A SUPPORT PROPERTY FOR INFINITE DIMENSIONAL INTERACTING DIFFUSION PROCESSES.

UNE PROPRIÉTÉ DE SUPPORT POUR DES PROCESSUS DE DIFFUSION EN DIMENSION INFINIE AVEC INTERACTION.

MICHAEL RÖCKNER AND BYRON SCHMULAND

ABSTRACT. The Dirichlet form associated with the intrinsic gradient on Poisson space is known to be quasi-regular on the complete metric space $\ddot{\Gamma} = \{Z_+\text{-valued}$ Radon measures on $I\!R^d\}$. We show that under mild conditions, the set $\ddot{\Gamma} \setminus \Gamma$ is $\mathcal{E}$-exceptional, where Γ is the space of locally finite configurations in $I\!R^d$, that is, measures $\gamma \in \ddot{\Gamma}$ satisfying $\sup_{x\in I\!R^d} \gamma(\{x\}) \leq 1$. Thus, the associated diffusion lives on the smaller space Γ. This result also holds for Gibbs measures with superstable interactions.

RÉSUMÉ. Il est connu que la forme de Dirichlet associée au gradient intrinsèque sur l'espace de Poisson est quasi-régulière sur l'espace métrique complet $\ddot{\Gamma} = \{$ mesures de Radon sur $I\!R^d$ à valeurs dans $Z_+\}$. Sous des conditions faibles, on montre que l'ensemble $\ddot{\Gamma} \setminus \Gamma$ est $\mathcal{E}$-exceptionnel, où Γ désigne l'ensemble de toutes les configurations localement finies dans $I\!R^d$, c'est-à-dire, les mesures $\gamma \in \ddot{\Gamma}$ qui verifient $\sup_{x\in I\!R^d} \gamma(\{x\}) \leq 1$. La diffusion associée prend donc ses valeurs dans l'espace des phases Γ. Ce résultat est valable meme si μ est une mesure de Gibbs associée à un potentiel superstable.

Version française abrégée

Soit Γ l'ensemble de toutes les configurations localement finies dans $I\!R^d$. Γ est vu comme un sous-ensemble de l'espace des mesures de Radon sur $I\!R^d$ muni de la topologie vague. Soit $\sigma(dx) = \rho(x)m(dx)$, m étant la mesure de Lebegue sur $I\!R^d$ et $\rho^{1/2} \in H^{1,2}_{\mathrm{loc}}(I\!R^d)$, $\rho(x) > 0$ m-presque tout $x \in I\!R^d$. Soit π_σ la mesure de Poisson d'intensité σ sur Γ. On appellera mesure de Poisson mixte toute mesure μ de la forme $\mu := \int_{I\!R_+} \pi_{z\sigma}\,\lambda(dz)$. Définissons l'espace des fonctions cylindriques lisses et bornées

$$\mathcal{F}C_b^\infty := \{u \mid u(\gamma) = g(\langle f_1, \gamma\rangle, \langle f_2, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle)\},$$

où $f_i \in C_0^\infty(I\!R^d)$, $g \in C_b^\infty(I\!R^n)$, et $\langle f, \gamma\rangle = \sum_{x\in\gamma} f(x)$. D'après Albeverio, Kondratiev, et Röckner [1] nous définissons le gradient intrinsèque d'une fonction $u \in \mathcal{F}C_b^\infty$ par

$$\left(\nabla^\Gamma u\right)(\gamma; x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\langle f_1, \gamma\rangle, \langle f_2, \gamma\rangle, \ldots, \langle f_n, \gamma\rangle\right) \nabla f_i(x).$$

Research at MSRI is supported in part by NSF grant DMS-9701755.

Donc on a l'opérateur carré du champ

$$\Gamma(u,v)(\gamma) := \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(\gamma;x), (\nabla^\Gamma v)(\gamma;x)\rangle_{\mathbb{R}^d}\, \gamma(dx).$$

La forme de Dirichlet engrendreé par cet opérateur carré du champ est la fermeture $(\mathcal{E}, D(\mathcal{E}))$ de la forme $(\mathcal{E}, \mathcal{F}C_b^\infty)$ définie pour $u, v \in \mathcal{F}C_b^\infty$ par

$$\mathcal{E}(u,v) := \int_\Gamma \Gamma(u,v)(\gamma)\, \mu(d\gamma).$$

On sait que cette forme est quasi-régulière sur l'espace des phases complet $\ddot{\Gamma}(\supset \Gamma)$. On va prouver ici que $(\mathcal{E}, D(\mathcal{E}))$ est quasi-régulière sur Γ; pour cela il est suffisant de voir que l'ensemble $\ddot{\Gamma} \setminus \Gamma$ est $\mathcal{E}$-exceptionnel. On a le résultat suivant:

PROPOSITION 1. *Si $d \geq 2$, $\rho \in L^2_{\mathrm{loc}}(dx)$, et $\int_{\mathbb{R}_+} z^2\, \lambda(dz) < \infty$, l'ensemble $\ddot{\Gamma} \setminus \Gamma$ est $\mathcal{E}$-exceptionnel.*

Ce résultat est valable meme si μ est une mesure de Gibbs associé à un potentiel superstable au sens de Ruelle, puisque l'existence des points avec masse ≥ 2 est un événement local et une telle mesure de Gibbs est localement absolument continue par rapport à une mesure de Poisson.

Let Γ be the space of locally finite configurations in $\mathbb{R}^d$,

$$\Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for every compact } K\}. \tag{1}$$

A configuration γ will be identified with the Radon measure $\sum_{x\in\gamma} \varepsilon_x$. The space Γ will be given the topology of vague convergence of measures, and measures on Γ are defined on the corresponding Borel sets. Define

$$\begin{aligned} \mathcal{F}C_b^\infty := \{\, u \ \ | \ \ & \gamma \mapsto u(\gamma) = g(\langle f_1,\gamma\rangle, \langle f_2,\gamma\rangle, \ldots, \langle f_n,\gamma\rangle) \\ & \text{for some } f_i \in C_0^\infty(\mathbb{R}^d) \text{ and } g \in C_b^\infty(\mathbb{R}^n)\big\}, \end{aligned} \tag{2}$$

where $\langle f,\gamma\rangle = \sum_{x\in\gamma} f(x)$.

For $u \in \mathcal{F}C_b^\infty$, we define the gradient $\nabla^\Gamma u$ at the point $\gamma \in \Gamma$ as an element of the "tangent space" $T_\gamma(\Gamma) := L^2(\mathbb{R}^d \to \mathbb{R}^d; \gamma)$ by

$$x \mapsto \left(\nabla^\Gamma u\right)(\gamma;x) := \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\langle f_1,\gamma\rangle, \langle f_2,\gamma\rangle, \ldots, \langle f_n,\gamma\rangle\right) \nabla f_i(x). \tag{3}$$

Here ∇ refers to the usual gradient on $\mathbb{R}^d$. It is not hard to prove that $\nabla^\Gamma u$ is well-defined, even though the representation of u as a cylinder function is not unique.

DEFINITION 1. *For $u, v \in \mathcal{F}C_b^\infty$ define the square field operator Γ as the real-valued function on Γ given by*

$$\begin{aligned} \gamma \mapsto \Gamma(u,v)(\gamma) &:= \langle \nabla^\Gamma u, \nabla^\Gamma v\rangle_{T_\gamma(\Gamma)} \\ &= \int_{\mathbb{R}^d} \langle (\nabla^\Gamma u)(\gamma;x), (\nabla^\Gamma v)(\gamma;x)\rangle_{\mathbb{R}^d}\, \gamma(dx). \end{aligned} \tag{4}$$

We will often use the abbreviation $\Gamma(u) := \Gamma(u,u)$.

Let σ be a measure on $\mathbb{R}^d$ that has a density ρ with respect to Lebesgue measure satisfying $\rho > 0$ almost everywhere, and $\rho^{1/2} \in H^{1,2}_{loc}(\mathbb{R}^d)$. Here $H^{1,2}_{loc}(\mathbb{R}^d)$ denotes the local Sobolev space of order 1 in $L^2_{loc}(\mathbb{R}^d; dx)$. The *Poisson measure* π_σ with intensity measure σ is the probability measure on Γ characterized by

$$\int_\Gamma \exp(\langle f, \gamma\rangle)\, \pi_\sigma(d\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{f(x)} - 1)\, \sigma(dx)\right), \tag{5}$$

for $f \in C_0(\mathbb{R}^d)$. A *mixed Poisson measure* is given by

$$\mu := \int_{\mathbb{R}_+} \pi_{z\sigma}\, \lambda(dz), \tag{6}$$

where λ is a probability measure such that $\int_{\mathbb{R}_+} z\, \lambda(dz) < \infty$.

DEFINITION 2. *For* $u, v \in \mathcal{F}C_b^\infty$ *we define the pre-Dirichlet form*

$$\mathcal{E}(u,v) := \int_\Gamma \Gamma(u,v)(\gamma)\, \mu(d\gamma). \tag{7}$$

Albeverio, Kondratiev, and Röckner proved an integration by parts formula [3, Corollary 4.1 and Remark 4.3] which implies that the form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable, and that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric, local, Dirichlet form. The quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$ has been proven for certain cases by Yoshida [9], and in general by Ma and Röckner [4]. However, Γ is not completely metrizable with respect to the vague topology so it is necessary to use the Polish state space

$$\ddot{\Gamma} := \{Z_+\text{-valued Radon measures on } \mathbb{R}^d\}. \tag{8}$$

Since $\Gamma \subset \ddot{\Gamma}$ and $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}; \mu)$. The associated Markov process $((X_t)_{t\geq 0}, (P_\gamma)_{\gamma\in\ddot{\Gamma}})$ has vaguely continuous sample paths since $(\mathcal{E}, D(\mathcal{E}))$ is a local form (cf. [6, Chapter V, Theorem 1.11]).

We will use the following lemma from Dirichlet form theory.

LEMMA 1. *Let* $u_n \in D(\mathcal{E})$ *be continuous functions,* $\sup_n \mathcal{E}(u_n, u_n) < \infty$, *and* $u_n \to u$ *pointwise. Then* u *is an* $\mathcal{E}$*-quasi-continuous function, in particular, for* μ*-almost every* $\gamma \in \ddot{\Gamma}$,

$$P_\gamma\left(t \to u(X_t) \text{ is continuous }\right) = 1. \tag{9}$$

For example, if 1_N is $\mathcal{E}$-quasi-continuous and $\mu(N) = 0$, then for μ-almost every $\gamma \in \ddot{\Gamma}$,

$$P_\gamma\left(X_t \notin N \text{ for all } 0 < t < \infty\right) = 1. \tag{10}$$

Such a set N is called $\mathcal{E}$*-exceptional.*

PROPOSITION 1. *If* $d \geq 2$, $\rho \in L^2_{\text{loc}}(dx)$, *and* $\int_{\mathbb{R}_+} z^2\, \lambda(dz) < \infty$, *then the set* $\ddot{\Gamma} \setminus \Gamma$ *is* $\mathcal{E}$*-exceptional.*

Proof. It suffices to prove the result locally, that is, to show that for every positive integer a, the function $u := 1_N$ is quasi-continuous, where

$$N := \{\gamma : \sup(\gamma(\{x\}) : x \in [-a,a]^d) \geq 2\}. \tag{11}$$

Our analysis begins with a smooth partition of $\mathbb{R}^d$ into small pieces. Let ϕ be a $C_0^\infty(\mathbb{R})$ function satisfying $1_{[0,1]} \leq \phi \leq 1_{[-1/2,3/2)}$ and $|\phi'| \leq 3 \times 1_{[-1/2,3/2)}$, and for any $n \in \mathbb{N}$ and $i = (i_1,\ldots,i_d) \in Z^d$, define a $C_0^\infty(\mathbb{R}^d)$ function by

$$\phi_i(x) := \prod_{k=1}^d \phi(nx_k - i_k). \tag{12}$$

We also let $I_i(x) := \prod_{k=1}^d 1_{[-1/2,3/2)}(nx_k - i_k)$ and note that $\phi_i \leq I_i$. Taking the j^{th} partial derivative of ϕ_i gives

$$\partial_j\phi_i(x) = n\phi'(nx_j - i_j)\prod_{k\neq j}\phi(nx_k - i_k), \tag{13}$$

and so $(\partial_j\phi_i(x))^2 \leq 9n^2 I_i(x)$. Adding over j from 1 to d gives us

$$|\nabla\phi_i(x)|^2 \leq 9n^2 d I_i(x). \tag{14}$$

Let ψ be a smooth function on $\mathbb{R}$ satisfying $1_{[2,\infty)} \leq \psi \leq 1_{[1,\infty)}$ and $|\psi'| \leq 2 \times 1_{(1,\infty)}$. Choosing $A := Z^d \cap [-na,na]^d$, define a continuous element of $D(\mathcal{E})$ by

$$u_n(\gamma) := \psi\left(\sup_{i\in A}\langle\phi_i,\gamma\rangle\right). \tag{15}$$

Then $u_n \to u$ pointwise as $n \to \infty$, so to apply Lemma 1 we only need prove that $\sup_n \mathcal{E}(u_n,u_n) < \infty$. We begin by bounding $\Gamma(u_n)$, the square field operator applied to u_n. First note that

$$\left(\psi'\left(\sup_{i\in A}\langle\phi_i,\gamma\rangle\right)\right)^2 \leq 4 \times 1_{(\sup_{i\in A}\langle\phi_i,\gamma\rangle > 1)} \leq 4 \times 1_{(\sup_{i\in A}\langle I_i,\gamma\rangle \geq 2)}, \tag{16}$$

where for the final inequality we use the fact that $\langle I_i,\gamma\rangle$ is an integer. Therefore, first using the inequality $\Gamma(u \vee v) \leq \Gamma(u) \vee \Gamma(v)$, and then using (14) and (16), we get

$$\begin{aligned}
\Gamma(u_n)(\gamma) &= \left(\psi'\left(\sup_{i\in A}\langle\phi_i,\gamma\rangle\right)\right)^2 \Gamma(\sup_{i\in A}\langle\phi_i,\cdot\rangle)(\gamma) \\
&\leq \left(\psi'\left(\sup_{i\in A}\langle\phi_i,\gamma\rangle\right)\right)^2 \sup_{i\in A}\Gamma(\langle\phi_i,\cdot\rangle)(\gamma) \\
&= \left(\psi'\left(\sup_{i\in A}\langle\phi_i,\gamma\rangle\right)\right)^2 \sup_{i\in A}\int |\nabla\phi_i(x)|^2\,\gamma(dx) \\
&\leq 4 \times 1_{(\sup_{i\in A}\langle I_i,\gamma\rangle\geq 2)}\, 9\,n^2\,d\,\sup_{i\in A}\langle I_i,\gamma\rangle \\
&\leq 36n^2 d\sum_{i\in A} 1_{(\langle I_i,\gamma\rangle\geq 2)}\langle I_i,\gamma\rangle.
\end{aligned} \tag{17}$$

From equation (6) we have

$$\int_{(\langle I_i,\gamma\rangle\geq 2)} \langle I_i,\gamma\rangle \mu(d\gamma) = \int_{I\!R_+} z\langle I_i,\sigma\rangle \left(1-e^{-z\langle I_i,\sigma\rangle}\right) \lambda(dz) \tag{18}$$

$$\leq \langle I_i,\sigma\rangle^2 \int_{I\!R_+} z^2\,\lambda(dz), \tag{19}$$

and combined with (17) this gives

$$\mathcal{E}(u_n,u_n) \leq c\,n^2 \sum_{i\in A} \langle I_i,\sigma\rangle^2. \tag{20}$$

Although the supports of the indicator functions I_i are not disjoint, each point belongs to at most 2^d of the sets $\{I_i=1\}$ for $i\in A$. Therefore the Cauchy-Schwarz inequality gives us

$$\begin{aligned} \sum_{i\in A}\langle I_i,\sigma\rangle^2 &= \sum_{i\in A}\left(\int I_i(x)\,\rho(x)\,dx\right)^2 \\ &\leq \sum_{i\in A}\left(\int I_i(x)\,\rho(x)^2\,dx\right)\left(\int I_i(x)\,dx\right) \\ &\leq 2^d \int_{[-(a+1),a+1]^d} \rho(x)^2\,dx\,(2/n)^d, \end{aligned} \tag{21}$$

and combining this with (20) we find that

$$\mathcal{E}(u_n,u_n) \leq cn^{2-d}. \tag{22}$$

Since $d\geq 2$ we see that $\sup_n \mathcal{E}(u_n,u_n)<\infty$.

COROLLARY 1. *The conclusion of Proposition 1 holds if μ is replaced by any Gibbs measure ν, in the sense of Ruelle* [8], *with a pair potential Φ that is superstable, lower regular, and where $\int|\exp(-\Phi(x))-1|\,dx<\infty$.*

Proof. A Gibbs measure ν as in the statement admits a system of density distributions that satisfy $\sigma^n_\Delta(x_1,\ldots,x_n)\leq\xi^n$ for some constant ξ [8, Theorem 5.5].

Any measure ν on Γ admitting such density distributions is locally absolutely continuous with respect to the Poisson measure $\pi_{\xi m}$. The bounds obtained in the proof of Proposition 1 are valid for this Poisson measure ($\lambda=\varepsilon_\xi$, $\sigma=m=$ Lebegue measure) and carry over trivially to the measure ν.

Remarks.

1. More detailed analysis shows that the condition $\int z^2\,\lambda(dz)<\infty$ in Proposition 1 can be dropped if the measure σ satisfies the growth condition $\sigma(S_r)\leq a\exp(br)$ for some $a,b>0$, where S_r is the sphere of radius r centered at the origin.
2. We point out that the form $(\mathcal{E},\mathcal{D})$ in Osada [7] extends our $(\mathcal{E},D(\mathcal{E}))$, but it is not known whether the forms coincide. A form with a larger domain has more exceptional sets, so proofs of exceptionality are easier in Osada's setting, and do not imply exceptionality in our setting.

3. The results of this paper have been announced in lectures at Osaka, Seoul, and Taipei (July 1997), and at MSRI Berkeley (November 1997).

Acknowledgements.

The authors gratefully acknowledge the support of the DFG through SFB 343 Bielefeld, and of NSERC (Canada). We also thank the Mathematics Sciences Research Institute at Berkeley for a very pleasant stay.

References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: 'Differential geometry of Poisson spaces', *Comptes Rendus de L'Académie des Sciences Paris*, **323** 1129–1134, 1996.

[2] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: 'Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces', *Comptes Rendus de L'Académie des Sciences Paris*, **323** 1179–1184, 1996.

[3] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: 'Analysis and geometry on configuration spaces', to appear in the *Journal of Functional Analysis.*

[4] S. Albeverio, Yu. G. Kondratiev, and M. Röckner: 'Analysis and geometry on configuration spaces: The Gibbsian case', to appear in the *Journal of Functional Analysis.*

[5] Z.M. Ma and M. Röckner: 'Construction of diffusions on configuration spaces', Preprint.

[6] Z.M. Ma and M. Röckner: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Berlin: Springer 1992.

[7] H. Osada: 'Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions', *Communications in Mathematical Physics* **176** 117–131, 1996.

[8] D. Ruelle: 'Superstable interactions in classical statistical mechanics', *Communications in Mathematical Physics* **18** 127–159, 1970.

[9] M. W. Yoshida: 'Construction of infinite dimensional interacting diffusion processes through Dirichlet forms', *Probability Theory and Related Fields* **106** 265–297, 1996.

Michael Röckner, Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany

E-mail address: roeckner@mathematik.uni-bielefeld.de

Byron Schmuland, Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

E-mail address: schmu@stat.ualberta.ca