A QUANTUM OCTONION ALGEBRA

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To the memory of Alberto Izquierdo

Introduction

Using the representation theory of the quantum group $U_q(\mathrm{D}_4)$ of D_4 we construct a quantum analogue $\mathbb{O}_q$ of the split octonions and study its properties.

A unital algebra over a field with a nondegenerate bilinear form $(\,|\,)$ of maximal Witt index which admits composition,

$$(x \cdot y | x \cdot y) = (x|x)(y|y),$$

must be the field, two copies of the field, the split quaternions, or the split octonions. There is a natural q-version of the composition property that the algebra $\mathbb{O}_q$ of quantum octonions is shown to satisfy (see Prop. 4.12 below). We also prove that the quantum octonion algebra $\mathbb{O}_q$ satisfies the "q-Principle of Local Triality" (Prop. 3.12). Inside the quantum octonions are two nonisomorphic 4-dimensional subalgebras, which are q-deformations of the split quaternions. One of them is unital, and both of them give gl_2 when considered as algebras under the commutator product $[x, y] = x \cdot y - y \cdot x$.

By its construction, $\mathbb{O}_q$ is a nonassociative algebra with a Yang-Baxter operator action coming from the R-matrix of $U_q(\mathrm{D}_4)$. Associative algebras with a Yang-Baxter operator (or r-algebras) arise in Manin's work on noncommutative geometry and include such important examples as Weyl and Clifford algebras, quantum groups, and certain universal enveloping algebras (see for example [B]).

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In the process of constructing $\mathbb{O}_q$ we define an algebra $\mathbb{P}_q$, which is a q analogue of the 8-dimensional para-Hurwitz algebra. The quantum para-Hurwitz algebra $\mathbb{P}_q$ is shown to satisfy certain identities which become familiar properties of the para-Hurwitz algebra at $q = 1$. The para-Hurwitz algebra is a (non-unital) 8-dimensional algebra with a non-degenerate associative bilinear form admitting composition. The algebra $\mathbb{P}_q$ exhibits related properties (see Props. 5.1, 5.3).

While this paper was in preparation, we received a preprint of [Br], which constructs a quantized octonion algebra using the representation theory of $U_q(sl_2)$. Although both Bremner's quantized octonions and our quantum octonions reduce to the octonions at $q = 1$, they are different algebras. The quantized octonion algebra in [Br] is constructed from defining a multiplication on the sum of irreducible $U_q(sl_2)$-modules of dimensions 1 and 7. As a result, it carries a $U_q(sl_2)$-module structure and has a unit element. The quantum octonion algebra constructed in this paper using $U_q(\mathrm{D}_4)$ has a unit element only for the special values $q = 1, -1$.

An advantage to the $U_q(\mathrm{D}_4)$ approach is that allows properties such as the ones mentioned above to be derived from its representation theory. Using the fact that the product in $\mathbb{O}_q$ is a $U_q(\mathrm{D}_4)$-module homomorphism, we prove identities for $\mathbb{O}_q$ (see Section 4) and as a consequence obtain at $q = 1$ new "representation theory" proofs for very well-known identities satisfied by the octonions that had been established previously by other methods. The $U_q(\mathrm{D}_4)$ approach also affords connections with fixed points of graph automorphisms - although as we show in Section 7, the fixed point subalgebras of $U_q(\mathrm{D}_4)$ are not the quantum groups $U_q(\mathrm{G}_2)$ and $U_q(\mathrm{B}_3)$, because those quantum groups do not have Hopf algebra embeddings into $U_q(\mathrm{D}_4)$. In the final section of this paper we explore connections with quantum Clifford algebras. In particular, we obtain a $U_q(\mathrm{D}_4)$-isomorphism between the quantum Clifford algebra $C_q(8)$ and the endomorphism algebra $\mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$ and discuss its relation to the work of Ding and Frenkel [DF] (compare also [KPS]).

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§1. Preliminaries on Quantum Groups

The quantum group $U_q(\mathfrak{g})$.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ, and simple roots $\Pi = \{\alpha_i \mid i = 1, \ldots, r\} \subset \Phi$ relative to $\mathfrak{h}$. Let $\mathfrak{A} = (a_{\alpha,\beta})_{\alpha,\beta \in \Pi}$ be the corresponding Cartan matrix. Thus, $a_{\alpha,\beta} = 2(\beta,\alpha)/(\alpha,\alpha)$, and there exist relatively prime positive integers,

$$d_\alpha = \frac{(\alpha,\alpha)}{2}, \tag{1.1}$$

so the matrix $(d_\alpha a_{\alpha,\beta})$ is symmetric. These integers are given explicitly by

A_r, D_r, E_r $(r = 6, 7, 8)$	$d_{\alpha_i} = 1$ for all $1 \le i \le r$
B_r	$d_{\alpha_i} = 2$ for all $1 \le i \le r-1$, and $d_{\alpha_r} = 1$
C_r	$d_{\alpha_i} = 1$ for all $1 \le i \le r-1$, and $d_{\alpha_r} = 2$
F_4	$d_{\alpha_i} = 2, i = 1, 2$, and $d_{\alpha_i} = 1, i = 3, 4$
G_2	$d_{\alpha_1} = 1$ and $d_{\alpha_2} = 3$. (1.2)

Let $\mathbf{K}$ be a field of characteristic not 2 or 3. Fix $q \in \mathbf{K}$ such that q is not a root of unity, and such that $q^{\frac{1}{2}} \in \mathbf{K}$. For $m, n, d \in \mathbf{Z}_{>0}$, let

$$[m]_d = \frac{q^{md} - q^{-md}}{q^d - q^{-d}}, \qquad \text{and set} \tag{1.3}$$

$$[m]_d! = \prod_{j=1}^{m} [j]_d \tag{1.4}$$

and $[0]_d! = 1$. Let

$$\begin{bmatrix} m \\ n \end{bmatrix}_d = \frac{[m]_d!}{[n]_d![m-n]_d!}. \tag{1.5}$$

Definition 1.6. *The quantum group $U = U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbf{K}$ generated by elements $E_\alpha, F_\alpha, K_\alpha$, and K_α^{-1} (for all $\alpha \in \Pi$) and subject to the relations,*

(Q1) $K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \qquad K_\alpha K_\beta = K_\beta K_\alpha$

(Q2) $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$

(Q3) $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$

(Q4) $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \dfrac{K_\alpha - K_\alpha^{-1}}{q^{d_\alpha} - q^{-d_\alpha}}$

$$\textit{(Q5)}\ \sum_{s=0}^{1-a_{\alpha,\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha,\beta} \\ s \end{bmatrix}_{d_\alpha} E_\alpha^{1-a_{\alpha,\beta}-s} E_\beta E_\alpha^s = 0$$

$$\textit{(Q6)}\ \sum_{s=0}^{1-a_{\alpha,\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha,\beta} \\ s \end{bmatrix}_{d_\alpha} F_\alpha^{1-a_{\alpha,\beta}-s} F_\beta F_\alpha^s = 0.$$

Corresponding to each $\lambda = \sum_{\alpha\in\Pi} m_\alpha \alpha$ in the root lattice $\mathbf{Z}\Phi$ there is an element

$$K_\lambda = \prod_{\alpha\in\Pi} K_\alpha^{m_\alpha},$$

in $U_q(\mathfrak{g})$, and $K_\lambda K_\mu = K_{\lambda+\mu}$ for all $\lambda, \mu \in \mathbf{Z}\Phi$. Using relations (Q2),(Q3), we have

$$K_\lambda E_\beta K_\lambda^{-1} = q^{(\lambda,\beta)} E_\beta \qquad \text{and} \qquad K_\lambda F_\beta K_\lambda^{-1} = q^{-(\lambda,\beta)} F_\beta \tag{1.7}$$

for all $\lambda \in \mathbf{Z}\Phi$ and $\beta \in \Pi$.

The algebra $U_q(\mathfrak{g})$ has a noncommutative Hopf structure with comultiplication Δ, antipode S, and counit ϵ given by

$$\begin{gathered}
\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \\
\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha \\
\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha \\
S(K_\alpha) = K_\alpha^{-1}, \\
S(E_\alpha) = -K_\alpha^{-1} E_\alpha \qquad S(F_\alpha) = -F_\alpha K_\alpha \\
\epsilon(K_\alpha) = 1, \qquad \epsilon(E_\alpha) = \epsilon(F_\alpha) = 0.
\end{gathered} \tag{1.8}$$

The opposite comultiplication Δ^{op} has the property that if

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then} \quad \Delta^{\mathrm{op}}(a) = \sum_a a_{(2)} \otimes a_{(1)}.$$

(We are adopting the commonly used Sweedler notation for components of the comultiplication.) The algebra U with the same multiplication, same identity, and same counit, but with comultiplication given by Δ^{op} and with S^{-1} as its antipode is also a Hopf algebra. The comultiplication Δ^{op} can be regarded as the composition $\tau\circ\Delta$, where $\tau : U\otimes U \to U\otimes U$ is given by $\tau(a\otimes b) = b\otimes a$. The map $\psi : U \to U_q(\mathfrak{g})$ with $E_\alpha \mapsto F_\alpha$, $F_\alpha \mapsto E_\alpha$, $K_\alpha \mapsto K_\alpha^{-1}$ is an algebra automorphism, and a coalgebra anti-automorphism, so U is isomorphic to $U_q(\mathfrak{g})$ as a Hopf algebra (see [CP, p. 211]).

The defining property of the antipode S in any Hopf algebra is that it is an inverse to the identity map id_U with respect to the convolution product, so that for all $a \in U$,

$$\sum_a S(a_{(1)})a_{(2)} = \epsilon(a)\mathrm{id}_U = \sum_a a_{(1)}S(a_{(2)}). \tag{1.9}$$

Representations.

For any Hopf algebra U the comultiplication operator allows us to define a U-module structure on the tensor product $V \otimes W$ of two U-modules V and W, where

$$a(v \otimes w) = \sum_a a_{(1)}v \otimes a_{(2)}w.$$

The counit ϵ affords a representation of U on the one-dimensional module given by $a1 = \epsilon(a)1$. If V is a finite-dimensional U-module, then so is its dual space V^* where the U-action is given by the antipode mapping,

$$(af)(v) = f(S(a)v), \tag{1.10}$$

for all $a \in U$, $f \in V^*$, and $v \in V$.

A finite-dimensional module M for $U = U_q(\mathfrak{g})$ is the sum of weight spaces, $M = \bigoplus_{\nu,\chi} M_{\nu,\chi}$, where $M_{\nu,\chi} = \{x \in M \mid K_\alpha x = \chi(\alpha)q^{(\nu,\alpha)}x \text{ for all } \alpha \in \Pi\}$, and $\chi : \mathbf{Z}\Phi \to \{\pm 1\}$ is a group homomorphism. All the modules considered in this paper will be of type I, that is $\chi(\alpha) = 1$ for all $\alpha \in \Pi$, so we will drop χ from the notation and from our considerations.

Each finite-dimensional irreducible U-module M has a highest weight λ, which is a dominant integral weight for $\mathfrak{g}$ relative to Π, and a unique (up to scalar multiple) maximal vector v^+ so that $E_\alpha v^+ = 0$ and $K_\alpha v^+ = q^{(\lambda,\alpha)}v^+$ for all $\alpha \in \Pi$. In particular, the trivial U-module, $\mathbf{K}1$, has highest weight 0. We denote by $L(\lambda)$ the irreducible U-module with highest weight λ (and $\chi \equiv 1$.)

When V is a finite-dimensional U-module, we can suppose $\{x_{\nu,i}\}$ is a basis of V of weight vectors with $i = 1, \ldots, \dim V_\nu$. Let $\{x^*_{\nu,i}\}$ be the dual basis in V^* so that $x^*_{\mu,i}(x_{\nu,j}) = \delta_{\mu,\nu}\delta_{i,j}$. Then

$$\begin{aligned}(K_\alpha x^*_{\mu,i})(x_{\nu,j}) &= x^*_{\mu,i}(S(K_\alpha)x_{\nu,j}) = x^*_{\mu,i}(K_\alpha^{-1}x_{\nu,j}) \\ &= q^{-(\nu,\alpha)}x^*_{\mu,i}(x_{\nu,j}) = q^{-(\nu,\alpha)}\delta_{\mu,\nu}\delta_{i,j} = q^{-(\mu,\alpha)}\delta_{\mu,\nu}\delta_{i,j},\end{aligned} \tag{1.11}$$

from which we see that $x^*_{\mu,i}$ has weight $-\mu$.

Suppose M and N are U-modules for a quantum group U. Define a U-module structure on $\mathrm{Hom}_{\mathbf{K}}(M, N)$ by

$$(a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m). \tag{1.12}$$

(This just means that the natural map $N \otimes M^* \to \mathrm{Hom}_{\mathbf{K}}(M, N)$ is a U-module homomorphism.)

Note that if $\gamma \in \mathrm{Hom}_U(M, N)$, then

$$(a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m) = \sum_a a_{(1)}S(a_{(2)})\gamma(m) = \epsilon(a)\gamma(m)$$

so that $a\gamma = \epsilon(a)\gamma$. That says

$$\mathrm{Hom}_U(M, N) \subseteq \mathrm{Hom}_{\mathbf{K}}(M, N)^U = \{\gamma \in \mathrm{Hom}_{\mathbf{K}}(M, N) \mid a\gamma = \epsilon(a)\gamma\}.$$

Conversely, if γ belongs to the invariants $\mathrm{Hom}_{\mathbf{K}}(M, N)^U$, then we have

$$\begin{aligned}
0 = \epsilon(K_\alpha)\gamma = \gamma = K_\alpha \circ \gamma \circ K_\alpha^{-1}, &\quad \text{and hence,} \\
0 = K_\alpha \circ \gamma - \gamma \circ K_\alpha & \\
0 = \epsilon(E_\alpha)\gamma = E_\alpha\gamma = E_\alpha \circ \gamma - K_\alpha \circ \gamma \circ K_\alpha^{-1} \circ E_\alpha, &\quad \text{which implies} \\
0 = E_\alpha \circ \gamma - \gamma \circ E_\alpha & \\
0 = \epsilon(F_\alpha)\gamma = F_\alpha\gamma = F_\alpha \circ \gamma \circ K_\alpha - \gamma \circ F_\alpha \circ K_\alpha &\quad \text{which implies} \\
0 = F_\alpha \circ \gamma - \gamma \circ F_\alpha &
\end{aligned}$$

because of the relations in (1.8). Since these elements generate $U_q(\mathfrak{g})$, we see that $\gamma \in \mathrm{Hom}_{\mathbf{K}}(M, N)^U$ implies $\gamma \in \mathrm{Hom}_U(M, N)$, so the two spaces are equal:

$$\mathrm{Hom}_U(M, N) = \mathrm{Hom}_{\mathbf{K}}(M, N)^U. \tag{1.13}$$

Now if $M = N = L(\lambda)$, a finite-dimensional irreducible U-module of highest weight λ, then any $f \in \mathrm{Hom}_U(M, M)$ must map the highest weight vector v^+ to a multiple of itself. Since v^+ generates M as a U-module, it follows that f is a scalar multiple of the identity. Thus, $\mathrm{Hom}_U(M, M) = \mathbf{K}\mathrm{id}_M$. So the space of invariants $\mathrm{Hom}_{\mathbf{K}}(M, M)^U$, or equivalently $(M \otimes M^*)^U$, is one-dimensional.

R-matrices.

The quantum group $U = U_q(\mathfrak{g})$ has a triangular decomposition $U = U^-U^0U^+$, where U^+ (resp. U^-) is the subalgebra generated by the E_α, (resp. F_α), and U^0 is the subalgebra generated by the $K_\alpha^{\pm 1}$, for all $\alpha \in \Pi$. Then $U^+ = \sum_\mu U_\mu^+$ where $U_\mu^+ = \{a \in U^+ \mid K_\alpha a K_\alpha^{-1} = q^{(\mu,\alpha)}a\}$. It is easy to see using the automorphism ψ above, that U^- has the decomposition $U^- = \sum_\mu U_{-\mu}^-$, and $U_\mu^+ \neq (0)$ if and only if $U_{-\mu}^- \neq (0)$. There is a nondegenerate bilinear form $(\,,\,)$ on U (see [Jn, Lemma 6.16, Prop. 6.21]) which satisfies

(1.14)

(a) $(b, a) = q^{(2\rho,\mu-\nu)}(a, b)$ for all $a \in U_{-\nu}^-U^0U_\mu^+$ and $b \in U_{-\mu}^-U^0U_\nu^+$, where ρ is the half-sum of the positive roots of $\mathfrak{g}$.

(b) $(F_\alpha, E_\beta) = -\delta_{\alpha,\beta}(q_\alpha - q_\alpha^{-1})^{-1}$.

Choose a basis $a_1^\mu, a_2^\mu, \dots, a_{r(\mu)}^\mu$ of U_μ^+ and a corresponding dual basis $b_1^\mu, b_2^\mu, \dots, b_{r(\mu)}^\mu$ of $U_{-\mu}^-$ so that $(b_i^\mu, a_j^\mu) = \delta_{i,j}$. Set

$$\Theta_\mu = \sum_{i=1}^{r(\mu)} b_i^\mu \otimes a_i^\mu.$$

In particular, $\Theta_\alpha = -(q_\alpha - q_\alpha^{-1})F_\alpha \otimes E_\alpha$ for all $\alpha \in \Pi$.

Suppose $\pi : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ and $\pi' : U_q(\mathfrak{g}) \to \mathrm{End}(V')$ are two finite-dimensional $U_q(\mathfrak{g})$-modules. Since the set of weights of V is finite, there are only finitely many $\mu \in \mathbf{Z}\Phi$ which are differences of weights of V. Therefore, all but finitely many Θ_μ act as zero on V and V', and $\Theta_{V,V'} = (\pi \otimes \pi')(\Theta)$ for $\Theta = \sum_\mu \Theta_\mu$ is a well-defined mapping on $V \otimes V'$. By suitably ordering the basis of $V \otimes V'$, we see that each Θ_μ for $\mu \neq 0$ acting on $V \otimes V'$ has a strictly upper triangular matrix relative to that basis. Since $\Theta_0 = 1 \otimes 1$, the transformation $\Theta_{V,V'}$ is unipotent.

Consider the map $f = f_{V,V'}$, which is defined by $f_{V,V'} : V \otimes V' \to V \otimes V'$, $f(v \otimes w) = q^{-(\lambda,\mu)} v \otimes w$ for all $v \in V_\lambda$ and $w \in V'_\mu$.

Proposition 1.15. *(Compare [Jn, Thm. 7.3].) Let V, V' be finite-dimensional $U_q(\mathfrak{g})$-modules, and let $\sigma = \sigma_{V',V}$ be defined by*

$$\begin{aligned} \sigma : V' \otimes V &\to V \otimes V' \\ w \otimes v &\mapsto v \otimes w. \end{aligned}$$

Then the mapping $\check{R}_{V',V} : V' \otimes V \to V \otimes V'$ given by $\check{R}_{V',V} = \Theta \circ f \circ \sigma$ is a $U_q(\mathfrak{g})$-module isomorphism.

There is an alternate expression for Θ_μ which is more explicit and can be best understood using the braid group action. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators T_α, $\alpha \in \Pi$, and defining relations

$$(T_\alpha T_\beta)^{m_{\alpha,\beta}} = (T_\beta T_\alpha)^{m_{\alpha,\beta}}$$

where $m_{\alpha,\beta} = 2, 3, 4, 6$ if $a_{\alpha,\beta} a_{\beta,\alpha} = 0, 1, 2, 3$ respectively. It acts by algebra automorphisms on $U_q(\mathfrak{g})$ according to the following rules (see [Jn, Sec. 8.14]):

$$\begin{aligned} &T_\alpha(K_\beta) = K_{s_\alpha \beta}, \qquad T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \qquad T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha \\ &T_\alpha(E_\beta) = \sum_{t=0}^{-a_{\alpha,\beta}} (-1)^t q_\alpha^{-t} (E_\alpha)^{(-a_{\alpha,\beta}-t)} E_\beta (E_\alpha)^{(t)} \quad \alpha \neq \beta \\ &T_\alpha(F_\beta) = \sum_{t=0}^{-a_{\alpha,\beta}} (-1)^t q_\alpha^{t} (F_\alpha)^{(t)} F_\beta (F_\alpha)^{(-a_{\alpha,\beta}-t)} \quad \alpha \neq \beta, \end{aligned} \tag{1.16}$$

where s_α is the simple reflection corresponding to α in the Weyl group W of $\mathfrak{g}$, and

$$(E_\alpha)^{(n)} = \frac{E_\alpha^n}{[n]_{d_\alpha}!}, \qquad (F_\alpha)^{(n)} = \frac{F_\alpha^n}{[n]_{d_\alpha}!},$$

(the factorials are as defined in (1.4)).

Let $s_i = s_{\alpha_i}$ be the reflection in the hyperplane perpendicular to the simple root α_i and fix a reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ of the longest element of W. Every positive root occurs precisely once in the sequence

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \ \ldots, \ \beta_m = s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}). \tag{1.17}$$

The elements

$$\begin{aligned} E_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}}(E_{\alpha_{i_t}}) \\ F_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}}(F_{\alpha_{i_t}}) \end{aligned} \tag{1.18}$$

for $t = 1, \ldots, m$, are called the positive (resp. negative) root vectors of $U_q(\mathfrak{g})$. For any m-tuple $\underline{\ell} = (\ell_1, \ldots, \ell_m)$ of nonnegative integers let

$$\begin{aligned} E^{\underline{\ell}} &= E_{\beta_m}^{\ell_m} E_{\beta_{m-1}}^{\ell_{m-1}} \cdots E_{\beta_1}^{\ell_1} \\ F^{\underline{\ell}} &= F_{\beta_m}^{\ell_m} F_{\beta_{m-1}}^{\ell_{m-1}} \cdots F_{\beta_1}^{\ell_1}. \end{aligned}$$

Then the elements $E^{\underline{\ell}}$ (resp. $F^{\underline{\ell}}$) determine a basis of U^+ (resp. U^-). (See for example, [Jn, Thm. 8.24].)

Consider for $t = 1, \ldots, m$ the sum

$$\Theta^{[t]} = \sum_{j \geq 0} (-1)^j q_\alpha^{-j(j-1)/2} \frac{(q_\alpha - q_\alpha^{-1})^j}{[j]_{d_\alpha}!} F_{\beta_t}^j \otimes E_{\beta_t}^j$$

where $\alpha = \alpha_{i_t}$ and E_{β_t} and F_{β_t} are as in (1.18). This lies in the direct product of all the $U_{-\mu}^- \otimes U_\mu^+$. Then (as discussed in [Jn, Sec. 8.30]), Θ_μ is the $(U_\mu^- \otimes U_\mu^+)$-component of the product

$$\Theta^{[m]} \Theta^{[m-1]} \cdots \Theta^{[2]} \Theta^{[1]}.$$

That component involves only finitely many summands.

§2. 8-dimensional Representations of $U_q(\mathrm{D}_4)$

In this section we specialize to the case that $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra of type D_4. We can identify $\mathfrak{g}$ with the special orthogonal Lie algebra $so(8)$. The set of roots of $\mathfrak{g}$ is given by $\{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i \ne j \le 4\}$, where $\epsilon_i, i = 1, \dots, 4$, is an orthonormal basis of $\mathbf{R}^4$, and the simple roots may be taken to be $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \epsilon_3 + \epsilon_4$. The corresponding fundamental weights have the following expressions:

$$\begin{aligned}
&\omega_1 = \alpha_1 + \alpha_2 + (1/2)\alpha_3 + (1/2)\alpha_4 = \epsilon_1,\\
&\omega_2 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \epsilon_1 + \epsilon_2\\
&\omega_3 = (1/2)\alpha_1 + \alpha_2 + \alpha_3 + (1/2)\alpha_4 = (1/2)\Big(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4\Big)\\
&\omega_4 = (1/2)\alpha_1 + \alpha_2 + (1/2)\alpha_3 + \alpha_4 = (1/2)\Big(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4\Big),
\end{aligned} \tag{2.1}$$

and they satisfy $(\omega_i, \alpha_j) = \delta_{i,j}$ for all $i, j = 1, \dots, 4$.

The Lie algebra $\mathfrak{g}$ has three 8-dimensional representations - its natural representation and two spin representations with highest weights ω_1, ω_3, and ω_4 respectively, and so does its quantum counterpart $U_q(\mathrm{D}_4)$. (*Note we are writing $U_q(D_4)$ rather than $U_q(\mathfrak{g})$ to emphasize the role that D_4 is playing here.*) In particular, for the natural representation V of $U_q(\mathrm{D}_4)$, there is a basis $\{x_\mu \mid \mu = \pm\epsilon_i, i = 1, \dots, 4\}$ (compare [Jn, Chap. 5]) such that

$$\begin{aligned}
K_\alpha x_\mu &= q^{(\mu,\alpha)} x_\mu,\\
E_\alpha x_\mu &= \begin{cases} 0 & \text{if} \quad (\mu, \alpha) \ne -1\\ x_{\mu+\alpha} & \text{if} \quad (\mu, \alpha) = -1 \end{cases}\\
F_\alpha x_\mu &= \begin{cases} 0 & \text{if} \quad (\mu, \alpha) \ne 1\\ x_{\mu-\alpha} & \text{if} \quad (\mu, \alpha) = 1. \end{cases}
\end{aligned} \tag{2.2}$$

For D_4, the group of graph automorphisms of its Dynkin diagram can be identified with the symmetric group S_3 of permutations on $\{1, 2, 3, 4\}$ fixing 2. Thus, $\phi \in S_3$ permutes the simple roots $\phi(\alpha_i) = \alpha_{\phi i}$ and fixes α_2. Each graph automorphism ϕ induces an automorphism, which we also denote by ϕ, on $U_q(\mathrm{D}_4)$ defined by

$$E_{\alpha_i} \mapsto E_{\alpha_{\phi i}} \qquad F_{\alpha_i} \mapsto F_{\alpha_{\phi i}} \qquad K_{\alpha_i} \mapsto K_{\alpha_{\phi i}}.$$

Corresponding to ϕ there is new representation, $\pi_\phi : U_q(\mathrm{D}_4) \to \mathrm{End}(V)$ of $U_q(\mathrm{D}_4)$ on its natural representation V such that

$$\pi_\phi(a)x = \phi(a)x$$

for all $a \in U_q(\mathrm{D}_4)$ and $x \in V$. We denote the $U_q(\mathrm{D}_4)$-module with this action by V_ϕ. Then we have

$$\begin{aligned}
\pi_\phi(K_\alpha)x_\mu &= K_{\phi\alpha}x_\mu = q^{(\mu,\phi\alpha)}x_\mu = q^{(\phi^{-1}\mu,\alpha)}x_\mu \\
\pi_\phi(E_\alpha)x_\mu &= E_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) \neq -1 \\ x_{\mu+\phi\alpha} & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) = -1 \end{cases} \\
\pi_\phi(F_\alpha)x_\mu &= F_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) \neq 1 \\ x_{\mu-\phi\alpha} & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) = 1. \end{cases}
\end{aligned} \tag{2.3}$$

From these formulas the following is apparent:

Proposition 2.4. *V_ϕ is an irreducible $U_q(D_4)$-module with highest weight $\phi^{-1}\omega_1$. In particular, we have the following highest weights:*

(i) ω_1 for $\phi = id$ or $(3\ 4)$,
(ii) ω_3 for $\phi = (1\ 3)$ or $(1\ 4\ 3)$,
(iii) ω_4 for $\phi = (1\ 4)$ or $(1\ 3\ 4)$.

The operators E_α, F_α map elements in the basis $\{x_\mu\}$ to other basis elements or 0, and each vector x_μ is a weight vector of weight $\phi^{-1}\mu$. For the natural representation V, the basis elements are $x_{\pm\epsilon_i}, i = 1, \dots, 4$, which we abbreviate as $x_{\pm i}$. We can view the module V_ϕ as displayed below, where the index i on an arrow indicates that F_{α_i} maps the higher vector down to the lower vector and E_{α_i} maps the lower vector up to the higher one. The action of all F_{α_j}'s and E_{α_j}'s not shown is 0.

$$\begin{array}{ccc}
 & V_\phi & \\
 & & \\
 & x_1 \circ & \\
 & \downarrow \phi^{-1}1 & \\
 & x_2 \circ & \\
 & \downarrow 2 & \\
 & x_3 \circ & \\
\phi^{-1}3 \swarrow & & \searrow \phi^{-1}4 \\
x_4 \circ & & \circ\ x_{-4} \\
\phi^{-1}4 \searrow & & \swarrow \phi^{-1}3 \\
 & x_{-3} \circ & \\
 & \downarrow 2 & \\
 & x_{-2} \circ & \\
 & \downarrow \phi^{-1}1 & \\
 & x_{-1} \circ &
\end{array} \tag{2.5}$$

In particular, taking the identity, $\theta = (1\ 4\ 3)$, and $\theta^2 = (1\ 3\ 4)$ we have the following pictures for the natural representation V, and the spin representations V_- and V_+:

$$
\begin{array}{ccc}
V & V_- & V_+ \\[1em]
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 1 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
3 \swarrow & & \searrow 4 \\
x_4 \circ & & \circ\, x_{-4} \\
4 \searrow & & \swarrow 3 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 1 & \\
 & x_{-1}\circ &
\end{array}
&
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 3 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
4 \swarrow & & \searrow 1 \\
x_4 \circ & & \circ\, x_{-4} \\
1 \searrow & & \swarrow 4 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 3 & \\
 & x_{-1}\circ &
\end{array}
&
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 4 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
1 \swarrow & & \searrow 3 \\
x_4 \circ & & \circ\, x_{-4} \\
3 \searrow & & \swarrow 1 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 4 & \\
 & x_{-1}\circ &
\end{array}
\end{array}
\tag{2.6}
$$

There is a unique (up to scalar multiple) $U_q(\mathrm{D}_4)$-module homomorphism $* : V_- \otimes V_+ \to V$, which we can compute directly using the expressions for the comultiplication in (1.8). We display the result of this computation in Table 2.7 below. To make the table more symmetric, we scale the mapping by $q^{\frac{3}{2}}$.

Table 2.7

$*$	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	$q^{-\frac{3}{2}}x_1$	0	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_2$	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_3$	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	$-q^{\frac{3}{2}}x_1$	0	0	0	$-q^{-\frac{1}{2}}x_{-4}$	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-4}	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	$q^{\frac{1}{2}}x_4$	0	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	$q^{\frac{1}{2}}x_{-3}$	0	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-2}$	0	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	0	$-q^{\frac{3}{2}}x_{-1}$	0	0	0

It can be seen readily from (1.8) that the relations

$$\begin{aligned}(\phi \otimes \phi)\Delta(a) &= \Delta(\phi(a)) \\ \phi(S(a)) &= S(\phi(a))\end{aligned} \tag{2.8}$$

hold for all $a \in U_q(\mathrm{D}_4)$ and all graph automorphisms ϕ. Therefore, by identifying V_- with V_θ and V_+ with V_{θ^2} we obtain from

$$a(x * y) = \sum_a \theta(a_{(1)})x * \theta^2(a_{(2)})y$$

that the following relations hold:

$$\begin{aligned}\pi_\theta(a)(x * y) &= \sum_a \theta^2(a_{(1)})x * a_{(2)}y \\ \pi_{\theta^2}(a)(x * y) &= \sum_a a_{(1)}x * \theta(a_{(2)})y.\end{aligned} \tag{2.9}$$

They allow us to conclude

Proposition 2.10. *The product $* : V_- \otimes V_+ \to V$ displayed in Table 2.7 above is a $U_q(D_4)$-module homomorphism, and it gives $U_q(D_4)$-module homomorphisms,*

$$\begin{aligned} * &: V_+ \otimes V \to V_- \\ * &: V \otimes V_- \to V_+.\end{aligned}$$

Suppose now that $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Specializing (2.5) with ϕ equal to these permutations gives:

$$\begin{array}{ccccccccccc}
 & & V_\zeta & & & & & & V_\eta & & \\
 & & x_1\circ & & & & & & x_1\circ & & \\
 & & \downarrow 3 & & & & & & \downarrow 4 & & \\
 & & x_2\circ & & & & & & x_2\circ & & \\
 & & \downarrow 2 & & & & & & \downarrow 2 & & \\
 & & x_3\circ & & & & & & x_3\circ & & \\
1 \swarrow & & & & \searrow 4 & & 3 \swarrow & & & & \searrow 1 \\
x_4\circ & & & & \circ\, x_{-4} & & x_4\circ & & & & \circ\, x_{-4} \\
4 \searrow & & & & \swarrow 1 & & 1 \searrow & & & & \swarrow 3 \\
 & & x_{-3}\circ & & & & & & x_{-3}\circ & & \\
 & & \downarrow 2 & & & & & & \downarrow 2 & & \\
 & & x_{-2}\circ & & & & & & x_{-2}\circ & & \\
 & & \downarrow 3 & & & & & & \downarrow 4 & & \\
 & & x_{-1}\circ & & & & & & x_{-1}\circ & &
\end{array} \tag{2.11}$$

As these diagrams indicate, the transformation specified by

$$\begin{aligned} \jmath : V &\to V \\ x_i &\mapsto -x_i \qquad i \neq \pm 4 \\ x_4 &\mapsto -x_{-4} \\ x_{-4} &\mapsto -x_4 \end{aligned} \tag{2.12}$$

determines explicit $U_q(\mathrm{D}_4)$-isomorphisms,

$$\jmath : V_- \to V_\zeta \qquad \jmath : V_+ \to V_\eta. \tag{2.13}$$

This together with Proposition 2.10 enables to conclude the following:

Proposition 2.14. *The map* $\cdot : V_\zeta \otimes V_\eta \to V$ *given by* $x \otimes y \mapsto x \cdot y = \jmath(x) * \jmath(y)$ *is a* $U_q(D_4)$*-module homomorphism.*

Eigenvalues of $\check{R}_{V,V}$.

Suppose now $V = L(\omega_1)$, the natural representation of $U_q(\mathrm{D}_4)$. The module $V^{\otimes 2}$ is completely reducible: $V^{\otimes 2} = L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$. By determining the eigenvalues of $\check{R}_{V,V}$ on the corresponding maximal vectors, we obtain the eigenvalues of $\check{R}_{V,V}$ on the summands.

First, on the maximal vector $x_1 \otimes x_1$ of the summand $L(2\omega_1)$, $\check{R}_{V,V}(x_1 \otimes x_1) = q^{-(\epsilon_1,\epsilon_1)} x_1 \otimes x_1 = q^{-1}(x_1 \otimes x_1)$. The vector $x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2$ is a maximal vector for $L(\omega_2)$. Then

$$\begin{aligned} \check{R}_{V,V}(x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2) &= q^{-(\epsilon_1,\epsilon_2)} \Theta(x_1 \otimes x_2 - q^{-1} x_2 \otimes x_1) \\ &= x_1 \otimes x_2 - q^{-1} x_2 \otimes x_1 - (q - q^{-1}) x_2 \otimes x_1 \\ &= x_1 \otimes x_2 - q x_2 \otimes x_1 = -q(x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2), \end{aligned}$$

so that $\check{R}_{V,V}$ acts as $-q$ on $L(\omega_2)$. We will argue in the remark following Proposition 2.19 below that $\check{R}_{V,V}$ acts on $L(0)$ as q^7. Thus, the eigenvalues of $\check{R}_{V,V}$ are given by

$$\begin{array}{ll} q^{-1} & L(2\omega_1) \\ -q & L(\omega_2) \\ q^7 & L(0), \end{array} \tag{2.15}$$

(compare [D], [R]). Setting $\mathcal{S}^2(V) = L(2\omega_1) \oplus L(0)$ we see that $\check{R}_{V,V} + q\mathrm{id}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$, the *quantum symmetric space.* There are analogous decompositions for the two spin representations of $U_q(\mathrm{D}_4)$ obtained by replacing ω_1 by ω_3 and ω_4.

For each graph automorphism ϕ, we consider $\check{R} = \check{R}_{V_\phi,V_\phi}$ on $V_\phi \otimes V_\phi$. Note the bilinear form (,) on $U_q(\mathrm{D}_4)$ in [Jn, (6.12)] is $(\phi \otimes \phi)$-invariant, so

$$\Theta_{\phi(\mu)} = \sum_i \phi(b_i^\mu) \otimes \phi(a_i^\mu) = (\phi \otimes \phi)(\Theta_\mu).$$

Now

$$\begin{aligned} \check{R}(v \otimes w) &= (\Theta \circ f \circ \sigma)(v \otimes w) = q^{-(\phi^{-1}\lambda, \phi^{-1}\nu)}\Theta(w \otimes v) \\ &= q^{-(\lambda,\nu)}\Theta(w \otimes v) = \check{R}_{V,V}(v \otimes w), \end{aligned} \tag{2.16}$$

whenever w is in V_λ and $v \in V_\nu$. Consequently, the actions of the R-matrices agree. It follows that

$$\mathcal{S}^2(V) = \mathcal{S}^2(V_\phi) \tag{2.17}$$

for all graph automorphisms ϕ.

A "test" element of $\mathcal{S}^2(V)$.

In the next two chapters we will compute various $U_q(\mathrm{D}_4)$-module homomorphisms and prove that they are equal by evaluating them on a particular element of $\mathcal{S}^2(V)$. A good "test" element for these calculations is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$, but first we need to demonstrate that it does in fact belong to $\mathcal{S}^2(V)$. Since $\check{R}_{V,V} + q\mathrm{id}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$ for $V = L(\omega_1)$, the natural representation, it suffices to show that the image of $x_1 \otimes x_{-1}$ under $\check{R}_{V,V} + q\mathrm{id}_V$ is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$. Using the fact that $E_\gamma x_1 = 0$ for all positive roots γ we obtain

$$\begin{aligned} \check{R}_{V,V}(x_1 \otimes x_{-1}) &= (\Theta \circ f \circ \sigma)(x_1 \otimes x_{-1}) \\ &= q\Theta(x_{-1} \otimes x_1) = q(x_{-1} \otimes x_1). \end{aligned}$$

Consequently,

$$(\check{R}_{V,V} + q\mathrm{id}_V)(x_1 \otimes x_{-1}) = qx_{-1} \otimes x_1 + qx_1 \otimes x_{-1},$$

which implies the desired conclusion

$$x_{-1} \otimes x_1 + x_1 \otimes x_{-1} \in \mathcal{S}^2(V).$$

Bilinear Forms.

For the natural representation V of $U = U_q(\mathrm{D}_4)$, the dual space V^* has weights $\pm\epsilon_i$, $i = 1, \ldots, 4$ by (1.11). The vector x_{-1}^* is a maximal vector since it corresponds to the unique dominant weight, ϵ_1, and V^* is isomorphic to V. We have seen in (1.13) that the space of invariants $(V \otimes V^*)^U$ is one-dimensional. Since $V \cong V^*$, we have that the space of invariants $(V \otimes V)^U$ is one-dimensional. Thus, in $V \otimes V$ there is a unique copy of the trivial module, and hence a unique (up to scalar multiple) $U_q(\mathrm{D}_4)$-module homomorphism

$$(\,|\,) : V \otimes V \to \mathbf{K}.$$

The condition of $(\,|\,)$ being a $U_q(\mathrm{D}_4)$-module homomorphism is equivalent to

$$(ax|y) = (x|S(a)y) \tag{2.18}$$

for all $a \in U$ and $x, y \in V$ (see [Jn, p.122]). There is an analogous bilinear form giving the unique (up to multiples) $U_q(\mathrm{D}_4)$-module homomorphism, $V_\phi \otimes V_\phi \to \mathbf{K}$. All these bilinear forms can be regarded as $U_q(\mathrm{D}_4)$-module maps from the symmetric tensors $\mathcal{S}^2(V_\phi) \to \mathbf{K}$, since the trivial module is an irreducible summand of $\mathcal{S}^2(V_\phi)$ in each case.

Now for $a \in U_q(\mathrm{D}_4)$,

$$\begin{aligned}
\sum_a (\pi_\phi(a_{(1)})x|\pi_\phi(a_{(2)})y) &= \sum_a (\phi(a_{(1)})x|\phi(a_{(2)})y) \\
&= \sum_a (x|S(\phi(a_{(1)}))\phi(a_{(2)})y) \\
&= \sum_a (x|\phi(S(a_{(1)})a_{(2)})y \\
&= (x|\phi\Big(\sum_a S(a_{(1)})a_{(2)}\Big))y \\
&= (x|\phi(\epsilon(a)1)y) = \epsilon(a)(x|y).
\end{aligned} \tag{2.19}$$

(We have used in the third line the fact that ϕ and S commute (2.8).) Hence, by the calculation in (2.19), and by direct computation of $(\,|\,)$ on $V \otimes V$, we have

Proposition 2.20. *The $U_q(D_4)$-module homomorphism: $(\,|\,) : V \otimes V \to \mathbf{K}$ determines a $U_q(D_4)$-module homomorphism: $(\,|\,) : V_\phi \otimes V_\phi \to \mathbf{K}$ for any graph automorphism ϕ. It is given by*

$$(x_i|x_j) = \begin{cases} 0 & \text{if} \quad j \neq -i \\ (-1)^{i+1}\dfrac{q^{-(4-i)}}{q^3+q^{-3}} & \text{if} \quad i > 0 \text{ and } j = -i \\ (-1)^{i+1}\dfrac{q^{i+4}}{q^3+q^{-3}} & \text{if} \quad i < 0 \text{ and } j = -i. \end{cases}$$

We are free to scale the bilinear form $(\,|\,)$ on V_ϕ by any nonzero element in $\mathbf{K}$. In writing down the values of $(x_i|x_{-i})$ in Proposition 2.20, we have chosen a particular scaling with an eye towards results (such as Proposition 4.2) to follow. It is easy to see using the basis elements x_i that the following holds.

Proposition 2.21. *The transformation $\jmath$ is an isometry, ($(x|y) = (\jmath(x)|\jmath(y))$ for all $x, y \in V$), relative to the bilinear form in Proposition 2.20.*

Remark. Observe that $\check{R}_{V,V} \circ (\,|\,) = \xi(\,|\,)$, where ξ is the eigenvalue of $\check{R}_{V,V}$ on $L(0)$. Since $\check{R}_{V,V}(x_1 \otimes x_{-1}) = q(x_{-1} \otimes x_1)$, as our calculations for the "test element" show, we have $\xi(x_1|x_{-1}) = q(x_{-1}|x_1)$. Therefore, $\xi q^{-3} = qq^3$ or $\xi = q^7$, as asserted in (2.15).

For D_4, the half-sum of the positive roots is given by $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6} K_{\alpha_2}^{-10} K_{\alpha_3}^{-6} K_{\alpha_4}^{-6}$. Therefore, on the natural representation V, $K_{2\rho}^{-1}$ has a diagonal matrix relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ with corresponding diagonal entries given by $\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}$. It is easy to verify directly that

$$(x|y) = (y|K_{2\rho}^{-1}x). \tag{2.22}$$

§3. Octonions and Quantum Octonions

Suppose $\zeta = (1\ 3)$, $\eta = (1\ 4)$, and V_ϕ is the representation of $U_q(\mathrm{D}_4)$ coming from the graph automorphism ϕ. Then

$$\dim_{\mathbf{K}} \mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta), \mathbf{K}) = 1, \tag{3.1}$$

and we can realize these homomorphisms explicitly as multiples of the mapping

$$\begin{aligned} \mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta) \subset V_\zeta \otimes V_\zeta \otimes V_\eta \otimes V_\eta &\to \mathbf{K} \\ \sum_{k,\ell} u_k \otimes v_k \otimes w_\ell \otimes y_\ell &\mapsto \sum_{k,\ell} (u_k|v_k)(w_\ell|y_\ell). \end{aligned} \tag{3.2}$$

On the other hand, the mapping

$$\mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta) \subset V_\zeta \otimes V_\zeta \otimes V_\eta \otimes V_\eta \to V_\zeta \otimes V_\eta \otimes V_\zeta \otimes V_\eta \to V \otimes V \to \mathbf{K} \tag{3.3}$$

$$\begin{aligned} \sum_{k,\ell} u_k \otimes v_k \otimes w_\ell \otimes y_\ell &\mapsto \sum_{k,\ell} u_k \otimes \check{R}_{V_\zeta, V_\eta}(v_k \otimes w_\ell) \otimes y_\ell \\ &\mapsto \sum_{k,\ell} \sum_i u_k \cdot w_\ell^{(i)} \otimes v_k^{(i)} \cdot y_\ell \mapsto \sum_{k,\ell} \sum_i (u_k \cdot w_\ell^{(i)} | v_k^{(i)} \cdot y_\ell), \end{aligned}$$

is a $U_q(\mathrm{D}_4)$-module homomorphism, (where "$\cdot$" is the product in Proposition 2.14 and $\check{R}_{V_\zeta, V_\eta}(v_k \otimes w_\ell) = \sum_i w_\ell^{(i)} \otimes v_k^{(i)}$ is the braiding morphism). Thus, it must be a multiple of the mapping in (3.2).

We now consider what would happen if $q = 1$ and compute the multiple in that case. Here the braiding morphism may be taken to be $v \otimes w \mapsto w \otimes v$ since that is a module map in the $q = 1$ case. We evaluate the mappings at $e \otimes e \otimes e \otimes e$ where $e = x_4 - x_{-4}$. Now by Table 2.7, we have

$$\begin{aligned} e^2 &= \jmath(e) * \jmath(e) = (x_4 - x_{-4}) * (x_4 - x_{-4}) \\ &= q^{\frac{1}{2}} x_4 - q^{-\frac{1}{2}} x_{-4}, \end{aligned} \tag{3.4}$$

which shows that at $q = 1$ that $e^2 = e$. By Proposition 2.19,

$$(e|e) = -(x_4|x_{-4}) - (x_{-4}|x_4) = \frac{2}{q^3 + q^{-3}}, \tag{3.5}$$

which gives $(e|e)_1 = 1$ at $q = 1$. *(Here and throughout the paper we use* $(\,|\,)_1$ *to denote the bilinear form at* $q = 1$*.)* Therefore, $(e|e)_1(e|e)_1 = 1$ and $(e^2|e^2)_1 = (e|e)_1 = 1$. Thus, the two maps in (3.2) and (3.3) are equal at $q = 1$. By specializing $\sum_k u_k \otimes v_k = x \otimes x$ and $\sum_\ell w_\ell \otimes y_\ell = y \otimes y$ (which are symmetric elements at $q = 1$), we have as a consequence,

$$(x \cdot y | x \cdot y)_1 = (x|x)_1 (y|y)_1. \tag{3.6}$$

Using the multiplication table (Table 2.7) for $x * y$ and the formula $x \cdot y = \jmath(x) * \jmath(y)$, we can verify that $e = x_4 - x_{-4}$ is the unit element relative to the product "$\cdot$" when $q = 1$. (For further discussion, see the beginning of Section 5.) Thus, the product $\cdot : V_\zeta \otimes V_\eta \to V$ for $q = 1$ gives an 8-dimensional unital composition algebra over $\mathbf{K}$ whose underlying vector space is $V = V_\zeta = V_\eta$. Such an algebra must be the octonions. Thus, we have

Proposition 3.7. *Let V be the 8-dimensional vector space over* $\mathbf{K}$ *with basis* $\{x_{\pm i} \mid i = 1, \ldots, 4\}$*. Then V with the product $\cdot$ defined by $x \cdot y = \jmath(x) * \jmath(y)$, where $\jmath$ is as in (2.12) and $*$ is as in Table 2.7, is the algebra of octonions when $q = 1$.*

Because of this result we are motivated to make the following definition.

Quantum Octonions.

Definition 3.8. *Let V be the 8-dimensional vector space over* $\mathbf{K}$ *with basis* $\{x_{\pm i} \mid i = 1, \ldots, 4\}$*. Then V with the product "$\cdot$" defined by*

$$x \cdot y = \jmath(x) * \jmath(y),$$

where $\jmath$ is as in (2.12) and $*$ is given by Table 2.7, is the algebra of quantum octonions. We denote this algebra by $\mathbb{O}_q = (V, \cdot)$.

The multiplication table for the quantum octonions is obtained from Table 2.7 by interchanging the entries in the columns and rows labelled by x_{-4} and x_4. As a result, we have

Table 3.9

$\cdot$	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	0	$q^{-\frac{3}{2}}x_1$	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	$-q^{-\frac{1}{2}}x_2$	0	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_3$	0	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	0	$q^{\frac{1}{2}}x_4$	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-4}	$-q^{\frac{3}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_{-4}$	0	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-3}$	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	0	$q^{\frac{1}{2}}x_{-2}$	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	$-q^{\frac{3}{2}}x_{-1}$	0	0	0	0

The product map: $p : V_\zeta \otimes V_\eta \to V$, $p(x \otimes y) = x \cdot y$, of the quantum octonion algebra is the composition $p = * \circ (\jmath \otimes \jmath)$ of two $U_q(\mathrm{D}_4)$-module maps, so is itself a $U_q(\mathrm{D}_4)$-module homomorphism.

Let $\pi : U_q(\mathrm{D}_4) \to \mathrm{End}(V)$ be the natural representation of $U_q(\mathrm{D}_4)$. Then we have

$$\begin{aligned}\pi(a)(x \cdot y) &= a \cdot p(x \otimes y) = p(\Delta(a)(x \otimes y)) \\ &= p\Big(\sum_a \zeta(a_{(1)})x \otimes \eta(a_{(2)})y\Big) \\ &= \sum_a \jmath(\zeta(a_{(1)}x)) * \jmath(\eta(a_{(2)}y)) \\ &= \sum_a \Big(\zeta(a_{(1)})x\Big) \cdot \Big(\eta(a_{(2)})y\Big).\end{aligned} \tag{3.10}$$

Now when $q = 1$ we can identify $U_q(\mathrm{D}_4)$ with the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ of type D_4. The comultiplication specializes in this case to the usual comultiplication on $U(\mathfrak{g})$, which has $\Delta(a) = a \otimes 1 + 1 \otimes a$ for all $a \in \mathfrak{g}$. In particular, for elements $a \in \mathfrak{g}$, (3.10) becomes

$$\pi(a)(x \cdot y) = \big(\zeta(a)x\big) \cdot y + x \cdot \big(\eta(a)y\big). \tag{3.11}$$

Equation (3.11) is commonly referred to as the *Principle of Local Triality* (see for example, [J, p. 8] or [S, p. 88]). As a result we have,

Proposition 3.12. *The quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$ satisfies the q-Principle of Local Triality,*

$$\pi(a)(x \cdot y) = \sum_a \Big(\zeta(a_{(1)})x\Big) \cdot \Big(\eta(a_{(2)})y\Big)$$

for all $a \in U_q(D_4)$ and $x, y \in V$, where ζ and η are the graph automorphisms $\zeta = (1\ 3)$ and $\eta = (1\ 4)$ of $U_q(D_4)$, and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ is the comultiplication in $U_q(D_4)$.

A quantum para-Hurwitz algebra.

At $q = 1$, the map $\jmath : V \to V$ sending $x_i \mapsto -x_{-i}$ for $i \neq \pm 4$, $x_4 \mapsto -x_{-4}$, and $x_{-4} \mapsto -x_4$, agrees with the standard involution $x \to \overline{x}$ on the octonions so that

$$x * y = \jmath(x) \cdot \jmath(y) = \overline{x} \cdot \overline{y}. \tag{3.13}$$

Thus, the product "$*$" displayed in Table 2.7 gives what is called the *para-Hurwitz algebra* when $q = 1$. (Results on para-Hurwitz algebras can be found in [EM], [EP], [OO1], [OO2].)

Definition 3.14. *We say that the algebra $\mathbb{P}_q = (V, *)$ with the multiplication $*$ given by Table 2.7 is the quantum para-Hurwitz algebra. It satisfies*

$$x * y = \jmath(x) \cdot \jmath(y)$$

where $\mathbb{O}_q = (V, \cdot)$ is the quantum octonion algebra and $\jmath$ is the transformation on V defined in (2.12).

Our definitions have been predicated on using the mapping $\cdot : V_\zeta \otimes V_\eta \to V$. We can ask what would have happened if we had used the projection $V_\eta \otimes V_\zeta \to V$ instead? To answer this question, it is helpful to observe that the mapping $\jmath$ determines a $U_q(\mathrm{D}_4)$-module isomorphism $\jmath : V_{\zeta\eta\zeta} = V_{(3\ 4)} \to V$, which is readily apparent from comparing the following picture of $V_{(3\ 4)}$ with the diagram for V in (2.6).

$$
\begin{array}{ccc}
 & V_{(3\,4)} & \\
 & & \\
 & x_1\circ & \\
 & \downarrow 1 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
4\swarrow & & \searrow 3\\
x_4\circ & & \circ\, x_{-4}\\
3\searrow & & \swarrow 4\\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 1 & \\
 & x_{-1}\circ &
\end{array}
$$

Suppose as before that $p : V_\zeta \otimes V_\eta \to V$ is the product $p(x \otimes y) = x \cdot y$ in the quantum octonion algebra. Consider the mapping $\jmath \circ p \circ (\jmath \otimes \jmath) : V_\eta \otimes V_\zeta \to V$. We demonstrate that this is a $U_q(\mathrm{D}_4)$-module map. Bear in mind in these calculations that $\jmath : V_\zeta \to V_- = V_{\zeta\eta}$ and $\jmath : V_\eta \to V_+ = V_{\eta\zeta}$ for $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Then for $a \in U_q(\mathrm{D}_4)$ we have

$$
\begin{aligned}
\jmath \circ p \circ (\jmath \otimes \jmath)(a(x \otimes y)) &= \jmath \circ p \circ (\jmath \otimes \jmath)\Big(\sum_a \eta(a_{(1)})x \otimes \zeta(a_{(2)})y\Big)\\
&= \jmath \circ p\Big(\sum_a j(\eta(a_{(1)})x) \otimes j(\zeta(a_{(2)})y)\Big)\\
&= \jmath \circ p\Big(\sum_a \eta\zeta(a_{(1)})j(x) \otimes \zeta\eta(a_{(2)})j(y)\Big)\\
&= \jmath \circ p\Big(\sum_a \zeta^2\eta\zeta(a_{(1)})j(x) \otimes \eta^2\zeta\eta(a_{(2)})j(y)\Big)
\end{aligned}
$$

Now using the fact that $\eta\zeta\eta = \zeta\eta\zeta$ and that $(\phi \otimes \phi)\Delta = \Delta\phi$ for all graph automorphisms ϕ together with (3.10), we see that the last expression can be rewritten as

$$
\begin{aligned}
&\jmath \circ p\Big((\zeta \otimes \eta)\big(\Delta(\zeta\eta\zeta(a))\big)\big(\jmath(x) \otimes \jmath(y)\big)\Big)\\
&\qquad = \jmath\Big(\zeta\eta\zeta(a)p\big(\jmath(x) \otimes \jmath(y)\big)\Big) \quad (p \text{ is a } U_q(\mathrm{D}_4) - \text{homomorphism})\\
&\qquad = a(\jmath \circ p \circ (\jmath \otimes \jmath))(x \otimes y).
\end{aligned}
$$

Thus, $\jmath \circ p \circ (\jmath \otimes \jmath) : V_\eta \otimes V_\zeta \to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism. Since the space $\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\eta \otimes V_\zeta, V)$ is one-dimensional, we may assume the product

$p' : V_\eta \otimes V_\zeta \to V$ is given by the map we have just found, so that $p' = \jmath \circ p \circ (\jmath \otimes \jmath)$. Alternately, $\jmath \circ p' = p \circ (\jmath \otimes \jmath)$. This says that had we defined the quantum octonions using the product p' rather than p so that $x \cdot' y = p'(x \otimes y)$, the two algebras would be isomorphic via the map $\jmath$.

§4. Properties of Quantum Octonions

We use $U_q(\mathrm{D}_4)$-module homomorphisms to deduce various properties of the quantum octonions. At $q = 1$ these properties specialize to well-known identities satisfied by the octonions, which can be found for example in [ZSSS]. Since the vector spaces V, V_ζ, and V_η are the same, we will denote the identity map on them simply by "id" in what follows.

Proposition 4.1. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{S}^2(V_\zeta) \otimes V_\eta & \overset{(\,|\,) \otimes id}{\longrightarrow} & V_\eta \\ id \otimes p \downarrow & & \downarrow id \\ V_\zeta \otimes V & \overset{p \circ (\jmath \otimes id)}{\longrightarrow} & V_\eta, \end{array}$$

so that

$$\sum_k \jmath(u_k) \cdot \big(v_k \cdot y\big) = \sum_k (u_k | v_k) y \tag{4.2}$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q)$, $y \in \mathbb{O}_q$. In particular, at $q = 1$ this reduces to the well-known identity satisfied by the octonion algebra $\mathbb{O}$:

$$\overline{x} \cdot \big(x \cdot y\big) = (x|x)_1 y. \tag{4.3}$$

Proof. Since $p : V_\zeta \otimes V_\eta \to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism, we have that $\pi(a) \circ p = p\Big((\zeta \otimes \eta)\Delta(a)\Big)$, which implies that $\pi(\eta(a)) \circ p = p\Big((\zeta \otimes \eta)\Delta(\eta a)\Big) = p\Big((\zeta\eta \otimes \mathrm{id})\Delta(a)\Big)$. This says that $p : V_{\zeta\eta} \otimes V \to V_\eta$ is also a $U_q(\mathrm{D}_4)$-module homomorphism. Since $\jmath : V_\zeta \to V_{\zeta\eta}$ is a $U_q(\mathrm{D}_4)$-module homomorphism, it follows that $\big(p \circ (\jmath \otimes \mathrm{id})\big) \circ (\mathrm{id} \otimes p) \in \mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta) \otimes V_\eta, \mathbf{K})$. As that space of homomorphisms has dimension one, the maps in the diagram must be proportional.

To find the proportionality constant, let us consider the actions using the test element:

$$(x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1 \overset{(\,|\,)\otimes \mathrm{id}}{\to} x_1$$

and

$$p \circ (\jmath \otimes \mathrm{id}) \circ (\mathrm{id} \otimes p)\Big((x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1\Big) = \\ p \circ (\jmath \otimes \mathrm{id})\Big(- q^{\frac{3}{2}} x_1 \otimes x_4\Big) = q^{\frac{3}{2}} x_1 \cdot x_4 = x_1.$$

Therefore, the two maps are equal as claimed. At $q = 1$ we may substitute $x \otimes x \in \mathcal{S}^2(\mathbb{O})$ and $y \in \mathbb{O}$ into the above identity to obtain the corresponding one for the octonions. $\square$

Proposition 4.4. *The following diagram commutes*

$$\begin{array}{ccc} V_\zeta \otimes \mathcal{S}^2(V_\eta) & \overset{id \otimes (\,|\,)}{\longrightarrow} & V_\zeta \\ p \otimes id \downarrow & & \downarrow id \\ V \otimes V_\eta & \overset{p \circ (id \otimes \jmath)}{\longrightarrow} & V_\zeta, \end{array}$$

so that

$$\sum_k \big(y \cdot u_k\big) \cdot \jmath(v_k) = \sum_k (u_k|v_k) y \tag{4.5}$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q),\ y \in \mathbb{O}_q$. *In particular, at* $q = 1$ *this gives the identity*

$$\big(y \cdot x\big) \cdot \overline{x} = (x|x)_1 y \tag{4.6}$$

which is satisfied by the octonions.

Proof. Both maps are $U_q(\mathrm{D}_4)$-module homomorphisms and so must be multiples of each other. We compute their images on $x = x_1 \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1)$:

$$\big(\mathrm{id} \otimes (\,|\,)\big)x = \Big((x_1|x_{-1}) + (x_{-1}|x_1)\Big)x_1 = x_1,$$
$$p \circ (\mathrm{id} \otimes \jmath) \circ (p \otimes \mathrm{id})x = p \circ (\mathrm{id} \otimes \jmath)(q^{-\frac{3}{2}} x_{-4} \otimes x_1) = -q^{-\frac{3}{2}} x_{-4} \cdot x_1 = x_1.$$

Hence we have the desired equality. $\square$

Proposition 4.7. *The following diagram commutes*

$$\begin{array}{cccc} & V_\zeta \otimes V_\eta \otimes V & \overset{p \otimes id}{\longrightarrow} & V \otimes V \\ id \otimes \Big(\jmath \circ p \circ (id \otimes \jmath)\Big) & \downarrow & & \downarrow (\,|\,) \\ & V_\zeta \otimes V_\zeta & \overset{(\,|\,)}{\longrightarrow} & \mathbf{K} \end{array}$$

so that

$$(x \cdot y|z) = (x|\jmath(y \cdot \jmath(z))) = (\jmath(x)|y \cdot \jmath(z)) \tag{4.8}$$

for all x, y, z in the quantum octonions. At $q = 1$ this reduces to the identity

$$(x \cdot y|z)_1 = (x|z \cdot \overline{y})_1. \tag{4.9}$$

satisfied by the octonions.

Proof. Recall we have shown that $\jmath \circ p \circ (\jmath \otimes \jmath) : V_\eta \otimes V_\zeta \to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism, which is equivalent to saying

$$a \circ \jmath \circ p \circ (\jmath \otimes \jmath) = \jmath \circ p \circ (\jmath \otimes \jmath)\Big((\eta \otimes \zeta)\Delta(a)\Big)$$

for all $a \in U_q(\mathrm{D}_4)$. This implies

$$\begin{aligned}\zeta(a) \circ \jmath \circ p \circ (\jmath \otimes \jmath) &= \jmath \circ p \circ (\jmath \otimes \jmath)\Big((\eta \otimes \zeta)\Delta(\zeta(a))\Big)\\ &= \jmath \circ p \circ (\jmath \otimes \jmath)\Big((\eta \otimes \zeta)(\zeta \otimes \zeta)\Delta(a)\Big)\\ &= \jmath \circ p \circ (\jmath \otimes \jmath)\Big((\eta\zeta \otimes \mathrm{id})\Delta(a)\Big)\end{aligned}$$

so that $\jmath \circ p \circ (\jmath \otimes \jmath) : V_{\eta\zeta} \otimes V \to V_\zeta$ is a $U_q(\mathrm{D}_4)$-module homomorphism. From this we deduce that

$$\jmath \circ p \circ (\mathrm{id} \otimes \jmath) = \jmath \circ p \circ (\jmath \otimes \jmath) \circ (\jmath \otimes \mathrm{id}) : V_\eta \otimes V \to V_\zeta$$

is a $U_q(\mathrm{D}_4)$-module homomorphism as well. Thus, the mappings in the above diagram are scalar multiples of one another. The scalar can be discovered by computing the values of them on $x_1 \otimes x_{-3} \otimes x_{-2}$:

$$(x_1 \cdot x_{-3}|x_{-2}) = q^{-\frac{3}{2}}(x_2|x_{-2}) = \frac{-q^{-\frac{3}{2}}q^{-2}}{q^3 + q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3 + q^{-3}}$$

$$(x_1|\jmath\big(x_{-3} \cdot \jmath(x_{-2})\big)) = (x_1|\jmath(q^{-\frac{1}{2}}x_{-1})) = \frac{-q^{-\frac{1}{2}}q^{-3}}{q^3 + q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3 + q^{-3}}.$$

Hence, they agree and we have the result. The fact that $\jmath$ is an isometry gives last equality in (4.8). □

Proposition 4.10. $(x \cdot y|z) = (\jmath(y)|\jmath(z) \cdot K_{2\rho}^{-1}x)$ *for all $x, y, z \in \mathbb{O}_q = (V, \cdot)$, which at $q = 1$ gives*

$$(x \cdot y|z)_1 = (y|\overline{x} \cdot z)_1 \tag{4.11}$$

for the octonions.

Proof. This is an immediate consequence of the calculation,

$$\begin{aligned}(x\cdot y|z)&\overset{1}{=}(x|\jmath(y\cdot\jmath(z)))\overset{2}{=}(\jmath(y\cdot\jmath(z))|K_{2\rho}^{-1}x)\\&=(y\cdot\jmath(z)|\jmath(K_{2\rho}^{-1}x))\overset{3}{=}(\jmath(y)|\jmath(z)\cdot K_{2\rho}^{-1}x),\end{aligned}$$

where (1) is from Proposition 4.7, (2) uses (2.20), and (3) is Proposition 4.7. □

Note $K_{2\rho}^{-1}$ acts on $\mathbb{O}_q$ as an algebra automorphism because it is group-like.

Proposition 4.12. *The following diagram commutes*

$$\begin{array}{cccc} & V_\eta\otimes\mathcal{S}^2(V_\zeta)\otimes V_\eta & \overset{id\otimes(\,|\,)\otimes id}{\longrightarrow} & V_\eta\otimes V_\eta\\ \Big(\jmath\circ p\circ(\jmath\otimes\jmath)\Big)\otimes p & \downarrow & & \downarrow(\,|\,)\\ & V\otimes V & \overset{(\,|\,)}{\longrightarrow} & \mathbf{K}.\end{array}$$

When $q=1$, this gives the identity

$$(x\cdot y|x\cdot z)_1=(x|x)_1(y|z)_1 \tag{4.13}$$

for the octonions.

Proof. First, it is helpful to observe that

$$\begin{aligned}\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\eta\otimes\mathcal{S}^2(V_\zeta)\otimes V_\eta,\mathbf{K})&\cong\mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta),V_\eta\otimes V_\eta)\\&=\mathrm{Hom}_{U_q(\mathrm{D}_4)}(L(0),L(0)),\end{aligned}$$

so these spaces have dimension 1. We evaluate the mappings on $x=x_{-1}\otimes(x_1\otimes x_{-1}+x_{-1}\otimes x_1)\otimes x_1$:

$$\begin{aligned}(\,|\,)\circ\Big(\mathrm{id}\otimes(\,|\,)\otimes\mathrm{id}\Big)(x)&=(x_{-1}|x_1)=\frac{q^3}{q^3+q^{-3}}\\(\,|\,)\circ\Big(\big(\jmath\circ p\circ(\jmath\otimes\jmath)\big)\otimes p\Big)(x)&=(\jmath(x_{-1}\cdot x_1)|x_{-1}\cdot x_1)\\&=-q^3(x_{-4}|x_4)=\frac{q^3}{q^3+q^{-3}}.\end{aligned}$$ □

Proposition 4.14. *The following diagram commutes*

$$\begin{array}{cccc} & V_\zeta \otimes \mathcal{S}^2(V_\eta) \otimes V_\zeta & \overset{id \otimes (\,|\,) \otimes id}{\longrightarrow} & V_\zeta \otimes V_\zeta \\ p \otimes \Big(\jmath \circ p \circ (\jmath \otimes \jmath)\Big) & \downarrow & & \downarrow (\,|\,) \\ & V \otimes V & \overset{(\,|\,)}{\longrightarrow} & \mathbf{K} \end{array}$$

In particular, at $q = 1$ this gives the following identity satisfied by the octonions:

$$(y \cdot x | z \cdot x)_1 = (y|z)_1 (x|x)_1 \tag{4.15}$$

Proof. The proof is virtually identical to that of Proposition 4.12. We check the maps on $x = x_{-1} \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$:

$$\begin{aligned} (\,|\,) \circ \Big(p \otimes \big(\jmath \circ p \circ (\jmath \otimes \jmath)\big)\Big)(x) &= (x_{-1} \cdot x_1 | \jmath(x_{-1} \cdot x_1)) = (\jmath(x_{-1} \cdot x_1) | x_{-1} \cdot x_1) \\ &= -q^3 (x_{-4} | x_4) = \frac{q^3}{q^3 + q^{-3}}, \end{aligned}$$

which equals $(\,|\,) \circ \big(\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}\big)(x)$. $\square$

r-algebras.

The notion of an r-algebra (that is, an algebra equipped with a Yang-Baxter operator) arises in Manin's work [M] on noncommutative geometry. Noteworthy examples of r-algebras are Weyl and Clifford algebras, noncommutative tori, certain universal enveloping algebras, and quantum groups (see for example, Baez [B]).

Definition 4.16. *Suppose A is an algebra over a field $\mathbf{F}$ with multiplication $p : A \otimes A \to A$. Assume $R \in End(A^{\otimes 2})$ is a Yang-Baxter operator, i.e. $R_{1,2} R_{2,3} R_{1,2} = R_{2,3} R_{1,2} R_{2,3}$, where $R_{1,2} = R \otimes id \in End(A^{\otimes 3})$ and $R_{2,3} = id \otimes R \in End(A^{\otimes 3})$. Then A is said to be an r-algebra if*

$$\begin{aligned} R \circ (p \otimes id_A) &= (id_A \otimes p) \circ R_{1,2} \circ R_{2,3} \\ R \circ (id_A \otimes p) &= (p \otimes id_A) \circ R_{2,3} \circ R_{1,2}. \end{aligned} \tag{4.17}$$

When A has a unit element 1, the Yang-Baxter operator R is also required to satisfy $R(1 \otimes x) = x \otimes 1, \quad R(x \otimes 1) = 1 \otimes x \quad$ for all $x \in A$.

Our quantum octonions satisfy r-algebra properties similar to those in (4.17) with respect to the R-matrix of $U_q(\mathrm{D}_4)$.

Proposition 4.18. *(a) In* $Hom_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, V \otimes V)$ *we have*

$$\check{R}_{V,V} \circ (p \otimes id) = (id \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}.$$

(b) In $Hom_{U_q(\mathrm{D}_4)}(V \otimes V_\zeta \otimes V_\eta, V \otimes V)$ *we have*

$$\check{R}_{V,V} \circ (id \otimes p) = (p \otimes id) \circ \check{R}_{V,V_\eta} \circ \check{R}_{V,V_\zeta}.$$

Proof. Suppose that $\pi_1 : V \otimes V \to L(2\omega_1)$, $\pi_2 : V \otimes V \to L(\omega_2)$, and $\pi_3 : V \otimes V \to L(0)$ are the projections of $V \otimes V$ onto its irreducible summands. Let T_ℓ and T_r denote the transformations on the left and right in (a). We compute T_ℓ and T_r on several elements of $V_\zeta \otimes V_\eta \otimes V$ which have been chosen so that Θ acts as the identity. We use the fact that $\eta(\epsilon_1) = 1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$, $\zeta(\epsilon_1) = 1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$, $\eta(\epsilon_2) = 1/2(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$, and $\zeta(\epsilon_3) = 1/2(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)$.

We begin by evaluating the action of T_ℓ and T_r on $x_{-1} \otimes x_1 \otimes x_{-1}$:

$$\begin{aligned}
\Big(\check{R}_{V,V} \circ (p \otimes \mathrm{id})\Big)(x_{-1} \otimes x_1 \otimes x_{-1}) &= -q^{\frac{3}{2}} \check{R}_{V,V}(x_4 \otimes x_{-1}) = -q^{\frac{3}{2}}(x_{-1} \otimes x_4) \\
\Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}\Big)(x_{-1} \otimes x_1 \otimes x_{-1}) &= q^{\frac{1}{2}} \Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V}\Big)(x_{-1} \otimes x_{-1} \otimes x_1) \\
&= (\mathrm{id} \otimes p)(x_{-1} \otimes x_{-1} \otimes x_1) \\
&= -q^{\frac{3}{2}}(x_{-1} \otimes x_4).
\end{aligned}$$

Now $(x_{-1}|x_4) = 0$ implies that $x_{-1} \otimes x_4$ lies in $L(2\omega_1) \oplus L(\omega_2)$. The vector $x_{-1} \otimes x_4$ is not an eigenvector of $\check{R}_{V,V}$ because $\check{R}_{V,V}(x_4 \otimes x_{-1}) = x_{-1} \otimes x_4$. Therefore, $\pi_1(x_{-1} \otimes x_4) \neq 0$ and $\pi_2(x_{-1} \otimes x_4) \neq 0$. Since $\dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(2\omega_1)) = 1 = \dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(\omega_2)$, we see from the above calculation that $\pi_1 T_\ell = \pi_1 T_r$ and $\pi_2 T_\ell = \pi_2 T_r$.

Consider the action of T_ℓ and T_r on $x_3 \otimes x_2 \otimes x_{-1}$:

$$\begin{aligned}
\Big(\check{R}_{V,V} \circ (p \otimes \mathrm{id})\Big)(x_3 \otimes x_2 \otimes x_{-1}) &= q^{\frac{1}{2}} \check{R}_{V,V}(x_1 \otimes x_{-1}) = q^{\frac{3}{2}}(x_{-1} \otimes x_1) \\
\Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}\Big)(x_3 \otimes x_2 \otimes x_{-1}) &= q^{\frac{1}{2}} \Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V}\Big)(x_3 \otimes x_{-1} \otimes x_2) \\
&= q(\mathrm{id} \otimes p)(x_{-1} \otimes x_3 \otimes x_2) \\
&= q^{\frac{3}{2}}(x_{-1} \otimes x_1).
\end{aligned}$$

The projection π_3 comes from the form $(\,|\,)$, and so it is nonzero when it acts on $x_{-1} \otimes x_1$. Consequently, applying π_3 to both sides of $T_\ell(x_3 \otimes x_2 \otimes x_{-1}) = T_r(x_3 \otimes x_2 \otimes x_{-1})$ and using $\dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(0)) = 1$, we obtain $\pi_3 T_\ell = \pi_3 T_r$. Therefore, $T_\ell = (\pi_1 + \pi_2 + \pi_3)T_\ell = (\pi_1 + \pi_2 + \pi_3)T_r = T_r$. The calculations for part (b) are almost identical and are left as an exercise for the reader. $\square$

§5. Properties of the Quantum Para-Hurwitz Algebra

Recall that the quantum para-Hurwitz algebra $\mathbb{P}_q = (V, *)$ is obtained from identifying the underlying vector spaces of V_θ, V_{θ^2} and V, where V is the natural representation of $U_q(\mathrm{D}_4)$ and θ is the graph automorphism $\theta = (1\,4\,3)$, and using the multiplication $p^* : V_\theta \otimes V_{\theta^2} \to V$, $x \otimes y \mapsto x * y = \jmath(x) \cdot \jmath(y)$. The resulting product is displayed in Table 2.7. As we saw in the last chapter, the quantum octonion algebra enjoys very nice composition properties relative to the bilinear $(\,|\,)$. It is known that the para-Hurwitz algebra $\mathbb{P}$ (that is the $q = 1$ version of $\mathbb{P}_q$), which is obtained from the octonions using the product $x * y = \overline{x} \cdot \overline{y}$, has an associative bilinear form admitting composition. Here we establish similar results for the bilinear form on $\mathbb{P}_q = (V, *)$

Proposition 5.1. *For the quantum para-Hurwitz algebra $\mathbb{P}_q = (V, *)$,*

$$(x * y|z) = (x|y * z) \qquad \text{and} \qquad (x * y|z) = (y|z * K_{2\rho}^{-1}x)$$

for all $x, y, z \in \mathbb{P}_q$. At $q = 1$ these relations reduce to the well-known properties of the para-Hurwitz algebra $(x * y|z)_1 = (x|y * z)_1$ and $(x * y|z)_1 = (y|z * x)_1$.

Proof. The proof amounts to using the fact that $\jmath$ is an isometry together with the fact that $\jmath$ commutes with $K_{2\rho}^{-1}$. So we have

$$(x*y|z) = (\jmath(x)\cdot\jmath(y)|z) = \begin{cases} (x|\jmath(y)\cdot\jmath(z)) = (x|y*z) \\ (y|\jmath(z)\cdot K_{2\rho}^{-1}\jmath(x)) = (y|\jmath(z)\cdot\jmath(K_{2\rho}^{-1}x)) = (y|z*K_{2\rho}^{-1}x). \end{cases}$$

□

Proposition 5.2. *Ignoring the $U_q(D_4)$-module structures we have*

$$(\jmath \circ p^* \circ (\jmath \otimes \jmath)) \otimes p^* \qquad \begin{array}{ccc} V \otimes \mathcal{S}^2(V) \otimes V & \xrightarrow{id \otimes (\,|\,) \otimes id} & V \otimes V \\ \downarrow & & \downarrow (\,|\,) \\ V \otimes V & \xrightarrow{(\,|\,)} & \mathbf{K} \end{array}$$

$$p^* \otimes \Big(\jmath \circ p^* \circ (\jmath \otimes \jmath)\Big) \qquad \begin{array}{ccc} V \otimes \mathcal{S}^2(V) \otimes V & \xrightarrow{id \otimes (\,|\,) \otimes id} & V \otimes V \\ \downarrow & & \downarrow (\,|\,) \\ V \otimes V & \xrightarrow{(\,|\,)} & \mathbf{K} \end{array}$$

Proof. These diagrams commute when p is used in place of p^*. Now when $p^* = p \circ (\jmath \otimes \jmath)$ is taken, we have

$$\begin{aligned}(\,|\,) \circ (\jmath \circ p^* \circ (\jmath \otimes \jmath)) \otimes p^* &= (\,|\,) \circ \Big((\jmath \circ p) \otimes (p \circ (\jmath \otimes \jmath)) \Big) \\ &\overset{1}{=} (\,|\,) \circ p \otimes (j \circ p \circ (\jmath \otimes \jmath)) \\ &= (\,|\,) \circ \mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}\end{aligned}$$

where (1) follows from the fact that $\jmath$ is an isometry. The other diagram can be shown to commute in a similar way. $\square$

Proposition 5.3. *The following diagram commutes and shows at $q = 1$ that the relation $(x \cdot y) \cdot x = (x|x)_1 y = x \cdot (y \cdot x)$ holds for the para-Hurwitz algebra.*

$$\begin{array}{ccccc} V_{\theta^2} \otimes V & \xleftarrow{\mathrm{id} \otimes \left(\jmath \circ p^* \circ (\jmath \otimes \jmath)\right)} & \mathcal{S}^2(V_{\theta^2}) \otimes V_\theta = \mathcal{S}^2(V) \otimes V = \mathcal{S}^2(V_\theta) \otimes V_{\theta^2} & \xrightarrow{\mathrm{id} \otimes p^*} & V_\theta \otimes V \\ \downarrow p^* & & (\,|\,) \otimes id \downarrow & & \jmath \circ p^* \circ (\jmath \otimes \jmath) \quad \downarrow \\ V_\theta & = & V & = & V_{\theta^2} \end{array}$$

Proof. Again it suffices to check the maps on the element $x = (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$, and for it we have

$$\begin{aligned}\Big(\jmath \circ p^* \circ (\jmath \otimes \jmath)\Big) \circ (\mathrm{id} \otimes p^*)(x) &= \jmath \circ p^* \circ (\jmath \otimes \jmath)(x_1 \otimes (-q^{\frac{3}{2}}) x_4) \\ &= -\jmath(x_1) = x_1 = (\,|\,) \otimes \mathrm{id}(x), \quad \text{while}\end{aligned}$$

$$p^* \circ \Big(\mathrm{id} \otimes \big(\jmath \circ p^* \circ (\jmath \otimes \jmath)\big)\Big)(x) = p^*(x_1 \otimes q^{\frac{3}{2}} x_{-4}) = x_1 = (\,|\,) \otimes \mathrm{id}(x). \quad \square$$

§6. Idempotents and Derivations of Quantum Octonions

It is apparent from Table 3.9 that $e_1 = q^{-\frac{1}{2}} x_4$ and $e_2 = -q^{\frac{1}{2}} x_{-4}$ are orthogonal idempotents in the quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$. Their sum $e = e_1 + e_2$ acts as a left and right identity element on the span of $e_1, e_2, x_j, x_{-j}, j = 2, 3$. Moreover, the relations $e \cdot x_1 = q^2 x_1$, $x_1 \cdot e = q^{-2} x_1$, $e \cdot x_{-1} = q^{-2} x_{-1}$, and $x_{-1} \cdot e = q^2 x_{-1}$ hold in $\mathbb{O}_q$. Hence, at $q = 1$ or -1, the element e is an identity element. Relative to the basis $e_1, e_2, x_{\pm j}, j = 1, 2, 3$, the left and right multiplication operators $L_{e_1}, R_{e_1}, L_{e_2}, R_{e_2}$ can be simultaneously diagonalized. The operators L_{e_1}, R_{e_1} determine a Peirce decomposition of $\mathbb{O}_q$,

$$\mathbb{O}_q = \oplus_{(\gamma,\delta)}(\mathbb{O}_q)_{\gamma,\delta} \quad \text{where} \quad (\mathbb{O}_q)_{\gamma,\delta} = \{v \in \mathbb{O}_q \mid e_1 \cdot v = \gamma v, \text{ and } v \cdot e_1 = \delta v\}. \tag{6.1}$$

Now $(\mathbb{O}_q)_{0,q^{-2}} = \mathbf{K}x_1$, $(\mathbb{O}_q)_{1,0} = \mathbf{K}x_2 + \mathbf{K}x_3$, $(\mathbb{O}_q)_{1,1} = \mathbf{K}e_1$, $(\mathbb{O}_q)_{0,0} = \mathbf{K}e_2$, $(\mathbb{O}_q)_{0,1} = \mathbf{K}x_{-2} + \mathbf{K}x_{-3}$, and $(\mathbb{O}_q)_{q^{-2},0} = \mathbf{K}x_{-1}$. The idempotent e_2 gives a similar Peirce decomposition.

Suppose d is a derivation of $\mathbb{O}_q$ such that $d(e_1) = 0 = d(e_2)$. It is easy to see that d must map each Peirce space $(\mathbb{O}_q)_{\gamma,\delta}$ into itself. Thus, we may suppose that

$$\begin{aligned} d(x_1) &= b_1 x_1 & d(x_{-1}) &= b_{-1}x_{-1} \\ d(x_2) &= c_{2,2}x_2 + c_{3,2}x_3 & d(x_3) &= c_{2,3}x_2 + c_{3,3}x_3 \\ d(x_{-2}) &= d_{2,2}x_{-2} + d_{3,2}x_{-3} & d(x_{-3}) &= d_{2,3}x_{-2} + d_{3,3}x_{-3}. \end{aligned} \tag{6.2}$$

Applying d to the relation $x_1 \cdot x_{-1} = q^{-\frac{3}{2}}x_{-4} = -q^{-2}e_2$ (or to the relation $x_{-1} \cdot x_1 = -q^{\frac{3}{2}}x_4 = -q^2 e_1$) shows that $b_{-1} = -b_1$. Now from $x_k \cdot x_k = 0$, we deduce that $c_{j,k} = 0$ for $j \neq k$, and similarly from $x_{-k} \cdot x_{-k} = 0$ it follows that $d_{j,k} = 0$. There are relations $x_1 \cdot x_{-j} = \xi x_k$, $x_{-j} \cdot x_1 = \xi x_k$, $x_{-1} \cdot x_j = \xi x_{-k}$, and $x_j \cdot x_{-1} = \xi x_{-k}$ (where $j \neq k$ and ξ in each one indicates some appropriate scalar that can be found in Table 3.9), and applying d to those equations gives $b_1 + d_{j,j} = c_{k,k}$ and $b_{-1} + c_{j,j} = d_{k,k}$. The second is equivalent to the first since $b_{-1} = -b_1$. From $x_{\pm j} \cdot x_{\pm k} = \xi x_{\pm 1}$ we see that $c_{2,2} + c_{3,3} = b_1$ and $d_{2,2} + d_{3,3} = -b_1$. Finally, applying d to $x_j \cdot x_{-j} = \xi e_1$ or $x_{-j} \cdot x_j = \xi e_2$, we obtain $d_{j,j} = -c_{j,j}$ for $j = 2, 3$. Every nonzero product between basis elements in the quantum octonions is one of these types, or it is one involving the idempotents e_1, e_2, so we have determined all the possible relations. Consequently, we can conclude that for any derivation d such that $d(e_1) = 0 = d(e_2)$, there are scalars $b = b_1$ and $c = c_{2,2}$ so that the following hold:

$$\begin{aligned} & d(e_1) = 0 = d(e_2), \\ d(x_1) &= bx_1 & d(x_{-1}) &= -bx_{-1} \\ d(x_2) &= cx_2 & d(x_{-2}) &= -cx_{-2} \\ d(x_3) &= (b-c)x_3 & d(x_{-3}) &= -(b-c)x_{-3}. \end{aligned} \tag{6.3}$$

The same calculations show that any transformation defined by (6.3) where b, c are arbitrary scalars is a derivation of $\mathbb{O}_q$.

Now suppose that d is an arbitrary derivation of $\mathbb{O}_q$, and let $d(e_1) = a_0 e_1 + a_0' e_2 + \sum_{j=-3, j\neq 0}^{3} a_j x_j$ and $d(e_2) = b_0 e_1 + b_0' e_2 + \sum_{j=-3, j\neq 0}^{3} b_j x_j$. Computing $d(e_1) \cdot e_1 + e_1 \cdot d(e_1) = d(e_1)$, we obtain

$$2a_0e_1 + q^{-2}a_1x_1 + a_2x_2 + a_3x_3 + q^{-2}a_{-1}x_{-1} + a_{-2}x_{-2} + a_{-3}x_{-3} = d(e_1),$$

which shows that $a_0 = a_0' = a_{-1} = a_1 = 0$. A similar calculation with the relation $d(e_2^2) = d(e_2)$ gives $b_0 = b_0' = b_{-1} = b_1 = 0$. From $d(e_1)\cdot e_2 + e_1\cdot d(e_2) = 0$ it follows that $b_2 = -a_2$ and $b_3 = -a_3$, and from $d(e_2)\cdot e_1 + e_2\cdot d(e_1) = 0$, the relations $b_{-2} = -a_{-2}$, $b_{-3} = -a_{-3}$ can be determined.

Now $e_1 \cdot d(x_1) = -d(e_1)\cdot x_1$ shows that the coefficients of x_2 and x_3 in $d(x_1)$ are $q^{\frac{3}{2}}a_{-3}$ and $q^{\frac{3}{2}}a_{-2}$ respectively; while $d(x_1)\cdot e_2 = -x_1\cdot d(e_2)$ gives that those coefficients are $-q^{-\frac{3}{2}}b_{-3} = q^{-\frac{3}{2}}a_{-3}$ and $-q^{-\frac{3}{2}}b_{-2} = q^{-\frac{3}{2}}a_{-2}$. Thus, when q is not a cube root of unity, we obtain that $a_{-2} = 0 = a_{-3}$. Likewise the relations $a_2 = 0 = a_3$ can be found from analyzing $d(x_{-1})\cdot e_1 = -x_{-1}d(e_1)$ and $e_2\cdot d(x_{-1}) = -d(e_2)\cdot x_{-1}$. Consequently, when q is not a cube root of unity, every derivation of $\mathbb{O}_q$ must annihilate e_1 and e_2, and so must be as in (6.3) for suitable scalars b, c. To summarize we have

Theorem 6.4. *The derivation algebra of the quantum octonion algebra $\mathbb{O}_q = (V,\cdot)$ is two-dimensional. Every derivation d of $\mathbb{O}_q$ is given by (6.3) for some scalars $b, c \in \mathbf{K}$.*

Simplicity of the quantum octonions.

The idempotents also provide a proof of the following

Theorem 6.5. *The quantum octonion algebra $\mathbb{O}_q = (V,\cdot)$ is simple.*

Proof. Let I be a nonzero ideal of $\mathbb{O}_q$. Since $e_1I \subseteq I$ and $Ie_1 \subseteq I$, it follows that

$$I = \oplus_{\gamma,\delta} I_{\gamma,\delta},$$

where $I_{\gamma,\delta} = \{x \in I \mid e_1\cdot x = \gamma x, \text{ and } x\cdot e_1 = \delta x\}$. Assume first $x_i \in I$ for some $i = \{\pm1, \pm2, \pm3, \pm4\}$. If we multiply x_i by x_{-i} when $i = \pm1, \pm2$, or ±3, we see that e_1 or e_2 belongs to I. Therefore, either $e_1\mathbb{O}_q + \mathbb{O}_qe_1 \subseteq I$ or $e_2\mathbb{O}_q + \mathbb{O}_qe_2 \subseteq I$, and as a consequence, $I = \mathbb{O}_q$. It remains to consider the possibility that some nonzero linear combination $ax_2 + bx_3$ or $ax_{-2} + bx_{-3}$ is in I. Multiplying by one of the elements x_2, x_3, x_{-2}, or x_{-3}, we obtain that x_1 or x_{-1} is in I, and that implies $I = \mathbb{O}_q$ as before. Consequently, $\mathbb{O}_q$ is simple. $\square$

§7. Quantum Quaternions

The subalgebra $\mathbb{H}_q$ of $\mathbb{O}_q$ generated by $x = x_2$ and $y = x_{-2}$ is 4-dimensional and has x, y, e_1, e_2 as a basis, where $e_1 = q^{-\frac{1}{2}}x_4$ and $e_2 = -q^{\frac{1}{2}}x_{-4}$ are the orthogonal idempotents of Section 6. The sum $e = e_1 + e_2$ is the identity element of $\mathbb{H}_q$, and multiplication in this algebra is given by the following:

Table 7.1

$\cdot$	e_1	e_2	x	y
e_1	e_1	0	x	0
e_2	0	e_2	0	y
x	0	x	0	$q^{-1}e_1$
y	y	0	qe_2	0

It is easy to see that at $q = 1$ the algebra above is isomorphic to the algebra $M_2(\mathbf{K})$ of 2×2 matrices over $\mathbf{K}$ via the mapping $e_1 \mapsto E_{1,1}$, $e_2 \mapsto E_{2,2}$, $x \mapsto E_{1,2}$, $y \mapsto E_{2,1}$, which sends basis elements to standard matrix units. Since the split quaternion algebra is isomorphic to $M_2(\mathbf{K})$, this algebra reduces to the quaternions at $q = 1$. Another way to see this is to observe that the bilinear form on $\mathbb{O}_q$, when restricted to $\mathbb{H}_q$, satisfies (3.6) at $q = 1$. Thus, at $q = 1$ we have a 4-dimensional unital algebra with a nondegenerate bilinear form admitting composition, and so it must be the quaternions.

The algebra $\mathbb{H}_q$ is neither associative nor flexible if $q \neq \pm 1$ since

$$(x \cdot y) \cdot x = q^{-1}x \neq qx = x \cdot (y \cdot x).$$

However, it is *Lie admissible.* Indeed, if $h = q^{-1}e_1 - qe_2$, then

$$[h, x] = (q + q^{-1})x, \quad [h, y] = -(q + q^{-1})y, \quad \text{and} \quad [x, y] = h.$$

Therefore, the elements x, $\dfrac{2}{q+q^{-1}}y$, $\dfrac{2}{q+q^{-1}}h$ determine a standard basis for sl_2. Since $e = e_1 + e_2$ is the identity, we see that $\mathbb{H}_q$ is the Lie algebra gl_2 under the commutator product $[a, b] = ab - ba$.

The subalgebra generated by x_3 and x_{-3} is isomorphic to the algebra $\mathbb{H}_q$. However, that is not true if we consider the subalgebra $\mathbb{H}'_q$ generated by x_1 and x_{-1}. Let $u = x_{-1}$ and $v = x_1$. Then multiplication in $\mathbb{H}'_q$ is given by

Table 7.2

$\cdot$	e_1	e_2	u	v
e_1	e_1	0	$q^{-2}u$	0
e_2	0	e_2	0	q^2v
u	0	q^2u	0	q^2e_1
v	$q^{-2}v$	0	$q^{-2}e_2$	0

There is no identity element in $\mathbb{H}'_q$. But at $q = 1$, the element $e_1 + e_2$ becomes an identity element, and the algebra is isomorphic to $M_2(\mathbf{K})$. The bilinear form at $q = 1$ has the composition property, so the algebra $\mathbb{H}'_q$ too might rightfully be termed a quantum quaternion algebra. The elements $h' = q^2e_1 - q^{-2}e_2, u, v$ determine a standard basis for sl_2:

$$[h', u] = 2u, \qquad [h', v] = -2v, \qquad [u, v] = h'.$$

Since $[q^2e_1 + q^{-2}e_2, h'] = [q^2e_1 + q^{-2}e_2, u] = [q^2e_1 + q^{-2}e_2, v] = 0$, the algebra $\mathbb{H}'_q$ under the commutator product is isomorphic to gl_2. It is neither associative nor flexible because

$$\big((e_1 + v) \cdot e_1\big) \cdot (e_1 + v) = e_1 + q^{-4}v \neq e_1 + q^{-2}v = (e_1 + v) \cdot \big(e_1 \cdot (e_1 + v)\big).$$

§8. Connections with $U_q(\mathrm{B}_3)$ and $U_q(\mathrm{G}_2)$

In this section we assume that the field $\mathbf{K}$ has characteristic zero. We consider the subalgebras of $U_q(\mathrm{D}_4)$ that are described as the fixed points of graph automorphisms by

$$\begin{aligned} U_{\zeta,\eta} &= \{a \in U_q(\mathrm{D}_4) \mid \zeta(a) = a = \eta(a)\} \\ U_\delta &= \{a \in U_q(\mathrm{D}_4) \mid \delta(a) = a\}, \end{aligned} \tag{8.1}$$

where $\delta = (3\,4) = \zeta\eta\zeta$. Then $U_\delta \supseteq U_{\zeta,\eta}$. In the non-quantum case, the enveloping algebra $U(\mathrm{G}_2) \subseteq U_{\zeta,\eta} \subseteq U(\mathrm{D}_4)$ and $U(\mathrm{B}_3) \subseteq U_\delta \subseteq U(\mathrm{D}_4)$. We consider the structure of V, V_ζ, V_η as modules for the subalgebras given in (8.1).

Now for any $a \in U_\delta$, we have that

$$\jmath(ax) = \jmath(\delta(a)x) = a\jmath(x)$$

for all $x \in V$ so that any $a \in U_\delta$ commutes with $\jmath$. In particular, the eigenspaces of $\jmath$ are U_δ-invariant. But the eigenspaces of $\jmath$ are 1-dimensional (spanned by $x_4 - x_{-4}$), and 7-dimensional (spanned by $\{x_i \mid i = \pm 1, \pm 2, \pm 3\} \cup \{x_4 + x_{-4}\}$). Consider the 7-dimensional space under the action of $U_{\zeta,\eta}$. Since $F_{\alpha_2}, F_{\alpha_1} + F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_2}, E_{\alpha_1} + E_{\alpha_3} + E_{\alpha_4} \in U_{\zeta,\eta}$, it is easy to check that the 7-dimensional space is an irreducible $U_{\zeta,\eta}$-module, hence an irreducible U_δ-module as well. The action of $U_{\zeta,\eta}$ on V_ζ and V_η is the same as its action on V. As a result we have

Proposition 8.2. *The modules $V, V_\zeta \cong V_-$, and $V_\eta \cong V_+$ are isomorphic as modules for the subalgebra $U_{\zeta,\eta}$ in (8.1). They decompose into the sum of irreducible $U_{\zeta,\eta}$-modules of dimensions 1 and 7, which are the eigenspaces of $\jmath$.*

For $q = 1$, Proposition 8.2 amounts to saying that V, V_ζ, V_η are isomorphic $U(\mathrm{G}_2)$-modules which decompose into irreducible modules of dimension 1 and 7.

The elements $F_{\alpha_1}, F_{\alpha_2}, F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3} + E_{\alpha_4}$ belong to U_δ, so if we regard V_ζ or V_η as a module for U_δ, then we obtain a diagram

$$\begin{array}{cccccccc} & & & & & & x_4 & \\ & & & & & \overset{F_{\alpha_1}}{\nearrow} & & \\ x_1 & \overset{F_{\alpha_3}+F_{\alpha_4}}{\rightarrow} & x_2 & \overset{F_{\alpha_2}}{\rightarrow} & x_3 & & & \cdots \\ & & & & & \overset{F_{\alpha_3}+F_{\alpha_4}}{\searrow} & & \\ & & & & & & x_{-4} & \end{array}$$

which is gotten by superimposing the two diagrams of V_ζ and V_η. Using that, it is easy to argue that both these modules remain irreducible for U_δ. Moreover, for any $a \in U_\delta$, we have

$$\begin{aligned} \jmath(\zeta(a)x) &= \jmath((\zeta\delta)(a)x) = \jmath((\eta\zeta)(a)x) \\ &= \jmath(\theta^2(a)x) = \eta(a)\jmath(x) \end{aligned}$$

where $\theta = (1\,4\,3)$. Consequently, $V_\zeta \cong V_\eta$ as U_δ-modules.

At $q = 1$ these computations show that V_ζ and V_η (the two spin modules of $U(\mathrm{D}_4)$) remain irreducible for $U(\mathrm{B}_3)$, and in fact they are the spin representation of that algebra. To summarize we have

Proposition 8.3. *When regarded as modules for the subalgebra U_δ of $U_q(D_4)$, $V_\zeta \cong V_-$, and $V_\eta \cong V_+$ are irreducible and isomorphic.*

In light of these propositions it is natural to expect that these fixed point subalgebras of the graph automorphisms might be the quantum groups $U_q(\mathrm{G}_2)$ and $U_q(\mathrm{B}_3)$. However, that is not true. In fact the next "nonembedding" result shows that there are no Hopf algebra homomorphisms of them into $U_q(\mathrm{D}_4)$ except for the trivial ones given by the counit.

Proposition 8.4. *When q is not a root of unity, then any Hopf algebra homomorphisms*

$$U_q(G_2) \to U_q(B_3), \quad U_q(B_3) \to U_q(D_4), \quad U_q(G_2) \to U_q(D_4)$$

are trivial ($a \mapsto \epsilon(a)1$ for all a).

The method we use to establish this result is due originally to Hayashi [H]. We begin with a little background before presenting the proof.

Suppose $U_q(\mathfrak{g})$ is a quantum group and (X, π), $\pi : U_q(\mathfrak{g}) \to \mathrm{End}(X)$, is a representation of $U_q(\mathfrak{g})$. This determines another representation (X, π'), $\pi' = \pi \circ S^2$, of $U_q(\mathfrak{g})$ where S is the antipode. But it is well-known [Jn, p. 56] that S^2 has the following expression:

$$S^2(a) = K_{2\rho}^{-1} a K_{2\rho} \qquad \text{for all } a \in U_q(\mathfrak{g})$$

where $\rho = 1/2 \sum_{\gamma > 0} \gamma = \sum_{\alpha \in \Pi} c_\alpha \alpha$ (the half-sum of the positive roots of $\mathfrak{g}$) and $K_{2\rho} = \prod_{\alpha \in \Pi} K_\alpha^{c_\alpha}$ (where Π is the set of simple roots of $\mathfrak{g}$). The two representations (X, π) and (X, π') are in fact isomorphic by the following map

$$\phi : (X, \pi) \to (X, \pi'), \qquad x \mapsto \pi(K_{2\rho}^{-1})x$$

because $\phi(\pi(a)x) = \pi(K_{2\rho}^{-1})\pi(a)x = \pi(K_{2\rho}^{-1}a)x = \pi(S^2(a)K_{2\rho}^{-1})x = \pi(S^2(a))\pi(K_{2\rho}^{-1})x = \pi'(a)\phi(x)$ for all $a \in U_q(\mathfrak{g})$.

Examples 8.5. Let $\mathfrak{g} = \mathrm{D}_4$ and let $X = V, V_-, V_+$ (the natural or spin representations of $U_q(\mathrm{D}_4)$). Here $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$ so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6} K_{\alpha_2}^{-10} K_{\alpha_3}^{-6} K_{\alpha_4}^{-6}$. Relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ we have as before

$$\phi_{\mathrm{D}_4} = \phi_{\mathrm{D}_4}^- = \phi_{\mathrm{D}_4}^+ = \mathrm{diag}\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}. \tag{8.6}$$

Next let $\mathfrak{g} = \mathrm{G}_2$ and let X be the 7-dimensional irreducible $U_q(\mathrm{G}_2)$-representation. Here $\rho = 5\alpha_1 + 3\alpha_2$ so $K_{2\rho}^{-1} = K_{\alpha_1}^{-10} K_{\alpha_2}^{-6}$. With respect to the following basis (see [Jn], Chap. 5),

$$\{x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_1}, x_0, x_{-\alpha_1}, x_{-(\alpha_1+\alpha_2)}, x_{-(2\alpha_1+\alpha_2)}\}, \qquad \text{we have} \tag{8.7}$$

$$\phi_{\mathrm{G}_2} = \mathrm{diag}\{q^{-10}, q^{-8}, q^{-2}, 1, q^2, q^8, q^{10}\}. \tag{8.8}$$

Finally, assume X is the 7-dimensional natural representation or the 8-dimensional spin representation of $U_q(\mathrm{B}_3)$. In this case $2\rho = 5\alpha_1 + 8\alpha_2 + 9\alpha_3$ so that $K_{2\rho}^{-1} =$

$K_{\alpha_1}^{-5}K_{\alpha_2}^{-8}K_{\alpha_3}^{-9}$. If we assume as in [Jn] that a short root satisfies $(\alpha,\alpha)=2$, then setting $(\epsilon_i,\epsilon_i)=2$ for $i=1,2,3$, we have that the matrix of the bilinear form relative to the basis $\alpha_1=\epsilon_1-\epsilon_2$, $\alpha_2=\epsilon_2-\epsilon_3$, $\alpha_3=\epsilon_3$ of simple roots is

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

From this we can compute that $(-2\rho,\epsilon_i)=-10,-6,-2$ for $i=1,2,3$, respectively. Since we know that for any weight vector x_μ that $K_\alpha x_\mu = q^{(\mu,\alpha)}x_\mu$, we obtain that relative to the basis

$$\{x_{\epsilon_1},x_{\epsilon_2},x_{\epsilon_3},x_0,x_{-\epsilon_3},x_{-\epsilon_2},x_{-\epsilon_1}\}, \tag{8.9}$$

for the natural representation,

$$\phi_{\mathrm{B}_3}=\mathrm{diag}\{q^{-10},q^{-6},q^{-2},1,q^2,q^6,q^{10}\}. \tag{8.10}$$

The spin representation of $U_q(\mathrm{B}_3)$ has a basis

$$\{x_{\xi_1},x_{\xi_2},x_{\xi_3},x_{\xi_4},x_{-\xi_4},x_{-\xi_3},x_{-\xi_2},x_{-\xi_1}\},$$

where $\xi_1=(1/2)(\epsilon_1+\epsilon_2+\epsilon_3)$, $\xi_2=(1/2)(\epsilon_1+\epsilon_2-\epsilon_3)$, $\xi_3=(1/2)(\epsilon_1-\epsilon_2+\epsilon_3)$, and $\xi_4=(1/2)(-\epsilon_1+\epsilon_2+\epsilon_3)$. The corresponding matrix of $\phi^+_{\mathrm{B}_3}$ is given by

$$\phi^+_{\mathrm{B}_3}=\mathrm{diag}\{q^{-9},q^{-7},q^{-3},q^{-1},q,q^3,q^7,q^9\}.$$

Now suppose that $f:U_q(\mathfrak{g}_1)\to U_q(\mathfrak{g}_2)$ is a Hopf algebra homomorphism between two quantum groups. Consider representations (X,π) and (X,π') for $U_q(\mathfrak{g}_2)$ as above. Then $(X,\pi\circ f)$, $(X,\pi'\circ f)$ are representations for $U_q(\mathfrak{g}_1)$. Since $\pi'\circ f=\pi\circ S^2\circ f=\pi\circ f\circ S^2$, it follows that $\pi'\circ f=(\pi\circ f)'$. Moreover, the map

$$\phi_{\mathfrak{g}_2}:(X,\pi)\to(X,\pi')$$

satisfies

$$\phi_{\mathfrak{g}_2}((\pi\circ f)(a)x)=\pi'(f(a))\phi_{\mathfrak{g}_2}(x)=(\pi\circ f)'(a)\phi_{\mathfrak{g}_2}(x)$$

for all $a\in U_q(\mathfrak{g}_1)$ and $x\in X$ so it is an isomorphism as $U_q(\mathfrak{g}_1)$-modules: $\phi_{\mathfrak{g}_2}:(X,\pi\circ f)\to(X,(\pi\circ f)')$.

Proof of Proposition 8.4. Suppose $f:U_q(\mathrm{G}_2)\to U_q(\mathrm{B}_3)$ is a Hopf algebra homomorphism. Consider the natural representation $Y=(X,\pi)$ of $U_q(\mathrm{B}_3)$, and the corresponding representation

$$\phi_{\mathrm{B}_3}:(X,\pi\circ f)\to(X,(\pi\circ f)') \tag{8.11}$$

of $U_q(\mathrm{G}_2)$-modules. Either these are the 7-dimensional irreducible $U_q(\mathrm{G}_2)$-module or they are the sum of 7 trivial 1-dimensional modules. In the first case ϕ_{B_3} is a multiple of ϕ_{G_2}, so the eigenvalues of ϕ_{B_3} are $\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\}$. This is impossible since q is not a root of unity, so it must be that $\pi \circ f$ is trivial.

Analogously, consider the spin representation $Y^+ = (X^+, \pi^+)$ of $U_q(\mathrm{B}_3)$. Then as a $U_q(\mathrm{G}_2)$-module, such a representation should be the sum of representations of dimension 1 or 7. Reasoning as in (8.11), we see that if a 7-dimensional $U_q(\mathrm{G}_2)$-module occurs, then $\phi^+_{\mathrm{B}_3}$ restricted to the 7-dimensional submodule should be proportional to ϕ_{G_2}. This would mean that

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \subseteq \mathrm{diag}\{q^{\pm 9}, q^{\pm 7}, q^{\pm 3}, q^{\pm 1}\}$$

for some scalar λ, which is impossible. Thus $\pi^+ \circ f$ is trivial.

We have seen that the natural and spin representations for $U_q(\mathrm{B}_3)$ are trivial modules for $f(U_q(\mathrm{G}_2))$. Thus, $f(a) - \epsilon(f(a))1$ is in the annihilator of those representations. Since any finite-dimensional irreducible $U_q(\mathrm{B}_3)$-representation of type I can be obtained from the tensor product of those, and since f is a Hopf homomorphism, it follows that $f(U_q(\mathrm{G}_2))$ acts trivially on any finite-dimensional irreducible $U_q(\mathrm{B}_3)$-representation of type I. However, $\cap\, \mathrm{ann}(V) = (0)$ for all such representations (see [Jn, Prop. 5.11]). Consequently, $f(a) = \epsilon(f(a))1$ for all $a \in U_q(\mathrm{G}_2)$.

The other two cases in Proposition 8.4 are similar. For a Hopf homomorphism $f : U_q(\mathrm{G}_2) \to U_q(\mathrm{D}_4)$, it boils down to the fact that any finite-dimensional irreducible $U_q(\mathrm{D}_4)$-representation occurs in tensor products of copies of the natural or spin representations and

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \not\subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}.$$

In the final case, $f : U_q(\mathrm{B}_3) \to U_q(\mathrm{D}_4)$, the argument amounts to the impossibility of

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 6}, \lambda q^{\pm 2}, \lambda\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}, \quad \text{and}$$

$$\{\lambda q^{\pm 9}, \lambda q^{\pm 7}, \lambda q^{\pm 3}, \lambda q^{\pm 1}\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}. \quad \square$$

§9. Quantum Octonions and Quantum Clifford Algebras

In this section we investigate the representation of the quantum Clifford algebra $C_q(8)$ on $\mathbb{O}_q \oplus \mathbb{O}_q$. Our main reference for the quantum Clifford algebra is [DF], but we need to make several adjustments to make the results of [DF] conform to our work here. It is convenient to begin with the general setting of the quantum group of D_n and then to specialize to D_4.

Explicit form of $\check{R}_{V,V}$ for $U_q(\mathbf{D}_n)$.

The natural representation V of the quantum group $U_q(\mathrm{D}_n)$ has a basis

$$\{x_1, x_2, \ldots, x_{2n}\} = \{x_{\epsilon_1}, \ldots, x_{\epsilon_n}, x_{-\epsilon_n}, \ldots, x_{-\epsilon_1}\}, \tag{9.1}$$

where $E_\alpha, F_\alpha, K_\alpha$ act as in (2.2). A straightforward computation shows that relative to the basis $\{x_1, x_2, \ldots, x_{2n}\}$, the matrix of $\check{R} = \check{R}_{V,V}$ is given by

$$\begin{aligned}\check{R} = q^{-1}\sum_{i=1}^{2n} E_{i,i}\otimes E_{i,i} + \sum_{i\neq j,j'} E_{j,i}\otimes E_{i,j} + (q^{-1}-q)\sum_{j>i} E_{j,j}\otimes E_{i,i} \\ + q\sum_{i=1}^{2n} E_{i',i}\otimes E_{i,i'} + (q-q^{-1})\sum_{i<j}(-q)^{\rho_{j'}-\rho_{i'}} E_{i',j}\otimes E_{i,j'},\end{aligned} \tag{9.2}$$

where $i' = 2n+1-i$ and $\rho = (\rho_1, \ldots, \rho_{2n}) = (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, 2-n, 1-n)$. The projection $(\ ,\ ) : V\otimes V \to K$ is given by

$$(x_i, x_j) = \delta_{i,j'}(-1)^{n-1}(-q)^{\rho_{i'}}.$$

If we modify the basis slightly by defining

$$\begin{aligned} e_i &= (-1)^{n-1+\rho_{i'}} x_i \\ e_{i'} &= x_{i'} \end{aligned} \tag{9.3}$$

for $i = 1, \ldots, n$, then we obtain the usual expression for $\check{R}$:

$$\begin{aligned}\check{R} = q^{-1}\sum_{i=1}^{2n} E_{i,i}\otimes E_{i,i} + \sum_{i\neq j,j'} E_{j,i}\otimes E_{i,j} + (q^{-1}-q)\sum_{j>i} E_{j,j}\otimes E_{i,i} \\ + q\sum_{i=1}^{2n} E_{i',i}\otimes E_{i,i'} + (q-q^{-1})\sum_{i<j} q^{(\rho_{j'}-\rho_{i'})} E_{i',j}\otimes E_{i,j'},\end{aligned} \tag{9.4}$$

and the bilinear form is given by

$$(e_i, e_j) = \delta_{i,j'} q^{\rho_{i'}}.$$

The quantum Clifford algebra.

In [DF], Ding and Frenkel define the quantum Clifford algebra $C_q(2n)$ as the quotient of the tensor algebra $T(V)$ of V by the ideal generated by $\{v\otimes w + q\check{R}'_{2,1}(v\otimes w) - (w,v)1 \mid v, w \in V\}$ where $\check{R}'$ is $\check{R}$ with q and q^{-1} interchanged, and $\check{R}'_{2,1} = \sigma\circ\check{R}'\circ\sigma$ where $\sigma(v\otimes w) = (w\otimes v)$. This quantum Clifford algebra can be thought of as a unital associative algebra generated by $\psi_1,\dots,\psi_n,\psi_1^*,\dots,\psi_n^*$ with relations,

$$\begin{aligned}
&\psi_i\psi_j = -q^{-1}\psi_j\psi_i \qquad \psi_i^*\psi_j^* = -q\psi_j^*\psi_i^* \quad (i>j)\\
&\psi_i\psi_i = 0 = \psi_i^*\psi_i^* \qquad \psi_i\psi_j^* = -q^{-1}\psi_j^*\psi_i \quad (i\neq j)\\
&\psi_i\psi_i^* + \psi_i^*\psi_i = (q^{-2}-1)\sum_{i<j}\psi_j\psi_j^* + 1.
\end{aligned} \tag{9.5}$$

If we make the change of notation $Y_i = \psi_{n-i}^*$, $Y_i^* = \psi_{n-i}$, $i = 1,\dots,n$, then the relations in (9.5) become:

$$\begin{aligned}
&Y_iY_j = -q^{-1}Y_jY_i \qquad Y_i^*Y_j^* = -qY_j^*Y_i^* \quad (i>j)\\
&Y_iY_i = 0 = Y_i^*Y_i^* \qquad Y_iY_j^* = -qY_j^*Y_i \quad (i\neq j)\\
&Y_iY_i^* + Y_i^*Y_i = (q^{-2}-1)\sum_{i>j}Y_jY_j^* + 1.
\end{aligned} \tag{9.6}$$

In order to construct a representation of $C_q(8)$ on $\mathbb{O}_q\oplus\mathbb{O}_q$ we will need the following proposition:

Proposition 9.7. *Let I be the ideal of the tensor algebra $T(V)$ generated by $\{v\otimes w + q^{-1}\check{R}_{2,1}(v\otimes w) - (w,v)1 \mid v,w\in V\}$ where $\check{R}_{2,1} = \sigma\circ\check{R}\circ\sigma$. Then $C_q(2n)\cong T(V)/I$.*

Proof. Consider the following basis of V:

$$\{y_i = q^{\rho_i}e_i,\quad y_i^* = e_{i'}, \mid i = 1,\dots,n\} \tag{9.8}$$

which satisfies

$$(y_i, y_j^*) = \delta_{i,j} \qquad\qquad (y_j^*, y_i) = \delta_{i,j}q^{2\rho_i}. \tag{9.9}$$

Let $g: V\otimes V\to I$ be defined by

$$g(v\otimes w) = v\otimes w + q^{-1}\check{R}_{2,1}(v\otimes w) - (w,v)1. \tag{9.10}$$

From the explicit form of $\check{R}$ on the basis $\{e_i\}$, one can check that the following hold:

$$\begin{aligned}
&g(y_i\otimes y_j) = y_i\otimes y_j + q^{-1}y_j\otimes y_i \quad (i>j)\\
&g(y_j^*\otimes y_i^*) = y_j^*\otimes y_i^* + q^{-1}y_i^*\otimes y_j^* \quad (i>j)\\
&g(y_i\otimes y_i) = (1+q^{-2})y_i\otimes y_i \qquad g(y_i^*\otimes y_i^*) = (1+q^{-2})y_i^*\otimes y_i^*\\
&g(y_j^*\otimes y_i) = y_j^*\otimes y_i + q^{-1}y_i\otimes y_j^* \quad (i\neq j)\\
&g(y_i^*\otimes y_i) = y_i^*\otimes y_i + y_i\otimes y_i^* + (1-q^{-2})\sum_{i>j}y_j\otimes y_j^* - 1,
\end{aligned} \tag{9.11}$$

These give the relations in (9.6) in $T(V)/I$. However, we also have the relations:

$$\begin{aligned}
&g(y_j \otimes y_i) = q^{-1} y_i \otimes y_j + q^{-2} y_j \otimes y_i \quad (i > j) \\
&g(y_i^* \otimes y_j^*) = q^{-1} y_j^* \otimes y_i^* + q^{-2} y_i^* \otimes y_j^* \quad (i > j) \\
&g(y_i \otimes y_j^*) = q^{-1} y_j^* \otimes y_i + q^{-2} y_i \otimes y_j^* \quad (i \neq j) \\
&g(y_i \otimes y_i^*) = y_i^* \otimes y_i + q^{-2} y_i \otimes y_i^* \\
&\qquad\qquad + (1 - q^{-2}) q^{2\rho_i} \Big(\sum_{j=1}^{n} y_j \otimes y_j^* + \sum_{j>i} q^{-2\rho_j} y_j^* \otimes y_j \Big) - q^{2\rho_i}.
\end{aligned} \tag{9.12}$$

The first three identities don't say anything new. So to conclude the proposition, it suffices to verify that the relation $g(y_i \otimes y_i^*) = 0$ holds in $C_q(2n)$, that is:

Lemma 9.13. *In $C_q(2n)$ we have*

$$q^{-2} Y_i Y_i^* + Y_i^* Y_i + (1 - q^{-2}) q^{2\rho_i} \Big(\sum_{j=1}^{n} Y_j Y_j^* + \sum_{j>i} q^{-2\rho_j} Y_j^* Y_j \Big) = q^{2\rho_i}.$$

Proof. As a shorthand notation we set $a_i = Y_i Y_i^*$, $a_i' = Y_i^* Y_i$, and $b_i = a_i + a_i'$ for $i = 1, \dots, n$. From the relations in $C_q(2n)$ we know that $b_i = (q^{-2} - 1) \sum_{j<i} a_j + 1$ so that $b_{i+1} = (q^{-2} - 1) a_i + b_i$ or

$$a_{i+1} + a_{i+1}' = q^{-2} a_i + a_i'. \tag{9.14}$$

Letting $c_i = q^{-2} a_i + a_i'$ we have

$$c_i - q^2 c_{i+1} = q^{-2} a_i + a_i' - a_{i+1} - q^2 a_{i+1}' \stackrel{1}{=} (1 - q^2) a_{i+1}'$$

where (1) follows from (9.14). Therefore,

$$\begin{aligned}
c_i &= (1 - q^2) a_{i+1}' + q^2 (1 - q^2) a_{i+2}' + \cdots + q^{2(n-i-1)} (1 - q^2) a_n' + q^{2(n-i)} c_n \\
&= (q^{-2} - 1) q^{2\rho_i} \Big(\sum_{j>i} q^{-2\rho_j} a_j' \Big) + q^{2\rho_i} c_n.
\end{aligned} \tag{9.15}$$

On the other hand,

$$\begin{aligned}
c_n &= q^{-2} a_n + a_n' = (q^{-2} - 1) a_n + a_n + a_n' = (q^{-2} - 1) a_n + b_n \\
&= (q^{-2} - 1) a_n + (q^{-2} - 1) \Big(\sum_{j<n} a_j \Big) + 1 = (q^{-2} - 1) \Big(\sum_{j \leq n} a_j \Big) + 1.
\end{aligned} \tag{9.16}$$

The left-hand side of the identity in the statement of the lemma is

$$\begin{aligned} c_i + (1-q^{-2})q^{-2\rho_i}\Big(\sum_{j=1}^{n} a_j\Big) + (1-q^{-2})q^{2\rho_i}\Big(\sum_{j>i} q^{-2\rho_j} a'_j\Big) \\ \stackrel{1}{=} c_i + q^{2\rho_i}(1-c_n) + (1-q^{-2})q^{2\rho_i}\Big(\sum_{j>i} q^{-2\rho_j} a'_j\Big) \\ \stackrel{2}{=} q^{2\rho_i} \quad \text{(the right side)}, \end{aligned}$$

where (1) is (9.16) and (2) is (9.15). □

The $U_q(\mathbf{D}_4)$ case.

Let us return now to our quantum octonions. Consider the two "intertwining operators" Φ^0 and Φ^1 which are the $U_q(\mathrm{D}_4)$-module homomorphisms:

$$\begin{aligned} \Phi^0 &: V_\eta \otimes V \stackrel{\jmath\otimes \mathrm{id}}{\rightarrow} V_{\eta\circ\zeta} \otimes V \stackrel{\jmath\circ p \circ(\jmath\otimes\jmath)}{\rightarrow} V_\zeta \\ \Phi^1 &: V_\zeta \otimes V \stackrel{\jmath\otimes \mathrm{id}}{\rightarrow} V_{\zeta\circ\eta} \otimes V \stackrel{p}{\rightarrow} V_\eta, \end{aligned}$$

We scale these mappings by setting $\Psi^0 = q^{-\frac{3}{2}}\Phi^0$ and $\Psi^1 = q^{-\frac{3}{2}}\Phi^1$. Let $\mathfrak{S}_q = (\mathrm{id}{+}q^{-1}\check{R})$, where $\check{R} = \check{R}_{V,V}$, which we know maps $V\otimes V$ onto $\mathcal{S}^2(V)$.

Lemma 9.17. *(Compare [DF, Prop. 3.1.2])*

$$\begin{aligned} \Psi^1 \circ (\Psi^0 \otimes id) \circ (id \otimes \mathfrak{S}_q) &= id \otimes (\ ,\ ) \\ \Psi^0 \circ (\Psi^1 \otimes id) \circ (id \otimes \mathfrak{S}_q) &= id \otimes (\ ,\ ). \end{aligned}$$

Proof. It is enough to check the values on $x_1 \otimes x_1 \otimes x_{-1}$:

$$\begin{aligned} \Psi^1 \circ (\Psi^0 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{S}_q)(x_1 \otimes x_1 \otimes x_{-1}) &= \Psi^1 \circ (\Psi^0 \otimes \mathrm{id})(x_1 \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1)) \\ &= q^{-\frac{3}{2}}\Psi^1(q^{-\frac{3}{2}}x_4 \otimes x_1) = q^{-3}x_1. \end{aligned}$$

But $(x_1, x_{-1}) = (q^3 + q^{-3})(x_1|x_{-1}) = q^{-3}$, so both maps agree on $x_1 \otimes x_1 \otimes x_{-1}$. The computation for the other map is left as an exercise. □

Now for any $v \in V$ define

$$\begin{aligned} \Psi(v) : &V_\zeta \oplus V_\eta \rightarrow V_\zeta \oplus V_\eta \\ &\Psi(v)(x+y) = \Psi^0(y \otimes v) + \Psi^1(x \otimes v). \end{aligned} \tag{9.18}$$

The relations in the previous lemma say that

$$\Psi \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{S}_q) = \mathrm{id} \otimes (\ ,\ ),$$

so for any basis elements $e_i, e_j \in V$ and $x \in V_\zeta \oplus V_\eta$, we have

$$\begin{aligned}(e_j, e_i)x &= \Psi \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{S}_q)(x \otimes e_j \otimes e_i) \\ &= \Psi \circ (\Psi \otimes \mathrm{id})(x \otimes (e_j \otimes e_i + q^{-1}\check{R}(e_j \otimes e_i))) \\ &= \Psi(e_i)\Psi(e_j)(x) + q^{-1}\sum_{k,\ell}(\check{R} \circ \sigma)_{i,j}^{k,\ell}\Psi(e_\ell)\Psi(e_k)(x),\end{aligned}$$

where $(\check{R} \circ \sigma)_{i,j}^{k,\ell}$ is the coordinate matrix of $\check{R} \circ \sigma$ in the basis $\{e_i \otimes e_j\}$. So $\Psi(e_i)$ and $\Psi(e_j)$ satisfy

$$(e_j, e_i)\mathrm{id} = \Psi(e_i)\Psi(e_j) + q^{-1}\sum_{k,\ell}(\check{R} \circ \sigma)_{i,j}^{k,\ell}\Psi(e_\ell)\Psi(e_k).$$

Notice now that

$$\begin{aligned}e_i \otimes e_j + q^{-1}\check{R}_{2,1}(e_i \otimes e_j) &= e_i \otimes e_j + q^{-1}\sigma \circ \check{R} \circ \sigma(e_i \otimes e_j) \\ &= e_i \otimes e_j + q^{-1}\sum_{k,\ell}(\check{R} \circ \sigma)_{i,j}^{k,\ell} e_\ell \otimes e_k,\end{aligned}$$

so the operators $\Psi(v)$ verify the relations of the quantum Clifford algebra. This shows that the map

$$\begin{aligned}\Psi : V &\to \mathrm{End}(V_\zeta \oplus V_\eta) \\ v &\mapsto \Psi(v) : x + y \to q^{-3}\Big(\jmath(y \cdot \jmath(v)) + \jmath(x) \cdot v\Big)\end{aligned} \tag{9.19}$$

extends to a representation $\Psi : C_q(8) \to \mathrm{End}(V_\zeta \oplus V_\eta) = \mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$.

In [DF], Ding and Frenkel show that $C_q(2n)$ is isomorphic to the classical Clifford algebra, so $C_q(2n)$ is a central simple associative algebra of dimension 2^8. The same is true of $\mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$. As a result we have

Proposition 9.20. *The map* $\Psi : C_q(8) \to End(V_\zeta \oplus V_\eta) = End(\mathbb{O}_q \oplus \mathbb{O}_q)$ *in (9.19) is an algebra isomorphism.*

It is worth noting that at $q = 1$ the map in Proposition 9.20 reduces to

$$\begin{aligned}\Psi : &\mathbb{O} \to \mathrm{End}(\mathbb{O} \oplus \mathbb{O}) \\ &v \mapsto \Psi(v) : x + y \mapsto v \cdot \overline{y} + \overline{x} \cdot v\end{aligned} \tag{9.21}$$

so

$$\Psi(v)\Psi(v) : x + y \mapsto \Psi(v)(v \cdot \overline{y} + \overline{x} \cdot v) = v \cdot (\overline{v} \cdot x) + (y \cdot \overline{v}) \cdot v = (v|v)_1(x + y),$$

(see Props. 2.21, 4.3). This shows that Ψ extends to the standard Clifford algebra - that is, to the quotient $T(\mathbb{O})/ < v \otimes v - (v|v)_1 1 >$ of the tensor algebra, in this case as in [KPS].

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