WEAK TYPE ESTIMATES AND COTLAR INEQUALITIES FOR CALDERÓN-ZYGMUND OPERATORS IN NONHOMOGENEOUS SPACES

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ABSTRACT. In the paper we consider Calderón-Zygmund operators in nonhomogeneous spaces. We are going to prove the analogs of classical results for homogeneous spaces. Namely, we prove that a Calderón-Zygmund operator is of weak type if it is bounded in L^2. We also prove several versions of Cotlar's inequality for maximal singular operator. One version of Cotlar's inequality (a simpler one) is proved in Euclidean setting, another one in a more abstract setting when Besicovich covering lemma is not available. We obtain also the weak type of maximal singular operator from these inequalities.

Let μ be a measure on $\mathbb{C}$ satisfying the Ahlfors condition

$$\mu(B(x,r)) \le r \qquad \text{for every } x \in \mathbb{C}, r > 0$$

(as usual, $B(x,r) := \{y \in \mathbb{C} : |x-y| < r\}$).

Let $K(x,y)$ be a Calderon-Zygmund kernel, i.e.

1) $$|K(x,y)| \le \frac{1}{|x-y|}$$

and

2) $$|K(x,y) - K(x',y)|, |K(y,x) - K(y,x')| \le \frac{|x-x'|^{\varepsilon}}{|x-y|^{1+\varepsilon}} \text{ whenever } |x-x'| \le \frac{1}{2}|x-y|.$$

Theorem 1.

Assume that the operator

$$T\varphi(x) := \int_{\mathbb{C}} K(x,y)\varphi(y)\, d\mu(y)$$

(actually it would be better to write $T(\varphi d\mu)$ instead of $T\varphi$ but we hope that the reader will forgive us for a little bit inconsistent notation) is bounded in $L^2(\mu)$. Then for any (signed) measure ν on $\mathbb{C}$ and for any $t > 0$ one has

$$\mu\{|T\nu| > t\} \le \frac{A||\nu||}{t}.$$

For the Cauchy integral operator the weak type inequality was obtained first by X. Tolsa in [T1]. His method was specific for the Cauchy kernel and used the results

Partially supported by the NSF grant DMS 9622936, binational Israeli-USA grant BSF 00030, and research programs at MSRI in the Fall of 1995 and in the Fall of 1997.

from [Me]. Later the weak type estimate for general Calderón-Zygmund operators was proved in [NTV2], where also the weak type estimate for maximal singular operator were obtained along with certain Cotlar's inequalities for the maximal singular operator. Another proof of the weak type estimate for the maximal singular operator (and yet another Cotlar inequality) appeared in [T2].

Here and in the following we denote by A a constant depending on $||T||_{L^2(\mu)\to L^2(\mu)}$ and ε only (notice that we have already put the constants in the Ahlfors condition and in the properties of the Calderón-Zygmund kernel $K(x,y)$ to 1: it can always be achieved by multiplication of the measure and the kernel by a sufficiently small positive constant, which does not change anything in the problem).

Proof.

First of all we shall need the following standard

Obvious Lemma:

If η is a signed measure concentrated in some disk $B(x,\rho)$ and such that $\eta(\mathbb{C})=0$, then

$$\int_{\mathbb{C}\setminus B(x,2\rho)} |T\eta|d\mu \le A_1||\eta||.$$

Proof of Obvious Lemma:

For any $y\in\mathbb{C}\setminus B(x,2\rho)$ we have

$$|T\eta(y)| = \Big|\int_{B(x,\rho)} K(y,x')d\eta(x')\Big| = \Big|\int_{B(x,\rho)} [K(y,x')-K(y,x)]d\eta(x')\Big| \le$$
$$\le \int_{B(x,\rho)} |K(y,x')-K(y,x)|\, d|\eta|(x') \le ||\eta||\frac{\rho^\varepsilon}{|x-y|^{1+\varepsilon}}.$$

It remains to note that

$$\int_{\mathbb{C}\setminus B(x,2\rho)} \frac{\rho^\varepsilon}{|x-y|^{1+\varepsilon}}d\mu(y) \le 1+\tfrac{1}{\varepsilon}$$

due to the Ahlfors condition.

In order to proceed, we need to define the maximal operator

$$T^\sharp\varphi(x) := \sup_{r>0}|T_r\varphi(x)|$$

where

$$T_r\varphi(x) := T(\varphi\chi_{\mathbb{C}\setminus B(x,r)})(x) = \int_{\mathbb{C}\setminus B(x,r)} K(x,y)\varphi(y)d\mu(y).$$

We shall also need the Hardy-Littlewood maximal function operator

$$M\varphi(x) := \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|\varphi(y)|d\mu(y).$$

The crux of the proof is the following

Key lemma:

For any measurable set $F \in \mathbb{C}$ and any $x \in \mathbb{C}$

$$T^\sharp \chi_F(x) \le MT\chi_F(x) + A_2$$

This lemma was inspired by (the readable part of) [DM] and we had a strong temptation to attribute it to Guy David and Pertti Mattila. Finally we have suppressed this temptation and claim it to be our own result, but by no means we insist on the reader's doing the same. We will postpone the proof of the key lemma for a while and now let us derive the theorem from it.

Without loss of generality we may assume that ν is a finite linear combination of unit point masses with positive coefficients, i.e. that

$$\nu = \sum_{j=1}^N \alpha_j \delta_{x_j},$$

and what is more, the μ-measure of any circumference centered at any of the points x_j is equal to 0: an arbitrary measure ν can be obtained as a week limit of such measures; the discussion of why one always may pass to the limit is equivalent to the discussion of how to define $T\nu$ rigorously in the general case. We leave this headache to the reader to take care of.

Also we can always assume that $||\nu|| = \sum_j \alpha_j = 1$ (this is just the matter of normalization) and that the kernel $K(x,y)$ is real-valued (otherwise consider $Re\, K$ and $Im\, K$ separately). Thus we have to prove that

$$\mu\{|T\nu| > t\} \le \frac{A}{t}.$$

If $\mu(\mathbb{C}) < \frac{1}{t}$, there is nothing to do. Otherwise let $B(x_1, r_1)$ be a disk such that $\mu(E_1) := \mu(B(x_1, r_1) = \frac{\alpha_1}{t}$, $B(x_2, r_2)$ be a disk such that $\mu(E_2) := \mu(B(x_2, r_2) \setminus B(x_1, r_1) = \frac{\alpha_2}{t}$, and so on: $B(x_j, r_j)$ is a disk such that

$$\mu(E_j) := \mu\Big(B(x_j, r_j) \setminus \bigcup_{\ell=1}^{j-1} B(x_\ell, r_\ell)\Big) = \frac{\alpha_j}{t},$$

(the definition signs define E_j). Let $E = \bigcup_j E_j = \bigcup_j B(x_j, r_j)$ Clearly,

$$\mu(E) = \frac{1}{t}.$$

Now let us compare $T\nu$ to $t\sum_j \chi_{\mathbb{C}\setminus B(x_j, 2r_j)} \cdot T\chi_{E_j} =: t\sigma$ outside E. We have

$$T\nu - t\sigma = \sum_j \varphi_j$$

where

$$\varphi_j = \alpha_j T\delta_{x_j} - t\chi_{\mathbb{C}\setminus B(x_j, 2r_j)} \cdot T\chi_{E_j}.$$

Note now that

$$\int_{\mathbb{C}\setminus E} |\varphi_j| d\mu \le \int_{\mathbb{C}\setminus B(x_j, 2r_j)} |T[\alpha_j \delta_{x_j} - t\chi_{E_j} d\mu]| d\mu + \int_{B(x_j, 2r_j)\setminus B(x_j, r_j)} \alpha_j |T\delta_{x_j}| d\mu.$$

But, according to Obvious Lemma, the first integral does not exceed

$$A_1||\alpha_j\delta_{x_j} - t\chi_{E_j}d\mu|| = 2A_1\alpha_j,$$

while $|T\delta_{x_j}| \le r_j^{-1}$ outside $B(x_j, r_j)$ and therefore the second integral is not greater than $a_j r_j^{-1}\mu(B(x_j, 2r_j)) \le 2\alpha_j$. Finally we conclude that

$$\int_{\mathbb{C}\setminus E} |T\nu - t\sigma| d\mu \le 2(A_1+1)\sum_j \alpha_j = 2(A_1+1)$$

and thereby the absolute value of the difference $T\nu - t\sigma$ does not exceed t everywhere on $\mathbb{C}\setminus E$, except, maybe, a set of measure $\frac{2(A_1+1)}{t}$. To accomplish the proof of the theorem, it is enough to show that for sufficiently large A_3

$$\mu\{|\sigma| > A_3\} \le \frac{2}{t}.$$

We will do it by the standard Stein-Weiss duality trick. Assume that the inverse inequality holds. Then either $\mu\{\sigma > A_3\} > \frac{1}{t}$, or $\mu\{\sigma < -A_3\} > \frac{1}{t}$. Assume for definiteness that the first case takes place and choose some set $F \subset \mathbb{C}$ of measure exactly $\frac{1}{t}$ such that $\sigma > A_3$ everywhere on F. Then, clearly,

$$\int_{\mathbb{C}} \sigma\chi_F d\mu > \frac{A_3}{t}.$$

On the other hand this integral can be computed as

$$\sum_j \int_{\mathbb{C}} [T\chi_{E_j}]\cdot \chi_{F\setminus B(x_j,2r_j)}\, d\mu = \sum_j \int_{\mathbb{C}} \chi_{E_j}\cdot [T^*\chi_{F\setminus B(x_j,2r_j)}]\, d\mu$$

Note that for every point $x \in E_j \subset B(x_j, r_j)$

$$|T^*\chi_{F\setminus B(x_j,2r_j)}(x) - T^*\chi_{F\setminus B(x,r_j)}(x)| = \Big|\int_{B(x_j,2r_j)\setminus B(x,r_j)} K(y,x)d\mu(y)\Big| \le$$
$$\le r_j^{-1}\mu(B(x_j, 2r_j) \le 2$$

and thereby

$$|T^*\chi_{F\setminus B(x_j,2r_j)}(x)| \le (T^*)^\sharp\chi_F(x) + 2 \le MT^*\chi_F(x) + A_2 + 2$$

according to the key lemma.

Hence

$$\int_{\mathbb{C}} \sigma\chi_F d\mu \le (A_2+2)\mu(E) + \int_{\mathbb{C}} \chi_E\cdot MT^*\chi_F d\mu.$$

But the first term equals $\frac{A_2+2}{t}$ while the second one does not exceed

$$\|\chi_E\|_{L^2(\mu)}\|MT^*\chi_F\|_{L^2(\mu)} \le \frac{1}{t}||M||_{L^2(\mu)\to L^2(\mu)}||T^*||_{L^2(\mu)\to L^2(\mu)}.$$

Recalling that

$$||M||_{L^2(\mu)\to L^2(\mu)} \le 100$$

for an arbitrary measure μ, we see that it is enough to take

$$A_3 = A_2 + 2 + 100||T||_{L^2(\mu)\to L^2(\mu)}$$

to get a contradiction. The theorem is proved and it remains to prove the key lemma.

Proof of Key Lemma.

Let $x \in \mathbb{C}, r > 0$. Consider the sequence of disks $B_j := B(x, 3^j r)$ and the corresponding sequence of measures $\mu_j := \mu(B_j)$ $(j = 0, 1, \dots)$. Note that it is impossible that $\mu_j > 9\mu_{j-1}$ for every $j \geq 1$ (this would imply that $\mu(B(x,R)) > \text{const}\, R^2$ for large R, which contradicts the Alfors condition). Therefore there exists the smallest positive integer k for which $\mu_k \leq 9\mu_{k-1}$. Put $R = 3^{k-1}r$. Note that

$$|T_r\chi_F(x) - T_{3R}\chi_F(x)| \leq \int_{B(x,3R)\setminus B(x,r)} \frac{d\mu(y)}{|x-y|} \leq \sum_{j=1}^{k} \frac{\mu_j}{3^{j-1}r} \leq$$

$$\leq \sum_{j=1}^{k} \frac{\mu_k 9^{j+1-k}}{3^{j-1}r} = 27 \cdot 9^{-k}\mu_k r^{-1} \sum_{j=1}^{k} 3^j \leq 81 \frac{\mu_k}{3R} \leq 81.$$

And that's basically all, because now it is enough to pick up any standard proof based on the doubling condition to get the desired estimate for $T_{3R}\chi_F(x)$ (recall that $\mu(B(x,3R) \leq 9\mu(B(x,R))$!!!)

One of such standard ways is to compare $T_{3R}\chi_F(x)$ to the average

$$U_R(x) := \frac{1}{\mu(B(x,R))} \int_{B(x,R)} T\chi_F \, d\mu$$

(the quantity, which is clearly bounded by $MT\chi_F(x)$).

We have

$$T_{3R}\chi_F(x) - U_R(x) =$$

$$= \int_{F\setminus B(x,3R)} T^*[\delta_x - \tfrac{1}{\mu(B(x,R))}\chi_{B(x,R)}\, d\mu]\, d\mu - \frac{1}{\mu(B(x,R))} \int_{\mathbb{C}} \chi_{B(x,R)} \cdot T\chi_{F\cap B(x,3R)}\, d\mu.$$

The first term does not exceed $2A_1$ according to Obvious Lemma, while the second can be estimated by

$$\frac{1}{\mu(B(x,R))} ||T||_{L^2(\mu)\to L^2(\mu)} ||\chi_{B(x,R)}||_{L^2(\mu)} \cdot ||\chi_{F\cap B(x,3R)}||_{L^2(\mu)} \leq$$

$$\leq \frac{1}{\mu(B(x,R))} ||T||_{L^2(\mu)\to L^2(\mu)} \sqrt{\mu(B(x,R))}\sqrt{\mu(B(x,3R))} \leq 3||T||_{L^2(\mu)\to L^2(\mu)}.$$

Note at last, that we can restrict ourselves to "good" disks $B(x,r)$ in the definition of $M\varphi(x)$, namely, to the disks satisfying $\mu(B(x,3r)) \leq 81\mu(B(x,r))$ (we can replace 81 by 9 so far, but we will really need this larger constant in what follows). Thus the geometry of the space (the Besicovich maximal function theorem) is not involved in the proof: the Vitali covering theorem (that's why we used the stretching factor 3 instead of more natural 2 in the construction) is more than enough to show the boundedness of the restricted maximal function operator (we will denote it by $\widetilde{M}$) in $L^2(\mu)$.

As usual, by interpolation we conclude that T is bounded in $L^p(\mu)$ for every $1 < p \leq 2$ and then, by duality, this result automatically extends to all $p \in (1, +\infty)$.

Now we are ready to prove the boundedness of the maximal operator $T^\sharp$ in all spaces $L^p(\mu)$ with $1 < p < +\infty$.

Let us introduce one more maximal function

$$M'\varphi(x) := \sup_{r>0} r^{-1} \int_{B(x,r)} |\varphi| d\mu.$$

Note that M' is a bounded operator in $L^p(\mu)$ for all $p > 1$ (provided that μ satisfies the Alfors condition, of course) and that again the Vitali covering theorem is enough to prove this.

Let now $\beta > 0$. Define

$$M'_\beta \varphi := \left\{M'[|\varphi|^\beta]\right\}^{1/\beta}; \qquad \widetilde{M}_\beta \varphi := \left\{\widetilde{M}[|\varphi|^\beta]\right\}^{1/\beta}$$

Clearly, both M'_β and $\widetilde{M}_\beta$ are bounded operators in $L^p(\mu)$ for all $p > \beta$

The boundedness of the maximal operator $T^\sharp$ follows now from

Theorem 2.

For any $\beta > 1$

$$T^\sharp\varphi \le \widetilde{M}T\varphi + B\widetilde{M}_\beta\varphi + B'M'_\beta\varphi$$

where the constants B and B' depend on ε, β and $||T||_{L^2(\mu)\to L^2(\mu)}$ only.

Proof.

It is just a minor modification of the proof of Key Lemma. Let again $r > 0$. Define B_j and μ_j as before, but let now k be the smallest positive integer for which $\mu_{k+1} \le 81\mu_{k-1}$ (i.e. we look now two steps forward when checking for the doubling). Let $R = 3^{k-1}r$ exactly as before. We have

$$|T_r\varphi(x) - T_{3R}\varphi(x)| \le \int_{B(x,3R)\setminus B(x,r)} \frac{|\varphi(y)|d\mu(y)}{|x-y|} \le \sum_{j=1}^{k} \frac{I_j}{3^{j-1}r}$$

Where

$$I_j := \int_{B_j} |\varphi| d\mu \le \left\{\int_{B_j} |\varphi|^\beta\right\}^{1/\beta} \mu_j^\gamma \le M'_\beta\varphi(x)\,(3^j r)^{1/\beta}\mu_j^\gamma$$

where $\gamma = 1 - \frac{1}{\beta}$. Now we may continue the estimate:

$$\sum_{j=1}^{k} \frac{I_j}{3^{j-1}r} \le 3M'_\beta\varphi(x)\sum_{j=1}^{k}\left[\frac{\mu_j}{3^{j-1}r}\right]^\gamma \le 3M'_\beta\varphi(x)\left[\frac{\mu_k}{3^{k-1}r}\right]^\gamma \sum_{j=1}^{k} 3^{(j+2-k)\gamma} \le B'_1 M'_\beta\varphi(x).$$

So, again, we need only to estimate $T_{3R}\varphi(x)$. As before, define

$$U_R(x) := \frac{1}{\mu(B(x,R))}\int_{B(x,R)} T\varphi d\mu$$

(the quantity, which is clearly bounded by $\widetilde{M}T\varphi(x)$) and write

$$T_{3R}\varphi(x) - U_R(x) =$$
$$= \int_{\mathbb{C}\setminus B(x,3R)} T^*[\delta_x - \tfrac{1}{\mu(B(x,R))}\chi_{B(x,R)}d\mu]\,\varphi\,d\mu - \frac{1}{\mu(B(x,R))}\int_{\mathbb{C}} \chi_{B(x,R)} \cdot T[\varphi\chi_{B(x,3R)}]\,d\mu.$$

We leave it as an excercise for the reader to show that under the conditions of Obvious Lemma

$$\int_{\mathbb{C}\setminus B(x,2\rho)} |T\eta|\,|\varphi|\,d\mu \le A_4 M'\varphi(x)\,||\eta||,$$

and thereby the first term does not exceed $2A_4M'\varphi(x) \le 2A_4M'_\beta\varphi(x)$. As to the second term, we know that T acts in $L^\beta(\mu)$ and therefore this term is bounded by

$$\frac{1}{\mu(B(x,R))}||T||_{L^\beta(\mu)\to L^\beta(\mu)}||\chi_{B(x,R)}||_{L^{\beta'}(\mu)} \cdot ||\varphi\chi_{B(x,3R)}||_{L^\beta(\mu)}$$

where $\beta' := \frac{\beta}{\beta-1}$ is the conjugate exponent to β. But clearly

$$||\chi_{B(x,R)}||_{L^{\beta'}(\mu)} = \{\mu(B(x,R)\}^{1/\beta'}$$

while

$$||\varphi\chi_{B(x,3R)}||_{L^\beta(\mu)} \le \widetilde{M}_\beta\varphi(x)\{\mu(B(x,3R)\}^{1/\beta} \le \widetilde{M}_\beta\varphi(x)\{81\mu(B(x,R)\}^{1/\beta},$$

and, finally, the second term is not greater than

$$81||T||_{L^\beta(\mu)\to L^\beta(\mu)}\widetilde{M}_\beta\varphi(x)$$

(we can write $\widetilde{M}$ instead of M because due to the choice of k we have $\mu(B(x,9R)) \le 81\mu(B(x,R))$ and thereby both balls $B(x,R)$ and $B(x,3R)$ are used in the definition of $\widetilde{M}$ at the point x).

Now, to complete the "classical L^p-theory", it remains to prove that the maximal operator $T^\sharp$ satisfies a weak type L^1-estimate, namely, that

Theorem 3.

For any (signed) measure ν on $\mathbb{C}$ and for any $t > 0$ one has

$$\mu\{|T^\sharp\nu| > t\} \le \frac{A||\nu||}{t}.$$

Proof.

Let us again assume that $\nu = \sum_{j=1}^N \alpha_j\delta_{x_j}$, $\alpha_j > 0$, $||\nu|| = \sum_j \alpha_j = 1$. Fix some $t > 0$ and carry out the construction of Theorem 1. Define

$$m(x) := \sum_j \frac{\alpha_j}{r_j}\chi_{B(x_j,10r_j)}.$$

Let again

$$\varphi_j := T[\alpha_j\delta_{x_j}] - t\chi_{\mathbb{C}\setminus B(x_j,2r_j)} \cdot T\chi_{E_j}.$$

Let now $x \in \mathbb{C} \setminus E$, $r > 0$. Put

$$\sigma_r(x) := \sum_j \chi_{\mathbb{C}\setminus B(x_j,2r_j)} \cdot T\chi_{E_j\setminus B(x,r)}.$$

Our first aim is to estimate the difference $T_r\nu(x) - t\sigma_r(x)$. Clearly, it can be represented as

$$\sum_{j=1}^N \{T_r[\alpha_j\delta_{x_j}](x) - t\chi_{\mathbb{C}\setminus B(x_j,2r_j)}(x) \cdot T\chi_{E_j\setminus B(x,r)}(x) =: \sum_{j=1}^N \psi_j(x)\}.$$

There are 3 possibilities for each j:

1) $B(x_j, r_j) \cap B(x, r) = \emptyset$

Then $\psi_j(x) = \varphi_j(x)$ and thereby the sum of such terms does not exceed

$$\sum_{j=1}^N |\varphi_j(x)|.$$

2) $B(x_j, r_j) \cap B(x, r) \neq \emptyset$, $r \le r_j$

Since at any rate $|\psi_j(x)| \le \frac{2\alpha_j}{r_j}$ and since for every such j we have $x \in B(x_j, 2r_j) \subset B(x_j, 10r_j)$, we conclude that the sum of all such terms is not greater than $2m(x)$.

2) $B(x_j, r_j) \cap B(x, r) \neq \emptyset$, $r > 2r_j$

In this case we shall use the estimate $|\psi_j(x)| \le \frac{2\alpha_j}{r} = \frac{2t\mu(E_j)}{r}$. Since for every such j one has $E_j \subset B(x_j, r_j) \subset B(x, 3r)$ and since E_j are pairwise disjoint, we obtain that the sum of such terms can be estimated from above by

$$\frac{2t\mu(B(x, 3r)}{r} \le 6t.$$

Now let us recall that (due to Obvious Lemma)

$$\int_{\mathbb{C}\setminus E} \sum_{j=1}^N |\varphi_j| d\mu \le 2A_1$$

and note that

$$\int_{\mathbb{C}} m d\mu \le 10.$$

Thus, it remains only to check that for some large A_4

$$\mu\{x \in \mathbb{C} : \sigma^\sharp(x) > A_4\} \le \frac{A_4}{t},$$

where

$$\sigma^\sharp(x) := \sup_{r>0} |\sigma_r(x)|.$$

The proof will resemble a lot the proof of Key Lemma. We will show that

$$\sigma^\sharp(x) \le \widetilde{M}\sigma(x) + A_5 + \frac{2}{t}m(x) \qquad (*)$$

The last term is due to the cutting factors $\chi_{\mathbb{C}\setminus B(x,2r_j}$ in the definition of σ. Without them there would be no difference from Key Lemma at all either in the result or in the proof.

Let us demonstrate first of all that this inequality implies the needed estimate for $\sigma^\sharp$. In order to do that it is enough to check that

$$\int_{\mathbb{C}} |\sigma|^2 d\mu \le \frac{A}{t}.$$

We will do it by duality again. Let $\varphi \in L^2(\mu)$. We have

$$\int_{\mathbb{C}} \sigma\,\varphi\,d\mu = \sum_{j=1}^N \int_{E_j} T^*[\chi_{\mathbb{C}\setminus B(x_j,2r_j)}\varphi]d\mu =: \sum_j I_j.$$

But for every $z \in E_j$

$$T^*[\chi_{\mathbb{C}\setminus B(x_j,2r_j)}\varphi](z) = (T^*)_{r_j}[\chi_{\mathbb{C}\setminus B(x_j,2r_j)}\varphi](z) - \int_{B(x_j,2r_j)\setminus B(z,r_j)} K(y,z)\varphi(y)d\mu(y).$$

The absolute value of the first term does not exceed $(T^*)^\sharp\varphi(z)$ while the second one is not greater than $2M'\varphi(z)$. Hence (recall that both $(T^*)^\sharp$ and M' are bounded in $L^2(\mu)$)

$$\Big|\sum_j I_j\Big| \le \int_E [(T^*)^\sharp\varphi + 2M'\varphi]d\mu \le \sqrt{\mu(E)}\cdot A||\varphi||_{L^2(\mu)} \le \frac{A||\varphi||_{L^2(\mu)}}{\sqrt{t}},$$

and we are done.

Clearly, if J is some subset of the index set $\{1,\dots,N\}$ and

$$\sigma^{(J)} := \sum_{j\in J} \chi_{\mathbb{C}\setminus B(x_j,2r_j)} \cdot T\chi_{E_j},$$

the same reasoning yields the estimate

$$\int_{\mathbb{C}} |\sigma^{(J)}|^2 d\mu \le A\sum_{j\in J}\mu(E_j).$$

The first step in the proof of $(*)$ is the same as in Key Lemma: instead of $\sigma_r(x)$ it is enough to consider $\sigma_{3R}(x)$ with some R satisfying $\mu(B(x,9R)) \le 81\mu(x,R))$. Here the factors $\chi_{\mathbb{C}\setminus B(x_j,2r_j)}$ can only help and we leave this step to the reader as an excercise.

Now consider the difference

$$\sigma_{3R}(x) - \frac{1}{\mu(B(x,R))}\int_{B(x,R)} \sigma\,d\mu.$$

It can be represented as the sum

$$\sum_{j=1}^N \{\chi_{\mathbb{C}\setminus B(x_j,2r_j)}(x)\cdot T\chi_{E_j\setminus B(x,3R))}(x) - \frac{1}{\mu(B(x,R))}\int_{B(x,R)} \chi_{\mathbb{C}\setminus B(x,2r_j)}\cdot T\chi_{E_j} =: \sum_j D_j, d\mu\}.$$

We will split the index set $\{1,\dots,N\}$ into several subsets:

1) $J_1 := \{j : B(x,3R)\cap B(x_j,2r_j)=\emptyset\}$

For every $j\in J_1$ we have

$$D_j=\int_{E_j}T^*[\delta_x-\frac{1}{B(x,R)}\chi_{B(x,R)}d\mu]d\mu$$

and, since E_j are pairwise disjoint and since $E_j\subset\mathbb{C}\setminus B(x,3R)$ for every $j\in J_1$, Obvious Lemma shows that

$$\Big|\sum_{j\in J_1}D_j\Big|\le 2A_1.$$

2) $J_2 := \{j : B(x,3R)\cap B(x_j,2r_j)\neq\emptyset,\ R<2r_j\}$.

For any $j\in J_2$ we have $x\in B(x_j,10r_j)$ and, due to the trivial estimate $D_j\le 2\frac{\mu(E_j)}{r_j}$, we conclude that

$$\Big|\sum_{j\in J_2}D_j\Big|\le 2\frac{m(x)}{t}.$$

3) $J_3 := \{j : B(x,3R)\cap B(x_j,2r_j)\neq\emptyset,\ R\ge 2r_j\}$

Note that for every $j\in J_3$ one has $E_j\subset B(x,9R)$. Now observe that

$$\big|\chi_{\mathbb{C}\setminus B(x_j,2r_j)}(x)\cdot T\chi_{E_j\setminus B(X,3R))}(x)\big|\le\frac{\mu(E_j)}{3R}$$

and thereby

$$\sum_{j\in J_3}\big|\chi_{\mathbb{C}\setminus B(x_j,2r_j)}(x)\cdot T\chi_{E_j\setminus B(X,3R))}(x)\big|\le\sum_{j\in J_3}\frac{\mu(E_j)}{3R}\le\frac{\mu(B(x,9R))}{3R}\le 3.$$

On the other hand,

(1)

$$\Big|\frac{1}{\mu(B(x,R)}\int^{B(x,R)}\sigma^{(J_3)}d\mu\Big|\le\frac{1}{\mu(B(x,R)}\sqrt{\mu(B(x,R)}\Big\{\int^{B(x,R)}\big|\sigma^{(J_3)}\big|^2d\mu\Big\}^{1/2}\le$$
$$\le\sqrt{\frac{A\sum_{j\in J_3}\mu(E_j)}{\mu(B(x,R)}}\le\sqrt{\frac{A\mu(B(x,9R))}{\mu(B(x,R)}}\le 9\sqrt{A}.$$

Thus

$$\Big|\sum_{j\in J_3}D_j\Big|\le 3+9\sqrt{A}$$

and we are through.

Let M stands for a usual maximal function, that is let

$$M\varphi(x):=\sup_{r>0}\mu(B(x,r))^{-1}\int_{B(x,r)}|\varphi|d\mu.$$

This function is larger than $M'\phi$ introduced earlier. But in cases when this maximal function has weak type $(1,1)$ or is bounded in some other sense (which will be the case for Euclidean space setting by Besicovich covering lemma) we may give an easier proof of the weak boundedness of $T^\sharp$. The proof of the theorem itself does not require any geometry or any covering properties. It is the application that is sensitive to the type of covering theorems. Theorem below was obtained also by Xavier Tolsa in [T2]. He uses the earlier version [NTV2] to do that. Actually he uses a certain idea from [DM] but in the interpretation given in [NTV2].

In [T2] this result is used give a full characterization of positive measures μ on the complex plane such that the truncations of Cauchy integral of any complex measure would pointwisely converge with respect to μ. In particular, if μ is such that the Cauchy integral operator is bounded in $L^2(\mu)$ then this pointwise convergence takes place. These remarkable result finishes the series of works [NTV1], [T1], [NTV2], [T2], in which the nonhomogeneous Calderón-Zygmund theory was considered.

Coming back to the weak type estimates for maximal singular operator we first prove the following Cotlar's inequalities.

Theorem 4.

$$T^\sharp\varphi \le A_p\{M|T\varphi|^p\}^{1/p} + B_pM\varphi,$$

if $p \in (0,1)$. Also

$$T^\sharp\varphi \le AMT\varphi + BM\varphi$$

where the constants A and B depend on $||T||_{L^2(\mu)\to L^2(\mu)}$ only.

Proof.

It is just a minor modification of the proof of Key Lemma. Let again $r > 0$. First we are proving the following form of Cotlar's inequality. Fix $p < 1$.

Clearly the next inequality implies both inequalities in Theorem 4.

$$|(T_rF)(y)|^p \le M(|TF|^p)(y) + C(MF(y))^p.$$

Fix $r > 0$ and let R be the number from Key Lemma. Let us prove first that

$$|(T_rF)(y)| \le |(TF)(x)| + |(TF\chi_{B(y,3R)})(x)| + CMF(y)$$

for any $x \in B(y,R)$.

Suppose this is proved. Then we rais it to the power p and average over $B(y,R)$ with respect to measure μ. We will need the following estimate

$$\int_{B(y,R)} |(TF\chi_{B(y,3R)})(x)|^p d\mu(x) \le C\mu(B(y,R))^{1-p}(\int_{B(y,3R)} |F|d\mu)^p$$

Dividing by $\mu(B(y,R))$, and noticing that the choice of p gives $\mu(B(y,3R)) \le 9\mu(B(y,R))$ we get that the averaging of $|(TF\chi_{B(y,3R)})(x)|^p$ over $B(y,R)$ is bounded by

$$9^pC(\mu(B(y,3R))^{-1}\int_{B(y,3R))} |F|d\mu)^p \le 9^pC(MF(y))^p$$

. Then we get our first inequality, and we are done.

So we are left to prove two things

1) $|(T_rF)(y)| \le |(TF)(x)| + |(TF\chi_{B(y,3R)})(x)| + CMF(y)$,
2) $\int_{B(y,R)} |(TF\chi_{B(y,3R)})(x)|^p d\mu(x) \le C\mu(B(y,R))^{1-p}(\int_{B(y,3R)} |F|d\mu)^p$.

As for the first one, its proof repeats Key Lemma. Define B_j and μ_j as in the proof of Key Lemma. Then

$$|(T_rF)(y) - T_{3R}F)(y)| \leq \sum_{j=1}^{k} (3^{-j}R)^{-1}\mu(B(y,3R))3^{-2j}\mu(B_j)^{-1}\int_{B_j}|F|d\mu \leq CMF(y).$$

On the other hand, for $x \in B(y,R)$ obviously

$$|T_{3R}F)(y) - T_{3R}F)(x)|| \leq CMF(y)$$

and $|T_{3R}F)(x)| \leq |(TF)(x)| + |T(F\chi_{B(y,3R)})(x)|$. Thus the first inequality is proved.

Now the second one follows from the following elementary lemma.

Sublemma

Let $\varphi \in L^{1,\infty}$ and let $p \in (0,1)$. For a measurable set B we have

$$\int_B |\varphi|^p \leq C_p(\mu(B))^{1-p}(\|\varphi\|_{L^{1,\infty}})^p.$$

Proof

It is an obvious use of the formula

$$\int_B |\varphi|^p d\mu = \int_0^\infty ps^{p-1}\mu\{y \in B : \varphi > s\}ds \leq \int_0^\infty ps^{p-1}\min(\mu(B), s^{-1}\|\varphi\|_{L^{1,\infty}})ds$$

Now we apply Sublemma to $\varphi = T(F\chi_{B(y,3R)})$, and we get 2) immediately if we use already proven fact that T maps L^1 to $L^{1,\infty}$.

Theorem 4 is completely proved now.

Remark. It is clear from the proof that the maximal function in front of $|T\varphi|$ (or $|T\varphi|^p$ term is a restricted maximal function $\tilde{M}$. Meaning that it is taken only over the discs with doubling property. One need not Besicovich covering lemma to give the weak type estimate for such restricted maximal function. But the second maximal function–the one in front of φ–is the usual unrestricted centered maximal function. This is why Theorem 4 is applicable only in Euclidean spaces.

Let us show why Theorem 4 gives the weak type estimate for $T^\sharp$. We use the first inequality of Theorem 4. The second term is estimated as usual, and by Besicovich covering lemma it is in $L^{1,\infty}$. To estimate the first term let us notice that

$$\mu\{x : MF(x) > t\} \leq Ct^{-1}\int_{F>t/2} F d\mu$$

for positive functions F. To notice that we just split F to $F\chi_{\{x:F\leq t/2\}}$ and $F\chi_{\{x:F>t/2\}}$. Then apply the usual weak type estimate to $M(F\chi_{\{x:F>t/2\}})$. After this remark let us apply it to $F := |T\varphi|^p$ and $t := 2s^p$. Using Sublemma again (and also using twice the fact that T maps L^1 to $L^{1,\infty}$) we get

$$\mu\{x : M(|T\varphi|^p) > 2s^p\} \leq Cs^{-p}\int_{|T\varphi|>s}|T\varphi|^p d\mu$$

$$\leq Cs^{-p}\mu\{|T\varphi| > s\}^{1-p}\|\varphi\|_1^p \leq Cs^{-p}s^{p-1}\|\varphi\|_1^p\|\varphi\|_1^{1-p},$$

and we are done.

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