SINGULAR INTEGRALS ASSOCIATED TO HYPERSURFACES: L^2 THEORY

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Abstract. We consider singular integrals associated to a classical Calderón-Zygmund kernel K and a hypersurface given by the graph of $\varphi(\psi(t))$ where φ is an arbitrary C^1 function and ψ is a smooth convex function of finite type. We give a characterization of those Calderón-Zygmund kernels K and convex functions ψ so that the associated singular integral operator is bounded on L^2 for all C^1 functions φ.

1. Introduction

The main purpose of this paper is to investigate the L^2 boundedness of singular integral operators associated to hypersurfaces in $\mathbb{R}^n$, $n \geq 3$. Let $\Gamma(t)$ be a C^1 mapping from a neighborhood of the origin in $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ with $\Gamma(0) = 0$. For x in $\mathbb{R}^n$ and f a C^1 function with compact support in $\mathbb{R}^n$, we set

$$Hf(x) = \lim_{\epsilon \to 0} \int_{\epsilon \leq |t| \leq 1} f(x - \Gamma(t)) K(t)\, dt$$

where $K(t)$ is a Calderón-Zygmund kernel in $\mathbb{R}^{n-1}$. That is, K is smooth (C^∞) away from the origin,

$$\int_{a \leq |t| \leq b} K(t)\, dt = 0$$

for every $0 < a < b$, and

$$K(\lambda t) = \lambda^{-n+1} K(t)$$

for every $\lambda > 0$.

It is known that if $\Gamma(t)$ is smooth and the vectors $\{\frac{\partial^\alpha \Gamma}{\partial t^\alpha}(0)\}$, given by the derivatives of Γ at the origin, span $\mathbb{R}^n$, then

$$\|Hf\|_{L^p} \leq A_p \|f\|_{L^p}, \quad 1 < p < \infty.$$

See [St] for this result. Our main interest is studying what happens when the vectors $\frac{\partial^\alpha \Gamma}{\partial t^\alpha}(0)$ do *not* span $\mathbb{R}^n$. We shall consider surfaces of the form

$$\Gamma(t) = \big(t, \varphi(\psi(t))\big)$$

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where $t = (t_1, \ldots, t_{n-1})$ and $\psi(t)$ is a smooth convex function of finite type with $\psi(0) = \nabla\psi(0) = 0$. (We say that $\psi(t)$ is of finite type if the graph $t_n = \psi(t)$ has no lines tangent to infinite order.) If $\psi(t) = |t|^2 = t_1^2 + \cdots + t_{n-1}^2$, then

$$\|Hf\|_{L^2} \le A\,\|f\|_{L^2}$$

for any φ. The details of this easy calculation can be found in [KWWZ].

The main purpose of this paper is to decide for what convex functions of finite type ψ and Calderón-Zygmund kernels K do we have

$$\|Hf\|_{L^2} \le A\|f\|_{L^2}$$

for all C^1 functions φ with $\varphi(0) = 0$. To give the answer to this problem we introduce certain sets which were considered by Schulz, [Sch]. Let

$$E_\ell = \{v \in \mathbb{R}^{n-1} \mid \psi(sv) = \mathcal{O}(s^{\ell+1}) \text{ for small } s > 0\}.$$

From the convexity of ψ, each E_ℓ is a linear subspace of $\mathbb{R}^{n-1}$. Clearly $E_1 = \mathbb{R}^{n-1}$, $E_{\ell+1} \subseteq E_\ell$ and $\bigcap E_\ell = \{0\}$ (from the finite type condition). We let ℓ_0 be the smallest value of ℓ such that E_ℓ is not all of $\mathbb{R}^{n-1}$. We then have the following theorem.

Theorem 1. *If the codimension of E_{ℓ_0} in $\mathbb{R}^{n-1}$ is at least 2, then*

$$\|Hf\|_{L^2} \le A\,\|f\|_{L^2}$$

for all C^1 functions φ with $\varphi(0) = 0$.

If the codimension of E_{ℓ_0} is 1, then H is bounded on L^2 for all C^1 functions φ if and only if K satisfies an additional cancellation condition.

Theorem 2. *Suppose the codimension of E_{ℓ_0} is 1, and let v be a non-zero vector in $E_{\ell_0}^{\perp}$. Then*

$$\|Hf\|_{L^2} \le A\,\|f\|_{L^2}$$

for all C^1 functions φ with $\varphi(0) = 0$ if and only if $K(t)$ satisfies the additional cancellation condition

$$\int_{\substack{v\cdot t \ge 0 \\ a \le |t| \le b}} K(t)\,dt = 0 \tag{1.1}$$

for all $0 < a < b$.

Remarks.

(1) The positive assertions in Theorems 1 and 2 hold for the more general operators

$$Hf(x) = \int b(\psi(t))K(t)f(x - \Gamma(t))\,dt$$

for any bounded function b, with no change in the proof.

(2) Theorem 1 is vacuous and Theorem 2 is trivially true when $n = 2$, and so nothing new is being proved for singular integrals along curves in the plane.

Examples.

(1) $\psi(x,y,z) = x^2 + y^2 + z^4$ is a convex function of finite type where $\ell_0 = 2$ and
$$E_{\ell_0} = \{(0,0,z) \mid z \in \mathbb{R}\}$$
has codimension 2 and so Theorem 1 applies.

(2) $\psi(x,y,z) = x^2 + y^4 + z^4$ is also a convex function of finite type where $\ell_0 = 2$ but
$$E_{\ell_0} = \{(0,y,z) \mid y,z \in \mathbb{R}\}$$
has codimension 1 and so Theorem 2 applies with $v = (1,0,0)$.

Next we turn to examine what happens when the cancellation condition (1.1) is not satisfied.

Theorem 3. *Let $\bar{\varphi}(s) = \varphi(s^{\ell_0})$. Assume the codimension of E_{ℓ_0} is* 1*, and the cancellation condition* (1.1) *fails. Then if $\bar{\varphi}(s)$ is convex,*
$$\|Hf\|_{L^2} \le A\,\|f\|_{L^2}$$
if and only if
$$\bar{\varphi}'(Cs) \ge 2\bar{\varphi}'(s)$$
for some $C \ge 1$ and all $0 < s \le 1$.

Remarks.

(1) The significance of the power ℓ_0 is that $\frac{1}{\ell_0}$ is the smallest power α such that $[\psi(t)]^\alpha$ is a convex function.

(2) When $\phi(s) = |s|^2$ and so $n = 1$, Theorem 3 was proved in [NVWW].

If $E_{\ell_0} = \{0\}$, which means that ψ is approximately homogeneous of degree ℓ_0, we obtain L^p results for H and the corresponding maximal function
$$Mf(x) = \sup_{0<h\le 1} \frac{1}{h^{n-1}} \int\limits_{|t|\le h} |f(x - \Gamma(t))|\, dt.$$
We again set $\bar{\varphi}(s) = \varphi(s^{\ell_0})$.

Theorem 4. *Suppose $n \ge 3$, $E_{\ell_0} = \{0\}$ and $\bar{\varphi}(s)$ is convex. Then*
$$\|Hf\|_{L^p} \le A_p\,\|f\|_{L^p}, \quad 1 < p < \infty,$$
and
$$\|Mf\|_{L^p} \le A_p\,\|f\|_{L^p}, \quad 1 < p \le \infty.$$

Remark. It is known that the assertion of Theorem 4 fails in general if the hypothesis that $\bar{\varphi}$ is convex is dropped, even if $\psi(t) = |t|^2$. See [SWWZ].

Finally we make one observation in $\mathbb{R}^3$ in the case that $\psi(t)$ is not of finite type. Let t_0 be a point on the curve $\psi(t) = 1$ and $\ell(t_0)$ denote the line tangent to $\psi(t) = 1$ at t_0. Set

$$E(t_0, \epsilon) = \{s \in \mathbb{R}^2 \mid \psi(s) = 1 \text{ and } \operatorname{dist}(s, \ell(t_0)) \leq \epsilon\}.$$

Theorem 5. *Assume $\psi(t)$ is convex and homogeneous of degree 1. Then if*

$$\sup_{\substack{t_0 \\ \psi(t_0)=1}} \int_0^1 |E(t_0, \epsilon)| \frac{d\epsilon}{\epsilon} < \infty,$$

$\|Hf\|_{L^2} \leq A \|f\|_{L^2}$ for every C^1 function φ.

Example. Consider a smooth convex function $\psi(x, y)$, homogeneous of degree 1, such that for $|x| << |y|$,

$$\psi(x, y) \ = \ \sqrt{x^2 + y^2} \ \exp\left(- \left(\frac{\sqrt{x^2 + y^2}}{|x|} \right)^{\alpha} \right).$$

Clearly ψ is not of finite type and the integrability condition in Theorem 5 is satisfied exactly when $\alpha < 1$.

In section 2 we will prove Theorems 1 and 2 in the special cases where $\psi(x, y, z) = x^2 + y^2 + z^4$ (for Theorem 1) and $\psi(x, y, z) = x^2 + y^4 + z^4$ (for Theorem 2), where the main direction of the proof is not clouded by intricate estimates. The proof for Theorem 1 in the general case will be given in section 3. Theorems 2 and 3 will be proved in section 4 and sections 5 and 6 contain the proofs of Theorems 4 and 5 respectively.

Our work is heavily dependent on ideas of Schulz, [Sch]. We would like to thank A. Iosevich for bringing the paper [Sch] to our attention. We would also like to thank Professor A. Carbery for evaluating a determinant for us.

2. Special Cases

In this section we will prove Theorems 1 and 2 in the special cases $\psi(x, y, z) = x^2 + y^2 + z^4$ and $\psi(x, y, z) = x^2 + y^4 + z^4$. We begin with $\psi(x, y, z) = x^2 + y^2 + z^4$. Here no further cancellation condition is required for the Calderón-Zygmund kernel K. We need to show

$$\left| \int\limits_{\epsilon \leq x^2+y^2+z^2 \leq 1} e^{i\gamma\varphi(x^2+y^2+z^4)} e^{i\eta z} e^{i(\xi_1 x + \xi_2 y)} K(x, y, z) \, dx \, dy \, dz \right| \leq B \tag{2.1}$$

uniformly in $\xi = (\xi_1, \xi_2), \eta, \gamma$ and $\epsilon > 0$. Introducing polar coordinates in the (x, y) integral, the integral in (2.1) becomes

$$\int_{\epsilon \le r^2+z^2 \le 1} e^{i\gamma\varphi(r^2+z^4)} e^{i\eta z} r \int_0^{2\pi} e^{ir(\xi_1 \cos\theta + \xi_2 \sin\theta)} K(r\cos\theta, r\sin\theta, z)\, d\theta\, dr\, dz. \tag{2.2}$$

We split the integral in (2.2) as a sum of two integrals $I_1 + I_2$ where the r integration in I_1 is restricted to $r|\xi| \ge 1$ and where the integration in I_2 is over the complementary range. Using the fact that the θ integral in (2.2) is the Fourier transform of a smooth density on the unit circle, we see that

$$|I_1| \le C \int_{r|\xi|\ge 1} \frac{r^2}{(r|\xi|)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{(r^2+z^2)^{4/2}} + \frac{1}{r(r^3+|z|^3)}\, dz\, dr = C \int_{r|\xi|\ge 1} \frac{1}{(r|\xi|)^{1/2}} \frac{dr}{r} \le C.$$

In I_2 we replace $e^{ir(\xi_1\cos\theta+\xi_2\sin\theta)}$ with 1, creating an error at most a multiple of

$$\int_{r|\xi|\le 1} r^2 |\xi| \int_{-\infty}^{\infty} \frac{1}{(r^2+z^2)^{3/2}}\, dz\, dr = C \int_{r|\xi|\le 1} |\xi|\, dr = C.$$

Therefore the integral in (2.2) is

$$\int_0^{2\pi}\!\!\int \int_{\substack{\epsilon\le r^2+z^2\le 1\\ r|\xi|\le 1}} e^{i\gamma\varphi(r^2+z^4)} e^{i\eta z} K(r\cos\theta, r\sin\theta, z) r\, dr\, dz\, d\theta \;+\; \mathcal{O}(1).$$

Furthermore the (r, z) integration may be further restricted to the region where $|z| \le \delta r^{1/2}$ since integrating K over the complementary region is at most

$$\int \int_{\delta r^{1/2} \le |z| \le 1} \frac{1}{(r^2+z^2)^{3/2}}\, r\, dr\, dz \le \int_{|z|\le 1} \frac{1}{|z|^3} \int_{r \le \left(\frac{1}{\delta}|z|\right)^2} r\, dr \le C.$$

With the restriction $|z| \le \delta r^{1/2}$ for small $\delta > 0$, we may make the change of variables $\lambda = \sqrt{r^2+z^4}$ (so that $\lambda \sim r$) in the r integral to reduce matters to showing that the integral

$$I = \int_0^{2\pi} \int_{\lambda|\xi|\le 1} e^{i\gamma\varphi(\lambda^2)} \lambda \int_{\substack{|z|\le \delta\lambda^{1/2}\\ \epsilon \le \lambda^2+z^2\le 1}} e^{i\eta z} K(\sqrt{\lambda^2-z^4}\cos\theta, \sqrt{\lambda^2-z^4}\sin\theta, z)\, dz\, d\lambda\, d\theta$$

is uniformly bounded in γ, η, ξ and $\epsilon > 0$. Replacing $\sqrt{\lambda^2 - z^4}$ by λ in I creates an error at most

$$C \int_0^1 \int_{|z| \le \lambda^{1/2}} \frac{z^4}{(z^2 + \lambda^2)^{4/2}} \, dz \, d\lambda \le C \int_0^1 \lambda^{1/2} \, d\lambda \le C$$

and so

$$I = \int_0^{2\pi} \int_{\lambda|\xi| \le 1} e^{i\gamma\varphi(\lambda^2)} \lambda \int_{\substack{|z| \le \delta\lambda^{1/2} \\ \epsilon \le \lambda^2 + z^2 \le 1}} e^{i\eta z} K(\lambda\cos\theta, \lambda\sin\theta, z) \, dz \, d\lambda \, d\theta \ + \ \mathcal{O}(1).$$

Next we will see that we can replace the oscillatory factor $e^{i\eta z}$ with 1 in the above integral if we further restrict the λ integration to $\lambda \le \frac{1}{|\eta|}$. In fact we can integrate by parts in the z integral to see that the part of the integral where $\lambda|\eta| \ge 1$ is at most

$$C \frac{1}{|\eta|} \int_{\lambda|\eta| \ge 1} \lambda \int_{-\infty}^{\infty} \frac{1}{(z^2 + \lambda^2)^{4/2}} \, dz \le C \, \frac{1}{|\eta|} \int_{\lambda|\eta| \ge 1} \frac{1}{\lambda^2} \le C.$$

For $\lambda|\eta| \le 1$, replacing $e^{i\eta z}$ by 1 creates an error at most

$$C \, |\eta| \int_{\lambda|\eta| \le 1} \lambda \int_{-\infty}^{\infty} \frac{|z|}{(\lambda^2 + z^2)^{3/2}} \, dz \, d\lambda \le C \, |\eta| \int_{\lambda|\eta| \le 1} d\lambda \le C.$$

Therefore

$$I = \int_0^{2\pi} \int_{\substack{\lambda|\xi|\le 1\\ \lambda|\eta|\le 1}} e^{i\gamma\varphi(\lambda^2)}\lambda \int_{\substack{|z|\le\delta\lambda^{1/2}\\ \epsilon\le\lambda^2+z^2\le 1}} K(\lambda\cos\theta,\lambda\sin\theta,z)\,dz\,d\lambda\,d\theta \;+\; \mathcal{O}(1)$$

$$= \int_0^{2\pi} \int_{\substack{\lambda|\xi|\le 1\\ \lambda|\eta|\le 1}} e^{i\gamma\varphi(\lambda^2)}\frac{1}{\lambda} \int_{\substack{|s|\le\delta\lambda^{-1/2}\\ \epsilon\le\lambda^2(1+s^2)\le 1}} K(\cos\theta,\sin\theta,s)\,ds\,d\lambda\,d\theta \;+\; \mathcal{O}(1)$$

$$= \int_0^{2\pi} \int_{\substack{\lambda|\xi|\le 1\\ \lambda|\eta|\le 1}} e^{i\gamma\varphi(\lambda^2)}\frac{1}{\lambda} \int_{\epsilon\le\lambda^2(1+s^2)\le 1} K(\cos\theta,\sin\theta,s)\,ds\,d\lambda\,d\theta \;+\; \mathcal{O}(1)$$

$$= - \int_{\substack{\lambda|\xi|\le 1\\ \lambda|\eta|\le 1}} e^{i\gamma\varphi(\lambda^2)}\frac{1}{\lambda} \int_0^{2\pi} \int_{\substack{0\\ \lambda\le\sin\psi\le\frac{\lambda}{\sqrt{\epsilon}}}}^{\pi} K(\sin\psi\cos\theta,\sin\psi\sin\theta,\cos\psi)\sin\psi\,d\psi\,d\theta\,d\lambda + \mathcal{O}(1).$$

Here we made the change of variables $z = s\lambda$ followed by $s = \cot\psi$ in the z integral. Using the fact that

$$0 \;=\; \int_0^{2\pi}\int_0^{\pi} K(\sin\psi\cos\theta,\sin\psi\sin\theta,\cos\psi)\sin\psi\,d\psi\,d\theta$$

we easily see (by splitting the λ integration at $\lambda = \sqrt{\epsilon}$) that I is uniformly bounded in γ,ξ,η and $\epsilon > 0$. This finishes the proof of Theorem 1 in the case where $\psi(x,y,z) = x^2 + y^2 + z^4$.

For the example $\psi(x,y,z) = x^2 + y^4 + z^4$ we will show that the integral

$$\int_{\epsilon\le x^2+y^2+z^2\le 1} e^{i\gamma\varphi(x^2+y^4+z^4)}e^{i\eta x}e^{i(\xi_1 y+\xi_2 z)}K(x,y,z)\,dx\,dy\,dz \tag{2.3}$$

is uniformly bounded in $\gamma,\eta,\xi = (\xi_1,\xi_2)$ and $\epsilon > 0$ under the additional hypothesis that for all $0 < a < b$

$$\int_{\substack{a\le x^2+y^2+z^2\le b\\ x\ge 0}} K(x,y,z)\,dx\,dy\,dz = 0. \tag{2.4}$$

We would like to make the change of variables $\lambda^2 = x^2 + y^4 + z^4$ in the x integral. In order to do this first observe that the integral in (2.3) over the region $\delta|x|^{1/2} \le$

$\sqrt{y^2+z^2}$ is uniformly bounded. In fact

$$\iint \int_{\delta|x|^{1/2}<\sqrt{y^2+z^2}\le 1} |K(x,y,z)|\,dx\,dy\,dz$$

$$\le C \int\int_{\sqrt{y^2+z^2}\le 1} dy\,dz \int_{\delta|x|^{1/2}\le\sqrt{y^2+z^2}} \frac{1}{(y^2+z^2)^{3/2}}\,dx$$

$$\le C \int\int_{\sqrt{y^2+z^2}\le 1} \frac{1}{\sqrt{y^2+z^2}}\,dy\,dz \le C.$$

Hence it suffices to show the uniform boundedness of

$$II \;=\; \int_{\substack{\epsilon\le x^2+|\bar{y}|\le 1\\ |\bar{y}|\le\delta|x|^{1/2}}} e^{i\gamma\varphi(x^2+y^4+z^4)} e^{i\eta x} e^{i\xi\cdot\bar{y}} K(x,\bar{y})\,dx\,d\bar{y}$$

where $\bar{y}=(y,z)$. We write $II = II_+ + II_-$ where the integration in II_+ is over positive values of x. We first concentrate on II_+, making the change of variables

$$\lambda = \sqrt{x^2+y^4+z^4}, \quad x = x(\lambda,\bar{y}) = \sqrt{\lambda^2-y^4-z^4}$$

in the x integral so that $x\sim\lambda$. Then

$$II_+ = \int_0^1 e^{i\gamma\varphi(\lambda^2)} \int_{\substack{\epsilon\le\lambda^2+|\bar{y}|^2\le 1\\ |\bar{y}|\le\delta\lambda^{1/2}}} e^{i\eta x(\lambda,\bar{y})} e^{i\xi\cdot\bar{y}} K(x(\lambda,\bar{y}),\bar{y})\,\frac{\partial x}{\partial\lambda}\,d\bar{y}\,d\lambda \;+\; \mathcal{O}(1).$$

In order to analyze this integral we make the following simple observations regarding $x(\lambda,\bar{y})$ in the region $|\bar{y}|\le\delta\lambda^{1/2}$:

(a)

$$x(\lambda,0) \;=\; \lambda, \qquad \frac{\partial x}{\partial\lambda}(\lambda,0) \;=\; 1,$$

(b)

$$\left|\frac{\partial x}{\partial\bar{y}}\right| \;\sim\; \frac{|\bar{y}|^3}{\lambda}, \qquad \left|\frac{\partial x}{\partial\lambda\partial\bar{y}}\right| \;\sim\; \frac{|\bar{y}|^3}{\lambda^2},$$

and

(c)

$$\frac{\partial^4 x}{\partial y^4} \;\sim\; \frac{1}{\lambda}, \qquad \frac{\partial^4 x}{\partial z^4} \;\sim\; \frac{1}{\lambda}.$$

Using (a) and (b) we may replace $\frac{\partial x}{\partial \lambda}$ by 1 in II_+ with an error at most

$$C\int_0^1 \frac{1}{\lambda^2} \int_{|\bar{y}|\le\lambda^{1/2}} \frac{|\bar{y}|^4}{\lambda^3+|\bar{y}|^3}\,d\bar{y}\,d\lambda \le C\int_0^1 \lambda \int_{|\bar{s}|\le\lambda^{-1/2}} \frac{|\bar{s}|^4}{1+|\bar{s}|^3}\,d\bar{s}\,d\lambda$$
$$\le C\int_0^1 \frac{1}{\sqrt{\lambda}}\,d\lambda \le C.$$

Also replacing $x(\lambda,\bar{y})$ with λ in the kernel K creates an error at most

$$C\int_0^1 \frac{1}{\lambda} \int_{|\bar{y}|\le\lambda^{\frac{1}{2}}} \frac{|\bar{y}|^4}{\lambda^4+|\bar{y}|^4}\,d\bar{y}\,d\lambda \le C\int_0^1 \lambda \int_{|\bar{s}|\le\lambda^{-1/2}} \frac{|\bar{s}|^4}{1+|\bar{s}|^4}\,d\bar{s}\,d\lambda$$
$$\le C\int_0^1 d\lambda = C.$$

Therefore

$$II_+ = \int_0^1 e^{i\gamma\varphi(\lambda^2)} \int_{\substack{\epsilon\le\lambda^2+|\bar{y}|^2\le1\\ |\bar{y}|\le\delta\lambda^{1/2}}} e^{i\eta x(\lambda,\bar{y})}e^{i\xi\cdot\bar{y}}K(\lambda,\bar{y})\,d\bar{y}\,d\lambda \;+\; \mathcal{O}(1).$$

Next we will show that we can replace the oscillation $e^{i\eta x(\lambda,\bar{y})}$ with $e^{i\eta\lambda}$ provided that the λ integration is restricted to where $\lambda|\eta|^{1/3}\le 1$. In fact using the fact that $\frac{\partial^4 x}{\partial y^4}\sim\frac{1}{\lambda}$ and Van der Corput's lemma (see e.g., [St]) in the y integral we see that the part of the integral where $\lambda|\eta|^{1/3}\ge 1$ is at most

$$C\frac{1}{|\eta|^{1/4}}\int_{\lambda|\eta|^{1/3}\ge1} \lambda^{1/4}\left[\int\frac{1}{\lambda^4+|\bar{y}|^4}\,d\bar{y}+\int\frac{1}{\lambda^3+z^3}\,dz\right]d\lambda \le C\,\frac{1}{|\eta|^{1/4}}\int_{\lambda|\eta|^{1/3}\ge1}\frac{\lambda^{1/4}}{\lambda^2}\,d\lambda\le C.$$

For the part where $\lambda|\eta|^{1/3}\le 1$ we expect only to replace $e^{i\eta x(\lambda,\bar{y})}$ with $e^{i\eta\lambda}$ in the region where $|\bar{y}|\le\left(\frac{\lambda}{|\eta|}\right)^{1/4}$ since using (b)

$$|e^{i\eta x(\lambda,\bar{y})}-e^{i\eta\lambda}| \;\le\; |\eta|\,|x(\lambda,\bar{y})-x(\lambda,0)| \;\le\; |\eta|\,\frac{|\bar{y}|^4}{\lambda}.$$

In the complementary region, $\lambda|\eta|^{1/3} \leq 1$ and $|\bar{y}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}$ we see that K is uniformly integrable. In fact

$$\int_{\lambda|\eta|^{1/3}\leq 1} \int_{|\bar{y}|\geq\left(\frac{\lambda}{|\eta|}\right)^{1/4}} |K(\lambda,\bar{y})|\, d\bar{y}\, d\lambda \leq \int_{\lambda|\eta|^{1/3}\leq 1} \int_{|\bar{y}|\geq\left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{1}{\lambda^3+|\bar{y}|^3}\, d\bar{y}\, d\lambda$$

$$\leq C \int_{\lambda|\eta|^{1/3}\leq 1} \frac{1}{\lambda} \int_{|\bar{s}|\geq\left(\frac{\lambda}{|\eta|}\right)^{1/4}\frac{1}{\lambda}} \frac{1}{1+|\bar{s}|^3}\, d\bar{s}\, d\lambda$$

$$\leq C|\eta|^{1/4} \int_{\lambda|\eta|^{1/3}\leq 1} \frac{1}{\lambda^{1/4}}\, d\lambda \leq C.$$

Replacing $e^{i\eta x(\lambda,\bar{y})}$ with $e^{i\eta\lambda}$ in the region $\lambda|\eta|^{1/3} \leq 1$ and $|\bar{y}| \leq \left(\frac{\lambda}{|\eta|}\right)^{1/4}$ creates an error at most

$$C|\eta| \int_{\lambda|\eta|^{1/3}\leq 1} \frac{1}{\lambda} \int_{|\bar{y}|\leq\left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{|\bar{y}|^4}{\lambda^3+|\bar{y}|^3}\, d\bar{y}\, d\lambda$$

$$\leq C|\eta| \int_{\lambda|\eta|^{1/3}\leq 1} \lambda^2 \int_{|\bar{s}|=\left(\frac{\lambda}{|\eta|}\right)^{1/4}\frac{1}{\lambda}} \frac{|\bar{s}|^4}{1+|\bar{s}|^3}\, d\bar{s}\, d\lambda$$

$$\leq C\frac{|\eta|}{|\eta|^{3/4}} \int_{\lambda|\eta|^{1/3}\leq 1} \frac{1}{\lambda^{1/4}}\, d\lambda \leq C.$$

Therefore

$$II_+ = \int_{\lambda|\eta|^{1/3}\leq 1} e^{i\gamma\varphi(\lambda^2)} e^{i\eta\lambda} \int_{\substack{\epsilon\leq\lambda^2+|\bar{y}|^2\leq 1\\ |\bar{y}|\leq\delta\lambda^{1/2}}} e^{i\xi\cdot\bar{y}} K(\lambda,\bar{y})\, d\bar{y}\, d\lambda \;+\; \mathcal{O}(1).$$

A similar but easier argument allows us to replace $e^{i\xi\cdot\bar{y}}$ with 1 if we further restrict the λ integration where $\lambda|\xi| \leq 1$. Hence making the change of variables $\bar{y} = \lambda\bar{s}$,

$$II_+ = \int_{\substack{0\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1},|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} e^{i\eta\lambda} \int_{\substack{\epsilon\leq\lambda^2+|\bar{y}|^2\leq 1\\ |\bar{y}|\leq\delta\lambda^{1/2}}} K(\lambda,\bar{y})\, d\bar{y}\, d\lambda \;+\; \mathcal{O}(1)$$

$$= \int_{\substack{0\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1},|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} e^{i\eta\lambda} \frac{1}{\lambda} \int_{\epsilon\leq\lambda^2(1+|\bar{s}|^2)\leq 1} K(1,\bar{s})\, d\bar{s}\, d\lambda \;+\; \mathcal{O}(1).$$

Here we used the fact that

$$\int_{|\bar{s}|\geq\lambda^{-1/2}} |K(1,\bar{s})|\, d\bar{s} \;=\; \mathcal{O}(\lambda^{1/2}).$$

Making a polar change of coordinates $\bar{s} = (r\cos\theta, r\sin\theta)$ followed by $r = \tan\psi$, $0 \leq \psi < \pi/2$, in the $\bar{s}$ integral allows us to write

$$II_+ = \int_{\substack{0\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1},|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} e^{i\eta\lambda}\,\frac{1}{\lambda}\int_0^{2\pi}\int_{\substack{0\leq\psi\leq\pi/2\\ \lambda\leq\cos\psi\leq\lambda/\sqrt{\epsilon}}} K(\cos\psi,\sin\psi\cos\theta,\sin\psi\sin\theta)$$
$$\sin\psi\, d\psi\, d\theta\, d\lambda + \mathcal{O}(1)$$
$$= \int_{\substack{\sqrt{\epsilon}\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1},|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} e^{i\eta\lambda}\,\frac{1}{\lambda}\int_0^{2\pi}\int_0^{\pi/2} K(\cos\psi,\sin\psi\cos\theta,\sin\psi\sin\theta)$$
$$\sin\psi\, d\psi\, d\theta\, d\lambda + \mathcal{O}(1).$$

A similar analysis for II_- shows

$$II_- = \int_{\substack{\sqrt{\epsilon}\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1}|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} e^{-i\eta\lambda}\,\frac{1}{\lambda}\int_0^{2\pi}\int_{\pi/2}^{\pi} K(\cos\psi,\sin\psi\cos\theta,\sin\psi\sin\theta)$$
$$\sin\psi\, d\psi\, d\theta\, d\lambda + \mathcal{O}(1).$$

Therefore

$$II = II_+ + II_-$$
$$= \int_{\substack{\sqrt{\epsilon}\leq\lambda\leq 1\\ \lambda\leq\min(|\xi|^{-1}|\eta|^{-1/3})}} e^{i\gamma\varphi(\lambda^2)} \sin(\eta\lambda)\,\frac{1}{\lambda}\int_0^{2\pi}\int_0^{\pi} K(\cos\psi,\sin\psi\cos\theta,\sin\psi\sin\theta)$$
$$\sin\psi\, dx\, d\theta\, d\lambda + \mathcal{O}(1)$$
$$= \;0\; + \;\mathcal{O}(1)$$

by (2.4). Note that when the additional cancellation condition for K is not satisfied, we are left with a truncated Hilbert transform along the curve $(\lambda, \varphi(\lambda^2))$ and so we might expect to be able to use the analysis in [NVWW] when $\varphi(\lambda^2)$ is convex.

3. Proof of Theorem 1

According to Schulz [Sch], after a rotation of coordinates, we may write

$$\psi(t) = P(t) + R(t) \tag{3.1}$$

where

$$P(t) = \sum_{j=1}^{r} a_j t_j^{\ell_0} + \sum_{j=r+1}^{n-1} a_j t_j^{m_j} + P_1(t).$$

$P(t)$ is a convex polynomial, $P(t) > 0$ for $t \neq 0$, $a_j > 0$ for $1 \leq j \leq n-1$, $\ell_0 < m_j$ for $r+1 \leq j \leq n-1$, $P_1(t)$ has no pure powers of t, and if $At_1^{\alpha_1} \dots t_{n-1}^{\alpha_{n-1}}$ is a monomial of $P_1(t)$,

$$\frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{n-1} \frac{\alpha_j}{m_j} = 1.$$

$R(t)$ is smooth and if $At_1^{a_1} \dots t_{n-1}^{a_{n-1}}$ is a term in the Taylor expansion of $R(t)$

$$\frac{1}{\ell_0} \sum_{j=1}^{r} a_j + \sum_{j=r+1}^{n-1} \frac{a_j}{m_j} > 1.$$

To prove our theorems, we may assume $\psi(t)$ has the form (3.1). The hypothesis of Theorem 1 asserts that $r \geq 2$. Let $H(t)$ be the part of $P(t)$ which is homogeneous of degree ℓ_0. Then $H(t)$ is a function of only $t_1, \dots, t_r$. In fact if

$$A\, t_1^{\alpha_1} \dots t_r^{\alpha_r} t_{r+1}^{\alpha_{r+1}} \dots t_{n-2}^{\alpha_{n-2}} t_{n-1}^{\ell_0 - (\alpha_1 + \cdots + \alpha_{n-2})}$$

were a monomial of H, then

$$\frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{n-2} \frac{\alpha_j}{m_j} + \frac{\ell_0 - (\alpha_1 + \cdots + \alpha_{n-2})}{m_{n-1}} = 1.$$

This identity would clearly hold if $m_{r+1} = \dots m_{n-1} = \ell_0$, so it could not hold if one of the m's were bigger than ℓ_0. Similarly every monomial of $P(t)$ which depends only on $t_1, \dots, t_r$ belongs to H. So

$$H(t) = H(t_1, \dots, t_r) = P(t_1, \dots, t_r, 0, \dots, 0)$$

is convex and positive if some t_j is nonzero.

We write $y = (t_1, \dots, t_r)$ in $\mathbb{R}^r$ and $x = (t_{r+1}, \dots, t_{n-1})$ in $\mathbb{R}^{n-1-r}$. We shall suppose $n - r - 1 \geq 1$, otherwise the proof is similar but simpler. We then write

$$P(x, y) = H(y) + P_2(x, y).$$

To prove Theorem 1, we must show for ξ in $\mathbb{R}^r$ and η in $\mathbb{R}^{n-1-r}$,

$$\left| \int_{\epsilon \leq |x|^2 + |y|^2 \leq 1} e^{i\gamma\varphi(\psi(x,y))} e^{i\eta \cdot x} e^{i\xi \cdot y} K(y, x)\, dy\, dx \right| \leq B \tag{3.2}$$

uniformly in ξ, η, γ and $\epsilon > 0$.

We begin by introducing polar coordinates in the y variables. That is, we write $y = s\omega$ where s goes from 0 to 1 and ω runs over the surface $H(w) = 1$. The integral in (3.2) becomes

$$\int_{H(\omega)=1}\int_{\epsilon\le|x|^2+s^2|\omega|^2\le 1} e^{i\gamma\varphi(\psi(x,s\omega))}e^{i(s\xi\cdot\omega+\eta\cdot x)}K(s\omega,x)s^{r-1}h(\omega)\,ds\,dx\,d\omega \tag{3.3}$$

where h is a smooth function. We let m be the smallest of the values among the m_j's, $r+1 \le j \le n-1$, and choose $\sigma > 0$ so that $\frac{\ell_0}{m}+\sigma < 1$. We may restrict the integration in (3.3) to $|x| \le s^{\ell_0/m+\sigma}$ since integrating K over the complementary region is at most

$$\begin{aligned} C\int\int_{s^{\frac{\ell_0}{m}+\sigma}\le|x|\le 1} \frac{1}{s^{n-1}+|x|^{n-1}}s^{r-1}\,ds\,dx &\le C\int_{|x|\le 1}\frac{1}{|x|^{n-1}}\int_{s\le|x|^\lambda} s^{r-1}\,ds\,dx \\ &\le C\int_{|x|\le 1}\frac{1}{|x|^{n-r\lambda-1}}\,dx \le C \end{aligned}$$

since

$$\lambda = \frac{1}{\frac{\ell_0}{m}+\sigma} > 1 \quad \text{and} \quad x\in\mathbb{R}^{n-r-1}.$$

Furthermore in the region $|x| \le s^{\frac{\ell_0}{m}+\sigma}$,

$$\psi(x,s\omega) = s^{\ell_0} \;+\; \mathcal{O}(s^{\ell_0+\sigma}) \tag{3.4}$$

and

$$\frac{\partial\psi}{\partial s}(x,s\omega) = \ell_0 s^{\ell_0-1} \;+\; \mathcal{O}(s^{\ell_0-1+\sigma}). \tag{3.5}$$

In fact

$$\psi(x,s\omega) = s^{\ell_0} + P_2(x,s\omega) + R(x,s\omega)$$

and every monomial in P_2 or any monomial in R of the form $x^\alpha(s\omega)^\beta$ with $|\alpha| > 0$ has the bound

$$|x^\alpha(s\omega)^\beta| \le s^{\left(\frac{\ell_0}{m}+\sigma\right)|\alpha|+|\beta|} \le s^{\ell_0+\sigma}.$$

Also any monomial in R of the form $(s\omega)^\beta$ is $\mathcal{O}(s^{\ell_0+1})$. Therefore we may make the change of variables

$$\lambda^{\ell_0} = \psi(x,s\omega) \tag{3.6}$$

in s for fixed x and ω and write (3.3) as

$$\int\limits_{H(\omega)=1} h(\omega) \int\limits_0^1 e^{i\gamma\varphi(\lambda^{\ell_0})} \int\limits_{\substack{\epsilon\le|x|^2+s^2|\omega|^2\le 1\\ |x|\le s^{\frac{\ell_0}{m}+\sigma}}} e^{i\eta\cdot x}e^{is\xi\cdot\omega}K(s\omega,x)s^{r-1}\frac{\partial s}{\partial\lambda}\,dx\,d\lambda\,d\omega + \mathcal{O}(1) \tag{3.7}$$

where $s = s(\lambda,\omega,x)$. From (3.4) and (3.5) we have, for some $\epsilon > 0$,

$$s = \lambda + \mathcal{O}(\lambda^{1+\epsilon}) \quad \text{and} \quad \frac{\partial s}{\partial\lambda} = 1 + \mathcal{O}(\lambda^\epsilon).$$

Also we have

$$\left|\frac{\partial s}{\partial x}\right| \le C\lambda^{1-\frac{\ell_0}{m}} \tag{3.8}$$

which follows by differentiating (3.6) with respect to x, giving

$$0 = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial s}\frac{\partial s}{\partial x}.$$

(3.5) implies $\frac{\partial\psi}{\partial s} \sim \lambda^{\ell_0-1}$ and a similar argument which established (3.4) and (3.5) shows $\frac{\partial\psi}{\partial x} = \mathcal{O}(\lambda^{\ell_0-\frac{\ell_0}{m}})$ and thus (3.8).

Arguing as in section 2, using the above estimates on the derivatives of $s(\lambda,\omega,x)$ and s itself, shows that we may replace $s(\lambda,\omega,x)$ by λ (except in the oscillation $e^{is\xi\cdot\omega}$) and $\frac{\partial s}{\partial\lambda}$ by 1 with a uniformly bounded error. Thus the integral in (3.7) is

(3.9)
$$\int\limits_0^1 e^{i\gamma\varphi(\lambda^{\ell_0})}\lambda^{r-1} \int\limits_{|x|\le\lambda^{\frac{\ell_0}{m}+\sigma}} e^{i\eta\cdot x} \int\limits_{\substack{H(\omega)=1\\ \epsilon\le|x|^2+\lambda^2|\omega|^2\le 1}} e^{is(\lambda,\omega,x)\xi\cdot\omega}K(\lambda\omega,x)h(\omega)\,d\omega\,dx\,d\lambda + \mathcal{O}(1).$$

Consider first the contribution to (3.9) from those values of λ where $\lambda|\xi| \ge 1$. Since $H(\omega) = 1$ is of finite type we may for each ω_0 on $H(\omega) = 1$ parametrize $H(\omega) = 1$ in a neighborhood of ω_0 as

$$\omega_0 + (\tau_1,\dots,\tau_{r-1}, g(\tau_1,\dots,\tau_{r-1}))$$

where $g(0) = 0$, $\nabla g(0) = 0$, and for some $j_0 \ge 2$,

$$\frac{\partial^j g}{\partial\tau_1^j}(0) = 0$$

for $1 \le j \le j_0 - 1$ and

$$\frac{\partial^{j_0} g}{\partial\tau_1^{j_0}}(0) \ne 0.$$

It follows that we may assume

$$\frac{\partial^{j_0} g}{\partial \tau_1^{j_0}} \neq 0$$

for all τ in a neighborhood of 0. Therefore since $s(\lambda, \omega, x) \sim \lambda$,

$$\left| \int_{\substack{|\omega-\omega_0|\leq\delta \\ H(\omega)=1}} e^{is(\lambda,\omega,x)\xi\cdot\omega} h(\omega)\, dw \right| \leq C \frac{1}{(\lambda|\xi|)^\delta}$$

for some positive δ by Van der Corput's lemma . Integrating by parts now shows that the contribution to the integral in (3.9) from those λ where $\lambda|\xi| \geq 1$ is at most

$$C \int_{\lambda|\xi|\geq 1} \lambda^{r-1} \frac{\lambda}{(\lambda|\xi|)^\delta} \int_{\mathbb{R}^{n-r-1}} \frac{1}{\lambda^n + |x|^n}\, dx\, d\lambda \leq C \int_{\lambda|\xi|\geq 1} \frac{1}{(\lambda|\xi|)^\delta} \frac{d\lambda}{\lambda} \leq C.$$

Thus the proof of Theorem 1 reduces to showing that the integral

$$I = \int_{\lambda|\xi|\leq 1} e^{i\gamma\varphi(\lambda^{\ell_0})} \lambda^{r-1} \int_{H(\omega)=1} \int_{\substack{\epsilon\leq|x|^2+\lambda^2|\omega|^2\leq 1 \\ |x|\leq\lambda^{\frac{\ell_0}{m}+\sigma}}} e^{i\eta\cdot x} K(\lambda\omega, x) h(\omega)\, dx\, d\omega\, d\lambda$$

is uniformly bounded in γ, ξ, η and $\epsilon > 0$. Putting $x = \lambda z$ makes

$$I = \int_{\lambda|\xi|\leq 1} e^{i\gamma\varphi(\lambda^{\ell_0})} \frac{1}{\lambda} \int_{H(\omega)=1} \int_{\substack{\epsilon\leq\lambda^2(|z|^2+|\omega|^2)\leq 1 \\ |z|\leq\lambda^{\frac{\ell_0}{m}+\sigma-1}}} e^{i\lambda\eta\cdot z} K(\omega, z) h(\omega)\, dz\, d\omega\, d\lambda$$

and using the fact

$$\int_{\left(\frac{C}{\lambda}\right)^\delta\leq|z|} |K(\omega, z)|\, dz \;=\; \mathcal{O}(\frac{\lambda}{C})^{r\delta}$$

three times, first with $\delta = 1$ and $C = \sqrt{\epsilon}$, then with $\delta = 1$ and $C = 1$, and finally with $\delta = 1 - \left(\left(\frac{\ell_0}{m}\right) + \sigma\right)$ and $C = 1$, we see that

$$I = \int_{A\sqrt{\epsilon}\leq\lambda\leq\frac{1}{|\xi|}} e^{i\gamma\varphi(\lambda_0^\ell)} \frac{1}{\lambda} \int_{H(\omega)=1} \int e^{i\lambda\eta\cdot z} K(\omega, z) h(\omega)\, dz\, d\omega\, d\lambda \;+\; \mathcal{O}(1)$$

if A is chosen large enough. An integration by parts in the z integral shows that the part of the integral where $\lambda|\eta| \geq 1$ is at most (up to boundary terms)

$$C\frac{1}{|\eta|} \int_{\lambda|\eta|\geq 1} \frac{1}{\lambda^2} \int_{\mathbb{R}^{n-r-1}} \sup_\omega |\nabla K(\omega, z)|\, dz\, d\lambda \leq C\frac{1}{|\eta|} \int_{\lambda|\eta|\geq 1} \frac{1}{\lambda^2} \int_{|z|\geq 1} \frac{1}{|z|^n}\, dz\, d\lambda \leq C,$$

and so

$$I = \int\limits_{\substack{A\sqrt{\epsilon}\le\lambda\le 1\\ \lambda\le\min(\frac{1}{|\xi|},\frac{1}{|\eta|})}} e^{i\gamma\varphi(\lambda^{\ell_0})}\,\frac{1}{\lambda}\int H(\omega)=1\int e^{i\lambda\eta\cdot z}K(\omega,z)h(\omega)\,dzd\omega\,d\lambda \;+\; \mathcal{O}(1).$$

The boundary terms are handled similarly. Replacing $e^{i\lambda\eta\cdot z}$ by 1 creates an error at most

$$C|\eta|\int\limits_{\lambda|\eta|\le 1}\int\limits_{H(\omega)=1}\int\frac{|z|}{|\omega|^{n-1}+|z|^{n-1}}\,dz\,d\omega\,d\lambda\le C$$

since $r\ge 2$. Therefore

$$I = \int\limits_{\substack{A\sqrt{\epsilon}\le\lambda\le 1\\ \lambda\le\min(\frac{1}{|\xi|},\frac{1}{|\eta|})}} e^{i\gamma\varphi(\lambda^{\ell_0})}\,\frac{1}{\lambda}\int\limits_{H(\omega)=1}\int K(\omega,z)h(\omega)\,dzd\omega\,d\lambda \;+\; \mathcal{O}(1),$$

and so it suffices to show

$$\int\limits_{H(\omega)=1}\int\limits_{\mathbb{R}^{n-r-1}}K(\omega,z)h(\omega)\,dz\,d\omega=0. \tag{3.10}$$

Now for any $\delta>0$

$$\begin{aligned}
0 &= \int\limits_{1-\delta\le|y|^2+|x|^2\le 1} K(y,x)\,dy\,dx\\
&= \int\limits_{H(\omega)=1}\int\limits_{1-\delta\le\lambda^2|\omega|^2+|x|^2\le 1}\lambda^{r-1}K(\lambda\omega,x)h(\omega)\,d\lambda\,d\omega\,dx\\
&= \int\limits_{H(\omega)=1}\int\limits_{1-\delta\le\lambda^2(|\omega|^2+|z|^2)\le 1}K(\omega,z)h(\omega)\,\frac{d\lambda}{\lambda}\,d\omega\,dz\\
&= \int\limits_{H(\omega)=1}h(\omega)\int\limits_{\mathbb{R}^{n-r-1}}K(\omega,z)\int\limits_{\frac{1-\delta}{|\omega|^2+|z|^2}\le\lambda^2\le\frac{1}{|\omega|^2+|z|^2}}\frac{d\lambda}{\lambda}\,dz\,d\omega.
\end{aligned}$$

Dividing by δ and letting $\delta\to 0$ gives (3.10) and this finishes the proof of Theorem 1. $\square$

4. The proof of Theorems 2 and 3.

We may again assume $\psi(t)$ is of the general form (3.1), where now $r = 1$. The cancellation condition (1.1) now becomes

$$\int_{\Sigma+} K(t)\, d\sigma(t) = 0$$

where $\Sigma^+ = \{t \in \mathbb{R}^{n-1} \mid |t| = 1, t_1 > 0\}$ is the "upper" hemisphere of S^{n-2}. It will be convenient to let t_1 be denoted by y and $(t_2, \dots, t_{n-1}) = x \in \mathbb{R}^{n-2}$. We are then concerned with the uniform boundedness of

$$\begin{aligned}\int\int e^{i\gamma\varphi(\psi(y,x))} e^{i\xi\cdot x} e^{i\eta y} K(y,x)\, dy\, dx &= \int\int_{y>0} + \int\int_{y<0} \\ &= I \;+\; II.\end{aligned}$$

Theorems 2 and 3 will then follow if we can prove for some b, $0 < b < 1$,

$$I = \int_{\substack{0\le\lambda\le 1\\ \lambda\le\min(\frac{1}{|\xi|},\frac{1}{|\eta|^b})}} e^{i\gamma\varphi(\lambda^{\ell_0})} e^{i\eta q(\lambda)}\, \frac{d\lambda}{\lambda} \int_{\Sigma+} K(\omega) d\sigma(\omega) \;+\; \mathcal{O}(1), \tag{4.1}$$

$$II = -\int_{\substack{o\le\lambda\le 1\\ \lambda\le\min(\frac{1}{|\xi|},\frac{1}{|\eta|^b})}} e^{i\gamma\varphi(\lambda^{\ell_0})} e^{-i\eta q(\lambda)}\, \frac{d\lambda}{\lambda} \int_{\Sigma^+} K(\omega) d\sigma(\omega) \;+\; \mathcal{O}(1), \tag{4.2}$$

and for $\bar\varphi(\lambda) = \varphi(\lambda^{\ell_0})$ convex,

$$\int_{\substack{0\le\lambda\le 1\\ \lambda\le\min(\frac{1}{|\xi|},\frac{1}{|\eta|^b})}} e^{i\gamma\bar\varphi(\lambda)} \sin(\eta q(\lambda))\, \frac{d\lambda}{\lambda} \tag{4.3}$$

is uniformly bounded in γ, η and ξ if and only if

$$\bar\varphi'(C\lambda) \ge 2\bar\varphi'(\lambda)$$

for some $C \ge 1$ and $0 < \lambda \le 1$. Here $q(\lambda) = \lambda + \mathcal{O}(\lambda^{1+\epsilon})$ and $q'(\lambda) = 1 + \mathcal{O}(\lambda^\epsilon)$.

We begin with the proof of (4.1). It will convenient to write (3.1) in the form

$$\psi(y,x) = Ay^{\ell_0} + \sum_{j=1}^{n-2} a_j x_j^{m_j} + \sum_{j=1}^{n-2} b_j x_j^{\alpha_j} y^{\beta_j} + P_2(y,x) + R(y,x) \tag{4.4}$$

where $\ell_0 < m_j$, $1 \le j \le n-2$, $A > 0$, $a_j > 0$, $b_j \ne 0$, $1 \le j \le n-2$ and each monomial of $P_2(y,x)$ has the form $x_j^\alpha y^\beta$ with $\alpha > \alpha_j$, or contains powers of at least two different x_j's. Let $m = \min_{1\le j\le n-2}(m_j)$ and choose $\sigma > 0$ such that $\frac{\ell_0}{m} + \sigma < 1$. Again

$$\int\limits_{|x|\ge y^{\frac{\ell_0}{m}+\sigma}} |K(x,y)|\, dx\, dy \ = \ \mathcal{O}(1)$$

and so it suffices to study I in the region $|x| \le y^{\frac{\ell_0}{m}+\sigma}$. In this region we wish to make a change of variables

$$\lambda^{\ell_0} = \psi(y,x) \tag{4.5}$$

in the y integral. As in section 3, $|x| \le y^{\frac{\ell_0}{m}+\sigma}$ implies that $y = y(x,\lambda)$ defined implicitly by (4.5) satisfies

$$y(x,\lambda) = A^{-\frac{1}{\ell_0}}\lambda \ + \ \mathcal{O}(\lambda^{1+\sigma}), \tag{4.6}$$

$$\frac{\partial y}{\partial \lambda} = A^{-\frac{1}{\ell_0}} \ + \ \mathcal{O}(\lambda^{\sigma}), \tag{4.7}$$

and

$$\left|\frac{\partial y}{\partial x}\right| \le C\lambda^{1-\frac{\ell_0}{m}}. \tag{4.8}$$

Therefore, as before, making the change of variables (4.5), shows

$$I = \int\limits_{0\le\lambda\le 1} e^{i\gamma\varphi(\lambda^{\ell_0})} \int\limits_{|x|\le y^{\frac{\ell_0}{m}+\sigma}} e^{ix\cdot\xi} e^{i\eta y(x,\lambda)} K(x,\lambda)\, dx\, d\lambda \ + \ \mathcal{O}(1). \tag{4.9}$$

To study (4.9), it is necessary to have information on the derivatives of y with respect to the x variables.

Lemma 6. *Suppose* $|x| \le \lambda^{\frac{\ell_0}{m}+\sigma}$.

(1) *For* $\delta > 0$ *small,*

$$\left|\frac{\partial^k y}{\partial x_j^k}(x,\lambda)\right| \le \delta\lambda^{1-k\frac{\ell_0}{m_j}}, \qquad 1 \le k \le \alpha_j - 1, \quad 1 \le j \le n-2.$$

(2)

$$\frac{\partial^{\alpha_j} y}{\partial x_j^{\alpha_j}}(x,\lambda) \ \sim \ \lambda^{1-\alpha_j\frac{\ell_0}{m_j}}, \quad 1 \le j \le n-2.$$

(3) *For every* $\beta = (\beta_1, \dots, \beta_{n-2})$ *with* $0 \le \beta_j \le \alpha_j, \quad 1 \le j \le n-2,$

$$\left|\frac{\partial^{\beta} y}{\partial x^{\beta}}(x,\lambda)\right| \ \le \ C\lambda^{1-\ell_0\sum_{j=1}^{n-2}\frac{\beta_j}{m_j}}.$$

(4) *For every* $\beta = (\beta_1, \dots, \beta_{n-2})$ *with* $0 \le \beta_j \le \alpha_j - 1$, $1 \le j \le n-2$, *either*

$$\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) \;\sim\; \lambda^{p_\beta}$$

for some $p_\beta > -|\beta|$, *or*

$$\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) \;=\; \mathcal{O}(\lambda^N)$$

for every N.

Proof of lemma. For M large, write

$$\psi(y,x) = Ay^{\ell_0} + \sum_{j=1}^{n-2} b_j x_j^{\alpha_j} y^{\beta_j} + \sum_{u,\beta} c_{u,\beta} x^u y^\beta + \mathcal{O}(|x|^M) + \mathcal{O}(y^M)$$

where $A > 0$, $b_1, \dots, b_{n-2} \neq 0$, $\frac{\alpha_j}{m_j} + \frac{\beta_j}{\ell_0} = 1$ for $1 \le j \le n-2$, each $x^u y^\beta$ satisfies $\sum_{j=1}^{n-2} \frac{u_j}{m_j} + \frac{\beta}{\ell_0} \ge 1$ and if $u = (0, \dots, u_j, \dots, 0)$, then $u_j > \alpha_j$. To prove (1), we will show inductively that in the larger region, $|x_j| \le \epsilon \lambda^{\frac{\ell_0}{m_j}}$, $1 \le j \le n-2$,

$$\left| \frac{\partial^k y}{\partial x_j^k} \right| \le \delta \lambda^{1 - k\frac{\ell_0}{m_j}}, \quad 1 \le k \le \alpha_j - 1, \tag{4.10}$$

provided $\epsilon = \epsilon(\delta) > 0$ is small enough. We first prove (4.10) for $k = 1$ (so $\alpha_j \ge 2$ or there is nothing to prove). If we differentiate (4.5) with respect to x_j, noting $y \sim \lambda$ from (4.6), we obtain

$$0 \;=\; C_1 \frac{\partial y}{\partial x_j} + C_2$$

where

$$C_1 \;=\; A\ell_0 y^{\ell_0 - 1} + \mathcal{O}(\lambda^{\ell_0+1}) + E$$

and E is a finite sum of terms of the form $x^u y^{\beta-1}$ where $u = (u_1, \dots, u_{n-2}) \neq 0$ and $\sum_{j=1}^{n-2} \frac{u_j}{m_j} + \frac{\beta}{\ell_0} \ge 1$. Hence for $\epsilon > 0$ small enough, $|x^u y^{\beta-1}| \le \delta \lambda^{\ell_0 - 1}$ and therefore

$$C_1 \sim \lambda^{\ell_0 - 1}.$$

C_2 is $\mathcal{O}(\lambda^{\ell_0})$ plus a finite sum of terms of the form $x_j^{-1} x^u y^\beta$ with $u_j \ge 2$ and $\frac{|u|}{m} + \frac{\beta}{\ell_0} \ge 1$. Thus for $\epsilon > 0$ small enough,

$$\begin{aligned} |x_j^{-1} x^u y^\beta| &\le \delta \lambda^{|u|\frac{\ell_0}{m} + \beta} \lambda^{-\frac{\ell_0}{m_j}} \\ &\le \delta \lambda^{\ell_0 - \frac{\ell_0}{m_j}} \end{aligned}$$

and so $C_2 = \mathcal{O}(\delta \lambda^{\ell_0 - \frac{\ell_0}{m_j}})$ which proves (4.10) with $k = 1$.

Next we assume (4.10) for $k \le k_0 - 1$ where $k_0 \le \alpha_j - 1$, and prove (4.10) for $k = k_0$. Differentiating (4.5) k_0 times with respect to x_j, we again obtain

$$0 \ = \ D_1 \frac{\partial^{k_0} y}{\partial x_j^{k_0}} \ + \ D_2$$

where as before $D_1 \sim \lambda^{\ell_0 - 1}$. D_2 consists of a finite sum of products of terms involving either a positive power of x or a derivative of order at most $k_0 - 1$ of y with respect to x_j. In the first case we pick up an ϵ from the powers of x and in the second case we pick up a δ from the induction hypothesis. So we only need to determine the magnitude of each term in D_2. Since each term in the expression for $\psi(y,x)$ is $\mathcal{O}(\lambda^{\ell_0})$, we only need to understand how the powers of λ decrease when we differentiate a product involving x^u and

$$\left(\frac{\partial^k y}{\partial x_j^k}\right)^p, \quad 1 \le k \le k_0 - 2,$$

with respect to x_j. Differentiating x^u gives $x_j^{-1} x^u$, losing $\lambda^{-\frac{\ell_0}{m_j}}$ and differentiating

$$\left(\frac{\partial^k y}{\partial x_j^k}\right)^p \text{ gives } \left(\frac{\partial^k y}{\partial x_j^k}\right)^{p-1} \frac{\partial^{k+1} y}{\partial x_j^{k+1}}, \ \text{ losing } \lambda^{-(1-k\frac{\ell_0}{m_j})} \lambda^{1-(k+1)\frac{\ell_0}{m_j}} = \lambda^{-\frac{\ell_0}{m_j}}$$

by induction. Therefore each term in D_2 is $\mathcal{O}(\delta \lambda^{\ell_0 - k_0 \frac{\ell_0}{m_j}})$ and this finishes the proof of (4.10) and thus (1) of the lemma.

The proof of (2) follows in the same way as the proof of (1). The only difference is that differentiating the term $b_j x_j^{\alpha_j} y^{\beta}$, $b_j \ne 0$, contributes a term $b_j \alpha_j! \, y^{\beta_j} \sim \lambda^{\beta_j}$ and so

$$D_2 \ \sim \ \lambda^{\beta_j} + \mathcal{O}(\delta \lambda^{\ell_0 - \alpha_j \frac{\ell_0}{m_j}}) \ \sim \ \lambda^{\ell_0 - \alpha_j \frac{\ell_0}{m_j}}$$

since $\frac{\alpha_j}{m_j} + \frac{\beta_j}{\ell_0} = 1$. This shows (2).

The proof of (3) follows similarly. We use induction on the partial ordering $u = (u_1, \dots, u_{n-2}) \le \beta = (\beta_1, \dots, \beta_{n-2})$ if and only if $u_j \le \beta_j$, $1 \le j \le n-2$. (1) and (2) show (3) is true for all pure derivatives, $\beta = (0, \dots, \beta_j, \dots, 0)$. The arguments used in proving (1) and (2) show that if (3) is true for all $u \lneqq \beta$, then differentiating (4.5) shows

$$0 \ = \ D_1 \frac{\partial^{\beta} y}{\partial x^{\beta}} \ + \ D_2$$

where $D_1 \sim \lambda^{\ell_0 - 1}$ and

$$D_2 = \mathcal{O}\left(\lambda^{\ell_0 - \ell_0 \sum_{j=1}^{n-2} \frac{\beta_j}{m_j}}\right),$$

proving (3).

Finally to prove (4), we first note that (3) implies that it is enough to show (4) for any power p_β. Again we use induction on the partial ordering $\leq$, supposing (4) is true for all $u \lneqq \beta$. Rewriting (4.4) expresses (4.5) as

$$\lambda^{\ell_0} = Ay^{\ell_0} + \sum_{\substack{u \\ u \leq \beta}} a_u(y)x^u + \mathcal{O}(|x|^{|\beta|+1}) \tag{4.11}$$

where $a_0(y) = \mathcal{O}(y^{\ell_0+1}), \quad a_0'(y) = \mathcal{O}(y^{\ell_0})$ and the a_u's are smooth. Taking the β-th derivative of (4.11) gives

$$0 = [A\ell_0 y^{\ell_0-1}(0,\lambda) + a_0'(y(0,\lambda))] \frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) + C(\lambda)$$

where $C(\lambda)$ is a finite sum of terms of the form

$$a^{(s)}(y(0,\lambda)) \prod_{\substack{u \\ u \lneqq \beta}} \left(\frac{\partial^u y}{\partial x^u}(0,\lambda) \right)^{q_u}$$

for some non-negative integers q_u. Here $a(y)$ is either a power of y or one of the a_u's. Using the fact that $y(0,\lambda) \sim \lambda$ and the inductive hypothesis, we see that $C(\lambda) \sim \lambda^p$ for some p or $C(\lambda) = \mathcal{O}(\lambda^N)$ for every N. Since

$$\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) = -\frac{C(\lambda)}{[A\ell_0 y^{\ell_0-1}(0,\lambda) + a_0'(y(0,\lambda))]}$$

and $a_0'(y) = \mathcal{O}(\lambda^{\ell_0})$, we have shown (4) and this finishes the proof of the lemma. $\square$

We now turn back to the proof of (4.1) where we are examining the integral in (4.9). Let us write

$$y(x,\lambda) = M_1(x,\lambda) + M_2(x,\lambda)$$

where $M_1(x,\lambda)$ is a polynomial in x_1 of degree $\alpha_1 - 1$ and M_2 is that part of the Taylor expansion of $y(x,\lambda)$ in the variable x_1 that is $\mathcal{O}(|x_1|^{\alpha_1})$. We wish to replace the integral in (4.9) by a similar integral where $y(x,\lambda)$ is replaced by $M_1(x,\lambda)$ and the λ integral is restricted to

$$\lambda \leq \left(\frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}$$

where $\kappa(\alpha_1) = 1 - \alpha_1 \frac{\ell_0}{m_1}$. Note that $\alpha_1 + \kappa(\alpha_1) > 1$. Since

$$\frac{\partial^{\alpha_1} y}{\partial x_1^{\alpha_1}} \sim \lambda^{1 - \alpha_1 \frac{\ell_0}{m_1}}$$

by part (2) of Lemma 1 and since $\alpha_1 \geq 2$, an application of Van der Corput's lemma together with integration by parts shows

$$\left| \int\limits_{\lambda \geq \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}} e^{i\gamma\varphi(\lambda^{\ell_0})} \int\limits_{|x| \leq y^{\frac{\ell_0}{m}+\sigma}} e^{i\xi\cdot x} e^{i\eta y(x,\lambda)} K(x_1\lambda)\, dx\, d\lambda \right|$$

$$\leq C \left(\frac{1}{|\eta|}\right)^{1/\alpha_1} \int\limits_{\lambda \geq \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}} \left(\frac{1}{\lambda}\right)^{\frac{1}{\alpha_1}\left[1-\alpha_1\frac{\ell_0}{m_1}\right]} \int\limits_{\mathbb{R}^{n-2}} \frac{1}{|x|^n+\lambda^n}\, dx\, d\lambda$$

$$\leq C \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1}} \int\limits_{\lambda \geq \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}} \frac{\lambda^{\frac{\ell_0}{m_1}}}{\lambda^{2+\frac{1}{\alpha_1}}}\, d\lambda \ \leq \ C \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1}} |\eta|^{\frac{1}{\alpha_1+\kappa(\alpha_1)}\left[1+\frac{1}{\alpha_1}-\frac{\ell_0}{m_1}\right]}$$

$$= C \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1}} |\eta|^{\frac{1}{\alpha_1}\left[\frac{\alpha_1+\kappa(\alpha_1)}{\alpha_1+\kappa(\alpha_1)}\right]} \ = \ C.$$

In the region

$$\lambda \leq \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}$$

we would like to replace $e^{i\eta y(x,\lambda)}$ by $e^{i\eta M_1(x,\lambda)}$. We expect to be able to replace $e^{i\eta y(x,\lambda)}$ by $e^{i\eta M_1(x,\lambda)}$ with a bounded error when

$$|x| \leq \left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}$$

since

$$|e^{i\eta y(x,\lambda)} - e^{i\eta M_1(x,\lambda)}| \ \leq \ C|\eta|\lambda^{1-\alpha_1\frac{\ell_0}{m_1}}|x|^{\alpha_1} \ \leq \ C$$

when $|x| \leq \left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}$. However in the complementary region, when

$$\lambda \leq \left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_2)}} \qquad \text{and} \qquad |x| \geq \left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}},$$

K is uniformly integrable. In fact

$$\int_{\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}}\int_{|x|\ge\left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}}|K(x,\lambda)|\,dx\,d\lambda$$

$$\le C\int_{\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}}\int_{|x|\ge\left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}}\frac{1}{|x|^{n-1}}\,dx\,d\lambda$$

$$\le\int_{\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}}(|\eta|\lambda^{\kappa(\alpha_1)})^{\frac{1}{\alpha_1}}\,d\lambda$$

$$=C\int_{|\eta|^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}\lambda\le 1}\left(|\eta|^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}\lambda\right)^{\frac{\kappa(\alpha_1)}{\alpha_1}+1}\frac{d\lambda}{\lambda}\;\le\;C$$

since

$$\frac{\kappa(\alpha_1)}{\alpha_1}=\frac{1-\alpha_1\frac{\ell_0}{m_1}}{\alpha_1}>-1.$$

Replacing $e^{i\eta y(x,\lambda)}$ by $e^{i\eta M_1(x,\lambda)}$ when

$$\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}\qquad\text{and}\qquad|x|\le\left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}$$

creates an error at most

$$C|\eta|\int_{\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}}\lambda^{\kappa(\alpha_1)}\int_{|x|\le\left(\frac{1}{|\eta|\lambda^{\kappa(\alpha_1)}}\right)^{\frac{1}{\alpha_1}}}\frac{|x|^{\alpha_1}}{|x|^{n-1}}\,dx\,d\lambda$$

$$\le C|\eta|\int_{\lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1}+\kappa(\alpha_1)}}\frac{\lambda^{\kappa(\alpha_1)}}{(|\eta|\lambda^{\kappa(\alpha_1)})^{\frac{1}{\alpha_1}(\alpha_1-1)}}\,d\lambda$$

$$\le C\int_{|\eta|^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}\lambda\le 1}\left(|\eta|^{\frac{1}{\alpha_1+\kappa(\alpha_1)}}\lambda\right)^{\frac{\alpha_1+\kappa(\alpha_1)}{\alpha_1}}\frac{d\lambda}{\lambda}\;\le\;C$$

since

$$\frac{\alpha_1+\kappa(\alpha_1)}{\alpha_1}=1-\frac{\ell_0}{m_1}+\frac{1}{\alpha_1}>0.$$

Therefore

$$I = \int\limits_{\substack{0\le\lambda\le 1 \\ \lambda\le\left(\frac{1}{|\eta|}\right)^{\frac{1}{\alpha_1+K(\alpha_1)}}}} e^{i\gamma\varphi(\lambda^{\ell_0})} \int\limits_{|x|\le y^{\frac{\ell_0}{m}+\sigma}} e^{i\xi\cdot x} e^{i\eta M_1(x,\lambda)}\, K(x,\lambda)\, dx\, d\lambda \ + \ \mathcal{O}(1).$$

Since

$$M_1(x,\lambda) = \sum_{k=0}^{\alpha_1-1} \frac{1}{k!}\frac{\partial^k y}{\partial x_1^k}(0,x_2,\ldots,x_{n-2})x_1^k,$$

we see that for $2\le j\le n-2$,

$$\frac{\partial^{\alpha_j} M_1}{\partial x_j^{\alpha_j}}(x,\lambda) = \frac{\partial^{\alpha_j} y}{\partial x_j^{\alpha_j}}(0,x_2,\ldots,x_{n-2}) + \sum_{k=1}^{\alpha_1-1}\frac{1}{k!}\frac{\partial^{k+\alpha_j} y}{\partial x_1^k\partial x_j^{\alpha_j}}(0,x_2,\ldots,x_{n-2})x_1^k.$$

Also since $|x_1|\le \epsilon\lambda^{\frac{\ell_0}{m_1}}$, we have by part (3) of Lemma 1,

$$\left|\sum_{k=1}^{\alpha_1-1}\frac{1}{k!}\frac{\partial^{k+\alpha_j} y}{\partial x_1^k\partial x_j^{\alpha_j}}(0,x_2,\ldots,x_{n-2})x_1^k\right| \le \epsilon\lambda^{k\frac{\ell_0}{m_1}}\lambda^{1-\ell_0\left(\frac{k}{m_1}+\frac{\alpha_j}{m_j}\right)} \le \epsilon\lambda^{1-\ell_0\frac{\alpha_j}{m_j}},$$

and since

$$\frac{\partial^{\alpha_j} y}{\partial x_j^{\alpha_j}} \ \sim\ \lambda^{1-\ell_0\frac{\alpha_j}{m_j}}$$

by part (2) of Lemma 1, we conclude that for $2\le j\le n-2$,

$$\frac{\partial^{\alpha_j} M_1}{\partial x_j^{\alpha_j}}(x,\lambda) \ \sim\ \lambda^{1-\ell_0\frac{\alpha_j}{m_j}}.$$

Therefore we may proceed in the same manner to find that up to a bounded error

$$I = \int\limits_{\lambda\le\frac{1}{|\eta|^\delta}} e^{i\gamma\varphi(\lambda^{\ell_0})} \int\limits_{|x|\le\lambda^{\frac{\ell_0}{m}+\sigma}} e^{i\xi\cdot x}e^{i\eta Q(x,\lambda)}\, K(x,\lambda)\, dx\, d\lambda$$

for some $0<\delta<1$ where

$$Q(x,\lambda) = \sum_{\substack{\beta \\ \beta_j\le\alpha_j-1}} \frac{1}{\beta!}\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda)\, x^\beta.$$

By part (4) of Lemma 1, we have for each β with $\beta_j\le\alpha_j-1$, $1\le j\le n-2$, either

$$\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) \ \sim\ \lambda^{p_\beta}$$

for some $p_\beta > -|\beta|$ or

$$\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda) \ = \ \mathcal{O}\left(\lambda^N\right)$$

for every N. If the latter occurs, then up to a bounded error, we may clearly replace $e^{i\eta \frac{\partial^\beta y}{\partial x^\beta}(0,\lambda)x^\beta}$ by 1. When the former occurs, that is, when $\frac{\partial^\beta y}{\partial x^\beta}(0,\lambda)$ behaves like a power of λ, we can repeat the above argument to see that for some (other) δ, $0<\delta<1$,

$$I = \int_{\lambda \le \frac{1}{|\eta|^\delta}} e^{i\gamma\varphi(\lambda^{\ell_0})} e^{i\eta y(0,\lambda)} \int_{|x|\le \lambda^{\frac{\ell_0}{m}+\sigma}} e^{i\sum_{j=1}^{n-2}(\xi_j+\lambda^{p_j}\eta_j)x_j} K(x,\lambda)\,dx\,d\lambda \;+\; \mathcal{O}(1).$$

For each x_j integral, by splitting the λ integral where λ is smaller or larger than $|\xi_j/\eta_j|^{\frac{1}{p_j}}$, we can once again repeat the same argument to conclude that

$$\begin{aligned} I &= \int_{\lambda \le \min(\frac{1}{|\xi|},\frac{1}{|\eta|^b})} e^{i\gamma\varphi(\lambda^{\ell_0})} e^{i\eta y(0,\lambda)} \int_{|x|\le \lambda^{\frac{\ell_0}{m}+\sigma}} K(x,\lambda)\,dx\,d\lambda \;+\; \mathcal{O}(1) \\ &= \int_{\lambda \le \min(\frac{1}{|\xi|},\frac{1}{|\eta|^b})} e^{i\gamma\varphi(\lambda^{\ell_0})} e^{i\eta y(0,\lambda)} \frac{1}{\lambda} \int_{\mathbb{R}^{n-2}} K(z,1)\,dz\,d\lambda \;+\; \mathcal{O}(1) \end{aligned}$$

for some $0<b<1$. Thus the proof of (4.1) will be finished once we establish the identity

$$\int_{\mathbb{R}^{n-2}} K(x,1)\,dx = \int_{\Sigma^+} K(\omega)\,d\sigma(\omega). \tag{4.12}$$

This is done by making the change of variables

$$x_j = \frac{s_j}{1-|s|^2}, \quad 1\le j\le n-2.$$

In evaluating the Jacobian of this change of variables, we need to observe that if an $r\times r$ matrix $(\alpha_{j,k})$ is defined by $\alpha_{j,k}=s_js_k$ for $j\ne k$ and $\alpha_{j,j}=1-|s|^2-s_j^2$, then

$$\det(\alpha_{j,k}) = (1-|s|^2)^{r-1}.$$

This calculation was shown to us by A. Carbery and is carried out in the appendix. This establishes (4.12) and finishes the proof of (4.1). The proof of (4.2) is similar. It remains to prove (4.3).

Suppose first that there is no constant C_0 so that $\bar\varphi'(C_0\lambda)\ge 2\bar\varphi'(\lambda)$ for $0<\lambda\le 1$. Then there exists a sequence of points $\lambda_j \searrow 0$ such that

$$\frac{\lambda_j\bar\varphi'(\lambda_j)}{\lambda_j\bar\varphi'(\lambda_j)-\bar\varphi(\lambda_j)} \to \infty.$$

See, e.g., [NVWW]. Let

$$\gamma_j = \frac{\pi}{4}\,\frac{1}{\lambda_j\bar\varphi^1(\lambda_j)-\bar\varphi(\lambda_j)}, \quad \eta_j=\gamma_j\bar\varphi'(\lambda_j),$$

and choose ξ_j so that

$$\frac{1}{\xi_j} = \min\left(\lambda_j, \frac{1}{\eta_j^b}, \frac{1}{\eta_j^{1+\epsilon}}\right)$$

where $\epsilon > 0$ is chosen so that $q(\lambda) = \lambda + \mathcal{O}(\lambda^{1+\epsilon})$. Then

$$\left|\int_0^{\frac{1}{\xi}} e^{i\gamma_j \bar{\varphi}(\lambda)} \sin(\eta_j q(\lambda))\, \frac{d\lambda}{\lambda}\right| = \left|\int_{\frac{1}{\eta_j}}^{\frac{1}{\xi}} e^{i(\gamma_j \bar{\varphi}(\lambda) - \eta_j \lambda)}\, \frac{d\lambda}{\lambda}\right| \; + \; \mathcal{O}(1)$$

$$\geq A \log\left(\frac{\eta_j}{\xi_j}\right) \geq A\log(\lambda_j \eta_j) \to +\infty$$

for some $A > 0$ since $\lambda_j \eta_j \to \infty$ and

$$0 \leq \eta_j \lambda - \gamma_j \bar{\varphi}(\lambda) \leq \frac{\pi}{4}$$

for all $0 \leq \lambda \leq \lambda_j$.

Finally let us turn to the proof of the sufficiency of (4.3) and assume $\bar{\varphi}(0) = \bar{\varphi}'(0) = 0$, $\bar{\varphi}'(C_0 t) \geq 2\bar{\varphi}'(t)$ for some $C_0 \geq 1$. It suffices to show that the integral

$$II = \int_{\frac{1}{\eta} \leq t \leq 1} e^{i\gamma \bar{\varphi}(t)} e^{-i\eta q(t)}\, \frac{dt}{t} \tag{4.13}$$

is uniformly bounded in $\gamma, \eta > 0$. First assume $10\gamma > \eta$. Choosing t_0 such that $\bar{\varphi}'(t_0) = \frac{\eta}{\gamma}$ we write

$$II = \int_{\frac{1}{\eta} \leq t \leq \frac{t_0}{C_0}} + \int_{\frac{t_0}{C_0} \leq t \leq C_0 t_0} + \int_{C_0 t_0 \leq t \leq 1} = A + B + D.$$

For $\frac{1}{\eta} \leq t \leq \frac{t_0}{C_0}$, $\frac{d}{dt}(\eta t - \gamma \bar{\varphi}(t)) = \eta - \gamma \bar{\varphi}'(t) \geq \eta - \gamma \bar{\varphi}'(\frac{t_0}{C_0}) \geq \frac{\eta}{2}$, and so integrating by parts shows

$$|A| \leq \frac{1}{\eta} \int_{\frac{1}{\eta} \leq t \leq 1} \eta |q'(t) - 1|\, \frac{dt}{t} + \frac{1}{\eta} \int_{\frac{1}{\eta} \leq t} \frac{1}{t^2}\, dt \; + \; C$$

$$\leq C \int_0^1 t^\epsilon\, \frac{dt}{t} + C \leq C.$$

Also

$$|B| \leq \int_{\frac{t_0}{C_0} \leq t \leq C_0 t_0} \frac{1}{t}\, dt \leq 2\log(C_0).$$

For $C_0 t_0 \le t$, $\frac{d}{dt}(\gamma\bar{\varphi}(t) - \eta t) = \gamma\bar{\varphi}'(t) - \eta \ge \frac{\gamma}{2}\bar{\varphi}'(t)$, and so integrating by parts show

$$|D| \le \frac{1}{\gamma} \int\limits_{\frac{1}{\eta} \le t_0 \le t} \left(\frac{\bar{\varphi}''(t)}{t[\bar{\varphi}'(t)]^2} + \frac{1}{\bar{\varphi}'(t) t^2} + \frac{\eta |q'(t) - 1|}{\bar{\varphi}'(t) t} \right) dt + \frac{\eta}{\gamma} \frac{1}{\bar{\varphi}'(t_0)}$$

$$\le \frac{\eta}{\gamma} \int\limits_{t_0 \le t} \frac{\bar{\varphi}''(t)}{[\bar{\varphi}'(t)]^2} \, dt + \frac{1}{\bar{\varphi}'(t_0)} \left[\frac{\eta}{\gamma} + C \frac{\eta}{\gamma} \int\limits_0^1 \frac{t^\epsilon}{t} \, dt \right] + \frac{\eta}{\gamma} \frac{1}{\bar{\varphi}'(t_0)}$$

$$\le C \frac{\eta}{\gamma} \frac{1}{\bar{\varphi}'(t_o)} \le C$$

since $\frac{\eta}{\gamma} = \bar{\varphi}'(t_0)$. Next suppose $10\gamma \le \eta$. Then in a neighborhood of the origin, $\frac{d}{dt}[\eta t - \gamma\bar{\varphi}(t)] \ge \frac{\eta}{2}$, and so integrating by parts shows

$$|II| \le \frac{1}{\eta} \left[\int\limits_{\frac{1}{\eta} \le t} \eta |q'(t) - 1| \, \frac{dt}{t} + \int\limits_{\frac{1}{\eta} \le t} \frac{1}{t^2} \right] + C$$

$$\le C \int\limits_0^1 t^\epsilon \frac{dt}{t} + C \le C.$$

This completes the proof of Theorems 2 and 3. □

5. Proof of Theorem 4

We will prove the L^p boundedness of the maximal function

$$Mf(x', x_n) = \sup_{0 < h \le 1} \frac{1}{h^{n-1}} \left| \int\limits_{|t| \le h} f(x' - t, x_n - \varphi(\psi(t)) \, dt \right|.$$

The proof for the singular integral is similar.

When $E_{\ell_0} = \{0\}$, the main term $P(t)$, in the decomposition (3.1) for $\psi(t)$, $\psi(t) = P(t) + R(t)$, is a positive homogeneous polynomial of degree ℓ_0. $R(t)$ consists of all the terms in the Taylor expansion of ψ with degree greater than ℓ_0. The proof of L^p boundedness for M in the case $P(t) = |t|^2$ and $R(t) \equiv 0$ is carried out in [KWWZ]. We will see that slight modifications of the arguments given in [KWWZ] work for the general case.

It will be convenient for us to use polar coordinates with respect to the surface $P(\omega) = 1$. That is, every $t \ne 0 \in \mathbb{R}^{n-1}$ can be written uniquely as $t = r\omega$ where $r > 0$ and $P(\omega) = 1$. We also introduce a norm $\|\cdot\|$ so that $\|t\| = \|rw\| = r$. Since the Euclidean norm of ω, $|\omega|$, is bounded above and below as ω runs over the surface

$P(\omega) = 1$, it is clear that the maximal function $Mf(x)$ is pointwise comparable to the maximal function defined in terms of averages with respect to the norm $\|\cdot\|$. Therefore it suffices to consider

$$\mathcal{M}f(x) = \sup_{k>0} 2^{k(n-1)} \left| \int \chi(2^k\|t\|) f(x-\Gamma(t))\, dt \right| \stackrel{\text{def}}{=} \sup_{k>0} |f * d\mu_k(x)|$$

where χ is a smooth cut-off function supported in $[1,2]$ and chosen so that

$$2^{k(n-1)} \int_{\mathbb{R}^{n-1}} \chi(2^k\|t\|)\, dt \equiv 1.$$

To prove L^p bounds for $\mathcal{M}$ we introduce dilations $\{\delta(t)\}_{t>0}$, defined by $\delta(t)(\xi,\gamma) = (t\xi, \bar{\varphi}(t)\gamma)$. Although the "balls" generated with respect to these dilations do not in general form a space of homogeneous type with respect to Lebesgue measure, an appropriate singular integral and Littlewood–Paley theory for the dilations $\{\delta(t)\}_{t>0}$ has been worked out in [CCVWW]. Using this theory and well-known techniques, following the arguments detailed in [KWWZ], we reduce ourselves to proving two basic estimates for the Fourier transform of the measures $\{d\mu_k\}$ defined above:

$$|\widehat{d\mu_k}(\xi,\gamma) - 1| \le C|\delta(2^{-k+3})(\xi,\gamma)|, \tag{5.1}$$

and

$$|\widehat{d\mu_k}(\xi,\gamma)| \le C|\delta(2^{-k-1})(\xi,\gamma)|^{-\epsilon} \tag{5.2}$$

for some $\epsilon > 0$. Using polar coordinates $t = r\omega$,

$$\widehat{d\mu_k}(\xi,\gamma) = 2^{k(n-1)} \int_{\mathbb{R}} \int_{P(\omega)=1} \chi(2^k r) e^{i\xi r\cdot\omega} e^{i\gamma\varphi(\psi(r\omega))} r^{n-2} h(\omega)\, d\omega\, dr \tag{5.3}$$

where $h(\omega)$ is some smooth function. Since $\psi(r\omega) = r^{\ell_0} + \mathcal{O}(r^{\ell_0+1})$, we have for $k > 0$ large, $\varphi(\psi(r\omega)) \le \bar{\varphi}(2^{-k+3})$ when $2^{-k} \le r \le 2^{-k+1}$. Therefore

$$|\widehat{d\mu_k}(\xi,\gamma) - 1| \le C[2^{-k}|\xi| + \bar{\varphi}(2^{-k+3})|\gamma|] \le C|\delta(2^{-k+3})(\xi,\gamma)|,$$

establishing (5.1). To prove (5.2) we make the change of variables

$$\lambda^{\ell_0} = \psi(r\omega) = r^{\ell_0} + R(r\omega) \tag{5.4}$$

in the r integral in (5.3) for fixed ω. For $k > 0$ large this is a good change of variables and so

(5.5)

$$\widehat{d\mu_k}(\xi,\gamma) = 2^{k(n-1)} \int_{\mathbb{R}} e^{i\gamma\bar{\varphi}(\lambda)} \int_{P(\omega)=1} e^{ir(\lambda,\omega)\xi\cdot\omega} \chi(2^k r(\lambda,\omega))\, \frac{\partial r}{\partial\lambda}\, r^{n-2}(\lambda,\omega)\, h(\omega)\, d\omega\, d\lambda.$$

From (5.4) one easily deduces the following estimates on the derivatives of $r(\lambda, \omega)$:

$$r(\lambda,\omega) = \lambda + \mathcal{O}(\lambda^2), \quad \frac{\partial r}{\partial \lambda} = 1 + \mathcal{O}(\lambda), \tag{5.6}$$

$$\nabla_\omega r = \mathcal{O}(\lambda), \quad \frac{\partial r}{\partial \lambda \partial \omega} = \mathcal{O}(1), \tag{5.7}$$

$$\frac{\partial^2 r}{\partial \lambda^2} = \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{5.8}$$

Since $P(\omega) = 1$ is of finite type, we can argue as in section 3 to find an $\epsilon > 0$ such that

$$\left| \int\limits_{P(\omega)=1} e^{ir(\lambda,\omega)\xi\cdot\omega} h(\omega)\, d\omega \right| \le C \frac{1}{|\lambda\xi|^{2\epsilon}}.$$

Now integrating by parts, using (5.6) and (5.7), shows

$$|\widehat{d\mu_k}(\xi,\gamma)| \le C \left(\frac{1}{2^{-k-1}|\xi|} \right)^{2\epsilon},$$

establishing (5.2) if

$$\sqrt{|\gamma|\bar{\varphi}(2^{-k-1})} \le C 2^{-k-1}|\xi|.$$

On the other hand, if

$$C 2^{-k-1}|\xi| \le \sqrt{|\gamma|\bar{\varphi}(2^{-k-1})},$$

we perform the λ integration first, writing (5.5) as

$$\widehat{d\mu_k}(\xi,\gamma) = 2^{k(n-1)} \int\limits_{P(\omega)=1} h(\omega) \int\limits_{\mathbb{R}} e^{i[\gamma\bar{\varphi}(\lambda)+\lambda\xi\cdot\omega]} e^{i[r(\lambda,w)-\lambda]\xi\cdot\omega} \chi(2^k r)\, \frac{\partial r}{\partial \lambda}\, r^{n-2}\, d\lambda\, d\omega.$$

For $2^{-k} \le r(\lambda,\omega) \le 2^{-k+1}$, we have

$$\left| \frac{\partial}{\partial \lambda} [\gamma\bar{\varphi}(\lambda) + \lambda\xi\cdot\omega] \right| \ge \frac{|\gamma|}{2} \bar{\varphi}'(2^{-k-1}) \ge |\gamma| \, \frac{\bar{\varphi}(2^{-k-1})}{2^{-k}}$$

since $2^{-k}|\xi| \ll |\gamma|\bar{\varphi}(2^{-k-1})$. Thus integrating by parts, using (5.6) and (5.8), shows

$$\begin{aligned} |\widehat{d\mu_k}(\xi,\gamma)| &\le C \left[\frac{1}{|\gamma|\bar{\varphi}(2^{-k-1})} + \frac{|\xi| 2^{-k}}{|\gamma|\bar{\varphi}(2^{-k-1})} \right] \\ &\le C \frac{1}{\sqrt{|\gamma|\bar{\varphi}(2^{-k-1})}} \\ &\le C \frac{1}{\sqrt{|\delta(2^{-k-1})(\xi,\gamma)|}} \end{aligned}$$

since $C 2^{-k}|\xi| \le \sqrt{|\gamma|\bar{\varphi}(2^{-k-1})}$.

This completes the proof of (5.1) and (5.2) from which the L^p boundedness of the maximal function follows as in [KWWZ].

6. Proof of Theorem 5

We need to show that the multiplier for H,

$$m(\xi,\gamma) = \int\int_{\substack{|t|\leq 1\\ t\in\mathbb{R}^2}} e^{i\gamma\varphi(\psi(t))} e^{i\xi\cdot t} K(t)\, dt \tag{6.1}$$

is uniformly bounded for $\xi \in \mathbb{R}^2$ and $\gamma \in \mathbb{R}$. Introducing polar coordinates with respect to the convex curve $\psi(t) = 1$, we may write (6.1) as

$$\int_0^1 e^{i\gamma\varphi(r)} \frac{1}{r} \int_{\psi(\omega)=1} e^{ir\xi\cdot\omega} K(\omega)\, h(\omega)\, d\omega\, dr \tag{6.2}$$

for some smooth function $h(\omega)$. The argument used in the proof of Theorem 1 to establish (3.10) shows

$$\int_{\psi(\omega)=1} K(\omega)\, h(\omega)\, d\omega = 0$$

and so the part of the integral in (6.2) where $r \leq \frac{1}{|\xi|}$ is at most

$$C \int_{r\leq\frac{1}{|\xi|}} \frac{1}{r} \left| \int_{\psi(\omega)=1} (e^{ir\xi\cdot\omega} - 1) K(\omega) h(\omega)\, d\omega \right| dr \leq C|\xi| \int_{r\leq\frac{1}{|\xi|}} dr \leq C.$$

For the region where $r \geq \frac{1}{|\xi|}$, we observe that the inner integral in (6.2) is the Fourier transform of a smooth density on the convex curve $\psi(\omega) = 1$ evaluated at $r\xi$. This Fourier transform can be estimated in terms of the "balls," $E(t,\epsilon)$, introduced in section 1. In fact

$$\left| \int_{\psi(\omega)=1} e^{ir\xi\cdot\omega} K(\omega) h(\omega)\, d\omega \right| \leq C \left[|E(t_1(\xi), \frac{1}{r|\xi|})| + |E(t_2(\xi), \frac{1}{r|\xi|})| \right]$$

where $t_1(\xi)$ and $t_2(\xi)$ are the two points on the curve $\psi(t) = 1$ whose tangent lines are normal to ξ. See [BNW]. Therefore the part of the integral in (6.2) where $r \geq \frac{1}{|xi|}$ can be estimated by

$$C \sup_{\substack{t\\ \psi(t)=1}} \int_{\frac{1}{|\xi|}\leq r} |E(t, \frac{1}{r|\xi|})| \frac{dr}{r} \leq C \sup_{\substack{t\\ \psi(t)=1}} \int_0^1 |E(t,\delta)| \frac{d\delta}{\delta}.$$

Hence the multiplier $m(\xi, \gamma)$ is uniformly bounded in ξ and γ if the quantity

$$\sup_{\substack{t \\ \psi(t)=1}} \int_0^1 |E(t,\delta)| \, \frac{d\delta}{\delta}$$

is finite. This completes the proof of Theorem 5. □

7. Appendix

In this appendix we will compute the determinant of an $r \times r$ matrix $A = \{\alpha_{j,k}\}$ of the form $A = cI + B$ where $B = \{b_{j,k}\}$ and $b_{j,k} = b\, s_j t_k$. We will show that

$$\det(A) \;=\; c^r \;+\; c^{r-1}\, b \sum_{j=1}^{r} s_j t_j. \tag{7.1}$$

For the example we need in this paper, $\alpha_{j,k} = s_j s_k$ for $j \neq k$ and $\alpha_{j,j} = 1 - |s|^2 + s_j^2$. Therefore taking $t_j = s_j$, $c = 1 - |s|^2$ and $b = 1$ in the above formula (7.1) gives us the desired result $\det(A) = (1 - |s|^2)^{r-1}$ in this case. To prove (7.1) first note that as a function of $s = (s_1, \dots, s_r)$, $\det(A)$ is an affine function in each of the variables s_j separately. Also computing any pure mixed derivative, e.g., $\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3}$, of $\det(A)$ gives rise to two or more rows being identical and therefore zero. Hence expanding $\det(A)$ in its Taylor series in s about the origin, we see that (7.1) follows from the fact that for each $1 \leq j \leq r$, the partial derivative of $\det(A)$ with respect to s_j at the origin is $c^{r-1}\, b\, t_j$. This is a straightforward computation. □

References

[BNW] J. Bruna, A. Nagel, and S. Wainger. Convex hypersurfaces and Fourier transforms. *Ann. of Math.*, 127:333–365, 1988.

[CCVWW] Anthony Carbery, Michael Christ, James Vance, Stephen Wainger, and David K. Watson. Operators associated to flat plane curves: L^p estimates via dilation methods. *Duke Math. J.*, 59(3):675–700, 1989.

[KWWZ] Weon-Ju Kim, Stephen Wainger, James Wright, and Sarah Ziesler. Singular integrals and maximal functions associated to surfaces of revolution. *Bull. London Math. Soc.*, 28(3):291–296, 1996.

[NVWW] A. Nagel, J. Vance, S. Wainger, and D. Weinberg. Hilbert transforms for convex curves. *Duke Math. J.*, 50:735–744, 1983.

[Sch] Helmut Schulz. Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms. *Indiana Univ. Math. J.*, 40(4):1267–1275, 1991.

[SWWZ] A. Seeger, S. Wainger, J. Wright, and S. Ziesler. Classes of singular integrals along curves and surfaces. *Trans. Amer. Math. Soc.* To appear.

[St] E. Stein. *Harmonic Analysis* Princeton University Press, Princeton New Jersey, 1993.

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