\rangle_J, v=v_J$. Imitating the considerations of the previous section we also fix $M=\frac{1}{|J|}\int_J\left(\sum_{I\subseteq J}\frac{1}{|I|}\chi_Iw_I\alpha_I\right)^2v\,dx$ and $N=\frac{1}{|J|}\int_J\left(\sum_{I\subseteq J}\frac{1}{|I|}\chi_Iv_I\alpha_I\right)^2w\,dx$. By doing this we are making the following function concave:

$$\Phi(X,x,w,Y,y,v,M,N)\stackrel{\text{def}}{=}\sup\frac{1}{|J|}\sum_{I\subset J}\langle fw^{1/2}\rangle_I\langle gv^{1/2}\rangle_I\alpha_I$$

where the supremum is taken over nonnegative functions with “main” averages fixed as above and such that

$$\forall I \in \mathcal{D},\, I \subset J\, \frac{1}{|I|}\int_I \left(\sum_{I' \subseteq I} \frac{1}{|I'|}\chi_{I'} x_{I'} \alpha_{I'}\right)^2 v\,dx \le C w_I$$

$$\forall I \in \mathcal{D},\, I \subset J\, \frac{1}{|I|}\int_I \left(\sum_{I' \subseteq I} \frac{1}{|I'|}\chi_{I'} v_{I'} \alpha_{I'}\right)^2 w\,dx \le C v_I$$

We could try to use Φ as Bellman function of our problem having hope that it gives our main inequality (5.1). Actually it does not. It gives a certain inequality, but it is weaker than the one we need. But on the other hand we used all possible variables (averages, conditions). There are no more averages involved, and there are no more conditions (if we believe that our bilinear imbedding theorem is true).

First let us find the way out of this impass by euristic considerations. Notice that in the previous sections once the Bellman function is found we apply a certain inequality not to the Bellman function itself but rather to its composition with martingales. In fact, all these $\langle f^2\rangle_J, ..., v = \langle v\rangle_J$ are martingales. There are also supermartingales. For example,

$$M_J = \frac{1}{|J|}\int_J \left(\sum_{I \subseteq J} \frac{1}{|I|}\chi_I w_I \alpha_I\right)^2 v\,dx$$

is a supermartingale. Let us pay attention to its "discrete Laplacian" $M_J - \frac{M_{J_-}+M_{J_+}}{2}$. If we consider a supermartingale $m_J \stackrel{\text{def}}{=} \frac{1}{|J|}\sum_{I\subseteq J}\langle w\rangle_I^2\alpha_I$ from the previous section we see that its "discrete Laplacian" $m_J - \frac{m_{J_-}+m_{J_+}}{2}$ is equal to $\frac{1}{|J|}\langle w\rangle_J^2\alpha_J$. We notice that it involves only martingale $\langle w\rangle_J$, which has been already chosen as a variable (called w) of the Bellman function.

What happens with "discrete Laplacian" of supermartingale M_J? When we calculate it, we get

$$M_J - \frac{M_{J_-}+M_{J_+}}{2} = \frac{1}{|J|}\langle w\rangle_J K_J \alpha_J, \tag{7.1}$$

where

$$\mathcal{K}_J \stackrel{\text{def}}{=} \frac{1}{|J|}\sum_{I\subset J}\langle w\rangle_I\langle v\rangle_I\alpha_I.$$

So the "second derivative" of supermartingale M_J involves not only the martingale $\langle w\rangle_J$ but also a new supermartingale K_J.

The supermartingale K_J, which appeared as "discrete Laplacian" of our "natural variable" supermartingale M also should be embodied by a variable–naturally let us call it K.

One may now try to continue this process by taking the "second derivative" of K_J:

$$K_J - \frac{K_{J_-} + K_{J_+}}{2} = \frac{1}{|J|}\langle w\rangle_J \langle v\rangle_J \alpha_J \tag{7.2}$$

to make sure that it involves only already considered martingales $\langle w\rangle_J, \langle v\rangle_J$. Fortunately no new variables appear after K.

The recipe. Let us repeat that we are calculating the "discrete Laplacian" of the composition of a certain (concave) function (Bellman function) and several martingales and supermartingales. It is only too natural that in this calculation the "discrete Laplacians" of all these martingales and supermartingales appear. For martingales they will be zero, for supermartingales they are positive and involve our martingales and may be some new (super)martingales. If this happens, this new (super)martingales should be added to the list of (super)martingales, which must become the variables.

In the situation we have now we have another supermartingale N_J to which we have to apply this algorithm. It is given by the formula:

$$N_J = \frac{1}{|J|}\int_J \left(\sum_{I\subseteq J} \frac{1}{|I|}\chi_I v_I \alpha_I\right)^2 w\, dx.$$

And so its "discrete laplacian" can be calculated as follows:

$$N_J - \frac{N_{J_-} + N_{J_+}}{2} = \frac{1}{|J|}\langle v\rangle_J K_J \alpha_J, \tag{7.3}$$

where K_J is the same supermartingale we found by calculating the "discrete Laplacian" of M_J above.

Now we have all the variables: $X, x, w, Y, y, v, K, M, N$. Let us make the notation for this 9-tuple and also for X, x, w, Y, y, v, M, N and X, x, w, Y, y, v, K.

$a \stackrel{\text{def}}{=} (X, x, w, Y, y, v, K, M, N)$, $b \stackrel{\text{def}}{=} (X, x, w, Y, y, v, M, N)$, $c \stackrel{\text{def}}{=} (X, x, w, Y, y, v, K)$.

For technical reason we will be looking for B in the form $B(a) = Q(b) + P(c)$.

The domain of definition is

$$D = D_B = \{a = (X, x, w, Y, y, v, K, M, N) : x^2 \le Xw, y^2 \le Yv,$$
$$M \le Cw, N \le Cv, K \le C\sqrt{wv}, a \ge 0\}.$$

The only thing requiring the clarification is why $K \le C\sqrt{wv}$ here? But K stands for supermartingale K_J for which this ineqality was proved in Lemma 3.3. So this ineqality represents a natural restriction on K.

What inequalities on B would be sufficient for us? It should be the inequality on the composition of B with all our (super)martingales. So consider $a_J = (\langle f^2\rangle_J, \langle fw^{1/2}\rangle_J, w_J, \langle g^2\rangle_J, \langle gv^{1/2}\rangle_J, v_J, K_J, M_J, N_J)$ and $a_{J_\pm}$ denoting the same thing for $J_\pm$ instead of J. Here are the inequality we want to obtain

$$B(a_J) - \frac{B(a_{J_-}) + B(a_{J_+})}{2} \ge c\frac{1}{|J|}\langle fw^{1/2}\rangle_J \langle gv^{1/2}\rangle_J \alpha_J \tag{7.4}$$

$$0 \le B(a) \le C(X+Y) \tag{7.5}$$

If we have both of these inequalities then moving from J to $J_\pm$ and then continuing these process to "sons" of $J_\pm$ et cetera... we obtain the inequality

$$\frac{1}{|J|}\sum_{I\subset J}\langle fw^{1/2}\rangle_I\langle gv^{1/2}\rangle_I\alpha_I \le C(\langle f^2\rangle_J+\langle g^2\rangle_J) \tag{7.6}$$

The use of homogenuity of the left part (it does not change under the replacement of f by tf, g by $t^{-1}g$) would finish the proof of our bilinear weighted imbedding theorem because we obtain

$$\frac{1}{|J|}\sum_{I\subset J}\langle fw^{1/2}\rangle_I\langle gv^{1/2}\rangle_I\alpha_I \le C\sqrt{\langle f^2\rangle_J\langle g^2\rangle_J} = C\|f\|_{L^2_J}\|g\|_{L^2_J}.$$

Lemma 7.1 *To have (7.4) it is sufficient to have the following inequalities*

$$B(a)-\frac{B(a_-)+B(a_+)}{2} \ge \gamma_1\frac{xy}{wv}(K-\frac{K_-+K_+}{2}) \tag{7.7}$$

if

$$\frac{x^2}{w}K+\frac{y^2}{v}K \le Cxy \tag{7.8}$$

and

$$B(a)-\frac{B(a_-)+B(a_+)}{2} \ge \gamma_2\frac{x^2}{w^2}(M-\frac{M_-+M_+}{2})+\gamma_3\frac{y^2}{v^2}(N-\frac{N_-+N_+}{2}), \tag{7.9}$$

if

$$\frac{x^2}{w}K+\frac{y^2}{v}K \ge Cxy \tag{7.10}$$

Here $a \ge \frac{a_-+a_+}{2}$. *And* γ_i, $i=1,2,3$, *are positive constants.*

Proof. Let us calculate the right parts of (7.7) and (7.9) for

$$a=(X,x,w,Y,y,v,K,M,N)=(\langle f^2\rangle_J,\langle fw^{1/2}\rangle_J,w_J,$$
$$\langle g^2\rangle_J,\langle gv^{1/2}\rangle_J,v_J,K_J,M_J,N_J),$$

and $a_\pm$ having the same meaning with J replaced by $J_\pm$. We do this calculation by using (7.1)-(7.3). In both cases, the right part is bigger than

$$\frac{1}{|J|}xy\alpha_J=\frac{1}{|J|}\langle fw^{1/2}\rangle_J\langle g^2\rangle_J\alpha_J.$$

Notice that we use (7.10) when we calculate the right part of (7.9). At any rate in both cases we get (7.4).

Now we are left to find B satisfying (7.7) and (7.9). As we mentioned we will be looking for B in the form $B(a) = Q(b)+P(c)$ where $a = (X, x, w, Y, y, v, K, M, N)$, $b = (X, x, w, Y, y, v, M, N)$, $c = (X, Y, x, w, y, v, K)$.

Lemma 7.2 *Function* $Q = CX + CY - \frac{x^2}{w+M} - \frac{y^2}{v+N}$ *satisfies the inequality*

$$Q(b) - \frac{Q(b_-) + Q(b_+)}{2} \geq \gamma_2 \frac{x^2}{w^2}(M - \frac{M_- + M_+}{2}) + \gamma_3 \frac{y^2}{v^2}(N - \frac{N_- + N_+}{2}), \tag{7.11}$$

if $b \in D_Q \stackrel{\text{def}}{=} \{b = (X, x, w, Y, y, v, M, N) : x^2 \leq Xw, y^2 \leq Yv,$
$M \leq Cw, N \leq Cv, b \geq 0\}$.

Proof. This is a direct computation. It easily follows from the following infinitesimal estimates for Q:

$$\begin{aligned} \frac{\partial Q}{\partial M} &\geq \frac{cx^2}{w^2}, \\ \frac{\partial Q}{\partial N} &\geq \frac{cy^2}{v^2}, \\ d^2 Q &\leq 0. \end{aligned}$$

Lemma 7.3 *Function*

$$P = CX + CY - \sup_{0<s<\infty} \left(\frac{x^2}{w + sK} + \frac{y^2}{v + s^{-1}K} \right)$$

satisfies the inequality

$$P(c) - \frac{P(c_-) + P(c_+)}{2} \geq \gamma_1 \frac{xy}{wv}(K - \frac{K_- + K_+}{2}) \tag{7.12}$$

if

$$\frac{x^2}{w}K + \frac{y^2}{v}K \leq Cxy$$

if $c \in D_P \stackrel{\text{def}}{=} \{b = (X, x, w, Y, y, v, K) : x^2 \leq Xw, y^2 \leq Yv,$
$K \leq C\sqrt{wv}, c \geq 0\}$.

Proof. We hope that the formula for this function is as much surprising for the reader as it was for us. But once the formula is written the rest is a direct computation. It easily follows from the following infinitesimal estimates for P.

Optimal s can be found from the equation

$$\frac{sK}{(w + sK)^2} x^2 = \frac{s^{-1}K}{(v + s^{-1}K)^2} y^2 \tag{7.13}$$

Now we can make the direct computation of $\frac{\partial P}{\partial K}$. If $sK \le cw$ and $s^{-1}K \le Cv$, then

$$\frac{\partial P}{\partial K} \asymp \frac{x^2}{w^2} s + \frac{y^2}{v^2} s^{-1} \ge \frac{xy}{wv}.$$

It is left to show that if $sK >> w$ or if $s^{-1}K >> v$ (the symbol $>>$ means "much larger than") then we are automatically in the area where

$$\left(\frac{x^2}{w} + \frac{y^2}{v}\right) K \ge c\,xy.$$

In fact, if $s^{-1}K >> v$ then $sK << w$. This is because we are in the domain where $K \le C\sqrt{wv}$ Then (7.12) becomes

$$\frac{sK}{w^2} x^2 = \frac{1}{s^{-1}K} y^2$$

Thus $\frac{x^2}{y^2} \asymp \frac{w^2}{K^2}$ and $\frac{x^2}{w} K \ge cxy \frac{w}{K}\frac{K}{w} = cxy$. Similarly, if $sK >> w$ and thus $s^{-1}K << v$, equality (7.12) gives $\frac{y^2}{x^2} \asymp \frac{v^2}{K^2}$ and so $\frac{y^2}{v} K \ge cxy \frac{v}{K}\frac{K}{v} = cxy$. Lemma 7.3 is completely proved.

So bilinear weighted imbedding theorem is fully proved.

8. 8. Hilbert transform. Sufficient conditions via Green's potentials.

Let H stands for the Hilbert transform on the circle $\mathbb{T}$. In other words, operator H (defined first on trigonometric polynomials) acts by the formula:

$$H(\sum a_k e^{i\theta k}) \stackrel{def}{=} -i \sum_{k \ge 0} a_k e^{i\theta k} + i \sum_{k<0} a_k e^{i\theta k}\,.$$

Throughout this section we use the notation $f(z)$ for the Poisson extension of the function $f \in L^1(\mathbb{T})$ to the disc $\mathbb{D}$ evaluated at the point $z \in \mathbb{D}$. So, for example, $f(z)^2$ and $f^2(z)$ are different in general.

The two weights estimates for the Hilbert transform appear naturally in the theory of Hankel and Toeplitz operators and in the perturbation theory of linear operators (including the perturbation of differential operators).

For the case of the Hilbert transform there exists the approach of M.Cotlar, C.Sadosky through Krein's moment theory. Their approach (which can be referred as Generalized Bochner Theorem) provides both integral representation and extension of forms and kernels invariant under the shift operator. Being applied to a special bilinear form built with the help of the Hilbert transform and two measures, this approach gives a necessary and sufficient condition for the Hilbert transform to be bounded between L^2-spaces with respect to these measures (see [CS1]). The approach of M.Cotlar, C.Sadosky is very intersting because it provides a direct link between the lifting theory of Sz.-Nagy and Foias and thus the scattering theory) and the continuity of the Hilbert transform in weighted spaces (see [S]).

In the case of equal measures Cotlar-Sadosky theorem is a direct analog of Helson-Szegö characterization of A_2 weights.

On the other hand Cotlar-Sadosky theory leaves many questions. It does not provide any analog of the Hunt-Muckenhoupt-Wheeden characterization of A_2 weights. And it seems impossible to provide in their frame the treatment of general Calderón-Zygmund operator between two weighted spaces.

There is an extensive literature consisting of separately necessary and sufficient conditions in the spirit of the Hunt-Muckenhoupt-Wheeden characterization of A_2 weights. In most cases authors assume some sort of A_∞ for one (or both) of the weights. Another kind of results is represented very well by the following theorem of Dechao Zheng in [Z]: H is bounded from $L^2(w^{-1})$ to $L^2(v)$ if for a positive number η

$$\sup_{z\in\mathbb{D}} v^{1+\eta}(z)w^{1+\eta}(z) < \infty\,.$$

Sarason asked a question essentially equivalent to the question whether one can get rid of η. The counterexample of Nazarov [N] shows that η cannot be made 0. On the other hand, in [TVZ] it is shown that one can weaken the above assumption of Dechao Zheng if one uses certain Orlic norms instead of $L^{1+\eta}$.

Here we add another list of sufficient conditions to the existing collection of such conditions. We do not know whether our list represent necessary conditions. Most probably it does not. However, our conditions are completely different of those found before. We found them by the means of the same Bellman functions we have used above to solve the two weights problem for the Haar multiplies (which we consider as a "discrete analog" of two weights problem for H).

Let us introduce further notations. Let $P_z(t) = \frac{1-|z|^2}{|1-\bar{z}t|}$ be the Poisson kernel with pole at $z \in \mathbb{D}$. Let $G(z,\zeta) = \log\left|\frac{1-\bar{z}\zeta}{z-\zeta}\right|$ be Green's function with pole at z. If μ is a measure on $\mathbb{D}$ then Green's potential of this measure is given by $G(\mu)(z) \stackrel{def}{=} \int_{\mathbb{D}} G(z,\zeta)\,d\mu(\zeta)$. It is positive if μ is positive and $\Delta(G(\mu)) = \mu$ in the sense of distributions. We always abbreviate $G(f\,dxdy)(z)$ to $G(f)(z)$.

If $u(z)$ is a harmonic function then $u^{'}(z)$ denotes the holomorphic function $\frac{\partial u}{\partial z}$.

Lemma 8.1. *Let $f = (f_1, ..., f_k)$ be a k-tuple of real valued $L^1(\mathbb{T})$-functions, $f(z)$ be a corresponding k-tuple of harmonic functions, $f^{'}(z)$ be a corresponding k-tuple of analytic functions. Let B be a function of k real variables, and let d^2B denote its Hessian. Put $b(z) = B(f(z))$. Then $\Delta b(z) = 4(d^2B(f(z))f^{'}(z), f^{'}(z))$, where $(\cdot,\cdot)$ means the scalar product in $\mathbb{C}^k$.*

Proof. This is the direct computation of $\frac{\partial^2 B}{\partial\bar{z}\partial z}$ using the harmonicity of $f(z)$. See [NT].

Our last notation concerns an operator with positive kernel, which is related to H the same way as T_0 was related to T_ε. We first write the bilinear form of this H_0. Let f, g be from $L^2(\mathbb{T})$, and let $(\cdot,\cdot)$ *means now the scalar product in* $L^2(\mathbb{T})$. Then we fefine H_0 as follows

$$(H_0 f, g) \stackrel{def}{=} \int_{\mathbb{D}} f(z)g(z)\left|\frac{w^{'}(z)}{w(z)}\right|\left|\frac{v^{'}(z)}{v(z)}\right|(1-|z|)\,dxdy\,.$$

One can easily see that H_0 is the operator with positive kernel ($s,t\in\mathbb{T}$)

$$h(s,t)=\int_{\mathbb{D}}P_z(t)P_z(s)\left|\frac{w'(z)}{w(z)}\right|\left|\frac{v'(z)}{v(z)}\right|(1-|z|)\,dxdy\,.$$

Now we are ready to formulate and to prove our result about H.

Theorem 8.2. *The following conditions together are sufficient for the boudedness of H from $L^2(w^{-1})$ to $L^2(v)$:*

$$\sup_{z\in\mathbb{D}} w(z)v(z)<\infty\,; \tag{8.1}$$

$$G(|v'|^2w)(z)\le Cv(z)\,; \tag{8.2}$$

$$G(|w'|^2v)(z)\le Cw(z)\,; \tag{8.3}$$

$$\|H_0\|_{L^2(w^{-1})\to L^2(v)}<\infty \tag{8.4}$$

Remark. Thus, the problem for singular integral operator is reduced to the problem for the operator with positive kernel. In the discrete case considered in Sections 1-7 the corresponding kernel k did not satify the regularity conditions usually imposed on the kernels to apply Sawyer's theory (see [SW]). However, we found (see Theorem 0.3) necessary and sufficient conditions for the operator T_0 with kernel k to be bounded. Now we can do the same. The kernel h of H_0 does not satisfy in general any regularity conditions unforunately. Still we can immitate Theorem 0.3 and prove

Theorem 8.3. *Put $K(z)=G(|w'v'|)(z)$, and put $M(z)=G(Kw\frac{|w'v'|}{wv})$, $N(z)=G(Kv\frac{|w'v'|}{wv})$. Then the following conditions together are sufficient for the boudedness of H from $L^2(w^{-1})$ to $L^2(v)$:*

$$M(z)\le Cw(z); \tag{8.5}$$

$$N(z)\le Cv(z). \tag{8.6}$$

We leave Theorem 8.3 to the reader. Now we prove Theorem 8.2.

Proof of Theorem 8.2. Let f,g be continuous real valued functions on $\mathbb{T}$. First of all we can assume $g(0)=0$. We also may assume that w,v are continuous because the estimates we are going to get are independent from these assumptions. Clearly,

$$\left|\int_{\mathbb{T}}Hf\cdot g dm\right|\le C\int_{\mathbb{D}}|f'(z)||g'(z)|(1-|z|)\,dxdy\,.$$

Notice that, immitating the sums from Section 2, we can write

$$\int_{\mathbb{D}} |f'(z)||g'(z)|(1-|z|)\,dxdy$$

$$\leq \int_{\mathbb{D}} |f(z)||g(z)| \left|\frac{f'(z)}{f(z)} - \frac{w'(z)}{w(z)}\right| \left|\frac{g'(z)}{g(z)} - \frac{v'(z)|}{v(z)}\right| (1-|z|)\,dxdy$$

$$+ \int_{\mathbb{D}} |f(z)||g(z)| \left|\frac{w'(z)|}{w(z)}\right| \left|\frac{g'(z)|}{g(z)} - \frac{v'(z)}{v(z)}\right| (1-|z|)\,dxdy$$

$$+ \int_{\mathbb{D}} |f(z)||g(z)| \left|\frac{v'(z)}{v(z)}\right| \left|\frac{f'(z)}{f(z)} - \frac{w'(z)}{w(z)}\right| (1-|z|)\,dxdy$$

$$+ \int_{\mathbb{D}} |f(z)||g(z)| \left|\frac{w'(z)}{w(z)}\right| \left|\frac{v'(z)}{v(z)}\right| (1-|z|)\,dxdy$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$$

Now let us think that $f = \phi w^{1/2}$, $g = \psi v^{1/2}$. We only need to prove that Σ_i, $i = 1, 2, 3$ are estimated by $C\|\phi\|_2^2 + \|\psi\|_2^2$. In fact,replacing f by tf and g by $t^{-1}g$ allows then to estimate these sums by $C\|\phi\|_2\|\psi\|_2$.

Let us first do it for Σ_1. Consider $B(X, x, w, Y, y, v) \stackrel{\text{def}}{=} X - \frac{x^2}{w} + Y - \frac{y^2}{v}$. Put $b(z) = B(\phi^2(z), \phi w^{1/2}(z), w(z), \psi^2(z), \psi v^{1/2}(z), v(z))$. Using Lemma 8.1 we can compute its Laplacian and see that it is superharmonic and moreover that ($f(z) = \phi w^{1/2}(z)$, $g(z) = \psi v^{1/2}(z)$)

$$-\Delta(b(z)) \geq c\frac{f(z)^2}{w(z)} \left|\frac{f'(z)}{f(z)} - \frac{w'(z)}{w(z)}\right|^2 + c\frac{g(z)^2}{v(z)} \left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right|^2$$

Thus,

$$-\Delta(b(z)) \geq c\frac{|f(z)g(z)|}{\sqrt{(w(z)v(z)}} \left|\frac{f'(z)}{f(z)} - \frac{w'(z)}{w(z)}\right| \left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right|$$

And so we obtain using (8.1):

$$-\Delta(b(z)) \geq c|f(z)g(z)| \left|\frac{f'(z)}{f(z)} - \frac{w'(z)}{w(z)}\right| \left|\frac{g'(z)|}{g(z)} - \frac{v'(z)}{v(z)}\right| \tag{8.7}$$

Notice also that function b vanishes on the circle $\mathbb{T}$. In fact, $\phi^2(z) = \frac{(\phi w^{1/2}(z))^2}{w(z)}$, $\psi^2(z) = \frac{(\psi v^{1/2}(z))^2}{v(z)}$ on $\mathbb{T}$. Notice also that $b(0) \leq \|\phi\|_2^2 + \|\psi\|_2^2$.

Applying Green's formula, we get

$$-\int_{\mathbb{D}} \Delta(b(z)) \log\frac{1}{|z|}\,dxdy = b(o) - \int_{\mathbb{T}} b\,dm \leq \|\phi\|_2^2 + \|\psi\|_2^2 \tag{8.8}$$

Combining (8.7) and (8.8), we obtain that

$$\Sigma_1 \leq C\|\phi\|_2^2 + \|\psi\|_2^2\,.$$

Estimate of Σ_2, Σ_3 are similar to each other.

Let us estimateΣ_2. To do that we need the following $p(z)$ (remind that $f = \phi w^{1/2}$) :

$$p(z) \stackrel{\text{def}}{=} \phi^2(z) - \frac{f(z)^2}{w(z) + G(|w'|^2 v)(z)} + \psi^2(z) - \frac{g(z)^2}{v(z)}\,.$$

Notice how small is the difference with b. But this small difference allows the following estimate of the Laplacian (we use here Lemma 8.1 again and also formula (6.10)):

$$-\Delta(p(z)) \geq c\frac{f(z)^2}{(w(z) + G(|w'|^2 v)(z))^2}\Delta(w(z)+G(|w'|^2 v)(z))+c\frac{g(z)^2}{v(z)}\left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right|^2$$

Thus,

$$-\Delta(p(z)) \geq c\frac{|w'(z)|^2}{w(z)^2} f(z)^2 v(z) + c\frac{g(z)^2}{v(z)}\left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right|^2$$

We used here our second assumption from Theorem 8.2: $G(|w'|^2 v)(z) \leq Cw(z)$. Thus,

$$-\Delta(p(z)) \geq c\frac{|w'(z)|}{w(z)}|f(z)g(z)|\left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right|$$

Again p vanishes on $\mathbb{T}$ and again obviously $p(0) \leq \|\phi\|_2^2 + \|\psi\|_2^2$.

Thus, using Green's formula as before, we get

$$\int_{\mathbb{D}} \frac{|w'(z)|}{w(z)}|f(z)g(z)|\left|\frac{g'(z)}{g(z)} - \frac{v'(z)}{v(z)}\right| \log\frac{1}{|z|}\,dxdy \leq c^{-1}(\|\phi\|_2^2 + \|\psi\|_2^2),$$

which is the estimate of Σ_2. Similarly we treat Σ_3 using the third assumption of Theorem 8.2: $G(|v'|^2 w)(z) \leq Cv(z)$. Theorem 8.2 is completely proved.

9. 9.Two weight norm inequalities for S-functions. Necessary and sufficient conditions.

Square functions play an important role in the theory of Singular integral operators. Often the estimate of Singular integral operators goes through the estimate of a certain S-function. The multitude of examples can be found in [St]. In one of the most recent examples in [V] this approach was applied to characterize the matrix $\mathbf{A}_p$ weights.

Let us consider

$$S(f)(x) \stackrel{\text{def}}{=} \sum_{I:x\in I} |\langle f\rangle_{I_-} - \langle f\rangle_{I_+}|^2\,.$$

So when $f \to S(f)$ is bounded from $L^2(w^{-1})$ to $L^2(v)$? The following theorem gives the answer. It says basically that this is so if and only if there is a uniform bound on test functions $f = w\chi_J$.

Theorem 9.1.$\|S(f)\|_{L^2(v)} \leq C\|f\|_{L^2(w^{-1})}$ *if and only if (1.1) and (1.2) hold.*

Proof. Necessity is simple. To prove the sufficiency we again use the Bellman function approach. Let us consider

$$B(X,x,w,M)=CX-\frac{x^2}{w}-\frac{x^2}{w+M}\,.$$

We are going to compose it with the following (super)martingales

$$X_I=\langle\phi^2\rangle_I,\ x_I=\langle\phi w^{1/2}\rangle_I,\ w_I=\langle w\rangle_I$$

$$M_I=\frac{1}{|I|}\sum_{\ell\subseteq I}|\langle w\rangle_{\ell_-}-\langle w\rangle_{\ell_+}|^2\langle v\rangle_\ell|\ell|\,.$$

By (1.2) $M_I\le Cw_I$. Then we have the supermartingale

$$b(I)=B(X_I,x_I,w_I,M_I)\,.$$

Its "discrete Laplacian" $\Delta(b)(I)\stackrel{\text{def}}{=}\frac{b(I_-)+b(I_+)}{2}-b(I)$ can be estimated using Lemma 9.2 bellow (we use the notations $f=\phi w^{1/2}$):

$$-\Delta(b)(I)=b(I)-\frac{b(I_-)+b(I_+)}{2}\ge$$
$$c\frac{\langle f\rangle_I^2}{\langle w\rangle_I}\inf_{c_1,c_2\in[1/2,2]}\left|c_1\frac{\langle f\rangle_{I_-}-\langle f\rangle_{I_+}}{\langle f\rangle_I}-c_2\frac{\langle w\rangle_{I_-}-\langle w\rangle_{I_+}}{\langle w\rangle_I}\right|+$$
$$c\frac{\langle f\rangle_I^2}{\langle w\rangle_I^2}|\langle w\rangle_{I_-}-\langle w\rangle_{I_+}|^2v_I$$

We want to continue the estimate of the negative Laplacian from below. To this end let us consider two cases.

First case: $\left|\frac{\langle w\rangle_{I_-}-\langle w\rangle_{I_+}}{\langle w\rangle_I}\right|\le\frac{1}{10}\left|\frac{\langle f\rangle_{I_-}-\langle f\rangle_{I_+}}{\langle f\rangle_I}\right|$. In this case we use the first term in the estimate of our negative Laplacian to see that

$$-\Delta(b)(I)\ge 0.09c\cdot\frac{\langle f\rangle_I^2}{\langle w\rangle_I}\left|\frac{\langle f\rangle_{I_-}-\langle f\rangle_{I_+}}{\langle f\rangle_I}\right|^2$$

And so,

$$-\Delta(b)(I)\ge c'\frac{|\langle f\rangle_{I_-}-\langle f\rangle_{I_-}|^2}{\langle w\rangle_I}\ge c''|\langle f\rangle_{I_-}-\langle f\rangle_{I_+}|^2\langle v\rangle_I\,. \tag{9.1}$$

In the last inequality we use (1.1).

Second case: $\left|\frac{\langle w\rangle_{I_-}-\langle w\rangle_{I_+}}{\langle w\rangle_I}\right|\ge\frac{1}{10}\left|\frac{\langle f\rangle_{I_-}-\langle f\rangle_{I_+}}{\langle f\rangle_I}\right|$. In this case we use the second term in the estimate of the negative Laplacian. Taking into account the ineqality above we get

$$-\Delta(b)(I)\ge 0.01\cdot|\langle f\rangle_{I_-}-\langle f\rangle_{I_+}|^2\langle v\rangle_I\,. \tag{9.2}$$

So we get the same estimate as in the first case. Now we can just apply Green's formula for our discrete Laplacian, which amounts to just applying (9.1),(9.2) to I, then to $I_\pm$, then to "sons" of $I_\pm$, et cetera... to obtain

$$\frac{1}{|J|}\sum_{I\subseteq J}|\langle f\rangle_{I_-}-\langle f\rangle_{I_+}|^2\langle v\rangle_I|I|\le CB(X_J,x_J,w_J,M_J)\le C\langle\phi^2\rangle_J\,.$$

But $\langle\phi^2\rangle_J=\|f\|^2_{L^2(w^{-1})}$, and Theorem 9.1 is completely proved.

We are left to prove the following lemma. Let us denote $a=(X,x,w)$, and let $P(a)\stackrel{\text{def}}{=}X-\frac{x^2}{w}$, $Q(a,M)\stackrel{\text{def}}{=}X-\frac{x^2}{w+M}$.

Lemma 9.2. *Let $a=\frac{a_-+a_+}{2}$, $M\ge\frac{M_-+M_+}{2}$. Then*

$$-d^2P=2\frac{x^2}{w}\left|\frac{dx}{x}-\frac{dw}{w}\right|^2\;;$$

$$Q(a,M)-\frac{Q(a_-)+Q(a_+)}{2}\ge c\frac{x^2}{w^2}(M-\frac{M_-+M_+}{2})\,;$$

$$P(a)-\frac{P(a_-)+P(a_+)}{2}\ge c\frac{x^2}{w}\inf_{c_1,c_2\in[1/2,2]}\left|c_1\frac{x_--x_+}{x}-c_2\frac{w_--w+}{w}\right|^2\,.$$

Proof. The first equality is obtained by direct computation. It shows that B is concave. Let $a(t)$ be the linear function on $[-1,1]$ assuming values a_-,a_+ at endpoints (and thus $a_0=a$). Let $M(t)$ be the piecewise linear function on $[-1,1]$ assuming values M_-,M,M_+ at $-1,0,1$ correspondingly. Then $M(t)$ is concave, and $M''(t)=(\frac{M_-+M_+}{2}-M)\delta_0$, where δ_0 is a Dirac measure at 0. Notice that Q is a composition of P and a linear function. So Q is concave. Thus, by (6.10) $-d^2Q\ge-\frac{\partial Q}{\partial M}d^2M$. In particular, if $q(t)\stackrel{\text{def}}{=}Q(a(t),M(t))$ then measure $q''(t)$ satisfies

$$-q''(t)\ge c\frac{x(t)^2}{w(t)^2}(M-\frac{M_-+M_+}{2})\delta_0\,.$$

Thus

$$q(0)-\frac{q(-1)+q(1)}{2}=-\int_{-1}^{1}(1-|t|)q''(t)\ge c\frac{x^2}{w^2}(M-\frac{M_-+M_+}{2})\,,$$

and the second inequality is proved.

To prove the third inequality of the lemma let us introduce $p(t)\stackrel{\text{def}}{=}P(a(t))$. Using the calculation for $-d^2P$ we obtain that then measure $p''(t)$ satisfies

$$-p''(t)=2\frac{x(t)^2}{w(t)}\left|\frac{x_--x_+}{x(t)}-\frac{w_--w_+}{w(t)}\right|^2\,.$$

On $[-1/2,1/2]$ we have that $\frac{x(t)}{x}\in[1/2,2]$ and $\frac{w(t)}{w}\in[1/2,2]$. Thus, on $[-1/2,1/2]$, we have

$$-p''(t)\ge c\frac{x^2}{w}\inf_{c_1,c_2\in[1/2,2]}\left|c_1\frac{x_--x_+}{x}-c_2\frac{w_--w_+}{w}\right|^2\,.$$

We finish the proof by combining this inequality with the following one

$$p(0) - \frac{p(-1) + p(1)}{2} = -\int_{-1/2}^{1/2} (1 - |t|)p''(t)\,.$$

10. Concluding remarks

1) Seems like the case $p = 2$ is a true miracle, because in this case we were able to give a finite list of simple conditions which are necessary and sufficient for two weight boundedness of our family $T_\pm$ of Calderón-Zygmund operators. As for $p \neq 2$ case, there are strong indications that the similar list of conditions (which can be actually copied from the "$p = 2$" case) will not be equivalent to two weights estimate.

Let us notice that there exists an approach through Cotlar-Sadosky theory to two weight estimate for the Hilbert transform even for $p \neq 2$ (see [CS2]).

2) One wonders whether the 5 conditions in our list in Section 1 are independent. This is most probably so, but the proof should be quite involved. At least it follows from [N] that (1.1) does not imply neither (1.2) nor (1.3).

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