VORTEX DYNAMICS FOR THE GINZBURG-LANDAU-SCHRÖDINGER EQUATION

J. E. COLLIANDER AND R. L. JERRARD

ABSTRACT. The initial value problem for the Ginzburg-Landau-Schrödinger equation is examined in the $\epsilon \to 0$ limit under two main assumptions on the initial data ϕ^ϵ. The first assumption is that ϕ^ϵ exhibits m distinct vortices of degree ± 1; these are described as points of concentration of the Jacobian $[J\phi^\epsilon]$ of ϕ^ϵ. Second, we assume energy bounds consistent with vortices at the points of concentration. Under these assumptions, we identify "vortex structures" in the $\epsilon \to 0$ limit of ϕ^ϵ and show that these structures persist in the solution $u^\epsilon(t)$ of GLS_ϵ. We derive ordinary differential equations which govern the motion of the vortices in the $\epsilon \to 0$ limit. The limiting system of ordinary differential equations is a Hamitonian flow governed by the renormalized energy of Bethuel, Brezis and Hélein. Our arguments rely on results about the structural stability of vortices which are proved in a separate paper.

1. INTRODUCTION

We investigate the $\epsilon \to 0$ behavior of the initial value problem

$$GLS_\epsilon \qquad \begin{cases} iu^\epsilon_t - \Delta u^\epsilon + \frac{1}{\epsilon^2}\big(|u^\epsilon|^2 - 1\big)u^\epsilon = 0, & u^\epsilon : \mathbb{T}^2 \times [0,T) \longmapsto \mathbb{R}^2 \\ u^\epsilon(x,0) = \phi^\epsilon(x), & x \in \mathbb{T}^2 \end{cases}$$

exhibiting a finite number of vortices, under appropriate assumptions on the associated Hamiltonian

$$I^\epsilon[u] = \int_{\mathbb{T}^2} E^\epsilon(u)dx; \;\; E^\epsilon(u) = \frac{1}{2}|Du^\epsilon|^2 + \frac{1}{4\epsilon^2}\big(|u^\epsilon|^2 - 1\big)^2. \tag{1.1}$$

The $\epsilon \to 0$ asymptotics of minimizers of $I^\epsilon[u]$ subject to Dirichlet boundary constraints were studied by Bethuel, Brezis and Hélein in [1]. The corresponding parabolic evolution problem was studied by Lin [13] and Jerrard and Soner [11]. These works identify "vortex structures" emerging in the $\epsilon \to 0$ limit whose location is governed by a renormalized energy. The renormalized energy is obtained by removing the divergent part of $I^\epsilon[u]$. This note and the forthcoming paper [4] establish similar results for the conservative evolution GLS_ϵ.

The Landau theory of second order phase transitions [12] consists of expanding the energy in terms of a parameter which encodes the "order" in the phase and then exploiting energy properties to determine the evolution of the "order parameter". This theory was applied by Ginzburg and Pitaevskii [8] and Pitaevskii [16] to argue that the order parameter describing superfluid helium II evolves according to GLS_ϵ. In this context, I^ϵ is the free energy and u^ϵ is the order parameter "which plays the role of 'the effective wave function' of the superfluid part of the liquid"

[8]. The motion of u^ϵ under the GLS_ϵ evolution conserves $I^\epsilon[u^\epsilon]$. If we express $u^\epsilon(x_1, x_2) = \rho(x_1, x_2)e^{i\theta(x_1,x_2)}$, with ρ, θ $\mathbb{R}$-valued, then ρ^2 represents the density of the superfluid and $D\theta$ is the velocity of the superfluid. Gross [9] also derived GLS_ϵ as the Schrödinger equation for a wave function describing a system of interacting bosons. The equation GLS_ϵ is often called the Gross-Pitaevskii equation in the physics literature.

Because we are interested in studying the motion of vortices in the $\epsilon \to 0$ limit, we choose initial data $\phi^\epsilon : \mathbb{T}^2 \longmapsto \mathbb{C}$ possessing vortices. This heuristic notion is implemented by imposing two conditions. First, we assume the measure $[J\phi^\epsilon] = \det(D\phi^\epsilon)\,dx$ is close to a weighted sum of Dirac masses. More precisely, we assume

$$[J\phi^\epsilon] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{\alpha_i} \tag{1.2}$$

where $d_i = \pm 1$ with $\sum_{i=1}^{m} d_i = 0$, and α_i are distinct points in $\mathbb{T}^2$.

Second, we assume energy bounds. For example, certain of our results require that ϕ^ϵ satisfy

$$I^\epsilon[\phi^\epsilon] \le \pi m \log \frac{1}{\epsilon} + \gamma_1. \tag{1.3}$$

for some $\gamma_1 \in \mathbb{R}$. It follows from results that are cited and discussed in Section 3 that assuption (1.2) implies the lower bound

$$I^\epsilon[\phi^\epsilon] \ge \pi m \log \frac{1}{\epsilon} - C, \tag{1.4}$$

where C is independent of ϵ. Thus (1.3) asserts that the energy of ϕ^ϵ is $O(1)$, given the prescribed configuration of vortices. Our strongest results need more stringent hypotheses, which amount to assuming that the energy is $o(1)$, relative to the prescribed configuration of vortices.

We briefly outline the contents of this paper. Since we have not yet defined many of the terms we are using, this description is necessarily a little impressionistic.

Section 2 introduces notation and records some useful identities satisfied by solutions of GLS_ϵ.

Section 3 states and briefly discusses some results providing conditions under which the Jacobian of a function u (which in our context may be interpreted as the vorticity) and the Ginzburg-Landau energy density concentrate around a collection of points. These results provide a key element in our analysis. The proofs are long and technical and will appear in [4]. Section 3 also states some refined estimates related to the renormalized energy introduced by Bethuel, Brezis, and Hélein [1]. These too are proven in [4].

In Section 4 we prove that solutions of GLS_ϵ exhibit Lipschitz paths along which the energy and the vorticity concentrate. The main point is to control vortex mobility. We also describe weak limits of the "supercurrent" associated with solutions.

Finally, in Section 5 we give a complete description of limiting vortex dynamics, under somewhat stronger assumptions on the initial data. We also sharpen our characterization of the limiting behavior of solutions away from the vortices.

We ultimately show that, under appropriate hypotheses, the limiting vortices solve a system of ODEs, which is exactly that satisfied by classical point vortices in an ideal two dimensional fluid. This result was predicted as early as 1965 by Fetter [7], based on physical arguments, and more recently by Neu [15] and E [5], who arrived at the same conclusion through matched asymptotics.

2. Notation and identities

We introduce some vector notation and the quantity $j(\phi^\epsilon)$. We then define measures $[J\phi^\epsilon]$ and $\mu^\epsilon_{\phi^\epsilon}$ which will be used to locate the singularities of ϕ^ϵ. Then a norm is introduced that permits us to say the singularities of two functions ψ^ϵ and ϕ^ϵ are close if the associated measur es $[J\psi^\epsilon]$, $\mu^\epsilon_{\psi^\epsilon}$ and $[J\phi^\epsilon]$, $\mu^\epsilon_{\phi^\epsilon}$ are close. Next, we derive some identities for $[Ju^\epsilon]$ and $\mu^\epsilon_{u^\epsilon}$ assuming u^ϵ evolves according to GLS_ϵ. Finally, we introduce the renormalized energy W and a canonical harmonic map H with singularities of degrees $\{d_1, \ldots, d_m\}$ located at the points $\{a_1, \ldots, a_m\}$ mapping $\mathbb{T}^2 \setminus \{a_1, \ldots, a_m\} \longmapsto S^1$.

We introduce some notation for vector fields $u : \mathbb{T}^2 \longmapsto \mathbb{R}^2$ mapping $(x_1, x_2) \in \mathbb{T}^2$ to $(u^1(x_1, x_2), u^2(x_1, x_2)) \in \mathbb{R}^2$. The scalar product in $\mathbb{R}^2$ is denoted by "$\cdot$",

$$u \cdot v = u^1 v^1 + u^2 v^2.$$

Sometimes we will view $u \in \mathbb{R}^2$ as $u \in \mathbb{C}$ when we write iu or $e^{i\alpha}u$. These expressions are interpreted in the obvious way, e.g.

$$iu = -\mathbb{J}u;\ \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We define for $u, v \in \mathbb{R}^2$,

$$u \times v = u^1 v^2 - u^2 v^1 = \mathbb{J}_{ij} u^i v^j,$$

$$\nabla \times u = \partial_{x_1} u^2 - \partial_{x_2} u^1.$$

Repeated indices are implicitly summed throughout this paper. Note that $(iu) \cdot v = u \times v$ and $(iu) \cdot u = 0$. For a scalar function ϕ we define

$$\nabla \times \phi = (\phi_{x_2}, -\phi_{x_1}).$$

For a sufficiently differentiable $u : \mathbb{T}^2 \longmapsto \mathbb{R}^2$, we define

$$j(u) = (u \times u_{x_1}, u \times u_{x_2}).$$

We sometimes write $j(u) = (iu) \cdot Du$. In the superfluid model, $j(u^\epsilon)$ is interpreted as a current. If we write $u = \rho e^{i\theta}$, for $\rho, \theta \in \mathbb{R}$, then $j(u) = \rho^2 D\theta$. In particular, if $|u| = 1$, $j(u)$ is the phase gr adient.

We define the signed Jacobian of u,

$$Ju = \det Du. \tag{2.1}$$

The quantities $j(u)$ and Ju are related by the identity

$$\nabla \times j(u) = 2u_{x_1} \times u_{x_2} = 2Ju, \tag{2.2}$$

so the signed Jacobian can also be interpreted as the vorticity. We write $[Ju]$ to denote the distributional signed Jacobian, defined if $j(u) \in L^1$ via

$$\int \phi [Ju] = \frac{1}{2}\int (\nabla \times \phi)\cdot j(u), \ \ \phi \in C^1_c(\mathbb{T}^2). \tag{2.3}$$

In particular, $[Ju]$ is well defined for $u \in H^1(\mathbb{R}^2)$.

For a given $u \in H^1(\mathbb{T}^2;\mathbb{R}^2)$, define the measure

$$\mu^\epsilon_u(A) = \frac{1}{|\log \epsilon|}\int_A E^\epsilon(u)\ dx;\ E^\epsilon(u) = \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}(|u|^2-1)^2,$$

for subsets $A \subset \mathbb{T}^2$. The renormalization factor $\frac{1}{|\log\epsilon|}$ appears naturally upon considering $u = \frac{x}{|x|}$ smoothly cutoff in a ball of radius ϵ centered at $x = 0$.

We let $M(\mathbb{T}^2)$ denote the dual of $C(\mathbb{T}^2)$, i.e. the space of signed Radon measures on $\mathbb{T}^2$. Similarly, we let $M^1(\mathbb{T}^2)$ denote the dual of $C^1(\mathbb{T}^2)$, equipped with the dual norm. We also define a seminorm

$$\|\mu\|_{\widehat{M}^1(\mathbb{T}^2)} = \sup\left\{ \int \phi\ d\mu \ :\ \|D\phi\|_{L^\infty} \le 1,\ \int \phi = 0 \right\}.$$

If $\mu(\mathbb{T}^2) = 0$, we can compute $\|\mu\|_{M^1(\mathbb{T}^2)}$ by testing μ against functions ϕ such that $\int \phi\ dx = 0$. In this case it follows that

$$c^{-1}\|\mu\|_{\widehat{M}^1(\mathbb{T}^2)} \le \|\mu\|_{M^1(\mathbb{T}^2)} \le \|\mu\|_{\widehat{M}^1(\mathbb{T}^2)}.$$

This is a consequence of the fact that $\|\phi\|_{C^1(\mathbb{T}^2)} \le C\|D\phi\|_{L^\infty(\mathbb{T}^2)}$ whenever $\int\phi = 0$.

In particular, if μ has the form

$$\mu = \sum_{i=1}^n \delta_{\xi_i} - \sum_{i=1}^n \delta_{\eta_i} \tag{2.4}$$

for some points $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n \in \mathbb{T}^2$, not necessarily distinct, then Brezis, Coron and Lieb [3] show that

$$\|\mu\|_{\widehat{M}^1(\mathbb{T}^2)} = \min_{\pi\in S_n}\sum_{i=1}^n |\xi_i - \eta_{\pi(i)}| \tag{2.5}$$

where S_n is the symmetric group of permutations of n objects. For measures of the form (2.4) we thus have

$$c^{-1}\min_{\pi\in S_n}\sum_{i=1}^n |\xi_i - \eta_{\pi(i)}| \le \|\mu\|_{M^1(\mathbb{T}^2)} \le \min_{\pi\in S_n}\sum_{i=1}^n |\xi_i - \eta_{\pi(i)}|.$$

This estimate permits us to prove a fact illustrating the usefulness of the M^1 norm.

Proposition 1. *Suppose that for every $t \in [0,T)$, μ_t is a measure of the form $\sum_{i=1}^{n} \delta_{a_i(t)}$, for certain points $a_1(t), \ldots, a_n(t)$. The $\mu_{(\cdot)}$ is a countiuous (resp. Lipschitz) function from $[0,T)$ into M^1 if and only if the points $a_i(t)$ can be labelled in such a way th at $a_i(\cdot)$ is continuous (resp. Lipschitz) for each i.*

Proof. For any s,t, the measure $\mu_t - \mu_s$ has integral zero and so satisfies

$$\|\mu_t - \mu_s\|_{M^1(\mathbb{T}^2)} \sim \min_{\pi \in S_n} \sum_i |a_i(t) - a_{\pi(i)}(t)|.$$

The lemma follows immediately. □

Remark 1. The $\widehat{M}^1$ seminorm can be interpreted as the minimum cost in a Monge-Kantorovich mass transfer problem. Indeed, when $\int d\mu = 0$ as above, μ can be written in the form $\mu = \nu_1 - \nu_2$, where ν_1 and ν_2 are positive, mu tually singular measures and $\nu_1(\mathbb{T}^2) = \nu_2(\mathbb{T}^2)$. Then $\|\mu\|_{\widehat{M}^1}$ is precisely the minimum cost of "transporting" ν_1 to ν_2, subject to an appropriate cost functional. See for example [6] f or a more precise statement and more details.

Remark 2. Any reasonable weak norm on measures would be equally suitable for our purposes. The M^1 norm is a convenient choice, but is certainly not the only possible choice.

We employ O and o notation in some of the analysis below. We write, for example, $O_{a,b,c}(1)$ to indicate a quantity is $O(1)$ with respect to the interesting limit $\epsilon \to 0$, with the implicit constant depending only upon the parameters a, b, c.

We assume in this section, for fixed $\epsilon > 0$, that u solves

$$iu_t - \Delta u + \frac{1}{\epsilon^2}\left(|u|^2 - 1\right) u = 0$$

and derive evolution equations for certain functions of u and its derivatives. These are mostly well known. We formally calculate the evolution equation for $E^\epsilon(u) = \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}\left(|u|^2 - 1\right)^2$.

$$\begin{aligned} \frac{d}{dt}E^\epsilon(u) &= u_{x_i} \cdot u_{x_i t} + \frac{1}{\epsilon^2}(|u|^2 - 1)u \cdot u_t \\ &= (u_{x_i} \cdot u_t)_{x_i} - (u_{x_i x_i} - \frac{1}{\epsilon^2}(|u|^2 - 1)) \cdot u_t \end{aligned}$$

so that

$$\frac{d}{dt}E^\epsilon(u) = (u_{x_i} \cdot u_t)_{x_i} \tag{2.6}$$

using the equation and $(iu_t) \cdot u_t = 0$.

Similarly, we calculate

$$\frac{d}{dt}\frac{1}{2}|u|^2 = u \cdot u_t = -(iu) \cdot \Delta u$$

to obtain

$$\frac{d}{dt}\frac{1}{2}|u|^2 = -((iu)\cdot u_{x_i})_{x_i} = -\mathrm{div}\ j(u). \tag{2.7}$$

Next, we consider the evolution of a component of $j(u)$.

$$\begin{aligned}
\frac{d}{dt}j^k(u) &= \frac{d}{dt}(iu)\cdot u_{x_k} = (iu_t)\cdot u_{x_k} + (iu)\cdot u_{tx_k} \\
&= (iu_t)\cdot u_{x_k} - u\cdot(iu_{tx_k}) \\
&= u_{x_jx_j}\cdot u_{x_k} - u\cdot u_{x_jx_jx_k} + \frac{1}{\epsilon^2}|u|^2|u|^2_{x_k} \\
&= (u_{x_k}\cdot u_{x_j} - u\cdot u_{x_kx_j})_{x_j} + \frac{1}{\epsilon^2}|u|^4_{x_k}.
\end{aligned}$$

We rewrite as

$$\frac{d}{dt}j^k(u) = 2(u_{x_k}\cdot u_{x_j})_{x_j} - (|u_{x_j}|^2 - u\cdot u_{x_jx_j})_{x_k} + \frac{1}{2\epsilon^2}|u|^4_{x_k}. \tag{2.8}$$

Note that this shows $\int_{\mathbb{T}^2} j(u)dx$ is time-independent.

We calculate the evolution equation of Ju using (2.2)

$$\frac{d}{dt}Ju = \nabla\times(Du\cdot u_{x_j})_{x_j},$$

which we write out explicitly,

$$\frac{d}{dt}Ju = \mathbb{J}_{jk}(u_{x_k}\cdot u_{x_l})_{x_lx_j}. \tag{2.9}$$

Finally, we express (2.9) in a weak form. Multiply (2.9) by $\eta \in C_c^\infty(\mathbb{T}^2)$ and integrate to find

$$\frac{d}{dt}\int_{\mathbb{T}^2}\eta\ Ju\ dx = \int_{\mathbb{T}^2}\eta_{x_jx_l}\mathbb{J}_{jk}u_{x_k}\cdot u_{x_l}\ dx. \tag{2.10}$$

This is the key identity in our analysis of vortex dynamics of GLS_ϵ.

Remark 3. The preceding calculations apply equally well to $u^\epsilon(\cdot) \in H^1(\mathbb{T}^2)$. For $\epsilon > 0$ fixed, GLS_ϵ is a defocussing nonlinear Schrödinger equation. Bourgain has established [2] global wellposedness for GLS_ϵ below H^1. We validate the preceding calculations for $u^\epsilon \in H^1(\mathbb{T}^2)$ as follows using various aspects of Bourgain's result. By continuous dependence on the data, a different solution $\widetilde{u^\epsilon}(t)$ is close to $u^\epsilon(t)$ in $H^1(\mathbb{T}^2)$ provided the corresponding initial data $\widetilde{\phi^\epsilon}$ and ϕ^ϵ are close in $H^1(\mathbb{T}^2)$. Let $\widetilde{\phi^\epsilon}$ be a smooth approximator to ϕ^ϵ. The preceding calculations apply to $\widetilde{u^\epsilon}$ since it remains smooth for all time. The various identities above, in particular (2.10), are then validated for $u^\epsilon \in H^1(\mathbb{T}^2)$ by considering a sequence of smooth approximators.

We reformulate for the torus $\mathbb{T}^2$ some of the notions introduced by Bethuel, Brezis and Hélein [1] for the study of Dirichlet minimizers of $I^\epsilon[u]$. Let F solve

$$\Delta F = 2\pi(\delta_0 - 1) \qquad \text{on } \mathbb{T}^2. \tag{2.11}$$

with the normalization

$$\lim_{x\to 0}(F(x) - \log|x|) = 0 \tag{2.12}$$

For points $a_1, ..., a_m \in \mathbb{T}^2$ (sometimes denoted by (a)) and nonzero integers $d_1, ..., d_m$ satisfying $\sum d_i = 0$, define the renormalized energy

$$W(a,d) := -\pi \sum_{i\neq j} d_i d_j F(a_i - a_j). \tag{2.13}$$

Define

$$\Phi = \sum_{i=1}^{m} d_i F(x - a_i) \tag{2.14}$$

and notice that

$$\Delta\Phi = 2\pi \sum_{i=1}^{m} d_i \delta_{a_i}, \quad \int \Phi = 0.$$

The following proposition is proved in [1] (Theorem 1.1., Step 2 of proof).

Proposition 2. *There is a map $H \in C^\infty_{loc}(\mathbb{T}^2 \setminus (a); S^1) \cap W^{1,1}(\mathbb{T}^2; S^1)$ satisfying*

$$j(H) = -\nabla \times \Phi. \tag{2.15}$$

Moreover, if $\widetilde{H}$ is any other function satisfying (2.15)*, then $\widetilde{H} = e^{i\alpha} H$ for some $\alpha \in \mathbb{R}$.*

It follows from (2.15) that H satisfies

$$\operatorname{div} j(H) \ = \ 0, \tag{2.16}$$

$$2[JH] = \nabla \times j(H) = 2\pi \sum_{i=1}^{m} d_i \delta_{a_i}. \tag{2.17}$$

in the sense of distributions and pointwise away from the singluarities (a). The map H also satisfies

$$\int_{\mathbb{T}^2} j(H) dx = 0. \tag{2.18}$$

The map H has singularities of degree (d) located at the points (a) as seen in (2.17). The identity (2.16) reveals that H is a harmonic map into S^1 since $j(H) = D\theta$ when we express $H = e^{i\theta}$. Therefore, we refer to H as the *canonical harmonic map*[1] with singularities of degree (d) located at the points (a). We will sometimes highlight the dependence of H upon (d) and (a) by writing $H(x; a, d)$.

For a collection of points $a_1, ..., a_m \in \mathbb{T}^2$ define

$$\mathbb{T}^2_\rho := \mathbb{T}^2 \setminus \cup_i B_\rho(a_i).$$

[1]We have taken the name "canonical harmonic map" from the work of Bethuel, Brezis, and Hélein on the Dirichlet problem, where it is entirely appropriate. In the periodic context it is clearly something of a misnomer, since H is not unique. Nonetheless, it seems easiest to use the familiar terminology.

We will be interested in the case

$$0 < \rho < \frac{1}{4}\min_{i\neq j}|a_i - a_j|. \tag{2.19}$$

A configuration of vortices $(a),(d)$ determines (up to a constant phase factor $e^{i\alpha}$) a canonical harmonic map $H(\cdot,a,d)$. The map H has logarithmically divergent Dirichlet energy. The renormalized energy $W(a,d)$ describes to leading orde r the finite part of the energy associated with the configuration $(a),(d)$. The following proposition, essentially proven in [1] (Theorem I.7), makes these observations precise.

Proposition 3. *Let $H = H(\cdot,a,d)$. For ρ satisfying* (2.19),

$$\int_{\mathbb{T}^2_\rho} \frac{1}{2}|DH|^2 dx = m\pi\log\frac{1}{\rho} + W(a,d) + O(\rho).$$

We define, following Bethuel, Brezis, and Hélein [1], the quantity

$$I(\epsilon,\rho) = \min\left\{\int_{B_\rho} E^\epsilon(u)dx \ : \ u\in H^1(B_\rho),\ u(x) = \frac{x}{|x|} \text{ for } x\in\partial B_\rho\right\}. \tag{2.20}$$

It is shown in [1] that

$$I(\epsilon,\rho) = \pi\log(\frac{\rho}{\epsilon}) + O(1) \tag{2.21}$$

as $\epsilon\to 0$, for ρ fixed.

An easy adaptation of the construction given in ([1], Lemma VIII.1) establishes

Proposition 4. *Given any distinct $a_1,...,a_m\in\mathbb{T}^2$ and $d_1,...,d_m\in\{\pm1\}$, there exists a family of functions $v^\epsilon\in H^1(\mathbb{T}^2;\mathbb{R}^2)$ such that*

$$[Jv^\epsilon] \rightharpoonup \sum_{i=1}^m \pi d_i\delta_{a_i} \qquad \textit{weakly in } M$$

and for every $\rho>0$

$$\int_{\mathbb{T}^2} E^\epsilon[v^\epsilon]dx \leq m(\pi\log\frac{1}{\rho} + I(\epsilon,\rho)) + W(a,d) + C\rho + o(1)$$

as $\epsilon\to 0$.

Remark 4. The condition $[Jv^\epsilon]\rightharpoonup\sum\pi d_i\delta_{a_i}$ weakly in M implies the energy lower bound

$$\int_{\mathbb{T}^2} E^\epsilon(u^\epsilon)dx \geq m\left(\pi\log\frac{1}{\rho} + I(\epsilon,\rho)\right) + W(a,d) + \widetilde{C}\rho + o(1)$$

for some constant $\widetilde{C}$ and every $\rho>0$ satisfying (2.19). This follows from Theorem 2 below. Therefore, the functions v_ϵ of Proposition 4 are asymptotically energy minimizing, for the given configuration of vortices.

3. Concentration and structure

This section identifies two hypotheses on a function u^ϵ which imply detailed structural properties of u^ϵ as $\epsilon \to 0$. The first hypothesis is that $[Ju^\epsilon]$ is somewhat close to a weighted sum of Dirac masses. The second is an energy upper bound consistent with the number of Dirac masses. Theorem 1 says that these hypotheses imply the measures $[Ju^\epsilon]$ and $\mu^\epsilon_{u^\epsilon}$ concentrate to Dirac masses and that u^ϵ is uniformly regular away from the points of concentration. A more stringent energy upper bound on u^ϵ controls the current $j(u^\epsilon)$ and $D|u^\epsilon|$ as stated in Theorem 2. The proofs, which appear in [4], rely heavily on techniques developed in [10]. Similar techniques have been used by Sandier [17].

Theorem 1. **(Global structure)** *Suppose that $u \in H^1(\mathbb{T}^2;\mathbb{R}^2)$, and that there exist points $x_1, ..., x_m \in \mathbb{T}^n$, integers $d_1, ..., d_m \in \{\pm 1\}$, and $\epsilon \le r := \frac{1}{4}\min_{i\neq j}|x_i - x_j|$ such that*

$$|| \, [Ju] - \pi \sum_{i=1}^m d_i \delta_{x_i} \, ||_{M^1(\mathbb{T}^2)} \le \frac{\pi}{200} r, \tag{3.1}$$

and

$$\int_{\mathbb{T}^2} E^\epsilon(u) \; dx \le \pi m \log\left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{3.2}$$

for some constant γ_1. Then there exists points $a_i \in B_{r/2}(x_i)$, $i = 1, ..., m$ such that

$$||\mu^\epsilon_u - \pi \sum_{i=1}^m \delta_{a_i}||_{M^1(\mathbb{T}^2)} \le O_{\gamma_1}\left(\frac{1}{|\log \epsilon|}\right), \tag{3.3}$$

$$||[Ju] - \pi \sum_{i=1}^m d_i \delta_{a_i}||_{M^1(\mathbb{T}^2)} \le o_{\gamma_1}(1), \tag{3.4}$$

Moreover, for ρ fixed and $0 < \rho < \frac{r}{2}$,

$$\int_{\mathbb{T}^2_\rho} E^\epsilon(u) dx \le O_{\rho,\gamma_1}(1). \tag{3.5}$$

$$\|Du\|_{L^2(\mathbb{T}^2_\rho)} \le O_{\rho,\gamma_1}(1), \tag{3.6}$$

Finally for any $p \in [1, 2)$,

$$||Du||_{L^p(\mathbb{T}^2)} \le O_{p,\gamma_1}(1) \tag{3.7}$$

and

$$||j(u)||_{L^p(\mathbb{T}^2)} \le O_{p,\gamma_1}(1) \tag{3.8}$$

We have some remarks before commenting on the proof.

Remark 5. The estimate (3.1) says $[Ju]$ is $O(1)$-close to a weighted sum of Dirac masses located at the points x_i. The energy upper bound (3.2) allows us to locate points a_i near x_i and conclude $[Ju]$ is $o(1)$-close to a weig hted sum of Dirac masses located at the points a_i. This observation is exploited in the proof of Theorem 3 below.

Remark 6. Theorem 1 implies several other useful estimates. For u satisfying the hypotheses of the theorem, we have by (3.2) that

$$\||u|^2 - 1\|_{L^2(\mathbb{T}^2)} \leq C\epsilon\Big(\log\frac{r}{\epsilon} + 1\Big)^{\frac{1}{2}}. \tag{3.9}$$

Interpolating with (3.7) gives for any $p < \infty$

$$\|u\|_{L^p(\mathbb{T}^2)} \leq C_p. \tag{3.10}$$

For any fixed ρ with $0 < \rho < \frac{r}{2}$, the estimate (3.6) implies

$$\|Ju\|_{L^1(\mathbb{T}^2_\rho)} \leq C_\rho. \tag{3.11}$$

Remark 7. Related results for vector fields from $\mathbb{R}^n \longmapsto \mathbb{R}^n$ generalizing the two dimensional results above appear in [4].

We briefly describe the idea of the proof of Theorem 1 appearing in [4]. The theorem follows from a localized version with the hypotheses

$$\|[Ju] - \pi d\delta_{x_i}\|_{M^1(B_r(x_i))} \leq \frac{\pi}{200}r$$

$$\int_{B_r(x_i)} E^\epsilon(u)dx \leq \pi\log\frac{r}{e} + \gamma_1.$$

The Jacobian condition implies u must exhibit nonzero degree in many subsets of B_r. One way this condition can be satisfied, which is consistent with the assumed upper bound, is if u has a single isolated vortex of degree d near the center x_i. Careful lower bounds show that, if u deviates significantly from this description — for example, if u has more than one vortex, or if the energy or vorticity of u is spread out too much — then the total energy of u would be too large, contradicting the energy bound. The main technical point is thus lower bounds. These are proved by identifying some set which contains all concentrations of vorticity, and then covering it by balls in some optimal fashion. One can use this covering to relate the energy of the function to the distribution of vorticity. As noted above, this sort of argument has appeared before in [10] and in the work of Sandier [17].

The next result provides additional information about u^ϵ under a stronger energy upper bound hypothesis.

Theorem 2. *Suppose that* $u^\epsilon \in H^1$ *is a sequence such that*

$$[Ju^\epsilon] \rightharpoonup \pi\sum_{i=1}^{m} d_i\delta_{a_i} \text{ weakly as measures,} \tag{3.12}$$

and that there exists some $\gamma_2 \geq 0$ *such that*

$$\int E^\epsilon(u^\epsilon)dx \leq m\left(\pi \log \frac{1}{\rho} + I(\epsilon,\rho)\right) + W(a,d) + C\rho + \gamma_2 + o(1) \tag{3.13}$$

as $\epsilon \to 0$*, for every* $\rho > 0$*. Then*

$$\limsup_{\epsilon\to 0} \|\frac{1}{|u^\epsilon|} j(u^\epsilon) - j(H)\|^2_{L^2(\mathbb{T}^2_\rho)} \leq C\gamma_2 \tag{3.14}$$

and

$$\limsup_{\epsilon\to 0} \|\ D|u^\epsilon|\ \|^2_{L^2(\mathbb{T}^2_\rho)} \leq C\gamma_2 \tag{3.15}$$

for every $\rho > 0$*. Here* $H = H(\cdot; a, d)$ *is the canonical harmonic map and* C *is a universal constant.*

Remark 8. Observe that the conclusion of the theorem is ridiculous if γ_2 is negative which implies the claim in Remark 4 .

We conclude this section with a brief discussion of the $m = 0$ case. Consider GLS_ϵ with initial data $\phi^\epsilon = \rho^\epsilon e^{i\theta}$ with $\rho^\epsilon, \theta \in \mathbb{R}$. The solution is

$$u^\epsilon(x,t) = \rho^\epsilon e^{i\left[\theta + \frac{1}{\epsilon^2}\left((\rho^\epsilon)^2 - 1\right)t\right]}.$$

If $|(\rho^\epsilon)^2 - 1| > O(\epsilon^\alpha)$ for $\alpha < 2$ then u^ϵ oscillates rapidly in the t variable and $u^\epsilon \rightharpoonup 0$ weakly in $L^2(dxdt)$ as $\epsilon \to 0$. The canonical harmonic map in this case is $H(x) = e^{i\theta_1}$ for some constant $\theta_1 \in [0, 2\pi)$. The rapid temporal oscillations require $|(\rho^\epsilon)^2 - 1| = O(\epsilon^2)$ to have $u^\epsilon \to H$ in any strong sense.

The assumptions (3.2) and $m = 0$ imply $|(\rho^\epsilon)^2 - 1| = O(\epsilon)$. Contrast this with assumption (3.13) under the conditions $\gamma_2 = 0$ and $m = 0$ which yields $|(\rho^\epsilon)^2 - 1| = o(\epsilon)$.

The quantity $j(u)$ is insensitive to the temporal oscillations in u since $j(u) = j(e^{i\alpha}u)$ for $\alpha \in \mathbb{R}$. It is therefore natural to expect $j(u)$ to retain more information in the $\epsilon \to 0$ limit. This expectation is verified later in Theorem 3(iv).

4. Vortex paths and convergence

Theorem 1 describes a function u^ϵ (through the measures $[Ju^\epsilon]$ and μ^ϵ_u) provided that u^ϵ satisfies the hypotheses (3.1) and (3.2). The theorem identifies Dirac masses as the $\epsilon \to 0$ limits of $[Ju^\epsilon]$, μ^ϵ_u. The hypotheses (3.1) and (3.2) are chosen to allow for a similar description of $u^\epsilon(t)$, the solution of GLS_ϵ with initial data $u^\epsilon(0) = \phi^\epsilon$, at each t in a time interval $[0, T)$. In this section, we exploit the identity (2.10) and the results stated in Section 3 to show the Dirac masses in Theorem 1 move along Lipschitz paths provided u^ϵ solves GLS_ϵ. The paths are characterized as solutions of certain ordinary differential equations in the next section under stronger assumptions on the initial data.

We assume the initial data $\phi^\epsilon \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ satisfies the hypotheses (1.2) and (1.3). So, there exist distinct points $\{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{T}^2$, integers $\{d_1, \ldots, d_m\}$ with $d_i = \pm 1$ and $\sum d_i = 0$, fixed independently of ϵ such that

$$[J\phi^\epsilon] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{\alpha_i} \tag{4.1}$$

$$\int_{\mathbb{T}^2} E^\epsilon(\phi^\epsilon)\ dx \leq \pi m \log\frac{1}{\epsilon} + \gamma_1, \tag{4.2}$$

We will also assume

$$\int_{\mathbb{T}^2} j(\phi^\epsilon) dx = 0. \tag{4.3}$$

In light of (2.8), this implies $\int j(u^\epsilon(t))dx = 0$ for all time. This assumption is not essential but simplifies the exposition. There are modifications to the statements of Theorems 3 and 4 if $\int j(u^\epsilon(t))dx \neq 0$ which appear in [4].

For $\epsilon > 0$, Bourgain's recent work [2] shows GLS_ϵ is globally (in time) wellposed below H^1. More precisely, there exists $s_0 < 1$ such that for all $s > s_0$ there is a uniquely defined continuous map

$$H^s \longmapsto X \subset C([0,\infty); H^s)$$

taking

$$\phi^\epsilon \longmapsto u^\epsilon,$$

where u^ϵ is the solution of GLS_ϵ.

We can now state our first main theorem.

Theorem 3. *Let u^ϵ be the solution of GLS_ϵ with initial data ϕ^ϵ satisfying* (4.1) *and* (4.2) *and* (4.3). *Then, after passing to a subsequence as $\epsilon \to 0$, there exists a $T > 0$ (independent of ϵ) and Lipschitz pat hs $a_i : [0,T) \longmapsto \mathbb{T}^2$, $a_i(0) = \alpha_i$ such that*
(i) $[Ju^\epsilon(t)] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{a_i(t)}$ weakly as measures for all $t \in [0,T)$.
(ii) $\mu^\epsilon_{u^\epsilon(t)} \rightharpoonup \pi \sum_{i=1}^{m} \delta_{a_i(t)}$ weakly as measures for all $t \in [0,T)$.
(iii) $|u^\epsilon(t)|^2 \to 1$ in $L^2(dx)$ for all $t \in [0,T)$.
(iv) $j(u^\epsilon) = (iu^\epsilon)\cdot Du^\epsilon \rightharpoonup j(H))$ weakly in $L^p(dxdt)$ for all $1 \leq p < 2$ where $H(\cdot,t) = H(\cdot,a(t),d)$ is the canonical harmonic map with singularities of degree (d) located at $(a(t))$.

Proof. 1. Define for $r = \min_{i\neq j}\frac{1}{4}|\alpha_i - \alpha_j|$ with $0 < \epsilon \ll r \leq 1$ the quantity

$$T^\epsilon = \sup\left\{t \geq 0 :\ \|[Ju^\epsilon(s)] - [J\phi^\epsilon]\|_{M^1(\mathbb{T}^2)} \leq \frac{\pi}{400} r \text{ for all } 0 \leq s \leq t\right\}.$$

Recall that $Ju = \det Du = u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1}$. The estimates

$$\|[Ju^\epsilon] - [J\phi^\epsilon]\|_{M^1} \leq \|Ju^\epsilon - J\phi^\epsilon\|_{L^1} \leq C\|u^\epsilon - \phi^\epsilon\|_{H^1}\|u^\epsilon + \phi^\epsilon\|_{H^1}$$

and continuity of the flow of $\phi^\epsilon \longmapsto u^\epsilon(t)$ through H^1 guarantee $T^\epsilon > 0$.
2. **Claim:** For s, t satisfying $0 \leq s, t \leq T^\epsilon$ we have

$$\|[Ju^\epsilon(s)] - [Ju^\epsilon(t)]\|_{M^1(\mathbb{T}^2)} \leq c|s-t| + o_{\gamma_1}(1).$$

Proof of Claim: The definition of T^ϵ guarantees for all $t \in [0, T^\epsilon)$ that the hypothesis (3.1) of Theorem 1 holds if ϵ is sufficiently small. Therefore, for each $t \in [0, T^\epsilon)$, we can find points $a_i(t) \in B_{\frac{r}{2}}(\alpha_i),\ i = 1, \ldots, m$, for which

$$\left\| [Ju^\epsilon(t)] - \pi \sum_{i=1}^{m} d_i \delta_{a_i(t)} \right\|_{M^1(\mathbb{T}^2)} \leq o_{\gamma_1}(1), \tag{4.4}$$

by (3.4). Of course, the $a_i(t)$ may depend upon ϵ. So we can estimate

$$\|[Ju^\epsilon(s)] - [Ju^\epsilon(t)]\|_{M^1(\mathbb{T}^2)} \leq \left\| \pi \sum_{i=1}^{d} d_i (\delta_{a_i(s)} - \delta_{a_i(t)}) \right\|_{M^1(\mathbb{T}^2)} + o_{\gamma_1}(1).$$

By (2.5), we can estimate by

$$\leq \pi \sum_{i=1}^{m} |a_i(s) - a_i(t)| + o_{\gamma_1}(1).$$

The claim will be established once we show for $i = 1, \ldots, m$,

$$|a_i(s) - a_i(t)| \leq c|s-t| + o_{\gamma_1}(1). \tag{4.5}$$

3. We prove (4.5) by using the identity (2.10). Fix i and observe that $a_i(s),\ a_i(t) \in B_{\frac{r}{2}}(\alpha_i)$ for all $s, t \in [0, T^\epsilon)$. There exists an $\eta \in C_c^\infty(B_r(\alpha_i))$ satisfying

$$\eta(x) = \nu \cdot x \text{ for } x \in B_{\frac{3r}{4}}(a_i(0)),\ \nu \in S^1$$

and

$$\pi |a_i(s) - a_i(t)| = \pi d_i \int \eta (\delta_{a_i(s)} - \delta_{a_i(t)}).$$

The conditions on η guarantee that $supp(D^2\eta) \subset B_r(\alpha_i) \setminus B_{\frac{3r}{4}}(\alpha_i)$. Notice that η depends upon the index i.

Insert the function η described above into (2.10) and again use (4.4) to observe

$$\pi |a_i(s) - a_i(t)| = \int_s^t \int_{B_r(\alpha_i)} \eta_{x_j x_l} \mathbb{J}_{jk}\ u_{x_k} \cdot u_{x_l}\ dx\ d\tau + o_{\gamma_1}(1).$$

The support properties of $\eta_{x_j x_l}$ permit us to replace $B_r(\alpha_i)$ by $B_r(\alpha_i) \setminus B_{\frac{3r}{4}}(\alpha_i)$. Finally, we estimate by

$$\leq |s-t|\ \|D^2\eta\|_{L^\infty} \sup_{\tau \in [s,t]} \|Du(\tau)\|_{L^2(B_r \setminus B_{\frac{3r}{4}})}.$$

The size of $\|D^2\eta\|_{L^\infty}$ depends upon r but is independent of ϵ and (3.6) permits us to control the Du term by a constant independent of ϵ, so (4.5) follows and the claim is proven. We also note that the claim implies T^ϵ may be taken independently of ϵ, so we denote this quantity by T from now on.

4. The remaining convergence claims follow from the bounds stated in Theorem 1 and passing to subsequences, except for (iii) which follows directly from (4.2). We prove (iv). Fix any $p \in [1,2)$. Since the conditions of Theorem 1 hold for every $t \in [0,T)$, we deduce from (3.8) that

$$\|j(u^\epsilon)\|_{L^p(\mathbb{T}^2\times[0,T))} \leq O_{p,\gamma_1,T}(1).$$

It follows, upon passing to a subsequence as $\epsilon \to 0$, that

$$j(u^\epsilon) \rightharpoonup \bar{j} \text{ weakly in } \ L^p(dxdt)$$

for some $\bar{j}$. We wish to identify $\bar{j}$.

Let $\phi \in C_0^\infty(\mathbb{T}^2 \times [0,T))$. The identity (2.7) implies

$$\begin{aligned} \int j(u^\epsilon)\cdot D\phi dxdt &= \int \phi_t \frac{|u^\epsilon|^2}{2} dxdt \\ &= \int \phi_t \frac{1}{2}(|u^\epsilon|^2 - 1) dxdt \\ &\to 0 \end{aligned}$$

as $\epsilon \to 0$ for every t, by (iii). Therefore div $\bar{j} = 0$. Moreover, from (i) we have $\nabla \times \bar{j} = 2[\bar{J}] \otimes dt = 2\pi \sum d_i \delta_{a_i(t)} \otimes dt$ weakly.

Let $H(x,t) = H(x,a(t),d)$. If we define $V = \bar{j} - j(H)$, we have

$$\text{div } V = \nabla \times V = 0$$

weakly. Let η^δ be a standard mollifier and set $V^\delta = V * \eta^\delta$. The convolution here is in space and time. The above considerations imply

$$\text{div } V^\delta = \nabla \times V^\delta = 0$$

in $\mathbb{T}^2$ for every $t < T$. Since V^δ is smooth, this implies $V^\delta(x,t) = g^\delta(t)$. Letting $\delta \to 0$, we find that V is also constant in x for each fixed t. For any fixed t we have

$$\begin{aligned} \int V(x,t)dx &= \int \left(\bar{j} - j(H)\right)(x,t)dx \\ &= \int \bar{j}(x,t)\ dx \\ &= \lim_{\epsilon\to 0} \int j(u^\epsilon)(x,t)dx \\ &= 0 \end{aligned}$$

using (4.3). □

Remark 9. The proof given above may be iterated until

$$T = \inf\{t > 0 : |a_i(t) - a_j(t)| \to 0 \text{ for some } i \neq j\}.$$

5. Vortex dynamics

Under the more restrictive upper bound on the energy (3.13), we characterize the vortex paths $a_i(\cdot)$ as solutions of a system of ordinary differential equations that conserves the renormalized energy. More precisely, we have the follo wing result.

Theorem 4. *Suppose ϕ^ϵ satisfies*

$$[J\phi^\epsilon] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{\alpha_i} \text{ weakly as measures,} \tag{5.1}$$

and for every $\rho > 0$ as $\epsilon \to 0$,

$$\int_{\mathbb{T}^2} E^\epsilon(\phi^\epsilon)dx \leq m\left(\pi \log \frac{1}{\rho} + I(\epsilon,\rho)\right) + W(\alpha, d) + C\rho + o(1). \tag{5.2}$$

Let $\{a_1(\cdot), \dots, a_m(\cdot)\}$ and H be as in Theorem 3. Then the following statements hold:
(i) For each i and for $t \in [0, T_1)$,

$$\begin{cases} \frac{d}{dt} a_i = 2 \sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) = -\frac{1}{\pi} d_i \mathbb{J} D_{a_i} W(a,d) \\ \qquad\qquad a_i(0) = \alpha_i. \end{cases}$$

Here $T_1 > 0$ can be taken to be the largest time such that the above ODE has a solution on $[0, T_1)$.
(ii) For every $t \in [0, T_1)$,

$$\frac{1}{|u^\epsilon|} j(u^\epsilon) \to j(H) \text{ strongly in } L^2_{loc}(\mathbb{T}^2 \setminus (a(t))).$$

(iii) For every $\rho > 0$ and $t \in [0, T_1)$,

$$\lim_{\epsilon \to 0} \min_{\alpha \in [0,2\pi)} \left\| u^\epsilon(\cdot, t) - e^{i\alpha} H(\cdot, a(t), d) \right\|_{H^1(\mathbb{T}^2_\rho)} = 0$$

Remark 10. For an example of a point vortex system which develops a singularity in finite time, see Marchioro and Pulvirenti [14]. We expect, however, that for generic initial data the ODE does not develop singularities, in which case the above result is valid globally in time.

Remark 11. The example given at the end of Section 3 shows that the sort of convergence in *(iii)* is more or less optimal, unless (5.2) is strengthened still further.

Proof. 1. Let $a_i(t)$, $i = 1, \dots, m$ denote the paths of Theorem 2. Let $b_i(t)$ denote the solution of the system

$$\begin{cases} \frac{d}{dt} b_i = 2 \sum_{j:j\neq i} d_j \nabla \times F(b_i - b_j) \\ \qquad b_i(0) = \alpha_i. \end{cases} \tag{5.3}$$

A calculation shows that

$$D_{a_i} W = -2\pi \sum_{j:j\neq i} d_i d_j DF(a_i - a_j).$$

Therefore, the ODE in (5.3) may be reexpressed, as in (i), in Hamiltonian form showing that the renormalized energy W is conserved. This ODE system has a unique solution on a nontrivial time interval $[0, T')$. Let

$$T_1 = \min(T, T')$$

where T is as in Remark 9. Note that T_1 is independent of ϵ. We wish to show for all i that $b_i(t)$ coincides with $a_i(t)$ on the time interval $[0, T_1)$. Observe that this will imply $T_1 = T'$.

For $t \in [0, T_1)$, let

$$\zeta(t) = \sum_i |b_i(t) - a_i(t)|.$$

It suffices to prove that, given any $\widetilde{T} < T_1$, we can find some small $\delta(\widetilde{T}) > 0$ and a constant $C = C(\widetilde{T})$ such that

$$\frac{d}{dt}\zeta(t) \leq C\zeta(t) \tag{5.4}$$

for *a.e.* $t \in [0, \widetilde{T}]$ whenever $\zeta(t) \leq \delta$. We will show that (5.4) holds at each point where $a_i(\cdot)$ is differentiable for all i; by Rademacher's theorem, this condition is satisfied on a set of full measure.

Fix $\widetilde{T} < T_1$. By Remark 9, there is some $r = r(\widetilde{T}) > 0$ for which

$$\min_{i \neq j,\ t \leq \widetilde{T}} |a_i(t) - a_j(t)| \geq 4r. \tag{5.5}$$

2. We use the fact that b_i solves (5.3) and the triangle inequality to estimate

$$\begin{aligned}
\frac{d\zeta}{dt} &\leq \sum_i |b_{i,t} - a_{i,t}| \\
&\leq 2\sum_i \left| \sum_{j:j\neq i} d_j \nabla \times F(b_i - b_j) - \sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) \right| \\
&+ \sum_i \left| a_{i,t} - 2\sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) \right| \\
&= Term\ 1 + Term\ 2.
\end{aligned}$$

We immediately dispose of $Term$ 1. Fix $s < \widetilde{T}$ and a pair of indices $i \neq j$. Let $h = |(b_i(s) - b_j(s)) - (a_i(s) - a_j(s))|$. Note that by assumption $h \leq \zeta(s) \leq \delta$. By Taylor's theorem, at the fixed time s, we have

$$\begin{aligned}
|\nabla \times F(b_i - b_j) - \nabla \times F(a_i - a_j)| &\leq h \max_{\{x:|x-(a_i-a_j)|\leq h\}} |D^2 F| \\
&\leq C\zeta(s).
\end{aligned}$$

The last inequality follows from (5.5) provided $\delta < r$. Therefore, $Term$ 1 satisfies the desired estimate (5.4).

3. We turn our attention to $Term$ 2, which is a sum of terms $(Term\ 2)_i$, with

$i = 1, \dots, m$. Suppose that each function $a_i(\cdot)$ is differentiable at $s \in [0, \widetilde{T})$. Fix $\eta \in C_0^\infty$ such that $supp(\eta) \subset B_{r(\widetilde{T})}(a_i(s))$ and $\eta(x) = \nu \cdot x$ in a neighborhood of $a_i(s)$. Here we take $\nu \in S^1$ to satisfy

$$(Term\ 2)_i = d_i \nu \cdot \left(a_{i,t}(s) - 2 \sum_{j:j \neq i} d_j \nabla \times F(a_i(s) - a_j(s)) \right). \tag{5.6}$$

We will write ϵ_n for an appropriate subsequence as $\epsilon \to 0$. Since

$$[Ju^{\epsilon_n}(t)] \rightharpoonup \pi \sum d_i \delta_{a_i(t)} \text{ weakly as measures}$$

we can rewrite

$$\begin{aligned} d_i \nu \cdot a_{i,t}(s) &= \lim_{h \to 0} d_i \nu \cdot \frac{1}{h}(a_i(s+h) - a_i(s)) \\ &= \lim_{h \to 0} \lim_{n \to \infty} \frac{1}{\pi h} \int_{\mathbb{T}^2} (\eta [Ju^{\epsilon_n}(s+h)] - \eta [Ju^{\epsilon_n}(s)]) \, dx \\ &= \lim_{h \to 0} \lim_{n \to \infty} \frac{1}{\pi h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} u^{\epsilon_n}_{x_k} \cdot u^{\epsilon_n}_{x_l} dx dt. \end{aligned}$$

We used (2.10) in the last step.

Let $H(x, t) = H(x, a(t), d)$. We reexpress the remaining term in (5.6) using Lemma 1 which is stated and proven below,

$$\begin{aligned} & d_i \nu \cdot \left(2 \sum_{j:j \neq i} d_j \nabla \times F(a_i(s) - a_j(s)) \right) \\ = {} & \lim_{h \to 0} \frac{2}{h} \int_s^{s+h} d_i \nu \cdot \left(\sum_{j:j \neq i} d_j \nabla \times F(a_i(t) - a_j(t)) \right) dt \\ = {} & \lim_{h \to 0} \frac{1}{\pi h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) dx dt. \end{aligned}$$

Therefore,

$$(Term\ 2)_i = \lim_{h \to 0} \lim_{n \to \infty} \frac{1}{\pi h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} \left(u^{\epsilon_n}_{x_k} \cdot u^{\epsilon_n}_{x_l} - j^k(H) j^l(H) \right) dx dt. \tag{5.7}$$

Inside the integral, $H = H(\cdot, a(t), d)$ and $Du^\epsilon = Du^\epsilon(t)$.

On any set where $|u| > 0$, we can decompose u_{x_k}

$$u_{x_k} = \left[\frac{iu}{|u|} \cdot u_{x_k} \right] \frac{iu}{|u|} + \left[\frac{u}{|u|} \cdot u_{x_k} \right] \frac{u}{|u|}$$

$$= \frac{j^k(u)}{|u|} \frac{iu}{|u|} + |u|_{x_k} \frac{u}{|u|}. \tag{5.8}$$

Since $|Du| = 0$ $a.e.$ on the set $|u| = 0$, the representation above holds $a.e.$ on $supp(Du)$ and

$$u^\epsilon_{x_k} \cdot u^\epsilon_{x_l} = \frac{1}{|u^\epsilon|^2} j^k(u^\epsilon) j^l(u^\epsilon) + |u^\epsilon|_{x_k} |u^\epsilon|_{x_l}. \tag{5.9}$$

We will have proved (5.4) when we show

$$\lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} \left(|u^{\epsilon_n}|_{x_k} |u^{\epsilon_n}|_{x_l} \right) dx dt \le C\zeta(s), \tag{5.10}$$

and

$$\lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} \left(\frac{1}{|u^{\epsilon_n}|^2} j^k(u^{\epsilon_n}) j^l(u^{\epsilon_n}) - j^k(H) j^l(H) \right) dx dt \le C\zeta(s). \tag{5.11}$$

These estimates will follow from the tight upper bound (5.2) (see also (3.13)) on the energy and energy conservation.

4. The renormalized energy W is conserved for solutions $b(\cdot)$ of (5.3). This is apparent from the fact that (5.3) can be written in Hamiltonian form. Also $\int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x) dx$ is conserved for solutions u^ϵ o f GLS_ϵ. Therefore, for every $t \le \widetilde{T}$ and every $\rho > 0$, the upper bound (5.2) gives

$$\begin{aligned} \int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x) dx &= \int_{\mathbb{T}^2} E^\epsilon(\phi^\epsilon(\cdot))(x) dx \\ &\le m(\pi \log \frac{1}{\rho} + I(\epsilon,\rho)) + W(a(0),d) + C\rho + o(1) \\ &= m(\pi \log \frac{1}{\rho} + I(\epsilon,\rho)) + W(b(t),d) + C\rho + o(1). \end{aligned}$$

Arguing as in the estimate of $Term$ 1, we see that

$$W(b(t)) - W(a(t)) \le C \sum |b_i(t) - a_i(t)| = C\zeta(t)$$

provided δ is small enough. Therefore

$$\int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x) dx \le m \left(\pi \log \frac{1}{\rho} + I(\epsilon,\rho) \right) + W(a(t),d) + C\rho + C\zeta(t) + o(1), \tag{5.12}$$

as $\epsilon \to 0$ for every $\rho > 0$. We have from Theorem 3 that

$$[Ju^{\epsilon_n}(t)] \rightharpoonup \pi \sum_{i=1}^m d_i \delta_{a_i(t)} \text{ weakly as measures.} \tag{5.13}$$

The conditions (5.12), (5.13) are precisely the hypotheses of Theorem 3 with $\gamma_2 = C\zeta(t)$. So, for every $t \in [s, s+h]$,

$$\limsup_{n\to\infty} \|D|u^{\epsilon_n}|(\cdot,t)\|^2_{L^2(\mathbb{T}^2_\rho)} \le C\zeta(t) \tag{5.14}$$

and

$$\limsup_{n\to\infty}\left\|\frac{1}{|u^{\epsilon_n}|}j(u^{\epsilon_n})-j(H)\right\|^2_{L^2(\mathbb{T}^2_\rho)}\le C\zeta(t) \tag{5.15}$$

These estimates allow us to prove (5.10), (5.11).

We quickly estimate (5.10) by observing

$$(5.10)\le\lim_{h\to 0}\lim_{n\to\infty}\|D^2\eta\|_{L^\infty(\mathbb{T}^2_\rho)}\frac{C}{h}\int_s^{s+h}\|D|u^{\epsilon_n}|(\cdot,t)\|^2_{L^2(\mathbb{T}^2_\rho)}dt \tag{5.16}$$

and using (5.14).

5. We now establish (5.11). First, we show that

$$\left(\frac{1}{|u^{\epsilon_n}|}j(u^{\epsilon_n})-j(H)\right)\rightharpoonup 0 \text{ weakly in } L^2(\mathbb{T}^2_\rho\times[s,s+h]). \tag{5.17}$$

To see this, note that $\frac{1}{|u^{\epsilon_n}|}j(u^{\epsilon_n})$ is uniformly bounded in $L^2(\mathbb{T}^2_\rho\times[s,s+h])$ and hence converges weakly to some limit $\bar{j}$. We know from Theorem 3(iv) that $j(u^{\epsilon_n})\rightharpoonup j(H)$ weakly in $L^p(dxdt)$ for all $1\le p<2$. We also know from Theorem 3(iii) that $|u^{\epsilon_n}|^2\ rightarrow 1$ strongly in $L^2(dxdt)$. Thus

$$\begin{aligned} j(H) &= \text{weak } L^1\lim_{n\to\infty}j(u^{\epsilon_n})\\ &= \text{weak } L^1\lim_{n\to\infty}\left(\frac{j(u^{\epsilon_n})}{|u^{\epsilon_n}|}|u^{\epsilon_n}|\right)\\ &= \left(\text{strong } L^2\lim_{n\to\infty}|u^{\epsilon_n}|\right)\left(\text{weak } L^2\lim_{n\to\infty}\frac{j(u^{\epsilon_n})}{|u^{\epsilon_n}|}\right)\\ &= \bar{j}\end{aligned}$$

which proves (5.17).

For fixed k,l, observe that the quadratic term in (5.11) can be written

$$\left(\frac{1}{|u^{\epsilon_n}|}j^k(u^{\epsilon_n})-j^k(H)\right)\left(\frac{1}{|u^{\epsilon_n}|}j^l(u^{\epsilon_n})+j^l(H)\right)$$

and reexpressed as

$$\left(\frac{1}{|u^{\epsilon_n}|}j^k(u^{\epsilon_n})-j^k(H)\right)\left(\frac{1}{|u^{\epsilon_n}|}j^l(u^{\epsilon_n})-j^l(H)\right)+2j^l(H)\left(\frac{1}{|u^{\epsilon_n}|}j^k(u^{\epsilon_n})-j^k(H)\right).$$

Since $\left(\frac{1}{|u^{\epsilon_n}|}j^k(u^{\epsilon_n})-j^k(H)\right)\rightharpoonup 0$ weakly in $L^2(\mathbb{T}^2_\rho\times[s,s+h])$ and $j^l(H)$ does not depend upon n, the second expression contributes nothing as $n\to\infty$. The first expressi on is controlled using (5.15).

6. Since we have appropriately bounded $Term\ 1$ and $Term\ 2$, we have proven (5.4). Gronwall's inequality implies $\zeta=0$ which gives (i) of the Theorem. Since $\zeta=0$, (5.15) implies (ii). We conclude by proving (iii). Fix t and $\rho>0$. Let u^ϵ be a

subsequence which converges in $L^2(\mathbb{T}^2_\rho)$ to some limit $\bar{u}$. We may assume, by (3.6), that $Du^\epsilon \rightharpoonup D\bar{u}$ weakly in $L^2(\mathbb{T}^2_\rho)$. Also, (ii) gives

$$\liminf_{\epsilon\to 0} \|Du^\epsilon\|_{L^2(\mathbb{T}^2_\rho)} \geq \lim_{\epsilon\to 0} \left\| \frac{j(u^\epsilon)}{|u^\epsilon|} \right\|_{L^2(\mathbb{T}^2_\rho)} = \|D\bar{u}\|_{L^2(\mathbb{T}^2_\rho)}.$$

Therefore $Du^\epsilon \to D\bar{u}$ strongly in $L^2(\mathbb{T}^2_\rho)$. Finally, since $j(\bar{u}) = j(H)$, Proposition 2 implies $\bar{u} = e^{i\alpha}H$ for some $\alpha \in \mathbb{R}$.

□

We state and prove the lemmas employed above.

Lemma 1. *Suppose that $\eta \in C^2$ and that*

$$supp(\eta) \cap \{a_1, ..., a_m\} = \{a_i\}; \qquad D^2\eta \equiv 0 \text{ in a neighborhood of } a_i.$$

Let $H := H(\cdot; a, d)$ be the canonical harmonic map. Then

$$\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) = d_i D\eta(a_i) \cdot \left(2\pi \sum_{j:j\neq i} d_j \; \nabla \times F(a_i - a_j) \right)$$

Remark 12. This computation remains valid if $d_1, ..., d_m$ assume arbitrary integer values, that is, if we lift the assumption that $d_i = \pm 1$ for all i.

Proof. 1. We reexpress the integral in the lemma. Recall that $j(H) = -\nabla \times \Phi$ where Φ satisfies

$$\Delta\Phi = \sum_{i=1}^{m} 2\pi d_i \delta_{a_i}, \tag{5.18}$$

and, using (2.14), we write

$$\Phi(x) = d_i F(x - a_i) + G(x) \tag{5.19}$$

where

$$G(x) = \sum_{j:j\neq i} d_j F(x - a_j).$$

Since $j^k(H) = \mathbb{J}_{km}\Phi_{x_m}$, we have

$$\mathbb{J}_{jk} j^k(H) j^l(H) = -\mathbb{J}_{ln}\Phi_{x_n}\Phi_{x_j}. \tag{5.20}$$

Fix any number ρ so small that $D^2\eta = 0$ on $B_\rho(a_i)$. We have

$$\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) dx = - \int_{\mathbb{T}^2 \setminus B_\rho(a_i)} \eta_{x_j x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j}$$

$$= \int_{\mathbb{T}^2 \setminus B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n x_j} \Phi_{x_j} dx + \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j} \nu^j dH^1$$

where $\nu = (\nu^1, \nu^2)$ is the outward unit normal to ∂B_ρ. We recognize $\Phi_{x_n x_j}\Phi_{x_j} = \frac{1}{2}(\Phi_{x_j}\Phi_{x_j})_{x_n}$ and integrate by parts again to find

$$= -\int\limits_{\mathbb{T}^2\setminus B_\rho} \eta_{x_l x_n}\mathbb{J}_{ln}\frac{1}{2}\Phi_{x_j}\Phi_{x_j}\,dx \quad - \int\limits_{\partial B_\rho} \eta_{x_l}\mathbb{J}_{ln}\frac{1}{2}\Phi_{x_j}\Phi_{x_j}\nu^n\,dH^1$$
$$+ \int\limits_{\partial B_\rho} \eta_{x_l}\mathbb{J}_{ln}\Phi_{x_n}\Phi_{x_j}\nu^j\,dH^1.$$

Since $\eta_{x_l x_n}\mathbb{J}_{ln} = 0$, the integral over $\mathbb{T}^2 \setminus B_\rho$ vanishes and we are left with two boundary integrals I_ρ, II_ρ.
2. We calculate the boundary integrals. We begin with

$$I_\rho = -\int\limits_{\partial B_\rho} \eta_{x_l}\mathbb{J}_{ln}\frac{1}{2}\Phi_{x_j}\Phi_{x_j}\nu^n\,dH^1.$$

By using (5.19), we observe

$$\Phi_{x_j}\Phi_{x_j} = F_{x_j}F_{x_j} + 2d_i F_{x_j}G_{x_j} + G_{x_j}G_{x_j}. \tag{5.21}$$

Since F is even, the first term integrates to zero. We exploit the fact that G_{x_j} is nearly constant on $B_\rho(a_i)$ to calculate the contribution to I_ρ arising from the remaining two terms in (5.21). The cross term contributes

(5.22)
$$-d_i\eta_{x_l}(a_i)G_{x_j}(a_i)\mathbb{J}_{ln}\int\limits_{\partial B_\rho} F_{x_j}\nu^n\,dH^1 - d_i\eta_{x_l}(a_i)\mathbb{J}_{ln}\int\limits_{\partial B_\rho} F_{x_j}[G_{x_j}(x) - G_{x_j}(a_i)]\nu^n\,dH^1.$$

Since $F \sim \log|x - a_i|$, $|F_{x_j}| \sim \frac{1}{\rho}$ on ∂B_ρ and G is C^1 on B_ρ, the second integral contributes $O(\rho)$. The $G_{x_j}G_{x_j}$ term contributes $O(\rho)$ as well.

Next, we calculate

$$II_\rho = \int\limits_{\partial B_\rho} \eta_{x_l}\mathbb{J}_{ln}\Phi_{x_n}\Phi_{x_j}\nu^j\,dH^1$$

by expanding using (5.19). The $F_{x_n}F_{x_j}$ term again vanishes by symmetry. The $G_{x_n}G_{x_j}$ term contributes $O(\rho)$ and the cross terms remain to be estimated. The first cross term gives

$$d_i\eta_{x_l}(a_i)G_{x_j}(a_i)\mathbb{J}_{ln}\int\limits_{\partial B_\rho} F_{x_n}\nu^j\,dH^1 + O(\rho). \tag{5.23}$$

The second cross term contributes

$$d_i\eta_{x_l}(a_i)G_{x_n}(a_i)\mathbb{J}_{ln}\int\limits_{\partial B_\rho} F_{x_j}\nu^j\,dH^1 + O(\rho). \tag{5.24}$$

3. Since $\int_{\partial B_\rho} F_{x_n} \nu^j dH^1 = \int_{\partial B_\rho} F_{x_j} \nu^n dH^1$, the first term in (5.22) and (5.23) cancel and the only remaining contribution is (5.24). Finally, observe that

$$\int_{\partial B_\rho} F_{x_j} \nu^j dH^1 = \int_{B_\rho} \Delta F dx = 2\pi - 2\pi^2\rho^2, \tag{5.25}$$

using (2.11). Therefore

$$\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) dx = 2\pi d_i \eta_{x_l}(a_i) \mathbb{J}_{ln} G_{x_n}(a_i) + O(\rho). \tag{5.26}$$

Since ρ can be taken arbitrarily small, we have proved the lemma.

□

6. Acknowledgments

Many of the ideas in this paper grow out of the second author's joint work with H.M. Soner and reflect the influence of numerous conversations over a period of years. The first author was partially supported by a grant from the University of Illinois Research Board and by a NSF Postdoctoral Fellowship. The second author was partially supported by NSF grant # DMS 96-00080.

References

[1] F. Bethuel, H. Brezis, and F Hélein. *Ginzburg-Landau Vortices.* Birkhäuser, Boston, 1994.
[2] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I,II. *Geom. Funct. Anal.*, 3:107–156, 209–262, 1993.
[3] H. Brézis, J. M. Coron, and E. Lieb. Harmonic maps with defects. *Comm. Math. Phys.*, 107(4):649–705, 1986.
[4] J. E. Colliander and R. L. Jerrard. Ginzburg-Landau Vortex Dynamics: weak stability and Schrödinger equation dynamics. preprint, 1997.
[5] W. E. Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity. *Physica D*, 77:383–404, 1994.
[6] L.C. Evans and W. Gangbo. Differential equations methods for the Monge- Kantorovitch mass transfer problem. to appear.
[7] A.L. Fetter. Vortices in an Imperfect Bose Gas. I. the Condensate. *Physical Review*, 138(2A):A429–A437, 1965.
[8] V.L. Ginzburg and L.P. Pitaevskii. On the theory of superfluidity. *Soviet Physics JETP*, 34(7)(5):858–861, 1958.
[9] E.P. Gross. Dynamics of interacting bosons. In E. Meeron, editor, *Physics of many particle systems*, pages 231–406. Gordon and Breech, New York, 1966.
[10] R.L. Jerrard. Lower bounds for generalized Ginzburg-Landau functionals. preprint, 1997.
[11] R.L. Jerrard and H.M. Soner. Dynamics of Ginzburg-Landau vortices. to appear, Arch. Rat. Mech. Anal., 1998.
[12] L.D. Landau. *Journal of Experimental and Theoretical Physics (U.S.S.R.)*, 11, 1941.
[13] F.H. Lin. Some dynamical properties of Ginzburg-Landau vortices. *Comm. Pure. Appl. Math.*, 49(4), 1996.
[14] C. Marchioro and M. Pulvirenti. *Mathematical theory of incompressible nonviscous fluids.* Springer, 1994.
[15] J.C. Neu. Vortices in complex scalar fields. *Physica D*, 43:385–406, 1990.

[16] L.P. Pitaevskii. Vortex lines in an imperfect Bose gas. *Soviet Physics JETP*, 13(2):451 –454, 1961.
[17] E. Sandier. Lower bounds for the energy of unit vector fields and applications. preprint, 1997.

J. E. Colliander, Mathematical Sciences Research Institute, Berkeley, CA 94720

R. L. Jerrard, University of Illinois, Urbana, IL 61801