# THE CALDERÓN COMMUTATOR ALONG A PARABOLA

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Abstract. We introduce an analogue of Calderón's first commutator along a parabola, and establish its $L^2$ boundedness under essentially sharp hypotheses.

## 1. Introduction

During the past 30 years, many authors have investigated the $L^p$ mapping properties of singular integral operators whose "kernels" are actually singular measures supported on lower dimensional subvarieties. One may consult the survey article of Wainger [19] for a history of the subject up to the mid 1980's. The interested reader may also find a synopsis of more recent developments, along with numerous references, in the monograph of Stein [18, Chapter XI, sections 4.5, 4.7, 4.17]. The prototypical example of of such operators is the "Hilbert transform along a curve":

$$H_\gamma f(x) \equiv p.v. \int_{\mathbb{R}} f(x-\gamma(t))\frac{dt}{t}, \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $\gamma : \mathbb{R} \to \mathbb{R}^n$ is the parametrization of a smooth curve in $\mathbb{R}^n$. This sort of operator was first introduced by Fabes [8] who established $L^2$ boundedness of $H_\gamma$ in the special case $n = 2$, $\gamma(t) = (t, (sgnt)t^2)$. The original motivation for studying operators of this sort is that they arise when one tries to develop a "method of rotations" (see [2] in the classical elliptic case) for parabolic singular integrals of convolution type. In turn, the method of rotations enables one to significantly relax the regularity hypotheses on kernels of singular integral operators.

In the present paper, motivated in part by formal analogy to Fabes' goal of extending the method of rotations to the parabolic case, and in part by recent developments in the theory and application of parabolic singular integrals which are not of convolution type [16], [12], [13], [15], we introduce a certain bilinear analogue of (1.1): the "Calderón commutator along a parabola". To describe this operator, suppose that $A : \mathbb{R}^2 \to \mathbb{R}$ satisfies the $Lip_{1,\frac{1}{2}}$ condition

$$|A(x+h) - A(x)| \le B_0 r, \tag{1.2}$$

for some constant $B_0$, whenever $h \equiv (h_1, h_2)$ satisfies $|h_1| \le r$, $|h_2| \le r^2$. We remark that the results of the present paper may be readily extended to $\mathbb{R}^n$, $n > 2$, by the same arguments which we shall give below, but to minimize technicalities we shall restrict our attention to the case $n = 2$. Since the story here is not yet complete (we

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do not yet know how to treat the higher order commutators, for example), it does not seem crucial at this point that we state our results in the greatest generality.

For $A$ as in (1.2) and for $\gamma(t) \equiv (t, t^2)$, we define

$$T_A^\gamma f(x) \equiv p.v. \int_{\mathbb{R}} [A(x) - A(x - \gamma(t))] f(x - \gamma(t)) \frac{dt}{t^2}. \tag{1.3}$$

As mentioned above, the operator $T_A$ has a connection with the results of [16], [12], [13], and [15], which may be understood as follows. Let $K : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ satisfy the parabolic homogeneity property

$$K(\rho x_1, \rho^2 x_2) \equiv \rho^{-4} K(x), \tag{1.4}$$

and further suppose that $K \in L^1(S^1)$, and that $K(x_1, x_2)$ is even in $x_1$, for each fixed $x_2$. Then one may use parabolic polar co-ordinates to obtain a representation of the parabolic Calderón commutator in terms of $T_A^\gamma$. Indeed,

$$\begin{aligned} C_A f(x) &\equiv p.v. \int_{\mathbb{R}^2} [A(x) - A(y)] K(x - y) f(y) dy \\ &\equiv p.v. \int_{S^1} K(\sigma) T_A^{\gamma_\sigma} f(x) (1 + \sigma_2^2) d\sigma, \end{aligned} \tag{1.5}$$

where $T_A^{\gamma_\sigma}$ is defined as in (1.3), with $\gamma_\sigma(t) \equiv (t\sigma_1, t^2\sigma_2)$, $(\sigma_1, \sigma_2) \in S^1$. We have used here the parabolic polar coordinates

$$\begin{aligned} x_1 = \rho\sigma_1, \quad x_2 = \rho^2 \sigma_2 \\ dx = \rho^2 d\rho (1 + \sigma_2^2) d\sigma. \end{aligned}$$

When $\sigma = (\pm 1, 0)$, $L^2$ boundedness of $T_A^{\gamma_\sigma}$ reduces to that of Calderón's original commutator on the line [1]; when $\sigma = (0, \pm 1)$, matters reduce to a result of Murray [17]. Otherwise, $L^2$ boundedness of $T_A^{\gamma_\sigma}$ is new and will be treated in this paper, although to simplify the notation we shall take $\sigma_1 = 1 = \sigma_2$. The proof in the general case is identical and one obtains bounds independent of $\sigma \in S^1$.

In [12], it was shown that for the special case

$$K(x_1, x_2) \equiv x_2^{-2} \exp \left\{ \frac{-|x_1|^2}{4x_2} \right\} \chi_{\{x_2 > 0\}}, \tag{1.6}$$

a necessary and sufficient condition for the $L^2$ boundedness of $C_A$ is that, for some $B < \infty$,

$$\begin{aligned} &\text{(i)} \quad |A(x_1 + h, x_2) - A(x_1, x_2)| \leq B|h| \\ &\text{(ii)} \quad \|\mathbb{D}_2 A\|_{BMO} \leq B, \end{aligned} \tag{1.7}$$

where

$$\mathbb{D}_2 A \equiv \left( \frac{\xi_2}{\sqrt{|\xi_1|^2 - i\xi_2}} \hat{A}(\xi) \right)^\vee,$$

and $\hat{f}$ and $f^{\vee}$ denote the Fourier and inverse Fourier transforms of $f$, respectively. The $BMO$ norm is defined as usual by

$$\|b\|_{BMO} = sup \int_I |b - m_I b|,$$

where $m_I b$ denotes the mean value of $b$ over $I$, and where the sup runs over all parabolic "cubes" of the form

$$I \equiv I_r(a,b) \equiv \left[a - \frac{r}{2}, a + \frac{r}{2}\right] \times \left[b - \frac{r^2}{2}, b + \frac{r^2}{2}\right]. \tag{1.8}$$

The higher dimensional case was also treated in [12]. The kernel (1.6) arises in the multilinear expansion of the parabolic double layer potential on a time varying graph $x_3 = A(x_1, x_2)$. A condition similar to (1.7) (later shown to be equivalent in [15]) had previously been shown to be sufficient for $L^2$ boundedness in [16]. It was also shown in [12] that (1.7) implies the $Lip_{1,1/2}$ condition (1.2), with bound $B_0 \equiv CB$.

In this paper we prove the following:

**Theorem 1.** *Suppose $A$ satisfies* (1.7). *Then $T_A^\gamma$ is bounded on $L^2$.*

**Remarks.**

(1) By our previous comments, one may then obtain a "method of rotations" which permits us to deduce $L^2$ boundedness of $C_A$ (defined in (1.5)) for $K$ as in (1.4) satisfying only that $K \in L^1(S^1)$, and that $K(x_1, x_2)$ is even in $x_1$, for each fixed $x_2$. Previously, it had been required that $K$ have some smoothness on $S^1$.

(2) One can also show that $T_A^\gamma$ is bounded on $L^p, 1 < p < \infty$, by using the key estimate (2.4) below and Littlewood-Paley arguments adapted to rough singular integral operators. See e.g., [14].

In the next section we begin the proof of Theorem 1, and complete it in Section 3.

## 2. Proof of Theorem 1

We may assume, and do, that the constant $B$ in (1.7) is one. By the parabolic version of [14] (whose proof is virtually identical that given in [14], and is therefore omitted here), it is enough to check the following "rough operator" $T1$ criterion, which consists of three parts. We need to prove that

$$T_A^\gamma 1, (T_A^\gamma)^* 1 \in BMO, \tag{2.1}$$

where $(T_A^\gamma)^*$ is defined as a mapping from test functions to distributions by

$$\langle (T_A^\gamma)^* f, g\rangle \equiv \langle f, T_A^\gamma g\rangle,$$

for all $f, g \in C_0^\infty(\mathbb{R}^2)$. Second, for all $x \in \mathbb{R}^2$, $r > 0$, let $\Phi(x, r)$ denote the class of all $\varphi \in C_0^\infty$, supported in $I_r(x)$ (see (1.7)), and satisfying

$$\begin{aligned} &\text{(i)} \qquad && \|\varphi\|_\infty \leq 1 \\ &\text{(ii)} && |\varphi(y+h) - \varphi(y)| \leq \rho/r, \end{aligned} \tag{2.2}$$

whenever $|h_1| \leq \rho$, $|h_2| \leq \rho^2$, $h \equiv (h_1, h_2)$, and

$$\text{(iii)} \qquad \sup_{k+m\leq 2} \| \left(\frac{\partial}{\partial x_1}\right)^k \left(\frac{\partial}{\partial x_2}\right)^m \varphi\|_\infty \leq 1.$$

We shall need to establish the Weak Boundedness Property (WBP):

$$|\langle \psi, T_A^\gamma \phi\rangle| \leq Cr^3, \tag{2.3}$$

for all $\psi, \phi \in \Phi(x, r)$, for any $r > 0$, $x \in \mathbb{R}^2$. Since the homogeneous dimension of parabolic $\mathbb{R}^2$ is $d = 3$, (2.1) and (2.3) are the usual David-Journé conditions [7] in this context. However, in lieu of the standard Calderón-Zygmund kernel conditons considered in [7],we shall establish instead the weak smoothness condition of [14], which had also appeared in a similar connection in [3]. Let $Q_s$ denote the operator defined by convolution with a smooth function $\psi_s$ which has mean value zero, is supported in $I_s(0)$, and is normalized so that $\|\psi_s\|_1 = 1$. We shall prove that, whenever $s \leq 2^j$,

$$\|Q_s T_j\|_{L^2\to L^2} \leq C\left(\frac{s}{2^j}\right)^\epsilon \tag{2.4}$$

for some $\epsilon > 0$, where

$$T_j f(x) \equiv \int_{\mathbb{R}} [A(x) - A(x - \gamma(t))] f(x - \gamma(t)) \eta\left(\frac{t}{2^j}\right) \frac{dt}{t^2},$$

and where $\eta \in C_0^\infty\left[\left(\frac{1}{2}, 2\right) \cup \left(-2, -\frac{1}{2}\right)\right]$, $0 \leq \eta \leq 1$, and $\sum_{j=-\infty}^{\infty} \eta\left(\frac{\cdot}{2^j}\right) \equiv 1$ away from 0.

By [14] (or rather its parabolic analogue), Theorem 1 follows immediately from (2.1), (2.3) and (2.4). In this section we shall prove (2.1) and (2.3). In the next section we prove (2.4).

To establish WBP (2.3), by dilation invariance we may assume $r = 1$, so it is clearly enough to prove that, for any $x_0 \in \mathbb{R}^2$,

$$T_A^\gamma \varphi \in L^2(I) + L^\infty(I), \quad \forall \varphi \in \Phi(x_0, 4), \tag{2.5}$$

where $I = I_1(x_0)$. We claim that (2.5) also implies that $T_A^\gamma 1 \in BMO$ (we omit the proof of the fact that $(T_A^\gamma)^* 1 \in BMO$, since it is identical). To prove the claim, we need to consider

$$\frac{1}{|I|} \int_I |T_A^\gamma 1 - C_I|, \tag{2.6}$$

where $I = I_r(x_0)$, and by dilation invariance we may take $r = 1$. We write

$$1 = \phi + (1 - \phi),$$

where $\phi \equiv 1$ in $I_3(x_0)$, and $\phi \in C_0^\infty(I_4(x_0))$. Then up to a normalizing constant, $\phi \in \Phi(x_2, 4)$, so by (2.5)

$$\int_I |T_A^\gamma \phi| \leq C.$$

We now set $C_I \equiv T_A^\gamma(1-\varphi)(x_0)$, so that, for all $x \in I$,

$$\begin{aligned}
&|T_A^\gamma(1-\phi)(x) - T_A^\gamma(1-\phi)(x_0)| \\
&\leq \int |A(x) - A(x-\gamma(t)) - [A(x_0) - A(x_0) - \gamma(t)]||1-\phi(x-\gamma(t))|\frac{dt}{t^2} \\
&+ \int |A(x_0) - A(x_0-\gamma(t))||\phi(x-\gamma(t) - \phi(x_0-\gamma(t))|\frac{dt}{t^2} \\
&\leq c\left(\int_{|t|>1} \frac{dt}{t^2} + \int_{1<|t|<5} \frac{dt}{t}\right) = C,
\end{aligned}$$

by (1.2) (with $B_0 \equiv C$), and the definition of $\phi$. Consequently, (2.6) is no larger than $C$, which proves the claim. Thus, we have reduced the proofs of (2.1) and (2.3) to that of (2.5).

We establish (2.5) under the a priori assumption that $A \in C^\infty$, but our quantitative estimates will depend only on the bounds in (1.7) (which we have normalized to be 1). Let $\phi \in \Phi(x_0, 4)$ and then integrate by parts; to do this rigorously requires truncation of the principal value integral in (1.3), but it is routine to verify that the boundary terms (which arise when integrating the truncated integrals by parts) are harmless. We shall therefore argue formally, and ignore all such truncations and boundary terms, in order not to tire the reader with minutiae. Formally then,

$$\begin{aligned}
T_A^\gamma\phi(x) &= \int_{\mathbb{R}} \frac{\partial A}{\partial x}(x-\gamma(t))\phi(x-\gamma(t))\frac{dt}{t} + 2\int_{\mathbb{R}} \frac{\partial A}{\partial t}(x-\gamma(t))\phi(x-\gamma(t))dt \\
&+ \int [A(x) - A(x-\gamma(t)] \left\{\frac{\partial \phi}{\partial x}(x-\gamma(t)) + 2\frac{\partial \phi}{\partial t}(x-\gamma(t))t\right\}\frac{dt}{t} \equiv I + II + III.
\end{aligned}$$

But $I$ is precisely the Hilbert transform along the curve $(t, t^2)$, acting on the $L^2$ function $\frac{\partial A}{\partial x}\phi$. Thus, $I \in L^2$. Also $III \in L^\infty(I)$, by (1.2) and the fact that $\phi \in \Phi(x_0, 4)$ (see 2.2). The only delicate term is II. We recall that $\frac{\partial A}{\partial x} \in L^\infty, \mathbb{D}_2 A \in BMO$, with norm $B = 1$ (given our normalization). Define $a \equiv \mathbb{D}A \equiv (\sqrt{|\xi_1|^2 - i\xi_2}\hat{A}(\xi))^\vee$, which is therefore in $BMO$, with norm $CB = C$ by parabolic Calderón-Zygmund Theory [9, 10]. Then

$$\frac{\partial A}{\partial t} \equiv \mathbb{D}_2(\mathbb{D}A) \equiv \mathbb{D}_2 a,$$

so that

$$II = \int \mathbb{D}_2 a(x-\gamma(t))\phi(x-\gamma(t))dt. \tag{2.7}$$

By [9, Lemma 1 and Lemma 2, pp. 111-113],

$$\mathbb{D}_2 a(x) \equiv p.v. \int k(x-y)a(y)dy,$$

where $k(x)$ is odd, belongs to $C^\infty(\mathbb{R}^2 \backslash \{0\})$ and satisfies the homogeneity property

$$k(\lambda x, \lambda^2 t) \equiv \lambda^{-d-1}k(x,t) \tag{2.8}$$

(we recall that $d = 3$ is the homogeneous dimension of parabolic $\mathbb{R}^2$). By the oddness of $k$, we may assume that a has mean value zero on $I_1(x_0) \equiv I$. Let $\|x\| \equiv |x_1| + |x_2|^{1/2}$ be the parabolic "norm" of $x$. Let $\eta \in C_0^\infty[-10, 10]$, $\eta \equiv 1$ on $[-9, 9]$, and set $a_1(x) = a(x)\eta(\|x - x_0\|)$, $a_2 = a - a_1$. By (2.8) and the parabolic version of a standard estimate of [11],

$$\begin{aligned} |\mathbb{D}_2 a_2(x)| &\le \int \frac{c}{1 + \|x - y\|^{d+1}} |a(y)| dy \\ &\le C\|a\|_{BMO} \le C, \end{aligned}$$

whenever $x \in \operatorname{supp} \phi \subseteq I_4(x_0)$. Consequently the contribution of $a_2$ to (2.7) yields a bounded term, as desired. The contribution of $a_1$ is

$$\int \phi(x - \gamma(t))\eta(t)\mathbb{D}_2 a_1(x - \gamma(t))dt$$

(since $x \in I_1(x_0)$, $x - \gamma(t) \in I_4(x_0)$)

$$\begin{aligned} &= \int \eta(t) \int k(x - \gamma(t) - y)[\phi(x - \gamma(t)) - \phi(y)]a_1(y)dydt \\ &\quad + \int \eta(t)\mathbb{D}_2(\phi a_1)(x - \gamma(t))dt \\ &\equiv II_1 + II_2. \end{aligned}$$

By [12] (or even [9], since $\phi$ is smooth), the commutator $[\mathbb{D}_2, \phi]$ defines a bounded operator on $L^2$. Hence, by Minkowksi's inequality,

$$\begin{aligned} \|II_1\|_2 &\le \int \eta(t)\|[\mathbb{D}_2, \phi]a_1(\cdot - \gamma(t)\|_2 dt \\ &\le c\|a_1\|_2 \le c\|a\|_{BMO} \le C. \end{aligned}$$

Finally, since $\phi a_1 \in L^2$, and $\mathbb{D}_2$ is given by the multiplier $\xi_2(|\xi_1|^2 - i\xi_2)^{-1/2}$, it is enough to show that the operator

$$f \to \int f(x - \gamma(t))\eta(t)dt$$

is smoothing on $L^2$ of parabolic order 1; i.e., that the multiplier

$$m(\xi) \equiv \int e^{i(\xi_1 t + \xi_2 t^2)}\eta(t)dt$$

satisfies

$$|m(\xi)| \le C(|\xi_1| + |\xi_2|^{1/2})^{-1}$$

This estimate is well known, and is an easy consequence of standard integration by parts arguments as may be found in [18, Chapter VIII]. For the sake of completeness, we sketch the argument here.

**Case 1.** $|\xi_1| > 40|\xi_2|$.

In this case,

$$|n(\xi)| = \left| \int e^{i(\xi_1 t + \xi_2 t^2)} \frac{d}{dt} \left( \frac{1}{\xi_1 + 2 + \xi_2} \eta^0(t) \right) dt \right| \leq C|\xi_1|^{-1},$$

Since $\operatorname{supp} \eta \leq [-10, 10]$.

**Case 2.** $|\xi_1| < 40|\xi_2|$.

In this case we seek the estimate $|m(\xi)| \leq C|\xi_2|^{-1/2}$. We write

$$m(\xi) = \int e^{i\lambda\varphi_\xi(t)} \eta(t) dt,$$

where $\lambda \equiv |\xi_2|$, and $\phi_\xi(t) \equiv \left( \frac{\xi_1}{|\xi_2|} t + sgn \xi_2 t^2 \right)$. Note that $|\phi_\xi''(t)| = 2$. The desired bound now follows immediately from Vander Corput's Lemma, or to be more precise, its corollary given in [18, p. 334, inequality (6)]. This concludes the proof of (2.5), and therefore also the proofs of (2.1) and (2.3). We finish the proof of our Theorem in the next section, in which we prove (2.4).

## 3. Proof of Theorem 1 (continued): estimate (2.4)

In this section, we give the proof of estimate (2.4), which will complete the proof of Theorem 1. This is the most technical part of the proof, and follows ideas from [6]; see also [4] and [5] .

By dilation invariance, we may take $j = 0$. We recall that $T_0 f$ is a sum of two terms

$$\tilde{T}_0 f(x) \equiv \int_{\mathbb{R}} [A(x) - A(x - \gamma(t))] f(x - \gamma(t)) \eta(t) \frac{dt}{t^2}$$

where $\eta$ is an even smooth cut-off function supported in the "half annulus" $\frac{1}{2} < t < 2$, plus another term with $-2 < t < -\frac{1}{2}$. By symmetry, we treat only the former. Let $K_0$ denote the kernel of $\tilde{T}_0^* \tilde{T}_0^*$, where $\tilde{T}_0^*$ denotes the adjoint of $\tilde{T}_*$. Following [6], we see that the desired bound (2.4) then follows easily (we omit the routine details) from

**Claim.**

$$\int |K_0(x + h, y) - K(x, y)| dy \leq C\lambda^{1/3}, \tag{3.1}$$

whenever $|h_1| \leq \lambda$, $|h_2| \leq \lambda^2$, $\lambda \leq 1/1000$. Now, it is a routine matter to see that

$$\tilde{T}_0^* g(x) = -\int [A(x) - A(x + \gamma(s))] g(x + \gamma(s)) \eta(s) \frac{ds}{s^2}$$

Thus,

(3.2)

$$
\begin{aligned}
\tilde{T}_0\tilde{T}_0^* f(w) &= \int [A(w) - A(w - \gamma(t))] T_0^* f(\omega - \gamma(t))\phi(t)dt \\
&= -\iint [A(\omega) - A(\omega - \gamma(t))] \\
&\quad [A(\omega - \gamma(t)) - A(\omega + \gamma(s) - \gamma(t)] f(\omega + \gamma(s) - \gamma(t))\phi(t)\phi(s)dtds,
\end{aligned}
$$

where for convenience of notation we have defined

$$\phi(t) \equiv \eta(t)t^{-2}.$$

Now, the claim (3.1) amounts to saying that the operator

$$f \to \int [K_0(\cdot + h, y) - K_0(\cdot, y)] f(y)dy$$

maps $L^\infty \to L^\infty$ with norm $C\lambda^{1/3}$. Hence, we may with impunity excise (in (3.2)) any set in $(s,t)$ space of measure no larger than $C\lambda^{1/3}$. Indeed, we shall restrict the domain of integration in (3.2) to the set

$$\left\{ (s,t) \in \left[\frac{1}{2}, 2\right]^2 : |s - t| \geq 15\lambda^{\frac{1}{3}} \right\}.$$

By symmetry it suffices to consider only the case $s > t$. Note that in the set where $s - t \geq \lambda^{\frac{1}{3}}$, the map

$$
\begin{aligned}
&\Phi_\omega : (s,t) \mapsto \omega + \gamma(s) - \gamma(t) = y \\
&\frac{1}{4} \leq s < t \leq 4 \mapsto \mathbb{R}^2
\end{aligned}
\tag{3.3}
$$

is injective, and that the Jacobian matrix is

$$\begin{bmatrix} 1 & -1 \\ 2s & -2t \end{bmatrix}$$

with determinant $2(s-t) \geq 2\lambda^{1/3}$. Let $S_0$ denote the part of the operator $\tilde{T}_0\tilde{T}_0^*$ in (3.2), with $(s-t) > 15\lambda^{1/3}$. We then have that

$$
\begin{aligned}
|S_0 f(x+h) - S_0 f(x)| = |\iint_{E\lambda} & \phi(s)\phi(t)\,\{[A(x+h) - A(x+h-\gamma(t)][A(x+h-\gamma(t)] \\
& [A(x+h-\gamma(t)) - A(x+h+\gamma(s)-\gamma(t))]f(x+h+\gamma(s)-\gamma(t)) \\
& - [A(x) - A(x-\gamma(t))] \\
& \quad [A(x-\gamma(t)) - A(x+\gamma(s)-\gamma(t)]f(x+\gamma(s)-\gamma(t))\}\,dsdt| \\
& \equiv |\iint_{E_\lambda} \phi(s)\phi(t)\,\{B(x+h,s,t)f(x+h+\gamma(s)-\gamma(t)) \\
& - B(x,s,t)f(x+\gamma(s)-\gamma(t)\}\,dsdt|
\end{aligned}
$$

(where $E_\lambda \equiv \{(s,t) : s - t \geq 15\lambda^{1/3}, \frac{1}{2} \leq t < s \leq 2\}$, and $B(x,s,t) \equiv [A(x) - A(x - \gamma(t))][A(x-\gamma(t)) - A(x+\gamma(s)-\gamma(t))]$)

$$
\begin{aligned}
&= |\iint_{E_\lambda} \phi(s)\phi(t)\,\{[B(x+h,s,t) - B(x,s,t)]f(x+h+\gamma(s)-\gamma(t)) \\
&\quad + B(x,s,t)[f(x+h+\gamma(s)-\gamma(t)) - f(x+\gamma(s)-\gamma(t))]dsdt| \\
&\equiv |\iint (I + II)dsdt|.
\end{aligned}
$$

Now suppose that $\|f\|_{L^\infty=1}$. Taking the supremum over all such $f$, we see that the contribution of $\iint I dsdt$ to (3.1) satisfies the claim, by virtue of the $Lip(1, \frac{1}{2})$ character of $A$.

Next,

$$
\begin{aligned}
|\iint II dsdt| = |\int_{\Phi_{x+h}(E_\lambda)} & B(x, \Phi_{x+h}^{-1}(y))f(y)J\Phi_{x+h}^{-1}(y)dy \\
& - \int_{\Phi_x(\xi_\lambda)} B(x, \Phi_x^{-1}(y))f(y)J\Phi_x^{-1}(y)dy
\end{aligned}
$$

(where $\Phi_x$ is the mapping defined in (3.3))

$$
\begin{aligned}
&= |\int_{\Phi_{x+h}(E_\lambda)} \frac{[B(x, \Phi_{x+h}^{-1}(y)) - B(x, \Phi_x^{-1}(y)]}{J\Phi_{x+h}(\Phi_{x+h}^{-1}(y)]} f(y)dy \\
&\quad + \int_{\Phi_{x+h}(E_\lambda)} \left[\frac{1}{J\Phi_x(\Phi_{x+h}^{-1}(y))} - \frac{1}{J\Phi_x(\Phi_x^{-1}(y))}\right] B(x, \Phi_x^{-1}(y))f(y)dy \\
&\quad + \int_{\Phi_{x+h}(E_\lambda)\setminus\Phi_x(E_\lambda)} \frac{B(x, \Phi_x^{-1}(y))}{J\Phi_x(\Phi_x^{-1}(y))} f(y)dy - \int_{\Phi_x(E_\lambda)\setminus\Phi_{x+h}(E_\lambda)} \frac{B(x, \Phi_\lambda^{-1}(y))}{J\Phi_x(\Phi_x^{-1}(y))} f(y)dy| \\
&\equiv |\int (III + IV + V + VI)dy|,
\end{aligned}
$$

where we have used that $J\Phi_x(s,t)$ is independent of $x$. Then

$$
\begin{aligned}
(3.4) \qquad |\int III dy| &\le c\|f\|_\infty \int_{\Phi_{x+h}(E_\lambda)} |B(x,\Phi_{x+h}^{-1}(y)) - B(x,\Phi_x^{-1}(y))| J\Phi_{x+h}^{-1}(y)dy \\
&= C\|f\|_\infty \int_{E_\lambda} |B(x,s,t) - B(x,s_h,t_h)| dsdt,
\end{aligned}
$$

where

$$
(3.5) \qquad (s_h, t_h) \equiv \Phi_x^{-1}(\Phi_{x+h}(s,t)),
$$

i.e.

$$
\begin{aligned}
& x + \gamma(s_h) - \gamma(t_h) = x + h + \gamma(s) - \gamma(t) \\
\Leftrightarrow \quad & \gamma(s_h) - \gamma(t_h) = h + \gamma(s) - \gamma(t) \\
\Leftrightarrow \quad & (s_h - t_h, s_h^2 - t_h^2) = (h_1 + s - t, h_2 + s^2 - t^2).
\end{aligned}
$$

(by definition of $\gamma$) i.e.,

$$
\begin{aligned}
s_h &= \frac{1}{2}\left\{h_1 + s - t + \frac{h_2 + s^2 - t^2}{h_1 + s - t}\right\} \\
t_h &= \frac{1}{2}\left\{t - s - h_1 + \frac{h_2 + s^2 - t^2}{h_1 + s - t}\right\}
\end{aligned}
$$

Now,

(3.6)

$$
\begin{aligned}
|B(x,s,t) - B(x,s_h,t_h)| &\equiv |[A(x) - A(x-\gamma(t))][A(x-\gamma(t)) - A(x+\gamma(s)-\gamma(t)] \\
&\quad - [A(x) - A(x-\gamma(t_h))][A(x-\gamma(t_h) - A(x+\gamma(s_h)-\gamma(t_h)]| \\
&= |[A(x) - A(x-\gamma(t)) - \{A(x) - A(x-\gamma(t_h))\}][A(x-\gamma(t)) \\
&\quad - A(x+\gamma(s)-\gamma(t)] \\
&\quad + [A(x) - A(x-\gamma(t_h)][A(x-\gamma(t)) - A(x+\gamma(s)-\gamma(t)) \\
&\quad - \{A(x-\gamma(t_h) - A(x+\gamma(s_h)-\gamma(t_h)\}]| \\
&\le C(\|\gamma(t_h) - \gamma(t)\| + \|\gamma(s_h) - \gamma(s)\|),
\end{aligned}
$$

where $\|\cdot\|$ denotes the parabolic metric $\|(u,v)\| \cong |u| + |v|^{1/2}$. Notice that

$$
|t_h^2 - t^2| \cong |t_h - t|,
$$

and similarly for $s_h^2 - s^2$, so that

$$
\begin{aligned}
\|\gamma(t_h) - \gamma(t)\| &\le C|t_h - t|^{1/2} \\
\|\gamma(s_h) - \gamma(s)\| &\le C|s_h - s|^{1/2}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
|t^{\cdot} - t_h| &= \left| \frac{1}{2}\left\{ t + s + h_1 - \left( \frac{h_2 + s^2 - t^2}{h_1 + s - t} \right) \right\} \right| \\
&= \left| \frac{h_1}{2} + \frac{1}{2}\left\{ \frac{(h_1 + (s-t))(t+s) - (h_2 + s^2 - t^2)}{h_1 + s - t} \right\} \right| \\
&= \left| \frac{h_1}{2} + \frac{1}{2}\left\{ \frac{h_1(t+s) - h_2}{h_1 + s - t} \right\} \right| \\
&\leq \lambda^{2/3},
\end{aligned} \tag{3.7}
$$

since $|h_1| \leq \lambda \leq 1$, $|h_2| \leq \lambda^2$, and $s - t > 15\lambda^{1/3}$, $\frac{1}{2} \leq s, t \leq 2$, on $E_\lambda$. A similar estimate holds for $|s - s_h|$. Hence $\int (III) dy \leq C\lambda^{1/3}$ as desired, by virtue of (3.4) and (3.6). Next, we observe that

$$
\begin{aligned}
\left| \int IV dy \right| &\leq C\|f\|_\infty \int_{\Phi_{x+h}(E_\lambda)} \left| \frac{1}{J\Phi_x(\Phi_{x+h}^{-1}(y))} - \frac{1}{J\Phi_x(\Phi_x^{-1}(y))} \right| dy \\
&= C\|f\|_\infty \iint_{E_\lambda} \left| \frac{1}{s-t} - \frac{1}{s_h - t_h} \right| (s-t) ds dt
\end{aligned}
$$

where $(s_h, t_h)$ is defined as above (see (3.5)), and where we have used that $J\Phi_x(s,t) = 2(s-t)$. As we have observed (see (3.7)), $|t - t_h| + |s - s_h| \leq 2\lambda^{2/3} << 15\lambda^{\frac{1}{3}} \leq s - t$, so that

$$
\begin{aligned}
\left| \int IV dy \right| &\leq C\|f\|_\infty \int\int_{E_\lambda} \frac{\lambda^{2/3}}{(s-t)^2}(s-t) ds dt \\
&\leq C\|f\|_\infty \lambda^{1/3}
\end{aligned}
$$

as desired.

Turning last to the term $|\int V dy|$ (the term $|\int VI dy|$ can be handled by similar arguments, which we omit) we see that, since $J\Phi_x(s,t) = 2(s-t)$,

$$
\begin{aligned}
\left| \int V dy \right| &\leq C\|f\|_\infty \int_{\Phi_{x+h}(E_\lambda) \setminus \Phi_x(E_\lambda)} \frac{1}{J\Phi_x(\Phi_x^{-1}(y))} dy \\
&\leq C\|f\|_\infty \iint_{F_\lambda} \frac{1}{s_h - t_h}(s-t) ds dt,
\end{aligned}
$$

where $F_\lambda = \Phi_{x+h}^{-1}(\Phi_{x+h}(E_\lambda) - \Phi_x(E_\lambda))$, and where $(s_h, t_h)$ is defined as above (see (3.5)). Since $s_h - t_h \geq c\lambda^{1/3}$, it is enough to show that $|F_\lambda| \leq C\lambda^{2/3}$.

To this end, suppose that $(s,t) \in F_\lambda$. Then, in particular, $(s,t) \in E_\lambda$, and $y \equiv \Phi_{x+h}(s,t) \in \Phi_{x+h}(E_\lambda) \setminus \Phi_x(E_\lambda)$, i.e., $y \neq \Phi_\lambda(s', t')$ for any $(s', t') \in E_\lambda$. On the other hand, we have observed previously that $y = \Phi_x(s_h, t_h)$ for some $(s_h, t_h)$ satisfying

$$
|s_h - s| + |t_h - t| \leq 12\lambda^{2/3}.
$$

Since $(s_h, t_h) \notin E_\lambda$, $(s,t) \in E_\lambda$, we must have $\operatorname{dist}((s,t), \partial E_\lambda) \leq C\lambda^{2/3}$. Thus $|F_\lambda| \leq C\lambda^{2/3}$ as desired, and the proof is done.

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