DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUP: A SURVEY

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ABSTRACT. We survey results arising from the study of domains in $\mathbb{C}^n$ with non-compact automorphism group. Beginning with a well-known characterization of the unit ball, we develop ideas toward a consideration of weakly pseudoconvex (and even non-pseudoconvex) domains with particular emphasis on characterizations of **(i)** smoothly bounded domains with non-compact automorphism group and **(ii)** the Levi geometry of boundary orbit accumulation points.

Particular attention will be paid to results derived in the past ten years.

0. INTRODUCTION

In any area of mathematics, one of the fundamental problems is to determine the equivalence of the structures under consideration—that is, to determine the morphisms in the relevant category. In complex analysis one is interested, for example, in the holomorphic equivalence of complex manifolds. The problem that we study here is somewhat more subtle: we wish to see to what extent a domain is determined by the group of its biholomorphic self-mappings.

In this paper we deal only with domains in $\mathbb{C}^n$, for even then the equivalence problem that we wish to study (described below) turns out to be highly non-trivial. It is a well-known fact that, if $n \geq 2$, then two given domains in $\mathbb{C}^n$ are most likely to be holomorphically inequivalent. This can be understood, for example, by examining the induced mapping between the boundaries of two given domains (in cases when the original mapping can be extended to a mapping between the closures of the domains). Poincaré was one of the first to notice the connection between the equivalence problem for domains and that for their boundaries. Considering domains with real analytic boundary in $\mathbb{C}^2$ and writing the equations of the boundaries in a special form, he showed that Taylor expansions that have different coefficients for monomials of sufficiently high degree define inequivalent boundaries. Thus Poincaré proved that there are infinitely many inequivalent domains [Po].

As shown in [Fe], any biholomorphic mapping between smoothly bounded strictly pseudoconvex domains *does* extend to a mapping between their closures, and therefore one can endeavor to find an analogue of Poincaré's argument in this case. It is possible, for example, to derive such an analogue for general real analytic strictly pseudoconvex hypersurfaces (as well as for any hypersurfaces with non-degenerate Levi form) from Moser's normal form for their defining functions [CM]. Moreover, it

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turns out that almost any two strictly pseudoconvex domains with only C^2-smooth boundary are inequivalent [GK2] (see also [GK1], [BSW]).

We have gone into some considerable detail on this point of generic domain inequivalence in order to emphasize the special nature of the function theory of several complex variables, and to stress the particular difficulties that we shall face. Note especially that there is no moduli space, nor anything like a Teichmüller space, for smoothly bounded domains in $\mathbb{C}^n$ (in fact this assertion has been proved in a strong sense in [LR]).

As a result of these considerations, we must restrict ourselves to special collections of domains that on the one hand are sufficiently small so that we may hope for a reasonable classification, and on the other hand are sufficiently large to possess a rich and interesting structure.

Let $D \subset \mathbb{C}^n$ be a bounded (or, more generally, Kobayashi-hyperbolic—see Section 1 for the definition) domain. Denote by $\mathrm{Aut}(D)$ the group (under composition) of holomorphic automorphisms of D. The group $\mathrm{Aut}(D)$ is a topological group with the natural topology of uniform convergence on compact subsets of D (the compact-open topology). It turns out that $\mathrm{Aut}(D)$ can be given the structure of a Lie group whose topology agrees with the compact-open topology (see [Kob1]). Many abstract Lie groups can be realized as the automorphism groups of bounded domains in complex space [SZ], [BD], [TS], but in this paper we deal only with domains for which $\mathrm{Aut}(D)$ is "large enough".

More precisely, we consider the class of domains for which $\mathrm{Aut}(D)$ is *non-compact*. By a classical theorem of H. Cartan (see [N]), for a bounded domain this condition is equivalent to the non-compactness of every orbit of the action of $\mathrm{Aut}(D)$ on D (which is in fact equivalent to the existence of only one non-compact orbit). For example, any homogeneous domain (i.e. domain on which $\mathrm{Aut}(D)$ acts transitively) has non-compact automorphism group. The study of bounded homogeneous domains was initiated by É. Cartan [Car] and eventually led to their complete classification [P-S] (for the more general case of complex spaces see e.g. [HO]). The technique by which these classifications were obtained is mostly algebraic. We will, however, be more interested in geometric and analytic methods that have been developed under the additional hypothesis of a certain regularity of the boundary of the domain (local or global C^∞-smoothness in many cases). The regularity of the boundary is indeed a crucial component of all the considerations below; to illustrate its importance, we only mention here that for homogeneous domains with C^2-smooth boundary the classification in [P-S] turns into a single domain, namely the unit ball. We also note that, when the boundary smoothness is less than C^2, then many of the basic ideas in this subject break down (see [GK2]).

Section 1 contains basic definitions, notation and elementary background material. The reader already familiar with this subject may safely skip Section 1 and refer back to it as needed. We begin our survey in Section 2 with the now classical Ball Characterization Theorem for strictly pseudoconvex domains; this is the first main result in the subject (from the point of view that we wish to promulgate); it, in turn, led to the Greene/Krantz theorems (and their generalizations) that were the first attempts to obtain a general result for weakly pseudoconvex domains. Section 3 is built around the domains of Bedford/Pinchuk. It is quite a plausible conjecture that these domains give a complete classification of smoothly bounded domains with non-compact automorphism group.

The techniques in Section 3 clearly show the importance of the hypothesis of finiteness of type in the sense of D'Angelo [D'A1] of the boundary of the domain at the boundary orbit accumulation points (by definition, a point $q \in \partial D$ is a boundary orbit accumulation point for the action of $\mathrm{Aut}(D)$ on D if there exist a point $p \in D$ and a sequence $\{f_j\} \subset \mathrm{Aut}(D)$ such that $f_j(p) \to q$ as $j \to \infty$). The conjecture that finiteness of type always obtains at the boundary orbit accumulation points of a smoothly bounded domain is known as the "Greene/Krantz conjecture" and is discussed in Section 4. Another hypothesis important for many results in Section 3 is the pseudoconvexity of the boundary near a boundary orbit accumulation point. It is also discussed in Section 4.

In Section 5 we deal with properties of the boundary orbit accumulation set (the set of all boundary orbit accumulation points) as a whole, in particular, certain extremal properties of some invariants of the boundary of the domain. In Section 6 we consider domains with less than C^∞-regularity of the boundary (e.g. finitely smooth or piecewise smooth) and also unbounded domains. Some of the results for unbounded domains are localizations of those mentioned in the preceding sections, but some of them are essentially global, and the domain is then required to be Kobayashi-hyperbolic. For domains with rough boundary, the results included in this section lead, in particular, to an analogue of the Bedford/Pinchuk domains in the finitely smooth case. Note again that, when the domain under study has *extremely rough boundary*—say fractal boundary—then the classification problem appears to be intractable (see [Kra4]).

To set the tone for this article, we now present four examples of domains in $\mathbb{C}^2$ which, taken together, tend to suggest some of the subtlety and beauty of the subject. They are the domains

$$\begin{aligned} B^2 &:= \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}, \\ E_{1,2} &:= \{(z_1, z_2) : |z_1|^2 + |z_2|^4 < 1\}, \\ E_{2,2} &:= \{(z_1, z_2) : |z_1|^4 + |z_2|^4 < 1\}, \\ E_{1,\infty} &:= \{(z_1, z_2) : |z_1|^2 + 2 \cdot e^{-1/|z_2|^2} < 1\}. \end{aligned}$$

The domain B^2 is the unit ball, and has transitive automorphism group—this follows, for example, from the explicit description of $\mathrm{Aut}(B^2)$ (see [Ru] or [Kra3]).

The domain $E_{1,2}$ has non-compact automorphism group; to wit, the automorphisms

$$(z_1, z_2) \longmapsto \left(\frac{z_1 - a}{1 - \overline{a} z_1}, \frac{\sqrt[4]{1 - |a|^2} z_2}{\sqrt{1 - \overline{a} z_1}} \right) , \qquad |a| < 1 ,$$

form a non-compact set of automorphisms of $E_{1,2}$. As the parameter a approaches -1, the above family moves any interior point of $E_{1,2}$ to the boundary point $(1, 0)$ which is therefore a boundary orbit accumulation point for $\mathrm{Aut}(E_{1,2})$. The automorphism group of $E_{1,2}$ cannot be transitive, because then $E_{1,2}$ would be holomorphically equivalent to the unit ball (by the Ball Characterization Theorem–Theorem 2.1 below), but it is not (for instance, by a theorem of Bell [Bel1]).

The domain $E_{2,2}$ has *compact* automorphism group. The assertion is not entirely obvious. This example differs from the preceding one in that the boundary of $E_{2,2}$ has *two* orthogonal circles of weakly pseudoconvex points (see Section 1 for the definition), while $E_{1,2}$ has just one. Any automorphism of $E_{2,2}$ must **(i)** extend smoothly to the boundary (see e.g. [Bel1]) and **(ii)** take weakly pseudoconvex points to weakly pseudoconvex points. The two circles of weakly pseudoconvex points therefore serve to harness any fixed compact subset of $E_{2,2}$; in particular, they prevent any orbit from accumulating at a point in the boundary. Thus, the automorphism group of $E_{2,2}$ is compact.

The domain $E_{1,\infty}$ also has compact automorphism group. This is the most subtle example of all. The questions that it raises will be the focus of much of the rest of the present survey. Briefly, $E_{1,\infty}$ has compact automorphism group for the following reason. If the automorphism group is non-compact then some orbit must accumulate at the boundary. If it accumulates at a point of the form $q = (q_1, q_2)$ with $q_2 \neq 0$ then q is a point of strong pseudoconvexity (see Section 1 for the definition). It then follows from the Ball Characterization Theorem that $E_{1,\infty}$ is holomorphically equivalent to the unit ball—which it is not. If instead the orbit accumulates at a point of the form $q = (q_1, 0)$, then q is infinitely flat in a sense to be made precise in the next section. It turns out (and there is a general conjecture to this effect—see Section 4) that infinitely flat points in this sense *cannot* be boundary orbit accumulation points. Further details on this example can be found in [GK7]. An alternative proof of compactness of $\mathrm{Aut}(E_{1,\infty})$ follows from Theorem 3.6 in Section 3 where we discuss the case of *Reinhardt domains*. Note that the domains $E_{2,2}$ and $E_{1,\infty}$ are *not* holomorphically equivalent (this follows, for instance, from [BKU]).

We have exhibited four domains of the same topological type—indeed the closure of each one is diffeomorphic to the closure of each of the others—yet with strikingly different holomorphic automorphism group characteristics (note that these domains are pairwise holomorphically inequivalent). The reasoning that we have sketched in this brief discussion sets the tone for the arguments that we shall present in the rest of this paper.

Before proceeding, we note that the reader may find it useful to compare the present paper with the earlier surveys [GK5], [Kra2], [Kra4].

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1. Preliminaries

In this section we give the definitions of the main concepts and some of the facts that we use later in the paper. We shall not discuss them here in any detail, nor shall we make any historical remarks; we refer the reader to [Kra3] for additional information and background.

Although it may be possible to profitably study manifolds with non-compact automorphism group, in this paper we restrict attention to domains in $\mathbb{C}^n$, where by a *domain* we mean a connected open set. Our domains are usually, but not always, bounded; we generally denote them by D or Ω.

A *holomorphic function* on a domain D is a function

$$\begin{array}{rcl} f : D & \to & \mathbb{C} \\ (z_1, \ldots, z_n) & \mapsto & f(z_1, \ldots, z_n) \end{array}$$

that is holomorphic in each variable separately. Such a function is automatically C^∞-smooth as a function of the $2n$ real variables $x_1, y_1, \ldots, x_n, y_n$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$. Such a function also has a locally convergent power series expansion near every point of D. If D_1, D_2 are domains in $\mathbb{C}^n$ then a *holomorphic mapping* of D_1 to D_2 is a mapping

$$F : D_1 \to D_2$$

such that $F(z_1, \ldots, z_n) = \big(f_1(z_1, \ldots, z_n), \ldots f_n(z_1, \ldots, z_n)\big)$ where each f_j is a holomorphic function. We say that F is *biholomorphic* if it is one-to-one and onto. A biholomorphic mapping has an inverse which is automatically biholomorphic itself. Two domains are called *holomorphically equivalent* if there is a biholomorphic mapping from one domain onto the other.

The collection of biholomorphic mappings of a domain D onto itself (often termed *biholomorphic self-maps* or *automorphisms*) of D clearly forms a group under composition of mappings. We denote this group by $\mathrm{Aut}(D)$. This group is given the topology of uniform convergence on compact subsets of D (the compact-open topology). So equipped, $\mathrm{Aut}(D)$ turns out to be a real Lie group when D is bounded. If the group is positive dimensional, then it is never a complex Lie group for any bounded D (see [Kob1]).

In the complex plane, a bounded domain with C^1-smooth boundary that is finitely connected (with connectivity at least one) has compact automorphism group—in fact if the connectivity is at least two then the automorphism group is finite (see [GK5] for details). The only bounded planar domain with C^1-smooth boundary and non-compact automorphism group is, up to a biholomorphism, the unit disc (see [Kra1]).

As we will see below, in dimensions 2 and higher, the collection of domains with non-compact automorphism group and regular boundary is much bigger. For the most part, our discussion will center on domains with *smooth boundary.* Let $D \subset \mathbb{C}^n$ be a domain. In a neighborhood U of a fixed point $p \in \partial D$ we can write

$$D \cap U = \{z \in U : \rho(z) < 0\}.$$

Such a function ρ is called a *defining function for D near p.* We say that, for $1 \leq k \leq \infty$, D has C^k-smooth or real analytic boundary near p if there is a defining function ρ for D near p which is, respectively, either C^k-smooth or real analytic and $\nabla \rho \neq 0$ on ∂D. The boundary is said to be globally C^k-smooth or real analytic if it is such at every point. When the boundary is globally C^k-smooth then it is easy to patch together local defining functions to obtain a single global defining function for the entire boundary. From now on, when speaking about defining functions of domains with smooth boundary, we will be assuming that these functions satisfy the conditions just discussed.

It is natural in our studies to pay special attention to *pseudoconvex domains* or *domains of holomorphy.* A domain D is called a domain of holomorphy if it is the natural domain of existence for some holomorphic function; in other words, if there exists a function holomorphic in D and such that it cannot be holomorphically continued past any boundary point of D. A more technical equivalent definition is as follows: a domain D is a domain of holomorphy if, for any compact set $K \subset D$, its *holomorphic hull* $\hat{K} := \{z \in D : |f(z)| \leq \max_{\zeta \in K}|f(\zeta)|$, for any f holomorphic in $D\}$ is also compact in D.

If D has at least C^2-smooth boundary near $p \in \partial D$, then ∂D is said to be *pseudoconvex at* p if there is a defining function ρ for D near p such that

$$\mathcal{L}_\rho(p)(w,w) := \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq 0 \tag{1.1}$$

for all $w := (w_1, \ldots, w_n) \in T_p^c(\partial D)$; here $T_p^c(\partial D)$ is the *complex tangent space to* ∂D *at* p, which is the maximal complex subspace of the ordinary real tangent space $T_p(\partial D)$:

$$T_p^c(\partial D) := \left\{ (w_1, \ldots, w_n) : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0 \right\}.$$

We call $p \in \partial D$ a point of *strong* or *strict* pseudoconvexity if the inequality (1.1) is strict for non-zero $w \in T_p^c(\partial D)$. The Hermitian form $\mathcal{L}_\rho(p)$ defined in (1.1) is called the *Levi form of* ∂D *at* p . It depends on the defining function ρ and is defined up to multiplication by a positive constant; therefore, the signs of the eigenvalues of $\mathcal{L}$ do *not* depend on the choice of ρ. In particular, the notions of pseudoconvexity and strict pseudoconvexity at p do *not* depend on ρ.

It turns out that a domain with C^2-smooth boundary is a domain of holomorphy if and only if its boundary is pseudoconvex at every point. A pseudoconvex domain whose boundary is strictly pseudoconvex at every point is called *strictly pseudoconvex.*

The other extremal situation—in contrast with strict pseudoconvexity—is when the Levi form is identically zero in a neighborhood of $p \in \partial D$. In this case ∂D is called *Levi-flat near* p and is then foliated near p by complex submanifolds of dimension $n-1$; conversely, if ∂D admits such a foliation, it is Levi-flat (see e.g. [T]).

We next turn to the notion of *type* in the sense of D'Angelo for C^∞-smooth real hypersurfaces in $\mathbb{C}^n$ [D'A1]. The type measures the order of tangency of (possibly singular) holomorphic curves with the hypersurface at a given point. Let $D \subset \mathbb{C}^n$ be a domain with C^∞-smooth boundary and let $p \in \partial D$. Then the type $\tau(p)$ of ∂D at p is defined as

$$\tau(p) := \sup_F \frac{\nu(\rho \circ F)}{\nu(F)},$$

where ρ is a defining function of D near p, the supremum is taken over all holomorphic mappings F defined in a neighborhood of $0 \in \mathbb{C}$ into $\mathbb{C}^n$ such that $F(0) = p$, and

$\nu(\phi)$ is the order of vanishing of a function ϕ at the origin. The boundary ∂D is said to be of *finite type at p* if $\tau(p) < \infty$. The domain D is a *domain of finite type* if ∂D is of finite type at every point. It is an important fact that, if D is a bounded domain of finite type, then the type is uniformly bounded on ∂D; this last fact follows from a weak semi-continuity property of τ (see [D'A2]). Examples of domains of finite type are bounded domains with real analytic boundary [D'A2], [DF], [L]—though in many ways these examples are atypically simple. We also note that the boundary of a domain is of finite type at the points of strict pseudoconvexity. Occasionally we will be using a weaker condition than that of finite type: we say that ∂D is *variety-free at $p \in \partial D$* if ∂D does not contain positive-dimensional complex varieties passing through p.

Domains of finite type are important for function theory. They have many of the attractive properties of strongly pseudoconvex domains; in particular, biholomorphic mappings of bounded domains of finite type extend to diffeomorphisms of the closures.

In this paper we mainly consider bounded domains. However, we will see that some of the results and techniques can be generalized to *Kobayashi-hyperbolic* domains. Hyperbolicity is geometrically a natural generalization of boundedness and is defined in terms of the *Kobayashi pseudometric*. Let M be a complex manifold, $p \in M$, $v \in T_p(M)$. The *Kobayashi pseudonorm* of v is the quantity

$$k(p,v) := \inf_F \left\{ \frac{1}{r} \right\},$$

where the infimum is taken over all holomorphic mappings F from discs $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$ to M such that $F(0) = p$, $F'(0) = v$. For a connected M the Kobayashi pseudometric $K(p,q)$, $p, q \in M$, can now be defined as

$$K(p,q) := \inf_\gamma \int_0^1 k(\gamma(t), \gamma'(t))\, dt,$$

where the infimum is taken over all smooth paths $\gamma : [0,1] \to M$ that join p and q [PoSh]. The Kobayashi pseudometric is a biholomorphic invariant and generalizes the Poincaré metric on the unit disc in $\mathbb{C}$.

A manifold M is called Kobayashi-hyperbolic if the Kobayashi pseudometric on M is in fact a metric. To illustrate that hyperbolicity is really a generalization of boundedness we mention here that hyperbolic manifolds possess the Liouville property which clearly holds for bounded domains: a holomorphic mapping from $\mathbb{C}$ into a hyperbolic manifold must be constant (see [Kob1] for an elegant discussion of the relation of hyperbolicity, the Liouville property, and curvature). A complex manifold M is called *complete hyperbolic*, if it is hyperbolic and, in addition, the Kobayashi metric on M is complete. Examples of complete hyperbolic manifolds are bounded strictly pseudoconvex domains in $\mathbb{C}^n$ [Gr].

Another invariant metric (which is going to be a Hermitian metric) that we mention here is the *Bergman metric*. Let $D \subset \mathbb{C}^n$ be a domain. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis in the space of holomorphic square integrable functions on D. The function

$$B(p,q) := \sum_{j=1}^{\infty} \phi_j(p)\overline{\phi_j(q)}, \qquad p,q \in D$$

is called the *Bergman kernel* of D. The Bergman metric is then defined as

$$ds_B^2 := \sum_{j,k=1}^{n} \frac{\partial^2 B(z,z)}{\partial z_j\, \partial \overline{z}_k} dz_j\, d\overline{z_k}.$$

We will also need some invariant volume elements. Let $D \subset \mathbb{C}^n$ be a domain and $p \in D$. The *Carathéodory volume element of D at p* is defined to be

$$V^C(p) := \sup_F\{|\det F'(p)|\},$$

where the supremum is taken over all holomorphic mappings F from D to the unit ball $B^n := \{(z_1,\dots,z_n) : |z_1|^2 + \cdots + |z_n|^2 < 1\}$ such that $F(p) = 0$. Likewise, the *Eisenman-Kobayashi volume element of D at p* is defined by

$$V^K(p) := \inf_F \left\{ \frac{1}{|\det F'(0)|} \right\},$$

where the infimum is taken over all holomorphic mappings F from B^n into D such that $F(0) = p$. The quotient $\frac{V^C}{V^K}$ is a biholomorphic invariant and will be called the *C/K-invariant* (see [Kra3], [GK5] for a detailed discussion of this invariant and its uses).

2. The Ball Characterization Theorem and Theorems of Greene/Krantz Type

The first result that we mention in this section is the famous Ball Characterization Theorem of Bun Wong and Rosay:

THEOREM 2.1 ([Ro]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with $Aut(D)$ non-compact. Assume that there exists a boundary orbit accumulation point in a neighborhood of which ∂D is C^2-smooth and strictly pseudoconvex. Then D is holomorphically equivalent to the unit ball B^n.*

This result was first proved in [W1] for globally strictly pseudoconvex domains. An alternative proof (in the case of C^∞-smooth boundary) based on an analysis of the holomorphic sectional curvature of the Bergman metric was obtained in [Kl] (for related results see also [KY]).

It is important here to realize that the Levi geometry of the boundary orbit accumulation point completely determines the entire domain. Thus (micro)local geometric information at the boundary orbit accumulation point gives global geometric information. This theme will be one of the unifying ideas in the remainder of the present paper. The original approach of Bun Wong (to construct a special metric using the hypotheses), the Bergman geometry approach of Klembeck, and the function-theoretic approach of Rosay all manifest this local/global dialectic in different ways.

Theorem 2.1 clearly implies the following alternative characterization of B^n (cf. [P-S]):

Corollary 2.2. *If $D \subset \mathbb{C}^n$ is a bounded homogeneous domain with C^2-smooth boundary, then D is biholomorphically equivalent to B^n.*

Proof: Any C^2-smooth bounded domain in Euclidean space has a boundary point that is strongly convex, hence strongly pseudoconvex (just take a fixed point p in space that is far away from the domain and then take a point in the boundary of D that is at the maximal distance from p). Now apply Theorem 2.1. □

Here is another way, besides non-compactness or homogeneity, to think about the concept of "large automorphism group". It turns out that one can use only the isotropy group of a single point to characterize B^n in a much more general situation. Namely, for a complex manifold M and $p \in M$, let $I_p := \{f \in \mathrm{Aut}(M) : f(p) = p\}$ be the isotropy group of p.

THEOREM 2.3 ([GK3]). *If M is a connected, non-compact manifold of complex dimension n, and if there is a point $p \in M$ such that for some compact subgroup $H \subset I_p$ the set $\{df(p) : f \in H\}$ acts transitively on real tangent directions at p, then M is holomorphically equivalent to B^n.*

We would be remiss not to mention that Bland, Duchamp and Kalka [BDK] have obtained an analogue of Theorem 2.3 when the manifold is compact. They have also weakened the hypothesis from transitivity on real tangent directions to transitivity on complex tangential directions (this weakening applies both in the compact and in the non-compact case). Then the conclusion is that the manifold is complex projective space. Their techniques are different from those in [GK3], and well worth learning. For related results see also [HO], [MN] and a discussion in [GK5].

The above results suggest several possible directions that one may follow to endeavor to obtain characterizations for different classes of domains with "large" automorphism group. In this survey, we concentrate on domains whose automorphism group is non-compact, thus the scope of the present paper is to explore the direction given by Theorem 2.1.

A natural generalization of this theorem would come from replacing the assumption of strict pseudoconvexity of ∂D near a boundary orbit accumulation point by a weaker condition, e.g. weak pseudoconvexity. The first results in this direction are due to Greene and Krantz [GK4] and concern the characterization of more general domains, namely, complex ellipsoids of the form

$$E_m := \{(z_1, \dots, z_n) : |z_1|^2 + \dots + |z_{n-1}|^2 + |z_n|^{2m} < 1\},$$

with m a positive integer.

THEOREM 2.4 ([GK4]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with $Aut(D)$ non-compact and C^{n+1}-smooth boundary. Suppose that for some boundary orbit accumulation point p, ∂D near p coincides with ∂E_m near the point $p_0 = (1, 0, \dots, 0) \in \partial E_m$ up to a local biholomorphism that takes p into p_0. Then D is holomorphically equivalent to E_m.*

If in Theorem 2.4 one allows ∂D to be C^{2n+2}-smooth, then the condition of local coincidence of ∂D and ∂E_m up to a local biholomorphism can be replaced by the condition that, for some local biholomorphism f defined near p and such that $f(p) = p_0$, $f(\partial D)$ and ∂E_m osculate to order $2m$ near p_0 (see [GK5]).

The proof of Theorem 2.4 uses the C/K-invariant that was also an important tool in [Ro], [W1]. Note that, historically, the first applications of the C/K-invariant to the study of domains with non-compact automorphism group were based on Bun Wong's results, e.g.: a complete hyperbolic bounded domain D is biholomorphically equivalent to the ball if and only if there is a point $p \in D$ such that $[C/K](p) = 1$ [W1]. A different proof of Theorem 2.4 based on the analysis of an invariant arising from the Bergman metric can be found in [GK6].

Kodama in [Kod4] considered the more general domains

$$E(k, \alpha) := \left\{ (z_1, \ldots, z_n) : \sum_{j=1}^{k} |z_j|^2 + \left(\sum_{j=k+1}^{n} |z_j|^2 \right)^{\alpha} < 1 \right\},$$

where $1 \leq k \leq n$ and $\alpha > 0$. Using methods that avoid the $\overline{\partial}$-technique (that was needed to obtain the main technical result of [GK4]—see Lemma 4.3 there), Kodama proved a version of Theorem 2.4 for $E(k, \alpha)$ in place of E_m without assuming any global regularity of ∂D. In his theorem, Kodama assumed that $p_0 \in \partial D$ and that the boundaries ∂D and $\partial E(k, \alpha)$ actually coincide near p_0. In this result, one can also allow ∂D and $\partial E(k, \alpha)$ to osculate near p_0 rather than literally coincide, but then a non-trivial extra condition on the way an orbit of $\mathrm{Aut}(D)$ approaches p_0 is needed.

Another generalization of Theorem 2.4 is also due to Kodama. Let

$$E_{m_1,\ldots,m_n} := \{(z_1, \ldots, z_n) : |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 1\},$$

where the m_j are positive integers.

THEOREM 2.5 ([Kod5]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with non-compact automorphism group, and $p \in \partial D$ a boundary orbit accumulation point for $Aut(D)$. Suppose that the boundary of D near p coincides with that of $E_{m_1,\ldots,m_n}$ near a point $p_0 \in \partial E_{m_1,\ldots,m_n}$, up to a local biholomorphism that takes p into p_0. Then D is holomorphically equivalent to $E_{m_1,\ldots,m_n}$.*

This result is implicit in [Kod5]. There, the local equivalence is assumed to be the identity, and the conclusion is that D is literally equal to $E_{m_1,\ldots,m_n}$. But an inspection of the proof shows that local holomorphic equivalence suffices to establish the conclusion of global holomorphic equivalence to $E_{m_1,\ldots,m_n}$ (see [GK7]). Theorem 2.5 for domains with C^∞-smooth boundary was obtained independently in [Ber1]. Next, in [Kod7] (see also [Kod6]) the version of Theorem 2.5 as in [Kod5] (stating the literal equality of the domains) was extended to the case where the m_i are arbitrary positive real numbers (of course when the m_i are not integral then the boundary is not C^∞-smooth). Further, in [Kod8] this version was proved for generalized complex ellipsoids of the form

$$E_{n_1,\ldots,n_s;m_1,\ldots,m_s} := \left\{ (z_1, \ldots, z_s) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s} : \|z_1\|^{2m_1} + \cdots + \|z_s\|^{2m_s} < 1 \right\},$$

where n_i, m_i are positive integers, $n_1 + \cdots + n_s = n$ and $\|z_i\|$ denotes the ordinary norm of the vector z_i in $\mathbb{C}^{n_i}$. In the situation where the m_i are arbitrary positive real numbers, an analogue of Theorem 2.5 for $E_{n_1,\dots,n_s;m_1,\dots,m_s}$ was obtained in [KKM].

Further, Kim in [Ki2] (see also [Ki1]) obtained a result along the lines of Theorems 2.4 and 2.5 for domains satisfying a special local condition called *Condition* (L). Namely, a bounded domain $D \subset \mathbb{C}^n$ with non-compact automorphism group is said to satisfy Condition (L) at a boundary orbit accumulation point $p \in \partial D$ if ∂D is real analytic near p, of finite type $2k$ at p in the sense of D'Angelo (k is a positive integer), and ∂D near p is convex up to a local biholomorphism. The proof of Kim is in the spirit of the convex scaling technique due to Frankel [Fr] (see also [Ki4]). As we will see later, many results below were obtained by using a different scaling technique due to Pinchuk [Pi1], [Pi3]. Since these two scaling techniques are important for the current development of the subject, at the end of our survey we provided a brief tutorial in the scaling methods (for a more detailed discussion and comparison of these methods see [Ki4]).

3. The Bedford/Pinchuk Domains and Related Results

In the preceding section we listed the results that extend the Ball Characterization Theorem primarily from the point of view of the methods and ideology suggested by its proof; in particular, we emphasized localization principles that followed the work in [GK4]. In this section we turn to direct generalizations of this theorem obtained by completely different techniques. Throughout this section all domains will be assumed to be smoothly bounded, i.e., bounded and having C^∞-smooth boundary. The first result here is due to Bedford and Pinchuk.

THEOREM 3.1 ([BP2]). *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type with non-compact automorphism group such that the Levi form of ∂D has no more than one zero eigenvalue at any point. Then D is holomorphically equivalent to a complex ellipsoid E_m with m a positive integer.*

Note that the condition on the rank of the Levi form is not a restriction in complex dimension 2. This condition is the first step towards allowing the domain to be weakly pseudoconvex rather than strictly pseudoconvex; it says that the degeneracy of the Levi form that may occur is the least possible. However, in contrast with Theorem 2.1, Theorem 3.1 is essentially *non-local*.

Theorem 3.1 was first proved in [BP1] for domains in $\mathbb{C}^2$ with real analytic boundary (see also [Bel3]). We note here that real analyticity implies *a fortiori* the finite type condition [D'A2], [DF], [L]. We also mention here that, before the paper [BP2] appeared, Bell and Catlin noticed that in [BP1] real analyticity can be replaced by the finite type condition [BeCa].

We will now give important examples of smoothly bounded domains with non-compact automorphism group that are also due to Bedford and Pinchuk.

EXAMPLE 3.2 ([BP2]). Fix positive integers $m_2, \dots, m_n$ and, for a multi-index $K = (k_2, \dots, k_n)$, define its *weight* by $\mathrm{wt}(K) = \sum_{j=2}^{n} \frac{k_j}{m_j}$. Consider real polynomials of the form

$$P(\tilde{z},\overline{\tilde{z}})=\sum_{\mathrm{wt}(K)=\mathrm{wt}(L)=1} a_{KL}\tilde{z}^K\overline{\tilde{z}}^L,$$

where $\tilde{z}:=(z_2,\dots,z_n)$, $a_{KL}\in\mathbb{C}$ and $a_{KL}=\overline{a_{LK}}$. For any such polynomial we define a domain in $\mathbb{C}^n$ by

$$D_P:=\left\{(z_1,\tilde{z}):|z_1|^2+P(\tilde{z},\overline{\tilde{z}})<1\right\}. \tag{3.1}$$

A domain D_P of the form (3.1) is bounded if and only if the section $D\cap\{z_1=0\}$ is a bounded domain in $\mathbb{C}^{n-1}$. In particular, if D_P is bounded, then $P\geq 0$. Further, $\mathrm{Aut}(D_P)$ is non-compact since it contains the mappings

$$\begin{aligned} z_1 &\mapsto \frac{z_1-a}{1-\overline{a}z_1},\\ z_j &\mapsto \frac{(1-|a|^2)^{\frac{1}{2m_j}}z_j}{(1-\overline{a}z_1)^{\frac{1}{m_j}}}, \qquad j=2,\dots,n, \end{aligned}$$

where $|a|<1$ (note that, if D_P is bounded, then $|z_1|<1$ in D_P). Another way of checking the non-compactness of $\mathrm{Aut}(D_P)$ is to notice that D_P is holomorphically equivalent to the domain

$$\left\{(z_1,\tilde{z}):\mathrm{Re}z_1+P(\tilde{z},\overline{\tilde{z}})<1\right\},$$

which is invariant under the translations

$$\begin{aligned} z_1 &\mapsto z_1+it, \qquad t\in\mathbb{R},\\ \tilde{z} &\mapsto \tilde{z}. \end{aligned} \tag{3.2}$$

□

In their next paper [BP3] Bedford and Pinchuk obtained the following result.

THEOREM 3.3 ([BP3]). *Any convex smoothly bounded domain of finite type in $\mathbb{C}^n$, having non-compact automorphism group, is holomorphically equivalent to a bounded domain of the form (3.1).*

The approach of Bedford and Pinchuk involves two steps. For example, the proof of Theorem 3.1 goes as follows. In the first step they use the method of scaling introduced in [Pi1] (see also [Pi3]) to show that the domain D in consideration is holomorphically equivalent to a domain Ω of the form

$$\Omega=\{(z_1,\tilde{z}):\mathrm{Re}z_1+Q(\tilde{z},\overline{\tilde{z}})<0\},$$

where Q is a polynomial. The domain Ω has a non-trivial holomorphic vector field since it is invariant under translations (3.2). In the second step this vector field is transported back to D, the result is analyzed at the parabolic fixed point, and this information is used to determine the original domain. In the second step scaling is applied two more times. The first scaling is needed to show that the smallest "weight" involved in the vector field is either 1 or $\frac{1}{2}$. Next, it is shown that the orbit is well-behaved as $t\to\pm\infty$ for each of these weights, and the final rescaling is carried out along the parabolic orbit. The case of weight 1 is the most difficult one in the final rescaling procedure.

There has been also certain progress, by other authors, on the first step of the above procedure of Bedford/Pinchuk. The following completely local result in dimension 2 (not requiring even the boundedness of the domain) was obtained by Berteloot in [Ber3] (see also [BeCo], [Ber2]).

THEOREM 3.4 ([Ber3]). *Let $D \subset \mathbb{C}^2$ be a domain, and $p \in \partial D$ a boundary orbit accumulation point for $Aut(D)$. Assume that ∂D is pseudoconvex and of finite type near p. Then D is holomorphically equivalent to a domain of the form*

$$\{(z_1, z_2) : Re\, z_1 + P(z_2, \overline{z_2}) < 0\},$$

where P is a homogeneous subharmonic polynomial without harmonic terms.

For convex domains Theorem 3.4 was recently generalized to all dimensions in [Ga]. Further, using the convex scaling technique of Frankel [Fr] Kim in [Ki2] (see also [Ki1]) obtained a related result for domains satisfying Condition (L) (see Section 2 for the definition).

The techniques relying on either of the two scaling principles mentioned above (see the Appendix at the end of this paper) seem to require the following two additional hypotheses: pseudoconvexity (or even convexity) and finiteness of type (or analyticity) of the boundary. It is interesting to notice here, however, that in their recent paper that we received while this survey was being prepared, Bedford and Pinchuk managed to eliminate the pseudoconvexity assumption in dimension 2 and prove the next remarkable theorem.

THEOREM 3.5 ([BP4]). *Let $D \subset \mathbb{C}^2$ be a bounded domain with non-compact automorphism group and real analytic boundary. Then D is holomorphically equivalent to a complex ellipsoid E_m where m is a positive integer.*

The above results give one the hope that any smoothly bounded domain with non-compact automorphism group should be holomorphically equivalent to a domain of the form (3.1). Many experts believe that this is true without extra assumptions such as the finiteness of type and pseudoconvexity. We will now cite a result that confirms this point of view for *Reinhardt domains*, i.e. domains invariant under the rotations

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R}, \qquad j = 1, \dots, n.$$

Note first that Reinhardt domains of the form (3.1) are given by

$$\left\{ (z_1, \tilde{z}) : |z_1|^2 + \sum_{\mathrm{wt}(K)=1} a_K \tilde{z}^K \overline{\tilde{z}}^K < 1 \right\}, \tag{3.3}$$

where $a_K \in \mathbb{R}$.

THEOREM 3.6 ([FIK2]). *Any smoothly bounded Reinhardt domain in $\mathbb{C}^n$ with non-compact automorphism group is holomorphically equivalent to a domain of the form (3.3), and the equivalence is given by dilations and a permutation of coordinates.*

To the best of our knowledge, Theorem 3.6 at the moment is the only classification result for a fairly large class of domains with non-compact automorphism group that does not require the hypotheses of pseudoconvexity and finiteness of type. We should note, however, that there are smoothly bounded domains with non-compact automorphism group that are essentially non-Reinhardt. Namely, it is shown in [FIK1] that, among bounded domains of the form (3.1), there are some that are not holomorphically equivalent to any Reinhardt domain whatsoever. The proofs in [FIK1], [FIK2] are based on the description of the automorphism groups of bounded Reinhardt domains due independently to Kruzhilin [Kru] and Shimizu [Sh] (Kruzhilin considered the more general case of Kobayashi-hyperbolic domains). The descriptions of Kruzhilin and Shimizu generalize that due to Sunada for Reinhardt domains containing the origin [Su]. It is appropriate to note here that Kodama in [Kod1] used the description in [Su] to prove the following

THEOREM 3.7 ([Kod1]). *Let $D \subset \mathbb{C}^n$ be a bounded Reinhardt domain containing the origin. Suppose that there exists a compact subset $K \subset D$ such that $Aut(D) \cdot K = D$. Then D is holomorphically equivalent to a product of unit balls.*

Although the situation considered in [Kod1] is quite different from that in [FIK2], the effect is essentially the same: by using the explicit descriptions of the automorphism groups of Reinhardt domains, one can avoid imposing any extra conditions on the boundary.

4. The Greene/Krantz Conjecture and Pseudoconvexity at Boundary Orbit Accumulation Points

As we saw in the preceding section, many of the classification results for smoothly bounded domains with non-compact automorphism group were proved, in particular, under the hypothesis that the domain is of finite type. The local results in [Ber2], [Ber3], [Ga], [Ki1], [Ki2] and local considerations in [BP1]–[BP4] demonstrate the particular importance for the boundary of the domain to be of finite type at a boundary orbit accumulation point (note that by [D'A1]—also see [D'A2]—this implies that the boundary is of finite type in a neighborhood of the point). The Greene/Krantz conjecture states that this geometric condition should in fact always obtain.

Greene/Krantz Conjecture ([GK7]) *Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain with non-compact automorphism group. Then ∂D is of finite type at any boundary orbit accumulation point.*

The conjecture in its full generality is open. The classification in Theorem 3.6 confirms the conjecture for Reinhardt domains, but it would be desirable to have proofs supporting the conjecture (even in special cases) other than those given by explicit classification results. Below we give a theorem of such a kind due to Kim (see also an interesting special argument presented in [GK7]).

THEOREM 4.1 ([Ki5]). *Let $D \subset \mathbb{C}^n$ be a smoothly bounded convex domain with non-compact automorphism group, and $p \in \partial D$. Suppose that ∂D is Levi-flat in a neighborhood of p. Then p is not a boundary orbit accumulation point for $Aut(D)$.*

Note that, as we mentioned in Section 1 above, the Levi-flatness of ∂D near p is equivalent to the existence of a foliation of ∂D near p by complex submanifolds of dimension $n-1$. Thus, the relation of Theorem 4.1 to the Greene/Krantz conjecture is that ∂D does not admit such a foliation near any boundary orbit accumulation point. However, this is, of course, a much weaker statement compared to the conjecture itself. Also, the conjecture is believed to be true without any extra conditions such as the convexity that is required in Theorem 4.1.

Another important hypothesis used in many results cited in Section 3 is the hypothesis of pseudoconvexity near a boundary orbit accumulation point. The next theorem relates this local pseudoconvexity to the global pseudoconvexity of the domain.

THEOREM 4.2 ([GK6]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with non-compact automorphism group, and $p \in \partial D$ a boundary orbit accumulation point. Suppose that ∂D is C^∞-smooth near p and variety-free at p. Then local pseudoconvexity of ∂D near p implies that D is pseudoconvex.*

Note that the variety-free assumption in Theorem 4.2 (in the case of *globally* smoothly bounded domains) would follow from the Greene/Krantz conjecture.

However, a smoothly bounded domain with non-compact automorphism group in fact need not be globally pseudoconvex (but see Theorem 4.4 below). An example can be found among the Bedford/Pinchuk domains (3.1) (see [FIK2]).

EXAMPLE 4.3. Let D be the following smoothly bounded domain (of the form (3.1)) in $\mathbb{C}^3$:

$$D := \left\{ (z_1, z_2, z_3) : |z_1|^2 + |z_2|^4 + |z_3|^4 - \frac{3}{2}|z_2|^2|z_3|^2 < 1 \right\}.$$

Consider the boundary point $p = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt[4]{2}})$. The complex tangent space at p is

$$\{(w_1, w_2, w_3) : w_1 + 2^{3/4} w_3 = 0\},$$

and the Levi form at p is

$$\mathcal{L}(p)(w, w) = \frac{1}{2^{\frac{3}{2}}}(-3|w_2|^2 + 16|w_3|^2),$$

and thus is clearly not non-negative. Theorem 4.2 now implies that ∂D is not pseudoconvex in a neighborhood of any boundary orbit accumulation point (but it is *a fortiori* pseudoconvex *at* each boundary orbit accumulation point—see Theorem 4.4 below). □

We note here that in Example 4.3 the point p is *not* a boundary orbit accumulation point. In contrast, for boundary orbit accumulation points the following general fact holds:

THEOREM 4.4 ([GK6]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with non-compact automorphism group, and $p \in \partial D$ a boundary orbit accumulation point. Suppose that ∂D is C^2-smooth near p. Then ∂D is pseudoconvex at p.*

For domains with rough boundary a version of Theorem 4.4 remains true. One can say, for instance, that if $K \subset D$ is a compact set, then its holomorphic hull $\hat{K}$ cannot escape to the boundary at a boundary orbit accumulation point [GK6]. For discussions of Theorems 4.2 and 4.4 see also [Kra4].

To summarize, in this section we have discussed the two most common hypotheses on the boundary of a smoothly bounded domain near a boundary orbit accumulation point that occur in the literature: finiteness of type and pseudoconvexity. The first of these is believed to always be the case (and there are partial results to support this belief) and the second always holds *at* the boundary orbit accumulation point itself; however it is not true that the boundary should be pseudoconvex *in a neighborhood* of the boundary orbit accumulation point. As a result of these considerations (especially the second one), it is critical to have techniques that assume neither pseudoconvexity nor finite type at the outset; this, in particular, is what makes Theorem 3.5 mentioned above so important.

5. The Boundary Orbit Accumulation Set

In the preceding section we dealt with individual boundary orbit accumulation points. Here we will be interested in the collection of *all* boundary orbit accumulation points, i.e. the *boundary orbit accumulation set* as a whole. If D is a bounded domain with non-compact automorphism group then denote its boundary orbit accumulation set by $S(D)$. Very little is known about the topological and other properties of $S(D)$, except for the classes of domains for which there exists a complete classification as in Section 3 above. Here we give some results on $S(D)$ from [IK1], [H]. The proofs use the main theorem of [Bel2] and therefore require extendability of the automorphisms to the boundary of the domain; thus, in addition to being smoothly bounded, domains in this section are assumed to be pseudoconvex and of finite type (see [Kra3]).

THEOREM 5.1 ([IK1]). *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type with non-compact automorphism group. Suppose that $S(D)$ contains at least three points. Then $S(D)$ is a compact, perfect set and thus has the power of the continuum. Moreover, in this case, $S(D)$ is either connected, or else the number of its connected components is uncountable.*

It follows from [Z] that if D is a bounded, pseudoconvex domain which is in addition *algebraic*, i.e. given in the form $D = \{z \in \mathbb{C}^n : P(z) < 0\}$, where $P(z)$ is a polynomial such that $\nabla P \neq 0$ on ∂D, then the set $S(D)$ has only finitely many connected components. Therefore, for such domains, Theorem 5.1 now implies that either $S(D)$ contains only one or two points, or $S(D)$ is connected and has the power of the continuum.

Theorem 5.1 raises a number of natural questions: for example, can $S(D)$ be a one- or two-point set or can $S(D)$ look like a Cantor-type set (thus having uncountably many connected components)? Another question is whether the set $S(D)$ is always a smooth submanifold of ∂D. Note that, for instance, (the proof of) Theorem 3.6 shows that for a smoothly bounded Reinhardt domain, $S(D)$ is diffeomorphic to a sphere of odd dimension. The reference [GK2] gives an example of a domain with

$C^{1-\epsilon}$-smooth boundary, for which $S(D)$ has only two points. It seems plausible that this example can be modified, using a parabolic group of automorphisms, so that $S(D)$ has just one point. Using similar ideas, we also seem to be able to produce for any $k \geq 1$ a domain D with C^k-smooth (but not C^∞-smooth) boundary so that $S(D)$ has precisely two points. Indications are that the case of finite boundary smoothness will be different from the case of infinite boundary smoothness (see also Section 6).

If D is a smoothly bounded pseudoconvex domain of finite type, then each automorphism of D extends smoothly to the boundary. Therefore $\mathrm{Aut}(D)$ acts on the boundary, and the set $S(D)$ is invariant under that action. The following result shows that $S(D)$ is generically the smallest invariant subset of ∂D.

THEOREM 5.2 ([IK1]). *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type with non-compact automorphism group. Suppose that $A \subset \partial D$ is non-empty, compact and invariant under $Aut(D)$. Assume further that A is not a one-point subset of $S(D)$. Then $S(D) \subset A$.*

In particular, if $Aut(D)$ does not have fixed points in ∂D, then $S(D)$ is the smallest compact subset of ∂D that is invariant under $Aut(D)$.

We now list several corollaries of Theorem 5.2 regarding particular sets A. Let a domain D be as in the theorem. Fix $0 \leq k \leq n-1$ and denote by $L_k(D)$ the set of all points from ∂D where the rank of the Levi form of ∂D does not exceed k. Clearly, each set $L_k(D)$ is a compact subset of ∂D and is invariant under any automorphism of D. Let l_1 denote the minimal rank of the Levi form on ∂D and l_2 the minimal rank of the Levi form on $\partial D \setminus L_{l_1}(D)$.

Corollary 5.3 ([IK1], [H]). *Let D be as in Theorem 5.2. Then either*

(i) *$S(D) \subset L_{l_1}(D)$,*

or

(ii) *$L_{l_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset L_{l_2}(D)$.*

We note that Corollary 5.3 was proved earlier by Huang [H], and its proof in [H] also relies on the paper [Bel2].

One can further endeavor to prove a property analogous to Corollary 5.3 for the type $\tau(q)$, $q \in \partial D$, in the sense of D'Angelo. Indeed, denote by $T_k(D)$ the set of all points $q \in \partial D$ where $\tau(q)$ is at least k. We choose t_1 and t_2 such that $T_{t_1}(D) \neq \emptyset$, $t_2 < t_1$, and there exists a point of type t_2 in $\partial D \setminus T_{t_1}(D)$. Since τ is invariant under automorphisms of D, so is every set $T_k(D)$. However, the sets $T_k(D)$ do not have to be closed, as the type function τ may *not* be upper-semicontinuous on ∂D (see e.g. an example in [D'A2], p. 136). Therefore, for the type we only have a somewhat weaker result.

Corollary 5.4 ([IK1]). *Let D be as in Theorem 5.2. Then either*

(i) *$S(D) \subset \overline{T_{t_1}(D)}$,*

or

(ii) *$T_{t_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset \overline{T_{t_2}(D)}$.*

Notice that, loosely speaking, Corollaries 5.3 and 5.4 state respectively that the rank of the Levi form is *"minimal"* and the type is *"maximal"* along $S(D)$. The next corollary below states that, in this respect, the multiplicity function μ (see [D'A2], p. 145 for the definition) is analogous to the type function τ. The multiplicity μ is invariant under the extensions of automorphisms to ∂D and, for $q \in \partial D$, $\tau(q)$ is finite if and only if $\mu(q)$ is finite. In contrast with τ, however, the function μ *is* upper-semicontinuous on ∂D.

Analogously to what we have done above for the function τ, denote by $M_k(D)$ the set of all points $q \in \partial D$, where $\mu(q)$ is at least k and choose m_1 and m_2 such that $m_1 = \max_{q \in \partial D} \mu(q)$, $m_2 < m_1$, and there exists a point of multiplicity m_2 in $\partial D \setminus M_{m_1}(D)$. Because of the upper-semicontinuity and invariance of μ, each set $M_k(D)$ is a compact subset of ∂D that is invariant under $\mathrm{Aut}(D)$. This observation gives the following analogue of Corollary 5.4 for M_{m_1}, M_{m_2}.

Corollary 5.5 ([IK1]). *Let D be as in Theorem 5.2. Then either*

(i) $S(D) \subset M_{m_1}(D)$,

or

(ii) *$M_{m_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset M_{m_2}(D)$.*

Note that one can make a statement analogous to Corollary 5.5 for the multitype introduced in [Cat], since the multitype function is upper-semicontinuous with respect to lexicographic ordering.

It follows from Theorem 3.1 that, in complex dimension 2, for a smoothly bounded pseudoconvex domain of finite type, the rank of the Levi form is constant and minimal and the type is constant and maximal along $S(D)$ (cf. Corollaries 5.3 and 5.4). Theorem 3.6 implies that this also holds for smoothly bounded Reinhardt domains in any dimension. If we denote the minimal rank of the Levi form by k, one can see from the proof of Theorem 3.6 that for a smoothly bounded Reinhardt domain D, the real dimension of any orbit of the action of $\mathrm{Aut}(D)$ on D is at least $2(k+1)$. Moreover, there is precisely one orbit of minimal dimension $2(k+1)$ (see [Kra4] for a discussion of this phenomenon). This orbit approaches every point of $S(D)$ non-tangentially, whereas any other orbit approaches every point of $S(D)$ only along tangential directions. It would be very interesting to know if similar statements hold for more general domains. For example, the fact that there exists an orbit that approaches $S(D)$ non-tangentially would be very important for a proof of the Greene/Krantz conjecture (cf. [FW1] and Theorem 4 in [Ki5]). It also could be used to show that $S(D)$ is a smooth submanifold of ∂D. Generally speaking, the existence of non-tangential orbits to boundary orbit accumulation points—in any (even very weak) sense—is one of the main difficulties arising in the study of domains with non-compact automorphism groups.

6. More General Situations: Domains with Rougher Boundary and Unbounded Domains

In this section we give certain generalizations of some of the results mentioned above. More precisely, we will be interested in generalizations in two directions:

relaxing the condition of the regularity of the boundary and allowing domains to be unbounded.

First we consider domains with rough boundary. Note that in Section 2 we already mentioned the results from [Kod7] and [KKM] on characterization of generalized complex ellipsoids $E_{n_1,\dots,n_s;m_1,\dots,m_s}$ whose boundaries are not C^∞-smooth if the m_i are not integers.

The theorem below extends Corollary 2.2 to domains with piecewise smooth boundary. Recall that a bounded domain $D \subset \mathbb{C}^n$ is said to have *C^k-piecewise smooth* boundary, for $k \geq 1$, if ∂D is a $(2n-1)$-dimensional topological manifold and for some neighborhood U of ∂D there exist real functions $\rho_j \in C^k(U)$, $j = 1, \dots, m$, such that:

(i) $D \cap U = \{z \in U : \rho_j(z) < 0, j = 1, \dots, m\}$;

(ii) For any subset $\{j_1, \dots, j_r\} \subset \{1, \dots, m\}$ with $1 \leq j_1 < \dots < j_r \leq m$, one has $d\rho_{j_1} \wedge \dots \wedge d\rho_{j_r} \neq 0$ on $\cap_{s=1}^r S_{j_s}$, where $S_j := \{z \in U : \rho_j = 0\}$, $j = 1, \dots, m$.

The domain D is said to have *generic C^k-piecewise smooth boundary* if in the above definition one in addition has:

(iii) For any subset $\{j_1, \dots, j_r\} \subset \{1, \dots, m\}$ with $1 \leq j_1 < \dots < j_r \leq m$, one has $\partial\rho_{j_1} \wedge \dots \wedge \partial\rho_{j_r} \neq 0$ on $\cap_{s=1}^r S_{j_s}$, where ∂ means differentiation only with respect to holomorphic variables.

Roughly speaking, these rather technical conditions specify that the boundary consists of finitely many smooth pieces that have transversal crossings.

THEOREM 6.1 ([Pi2]). *If $D \subset \mathbb{C}^n$ is a bounded homogeneous domain with piecewise C^2-smooth boundary, then D is holomorphically equivalent to a product of unit balls.*

Theorem 6.1 was proved by applying the scaling method of Pinchuk that we mentioned in Section 3 above. Note that this method also gives a short proof of Theorem 2.1 and therefore Corollary 2.2 (see [Pi3], [Ki6] and the Appendix at the end of this survey). Further, it was shown in [CS] that for a bounded domain with piecewise C^2-smooth generic boundary such that every set $D_j := \{z \in U : \rho_j(z) < 0\}$ is strictly pseudoconvex, the non-compactness of $\mathrm{Aut}(D)$ implies that D is in fact equivalent to B^n. In the case of non-tangential approach to boundary orbit accumulation points this result was obtained in [Kod3] (see also [Kod2]).

The following result is due to Kim and requires the extra hypotheses of convexity and Levi-flatness (the latter means that each of the sets S_j from the above definition is Levi-flat).

THEOREM 6.2 ([Ki3]). *Let $D \subset \mathbb{C}^n$ be a bounded, convex domain with piecewise C^∞-smooth Levi-flat boundary and non-compact automorphism group. Then D is holomorphically equivalent to the product of the unit disc and a convex domain in $\mathbb{C}^{n-1}$.*

The following local version of Theorem 6.2 is also due to Kim.

THEOREM 6.3 ([Ki5]). *Let $D \subset \mathbb{C}^n$ be a bounded convex domain with non-compact automorphism group. Suppose that ∂D is C^∞-smooth and Levi-flat in a neighborhood of some boundary orbit accumulation point. Then D is holomorphically equivalent to the product of the unit disc and a convex domain in $\mathbb{C}^{n-1}$.*

Note that Theorem 4.1 that we mentioned above in connection with the Greene/Krantz conjecture is a corollary of Theorem 6.3.

The proofs of Theorems 6.2, 6.3 rely on (an extension of) the scaling technique of Frankel. Note that, in complex dimension 2, these theorems give characterizations of the bidisc $\Delta^2 := \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$. Another characterization of Δ^2 is due to Bun Wong:

THEOREM 6.4 ([W2]). *Let $D \subset \mathbb{C}^2$ be a bounded domain with non-compact automorphism group. Suppose that there is a sequence $\{f_j\} \subset Aut(D)$ such that*

(i) *$W := (\lim_{j\to\infty} f_j)(D)$ is a complex variety of positive dimension in ∂D;*

(ii) *W is contained in an open subset $U \subset \partial D$ such that U is C^1-smooth and there is an open set $V \subset \mathbb{C}^2$ for which $V \cap \partial D = U$ and $V \cap D$ is convex;*

(iii) *There exists a point $p \in D$ such that $\{f_j(p)\}$ converges to a point $q \in W$ non-tangentially.*

Then D is holomorphically equivalent to Δ^2.

We note here that the hypothesis **(iii)** of non-tangential convergence along some orbit is one that recurs in the literature, but it is rather artificial. A theorem about domains with non-compact automorphism group should, ideally, make no hypothesis about the way that an orbit approaches the boundary—especially a hypothesis that is unverifiable in practice. In fact non-tangential approach of orbits to a boundary orbit accumulation point should be part of the *conclusion* of the sorts of theorems discussed here, not part of the hypothesis. This hypothesis is one of the main difficulties in problems related to domains with non-compact automorphism groups (cf. Section 5, for example).

While this survey was being prepared we received a recent preprint of Fu and Wong where the following result was obtained:

THEOREM 6.5 ([FW2]). *Let $D \Subset \mathbb{C}^2$ be a bounded simply-connected domain with generic piecewise C^∞-smooth (but not smooth) Levi-flat boundary and non-compact automorphism group. Then D is holomorphically equivalent to Δ^2.*

The next theorem deals with the case of Reinhardt domains in $\mathbb{C}^2$ and is in the spirit of Product Domain Theorems 6.2 and 6.3.

THEOREM 6.6. *Let $D \subset \mathbb{C}^2$ be a bounded Reinhardt domain with C^1-piecewise smooth (but not smooth) boundary and non-compact automorphism group. Then D is holomorphically equivalent to a product $\Delta \times G$, where $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, and G is either Δ or an annulus $\{1 < |z| < r\}$ for some $r > 1$.*

Theorem 6.6 easily follows from the proof of Theorem 3.6 in [FIK2] and holds even for domains with much rougher boundary. The next result extends Theorem 3.6 to Reinhardt domains with boundary of only finite smoothness.

THEOREM 6.7 ([IK2]). *Let $D \subset \mathbb{C}^n$ be a bounded Reinhardt domain with C^k-smooth boundary, $k \geq 1$, and non-compact automorphism group. Then, up to dilations and permutations of coordinates, D is a domain of the form*

$$\left\{(z_1, \dots, z_n) : |z_1|^2 + \psi(|z_2|, \dots, |z_n|) < 1\right\},$$

where $\psi(x_2, \dots, x_n)$ is a non-negative C^k-smooth function in $\mathbb{R}^{n-1}$ that is strictly positive in $\mathbb{R}^{n-1} \setminus \{0\}$ and such that $\psi(|z_2|, \dots, |z_n|)$ is C^k-smooth in $\mathbb{C}^{n-1}$, and

$$\psi\left(t^{\frac{1}{\alpha_2}} x_2, \dots, t^{\frac{1}{\alpha_n}} x_n\right) = t\psi(x_2, \dots, x_n) \tag{6.1}$$

in $\mathbb{R}^{n-1}$ for all $t \geq 0$. Here $\alpha_j > 0$, $j = 2, \dots, n$, and each α_j is either an even integer or $\alpha_j > 2k$.

In complex dimension 2, Theorem 6.7 gives the following classification:

Corollary 6.8 ([IK2]). *If $D \subset \mathbb{C}^2$ is a bounded Reinhardt domain with C^k-smooth boundary, $k \geq 1$, and if $Aut(D)$ is non-compact, then, up to dilations and permutations of coordinates, D has the form*

$$\{|z_1|^2 + |z_2|^\alpha < 1\},$$

where $\alpha > 0$ and either is an even integer or $\alpha > 2k$.

Note that, in complex dimension 3 and higher, Reinhardt domains from Theorem 6.6 may look much more complicated than in dimension 2 since, in contrast with the infinitely smooth case, there is not any simple description of finitely smooth function satisfying (6.1).

EXAMPLE 6.9 ([IK2]). The domain

$$D := \left\{|z_1|^2 + |z_2|^9 + |z_3|^9 + \frac{1}{\log|z_3|^2 - \log|z_2|^2}\left(|z_2|^4|z_3|^5 - |z_2|^5|z_3|^4\right) < 1\right\}$$

is bounded, has non-compact automorphism group (see Example 6.10) below for a proof), and its boundary is C^2-smooth. The corresponding function $\psi(|z_2|, |z_3|)$ possesses weighted homogeneity property (6.1) with $\alpha_2 = \alpha_3 = 9$.

□

Along the lines of Theorem 6.7, one can consider the following examples of domains with non-compact automorphism group and C^k-smooth boundary, $k \geq 1$, that are not necessarily Reinhardt.

EXAMPLE 6.10 ([IK2]). Consider the domain

$$\left\{(z_1, \dots, z_n) : |z_1|^2 + \psi(z_2, \dots, z_n) < 1\right\}, \tag{6.2}$$

where $\psi(z_2, \dots, z_n)$ is a C^k-smooth function on $\mathbb{C}^{n-1}$ and

$$\psi\left(t^{\frac{1}{\alpha_2}} z_2, \dots, t^{\frac{1}{\alpha_n}} z_n\right) = |t|\psi(z_2, \dots, z_n) \tag{6.3}$$

in $\mathbb{C}^{n-1}$ for all $t \in \mathbb{C} \setminus \{z : \operatorname{Re} z < 0\}$. Here $\alpha_j > 0$, $j = 2, \dots, n$, and $t^{\frac{1}{\alpha_j}} = e^{\frac{1}{\alpha_j}(\log|t| + i\arg t)}$, for $t \neq 0$ and $-\pi < \arg t < \pi$. Also, to guarantee that the domain given in (6.2) is bounded, one can assume that the domain in $\mathbb{C}^{n-1}$

$$\{(z_2, \dots, z_n) : \psi(z_2, \dots, z_n) < 1\}$$

is bounded.

For any domain D of the form (6.2), $\operatorname{Aut}(D)$ is indeed non-compact, since it contains the subgroup

$$\begin{aligned} z_1 &\mapsto \frac{z_1 - a}{1 - \overline{a} z_1}, \\ z_j &\mapsto \frac{(1 - |a|^2)^{\frac{1}{\alpha_j}} z_j}{(1 - \overline{a} z_1)^{\frac{2}{\alpha_j}}}, \qquad j = 2, \dots, n, \end{aligned}$$

where $|a| < 1$. □

If $n = 2$ then, by differentiating both parts of (6.3) with respect to t and $\overline{t}$ and setting $t = 1$, we obtain that $\psi(z_2) = c|z_2|^\alpha$, with $c > 0$. Therefore, for $n = 2$, the domain (6.2) is equivalent to a domain of the form

$$\{(z_1, z_2) : |z_1|^2 + |z_2|^\alpha < 1\}$$

which is Reinhardt. However, as examples in [FIK1] show, there exists a bounded domain in $\mathbb{C}^2$ with non-compact automorphism group whose boundary is **(i)** real analytic at all points except one, **(ii)** $C^{1,\beta}$-smooth at the exceptional point for some $0 < \beta < 1$, and that is biholomorphically inequivalent to any Reinhardt domain and thus to any domain of the form (6.2). It would be interesting to construct such examples for the case of C^k-smooth boundaries, $k \geq 2$ and in dimensions $n \geq 2$. The domains (6.2) seem to be reasonable generalizations of the Bedford/Pinchuk domains (3.1) to the case of finitely smooth boundaries.

We turn now to the case of unbounded domains. Note that in Section 3 we already gave some classification results that hold for unbounded domains because of their completely local nature (see e.g. Theorem 3.4 and [Ga]). Another local result that we mention here is due to Efimov [E] and generalizes Theorem 2.1 to the case of unbounded domains (note that Theorem 2.1, being local, still requires the domain to be bounded).

THEOREM 6.11 ([E]). *Let $D \subset \mathbb{C}^n$ be a domain (not necessarily bounded), and $p \in \partial D$ a boundary orbit accumulation point for $\operatorname{Aut}(D)$. Assume that ∂D is C^2-smooth and strictly pseudoconvex near p. Then D is holomorphically equivalent to B^n.*

The next theorem is not local, and the domain is assumed to be Kobayashi-hyperbolic.

THEOREM 6.12 ([IK3]). *Let $D \subset \mathbb{C}^2$ be a hyperbolic Reinhardt domain with C^k-smooth boundary, $k \geq 1$, and let D intersect at least one of the coordinate complex* *lines $\{z_j = 0\}$, $j = 1, 2$. Assume also that $Aut(D)$ is non-compact. Then D is holomorphically equivalent to one of the following domains:*

(i) $\left\{(z_1, z_2) : |z_1|^2 + |z_2|^\alpha < 1\right\}$,
where either $\alpha < 0$, or $\alpha = 2m$ for some $m \in \mathbb{N}$, or $\alpha > 2k$;

(ii) $\left\{(z_1, z_2) : |z_1| < 1, (1 - |z_1|^2)^\alpha < |z_2| < R(1 - |z_1|^2)^\alpha\right\}$,
where $1 < R \leq \infty$ and $\alpha < 0$;

(iii) $\left\{(z_1, z_2) : e^{\beta|z_1|^2} < |z_2| < Re^{\beta|z_1|^2}\right\}$,
where $1 < R \leq \infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$, and, if $R = \infty$, $\beta > 0$.

If $k < \infty$ and ∂D is not C^∞-smooth, then D is holomorphically equivalent to a domain of the form **(i)** *with $\alpha \neq 2m$ for any $m \in \mathbb{N}$ and $\alpha > 2k$.*

In case **(i)** *the equivalence is given by dilations and a permutation of the coordinates; in cases* **(ii)** *and* **(iii)** *the equivalence is given by a mapping of the form*

$$\begin{aligned} z_1 &\mapsto \lambda z_{\sigma(1)} z_{\sigma(2)}^a, \\ z_2 &\mapsto \mu z_{\sigma(2)}^{\pm 1}, \end{aligned}$$

where $\lambda, \mu \in \mathbb{C}^*$, $a \in \mathbb{Z}$ and σ is a permutation of $\{1, 2\}$.

Note that in Theorem 6.12 we do not assume the existence of a finite boundary orbit accumulation point (of course the domain may be unbounded) which is an important hypothesis in [Ber3], [Ga]. The condition for the domain to intersect a coordinate complex line gives that $\mathrm{Aut}(D)$ has only finitely many connected components and therefore the non-compactness of $\mathrm{Aut}(D)$ is equivalent to the non-compactness of its identity component, to which the description in [Kru] applies. It seems that without this condition there is not any reasonable classification, since one can produce many "exotic" hyperbolic domains for which the identity component of the automorphism group is compact whereas the whole group is non-compact and has infinitely many connected components. Domains with such a structure of the automorphism groups seem to be intractable. To support this claim we give one "exotic" example below.

EXAMPLE 6.13 ([IK3])**.** Consider the Reinhardt domain $D \subset \mathbb{C}^2$ given by

$$D := \left\{(z_1, z_2) : \sin\left(\log\frac{|z_1|}{|z_2|}\right) < \log|z_1 z_2| < \sin\left(\log\frac{|z_1|}{|z_2|}\right) + \frac{1}{2}\right\}.$$

The boundary of D is clearly C^∞-smooth. The group $\mathrm{Aut}(D)$ is not compact since it contains all the mappings

$$\begin{aligned} z_1 &\mapsto e^{\pi k} z_1, \\ z_2 &\mapsto e^{-\pi k} z_2, \end{aligned}$$

for $k \in \mathbb{Z}$.

To see that D is hyperbolic, consider the mapping $f : D \to \mathbb{C}$, $f(z_1, z_2) = z_1 z_2$. It is easy to see that f maps D onto the annulus $A := \left\{z \in \mathbb{C} : e^{-1} < |z| < e^{\frac{3}{2}}\right\}$, which is a hyperbolic domain in $\mathbb{C}$. The annuli

$$\begin{aligned}
A_1 &:= \left\{z \in \mathbb{C} : e^{-\frac{1}{4}} < |z| < e^{\frac{1}{2}}\right\}, \\
A_2 &:= \left\{z \in \mathbb{C} : e^{-1} < |z| < e^{-\frac{1}{8}}\right\}, \\
A_3 &:= \left\{z \in \mathbb{C} : e^{\frac{1}{4}} < |z| < e^{\frac{3}{2}}\right\}
\end{aligned}$$

obviously cover A, and each of the inverse images $D_j = f^{-1}(A_j)$, $j = 1, 2, 3$, is hyperbolic since D_j is contained in a union of bounded pairwise non-intersecting domains. It then follows (see [PoSh]) that D is hyperbolic. $\square$

The following example suggests that, in complex dimension $n \geq 3$, an explicit classification result in the hyperbolic case—analogous to Theorem 6.12—does not exist; in fact it does not exist if we do not impose extra conditions on the domain, even if the domain contains the origin. We note here that, to obtain the finiteness of the number of connected components of $\mathrm{Aut}(D)$ for a hyperbolic Reinhardt domain $D \subset \mathbb{C}^n$, one needs the assumption that D intersects at least $n - 1$ coordinate hyperplanes [IK3], which certainly holds for domains containing the origin. Therefore the problem suggested by the following example is of a different kind compared to the one arising from Example 6.12 and is specific for dimensions $n \geq 3$.

EXAMPLE 6.14 ([IK3]). Consider the domain $D \subset \mathbb{C}^3$ given by

$$\begin{aligned}
D := \Biggl\{ &(z_1, z_2, z_3) : \phi(z) := \\
&|z_1|^2 + (1 - |z_1|^2)^2 |z_2|^2 \rho\Bigl(|z_2|^2(1 - |z_1|^2), |z_3|^2(1 - |z_1|^2)\Bigr) \\
&+ (1 - |z_1|^2)^2 |z_3|^2 - 1 < 0 \Biggr\}, \qquad (6.4)
\end{aligned}$$

where $\rho(x_1, x_2)$ is a C^∞-smooth function on $\mathbb{R}^2$ such that $\rho(x_1, x_2) > c > 0$ everywhere, and the partial derivatives of ρ are non-negative for $x_1, x_2 \geq 0$.

To show that ∂D is smooth, we calculate

$$
\begin{aligned}
\frac{\partial\phi}{\partial z_1} &= \overline{z_1}\Bigg(1-(1-|z_1|^2)\Big(2|z_2|^2\rho+(1-|z_1|^2)|z_2|^4\frac{\partial\rho}{\partial x_1}\\
&\qquad\qquad +(1-|z_1|^2)|z_2|^2|z_3|^2\frac{\partial\rho}{\partial x_2}+2|z_3|^2\Big)\Bigg),\\
\frac{\partial\phi}{\partial z_2} &= (1-|z_1|^2)^2\overline{z_2}\left(\rho+(1-|z_1|^2)|z_2|^2\frac{\partial\rho}{\partial x_1}\right),\\
\frac{\partial\phi}{\partial z_3} &= (1-|z_1|^2)^2\overline{z_3}\left((1-|z_1|^2)|z_2|^2\frac{\partial\rho}{\partial x_2}+1\right). \qquad (6.5)
\end{aligned}
$$

It follows from (6.5) that not all the partial derivatives of ϕ can vanish simultaneously at a point of ∂D. Indeed, if $\frac{\partial\phi}{\partial z_3}(p)=0$ at some point $p\in\partial D$ then, at p, either $|z_1|=1$ or $z_3=0$. If $|z_1|=1$, then clearly $\frac{\partial\phi}{\partial z_1}(p)\neq 0$. If $|z_1|\neq 1$, $z_3=0$, and, in addition, $\frac{\partial\phi}{\partial z_2}(p)=0$, then $z_2=0$, and therefore $|z_1|=1$, which is a contradiction. Therefore, ∂D is C^∞-smooth.

To show that D is hyperbolic, consider the holomorphic mapping defined by $f(z_1,z_2,z_3)=z_1$ from D into $\mathbb{C}$. Clearly f maps D onto the unit disc Δ, which is a hyperbolic domain in $\mathbb{C}$. Further, the discs $\Delta_r:=\{z:|z|<r\}$ for $r<1$ form a cover of Δ, and $f^{-1}(\Delta_r)$ is a bounded open subset of D for any such r. Thus, as in Example 6.13 above, we see that D is hyperbolic (see [PoSh]).

Further, $\mathrm{Aut}(D)$ is non-compact since it contains the automorphisms

$$
\begin{aligned}
z_1 &\mapsto \frac{z_1-a}{1-\overline{a}z_1},\\
z_2 &\mapsto \frac{(1-\overline{a}z_1)z_2}{\sqrt{1-|a|^2}}, \qquad (6.6)\\
z_3 &\mapsto \frac{(1-\overline{a}z_1)z_3}{\sqrt{1-|a|^2}},
\end{aligned}
$$

for $|a|<1$. □

Examples similar to Example 6.14 can be constructed in any complex dimension $n\geq 3$. They indicate that, most probably, there is no reasonable classification of smooth hyperbolic Reinhardt domains with non-compact automorphism group for $n\geq 3$ even in the case when the domains contain the origin. Indeed, in Example 6.14 we have substantial freedom in choosing the function ρ. We note that the boundary of domain (6.4) contains the complex hyperplane $z_1=\alpha$ for any $|\alpha|=1$. It may happen that, by imposing the extra condition of the finiteness of type on the boundary of the domain, one would eliminate the difficulty arising in Example 6.14 and obtain an explicit classification. It also should be observed that any point of the boundary of domain (6.4) with $|z_1|=1$, $z_2=z_3=0$ is a boundary orbit accumulation point for $\mathrm{Aut}(D)$ (see (6.6)); therefore, it is plausible that one needs the finite type condition only at such points (cf. the Greene/Krantz conjecture for the bounded case).

7. Concluding Remarks

The study of automorphism groups has considerable intrinsic interest, and also has roots in several of the major themes of twentieth century mathematics. Because domains in higher dimensions are generically biholomorphically distinct, it is natural to seek some unifying properties that they enjoy. The automorphism group provides one such natural set of ideas.

The program we have described suggests that considerable progress has been made in understanding domains of finite type with "large" automorphism group. The Greene/Krantz conjecture, which at this point in time appears likely to be true, suggests that finite type domains are the only ones that require study.

However, it should be borne in mind that these last remarks apply only to smoothly bounded domains. Evidence suggests that each boundary smoothness class C^k has different automorphism group phenomena, and that the picture becomes more and more complicated as k becomes smaller. In particular, for domains with fractal boundary almost nothing is known (and the self-similarity of a fractal boundary suggests that this case is of particular interest for automorphism group symmetry). We look forward to new insights in the future, some perhaps inspired by the present article.

Appendix on the Scaling Methods

We now sketch the key ideas in the methods of scaling and some of their applications to the study of domains with non-compact automorphism groups. We begin with the method originated by S. Pinchuk in the late 1970's [Pi1]. This discussion will lead to a proof of the Ball Characterization Theorem (Theorem 2.1). We then conclude the Appendix with an outline of Frankel's scaling technique. Our discussion of scaling mainly follows the exposition in [Ki4].

In the discussion of Pinchuk's method, for simplicity, we restrict attention to scaling of strongly pseudoconvex domains. This will convey the main ideas without the added baggage that treating finite type points would entail. It should be clearly understood, however, that scaling is of greatest importance in the weakly pseudoconvex case because it is virtually the only technique available in that setting.

Fix a smoothly bounded domain D with strongly pseudoconvex boundary point q. We assume that q is a boundary orbit accumulation point for the action of the automorphism group on D. Therefore there are a point $p \in D$ and a sequence of automorphisms $\{f_j\} \subset \mathrm{Aut}(D)$ such that $f_j(p) \to q$ as $j \to \infty$. We may apply a quadratic holomorphic polynomial change of coordinates so that q is mapped into the origin and there is a ball U centered at the origin such that $U \cap \partial D$ is strongly *convex* (see Narasimhan's Lemma in [Kra3]). Denote $\tilde{z} := (z_2, \ldots, z_n)$, so that $z := (z_1, \ldots, z_n) = (z_1, \tilde{z})$. Now a simple holomorphic change of coordinates (we denote it by F) allows us to write a defining function on the set $U \cap D$ (with a possibly smaller ball U) as

$$\rho(z_1, \tilde{z}) := \mathrm{Re}\, z_1 + ||\tilde{z}||^2 + o\big(|\mathrm{Im}\, z_1| + ||\tilde{z}||^2\big). \qquad (A.1)$$

It follows then that ∂D is variety-free at q. Now a simple normal family argument implies the following result (see [Kra3] for details):

Lemma A.1 *Let notation be as above. Then there is a subsequence of $\{f_j\}$ that converges to the constant mapping q uniformly on compact subsets of D.*

Define $p^j = f_j(p)$ for each j. Of course $p^j \to p$ as $j \to \infty$. Set $p^j = (p^j_1, \ldots, p^j_n)$. For each j, we construct a holomorphic change of variables as follows:

$$\begin{cases} \widehat{z}_1 &= e^{i\theta_j} z_1 - p^*_j - \sum_{m=2}^n a_m(z_m - p^j_m), \\ \widehat{\tilde{z}} &= \tilde{z} - \tilde{p^j}. \end{cases} \tag{A.2}$$

Here $\theta_j \in \mathbb{R}$ and $p^*_j, a_m \in \mathbb{C}$ are selected so that in the coordinates $\widehat{z} := (\widehat{z}_1, \ldots, \widehat{z}_n)$ one has:

- $(0, \ldots, 0) \in \partial D$;
- $p^j = (-\epsilon_j, 0, \ldots, 0)$, $\epsilon_j > 0$, for each j;
- The tangent plane to ∂D at $(0, \ldots, 0)$ is given by $\{z : \operatorname{Re} z_1 = 0\}$.

In the $\widehat{z}$-coordinates the defining function in equation $(A.1)$ is given by

$$\widehat{\rho}_j(\widehat{z}) := \widehat{c}_j \operatorname{Re}\left(\widehat{z}_1 + \sum_{m=1}^n A^j_m \widehat{z}^2_m\right) + \sum_{k,m=1}^n B^j_{km} \widehat{z}_k \overline{\widehat{z}_m} + E_j(\widehat{z}),$$

where $E_j(\widehat{z}) = o(|\operatorname{Im} \widehat{z}_1| + ||\widehat{\tilde{z}}||^2)$ and the coefficients of the quadratic terms converge to the coefficients of the corresponding quadratic terms for the defining function ρ in $(A.1)$. Furthermore, $\widehat{c}_j \to 1$ as $j \to \infty$.

Now we come to the heart of the scaling process. Thus far we have been normalizing coordinates so that the scaling can be performed in a natural manner. The motivation for the scaling that we do is as follows: the natural geometry of a strongly pseudoconvex point is parabolic in nature. This can be seen by examining the boundary behavior of the Carathéodory or Kobayashi metrics (see [Kra3]), but can be also seen in a more elementary fashion by examining, for instance, the defining function in $(A.1)$. We see that an arbitrary strongly pseudoconvex point can be viewed as a perturbation of the domain

$$\widehat{D} := \{z \in \mathbb{C}^n : \widehat{\rho} := \operatorname{Re} z_1 + ||\tilde{z}||^2 < 0\}. \tag{A.3}$$

A moment's thought reveals this last domain to be holomorphically equivalent to the unit ball B^n (see e.g. [Ru]). And the parabolic nature of the boundary is self-evident from comparing the roles of $\operatorname{Re} z_1$ and $\tilde{z}$ in the defining function $\widehat{\rho}$.

Having said all this, we now set

$$\begin{aligned} z'_1 &= \frac{\widehat{z}_1}{\epsilon_j}, \\ \tilde{z'} &= \frac{\widehat{\tilde{z}}}{\sqrt{\epsilon_j}}, \end{aligned} \tag{A.4}$$

with ϵ_j defined by $(A.2)$. Given that a strongly pseudoconvex point is nearly like a ball, what we are doing is scaling that ball up to have radius about 1. But the magnitude of the scaling depends on the normal distance of p_j to the boundary.

Let D_j denote the image of $D \cap U$ under the composition of F, mapping $(A.2)$ and mapping $(A.4)$. Taking into account the fact that $\epsilon_j \to 0$ as $j \to \infty$, we may write (dropping primes) the defining function for D_j as

$$\rho_j(z) = c_j \mathrm{Re}\left(z_1 + \sum_m A^j_m z_m^2\right) + \sum_{k,m} B^j_{km} z_k \overline{z}_m + \epsilon_j^{-1} E_j(\epsilon_j z_1, \sqrt{\epsilon_j}\tilde{z}).$$

As $j \to \infty$, we see that the "limiting defining function" is then the function $\widehat{\rho}$ from $(A.3)$. Then the domains D_j converge (in the sense of Hausdorff set convergence) to the limiting domain $\widehat{D}$.

Now the crux of the matter is this: combining our various coordinate changes, we see that we have constructed, for each j, a biholomorphic mapping

$$g_j : U \cap D \longrightarrow D_j.$$

For any compact subset $K \subset \widehat{D}$, we have $K \subset D_j$ for j sufficiently large and thus $G_j := f_j^{-1} \circ g_j^{-1}$ is defined on K for large j. Since D is bounded and K was an arbitrary compact subset of $\widehat{D}$, a subsequential limit yields a holomorphic mapping

$$g : \widehat{D} \longrightarrow \overline{D}.$$

On the other hand, by Lemma A.1, passing if necessary to a subsequence, we also know that any compact subset of D is mapped to $D \cap U$ under f_j for large j. Therefore, $G_j^{-1} = g_j \circ f_j$ is defined on any compact subset of D for j large enough.

Using these two facts, it is possible to prove that the limit mapping g is in fact a biholomorphism from $\widehat{D}$ onto D (see e.g. [Ki4]). Since $\widehat{D}$ is holomorphically equivalent to B^n, so is D. This concludes the proof of the Ball Characterization Theorem by scaling.

At the level of strongly pseudoconvex domains, the scaling technique is largely formalistic. In the case of weakly pseudoconvex domains of finite type the argument just presented is only the beginning of the proof. The difficulty in this case is that the limit domain $\widehat{D}$ is not so easily found as for strictly pseudoconvex domains. To determine $\widehat{D}$ one needs an argument that involves further applications of the scaling process [BP1]–[BP4].

With this last thought in mind, we now say just a few words about a scaling technique introduced by Frankel [Fr]. It has proved to be important because, in the case when the domains under consideration are *convex*, the delicate limiting arguments described above are easier to handle. Note that in the proof of Theorem 3.3 convexity also helps to make scaling arguments—based on Pinchuk's method—easier (see [BP3]).

Now let $D \subset \mathbb{C}^n$ be a bounded, convex domain. Suppose, as before, that there are a point $p \in D$ and a sequence of automorphisms $\{f_j\}$ of D such that $f_j(p) \to q \in \partial D$.

Consider the mappings

$$\omega_j(z) = \left[\partial f_j(p)\right]^{-1}(f_j(z) - f_j(p)).$$

where ∂f_j is the holomorphic Jacobian matrix of f_j. The central point of the scaling procedure is the following result of Frankel.

THEOREM A.2 ([Fr]) *Let notation be as above. Then*
(i) $\{\omega_j\}$ *is a normal family (i.e. every subsequence of* $\{\omega_j\}$ *has a subsequence that uniformly converges on compact subsets of* D*);*
(ii) *Every subsequential limit of* $\{\omega_j\}$ *is a holomorphic embedding of* D *into* $\mathbb{C}^n$*.*

The following version of the above result of Frankel is due to Kim.

PROPOSITION A.3 ([Ki4]) *Let notation be as above. Suppose that* ∂D *is variety-free at* q*. Then, by passing to a subsequence of* $\{f_j\}$ *if necessary, one can construct a sequence* $\{q_j\} \subset \partial D$*,* $q_j \to q$ *as* $j \to \infty$*, such that*

(i) *The mappings*

$$\sigma_j(z) = \left[\partial f_j(p)\right]^{-1}(f_j(z) - q_j)$$

form a normal family;

(ii) *Every subsequential limit of* $\{\sigma_j\}$ *is a holomorphic embedding of* D *into* $\mathbb{C}^n$*.*

Note that Theorem A.2 and Proposition A.3 do not require any regularity of ∂D. The sequence of scaled domains that one has to consider is then the sequence $\{\sigma_j(D)\}$. Further, as in Pinchuk's method above, one has to understand what the limit domain is and why it is holomorphically equivalent to D, and this is where the regularity of ∂D becomes important.

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