STRONG UNIQUENESS FOR CERTAIN INFINITE DIMENSIONAL DIRICHLET OPERATORS AND APPLICATIONS TO STOCHASTIC QUANTIZATION

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Abstract. Strong and Markov uniqueness problems in L^2 for Dirichlet operators on rigged Hilbert spaces are studied. An analytic approach based on a–priori estimates is used. The extension of the problem to the L^p-setting is discussed. As a direct application essential self–adjointness and strong uniqueness in L^p is proved for the generator (with initial domain the bounded smooth cylinder functions) of the stochastic quantization process for Euclidean quantum field theory in finite volume $\Lambda \subset \mathbb{R}^2$.

1. Introduction

The theory of Dirichlet forms is a rapidly developing field of modern analysis which has intimate relationships with potential theory, probability theory, differential equations and quantum physics. We refer to the monographs [16], [17], [20] and [32] where the theory of Dirichlet forms with applications to different branches of analysis and probability theory is presented. Though the abstract general theory is well developed, specific analytic questions remain open when one studies concrete situations. In this paper we mainly discuss Dirichlet forms and corresponding Dirichlet operators on infinite-dimensional state spaces. In particular, we are concerned with the classical Dirichlet form of gradient type on a separable Hilbert space with a probability measure. Initially, the form is defined on some "minimal" domain $\mathcal{D}$.

The first analytic problem which arises when one studies such forms is closability. This problem is well understood and necessary and sufficient conditions have been found (cf. [7]). The operators associated to the Dirichlet forms generate Markov semigroups (see e.g. [17],[20], [32]) on the corresponding L^2-space. If one assumes that the same domain $\mathcal{D}$ is contained in the domain of the generator, the next natural question which arises is whether the extension in L^2 of the generator restricted to $\mathcal{D}$ is unique. There are at least two different statements of the uniqueness problem in this context. The first is the so–called Markov uniqueness problem when one asks whether the extension generating a Markov semigroup is unique. This problem is completely solved for the finite dimensional case in [41] where Markov uniqueness was obtained under the most general conditions. The situation is quite different in

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the infinite dimensional case. We refer to [11], [12] for the best results in this direction known so far.

One speaks of strong uniqueness for the Dirichlet form resp. operator when there is only one lower bounded self-adjoint extension of the Dirichlet operator originally defined on $\mathcal{D}$. As is well–known this is equivalent to essential selfadjointness. This problem was addressed in many papers such as e.g. [1], [2], [3], [14], [30], [43], [46] (see also the references therein). The results obtained in these papers turn out to be incomparable, although they are all expressed in terms of conditions on the logarithmic derivative of the measure.

The aim of this paper is to investigate several "uniqueness problems" connected with Dirichlet forms resp. Dirichlet operators on rigged Hilbert spaces. The main result (Theorem 1) concerns strong uniqueness in L^2. The method we use is inherited from [30] (and also [1], [2], [3]) and based upon an a–priori estimate of the cylindric smooth solutions to the corresponding parabolic equations. This estimate is a generalization of the estimate obtained in [30] to the case when the logarithmic derivative of the measure contains two terms: one of them satisfies the conditions of [30] and the other is modeled according to that in [3] (see also [1], [2]). The method of this paper is purely analytic, in contrast to that in [3]. It consists of the reduction of the problem to estimates (independent of dimension) of gradients of solutions of the finite dimensional projections of the problem. We use these estimates to show that any arbitrary semigroup generated by an extension of the original Dirichlet operator on $\mathcal{D}$ can be approximated by the same approximation sequence, and therefore is unique.

While the main result of this paper is a (strict) generalization of [30], it is, though similar in nature, still quite disjoint from those in [1], [2], [3] w.r.t. applications. Nevertheless, our main result can be applied to prove essential self–adjointness of the corresponding Dirichlet operator in a situation where this, despite several attempts, had been an open problem for quite some time. That is the so–called stochastic quantization of field theory in finite volume. More precisely, here the underlying space is a Sobolev space of distributions on an open bounded set in $\mathbb{R}^2$ and the measure is a two–dimensional Euclidean quantum field in finite volume with polynomial self–interaction. We refer to Section 5 below for the (extensive list of the) corresponding literature and more details. We only mention here that Markov uniqueness, however, had been shown already in [40, Section 7].

Furthermore, it should be mentioned that the approach of the present paper to the uniqueness problem naturally extends to the L^p-setting. The question is then whether the extension of the Dirichlet operator on $\mathcal{D}$ generating a C_0-semigroup on L^p is unique. For the first L^p-uniqueness result based upon the method of a–priori estimates we refer to [29]. In this paper we obtain the uniqueness in L^1 (Theorem 2) which requires much simpler a–priori estimates than that needed for the proof of essential selfadjointness. As a consequence we derive a new approximation criterium for Markov uniqueness (Corollary 1). We also discuss the uniqueness problem in L^p for $p > 1,\ p \neq 2$ generalizing the result from [29], although we do not present the details for the a–priori estimates needed for this in the present paper, in order to avoid overloading the reader with additional technicalities. But we emphasize that the corresponding results then also apply to the cases described in Section 5 mentioned above. For further results on strong uniqueness in L^p we refer to [45], where uniqueness results of "perturbative type" for $p = 1$ are proved, and to [19] where, in particular, strong uniqueness in L^p in the above mentioned situation of the stochastic quantization is also proved, but only for $1 \le p < 2$. The latter case corresponds to our Theorems 2 and 4.

The organization of this paper is as follows. In Section 2 we present the framework and the main uniqueness results. In Section 3 we derive the a–priori estimates for gradients of the solutions to parabolic equations with smooth cylindric coefficients. In Section 4 we give proofs of the uniqueness results. In Section 5 we discuss the said applications. Section 6 is devoted to the discussion of the uniqueness problem in L^p.

2. Framework and main results

Let $\mathcal{H}_0$ be a separable real Hilbert space with the inner product $(\cdot,\cdot)_0$ and norm $|\cdot|_0$. Let

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$$

be a rigging of $\mathcal{H}_0$ by the Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ with the assumption that the embeddings are dense, continuous and belong to the Hilbert–Schmidt class. Without loss of generality we can suppose then that there exists a selfadjoint operator $T = T^* \geqslant 1$ in $\mathcal{H}_0$ with $\mathcal{D}(T) = \mathcal{H}_+$ such that T^{-1} is Hilbert–Schmidt. We refer to [13, Chap. 6, §3] for the details. We use the orthonormal system $(e_i)_{i=1}^{\infty}$ of eigenvectors of T as a basis in $\mathcal{H}_0$: $Te_i = \lambda_i e_i$ $(i = 1, 2, 3, \dots)$. For $e_i \in \mathcal{H}_+$, $x \in \mathcal{H}_-$, we define $x_i := {}_+(e_i, x)_-$, where ${}_+(\ ,\ )_-$ denotes the dualization between $\mathcal{H}_-$ and $\mathcal{H}_+$. Clearly, the norms in $\mathcal{H}_\pm$ can be calculated as follows

$$|x|_-^2 = \sum_{i=1}^{\infty} \lambda_i^{-2} x_i^2, \qquad |x|_+^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2.$$

For $N \in \mathbb{N}$ define $P_N : \mathcal{H}_- \to \mathcal{H}_+ \subset \mathcal{H}_0$ by

$$P_N x := \sum_{i=1}^{N} x_i\, e_i\ ,\ x \in \mathcal{H}_-\ .$$

Below we mostly identify the linear span of $\{e_1, \dots, e_N\}$ with $\mathbb{R}^N$.

We will denote the space of k-times continuously differentiable bounded mappings from $\mathcal{H}_-$ into a Banach space X by $C_b^k(\mathcal{H}_-, X)$ $(C_b^0(\mathcal{H}_-, X) \equiv C_b(\mathcal{H}_-, X))$. $C_b^2(\mathcal{H}_-, X)$ is a Banach space with the norm

$$\|f\|_{C_b^2} = \sup_{x\in\mathcal{H}_-} (\|f(x)\|_X + \|f'(x)\|_{\mathcal{L}(\mathcal{H}_-,X)} + \|f''(x)\|_{\mathcal{L}(\mathcal{H}_-,\mathcal{L}(\mathcal{H}_-,X))}),$$

where $\mathcal{L}(\mathcal{H}_-, X)$ denotes the space of all bounded linear operators from $\mathcal{H}_-$ to X. When X is the set of complex numbers $\mathbb{C}$ we identify $f'(x) \in \mathcal{L}(\mathcal{H}_-, \mathbb{C})$ with $\widetilde{f'}(x) \in \mathcal{H}_+$ and $f''(x) \in \mathcal{L}(\mathcal{H}_-, \mathcal{L}(\mathcal{H}_-, \mathbb{C}))$ with the operator $\widetilde{f''}(x) \in \mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)$. In this case

$${}_+(\widetilde{f'}(x), \varphi)_- = f'(x)\varphi, \quad {}_+(\widetilde{f''}(x)\varphi, \psi)_- = (f''(x)\varphi)\psi,\ \varphi, \psi \in \mathcal{H}_-.$$

So we make the convention that $\nabla f := f' = \widetilde{f}'$ and $f'' = \widetilde{f}''$ for $f \in C_b^2(\mathcal{H}_-)$.

By $\mathcal{F}C_b^k$, $k \in \mathbb{N} \cup \{\infty\}$, let us denote the set of all functions f on $\mathcal{H}_-$ such that there exist $N \in \mathbb{N}$, $\{\phi_1, \ldots, \phi_N\} \subset \mathcal{H}_+$ and $G \in C_b^k(\mathbb{R}^N)$ such that

$$f(x) = G({}_+(\phi_1, x)_-, \ldots, {}_+(\phi_N, x)_-), \; x \in \mathcal{H}_-.$$

Let ν be a probability measure defined on the σ-algebra $\mathcal{F}$ of Borel subsets of $\mathcal{H}_-$ with $\operatorname{supp} \nu = \mathcal{H}_-$. Assume that ν has a logarithmic derivative in the sense that there exists an $\mathcal{F}$-measurable mapping $\beta : \mathcal{H}_- \mapsto \mathcal{H}_-$ and $\forall q \in \mathcal{H}_+$, $\beta_q(\cdot) := {}_+(q, \beta(\cdot))_- \in L^2(\mathcal{H}_-, \nu)$ and the following integration by parts formula holds

$$\int_{\mathcal{H}_-} \nabla_i f(x) d\nu(x) = - \int_{\mathcal{H}_-} \beta_i(x) f(x) d\nu(x) \text{ for all } f \in \mathcal{F}C_b^\infty. \tag{1}$$

where $\beta_i := {}_+(\beta, e_i)_-$. We also use the following notations

$$\nabla_i f := (\nabla f, e_i)_0, \qquad \Delta f := Tr_{\mathcal{H}_0} f''.$$

$\| \cdot \|_p$ is the norm in $L^p(\mathcal{H}_-, \nu) \equiv L^p$, $\langle \cdot, \cdot \rangle$ is the inner product in L^2, and $\langle \ \rangle$ denotes expectation w.r.t. ν.

We introduce the Dirichlet form in L^2 as a closure of the form

$$\mathcal{E}(f, g) \equiv \int_{\mathcal{H}_-} (f'(x), g'(x))_0 \, d\nu = \langle (\nabla f, \nabla g)_0 \rangle, \; f, g \in \mathcal{F}C_b^\infty.$$

Under the stated conditions the form is closable (see [7]). The integration by parts formula implies that

$$\langle \nabla_i f, g \rangle = \langle f, -(\nabla_i + \beta_i) g \rangle \; \forall f, g \in C_b^1.$$

Let A be the operator associated with the form $\mathcal{E}$. Then $A = A^* \geqslant 0$. The semigroup $(e^{-tA}, t \geqslant 0)$ is positivity preserving (i.e., $e^{-tA}[L_+^2] \subset L_+^2 := \{f \in L^2 : f \geqslant 0 \text{ a.e.}\}$) and L^∞-contractive (i.e., $\|e^{-tA} f\|_\infty \leqslant \|f\|_\infty$, $\forall f \in L^2 \cap L^\infty$). We also have that $e^{-tA} 1 = 1$ for all $t > 0$, since $A1 = 0$. Recall that such a semigroup is called Markov semigroup and its generator is called Markov generator. It is a standard fact that this semigroup defines a family of C_0-semigroups of contractions in L^p, $p \in [1, \infty)$:

$$e^{-tA_p} := (e^{-tA} \upharpoonright [L^2 \cap L^p])^{\sim}_{L^p \to L^p}.$$

In what follows we use a stronger condition on the logarithmic derivative, namely, we assume that $\beta_i \in L^2(\mathcal{H}_-)$ which enables us to identify the action of the operator A :

$$A \upharpoonright C_b^2 = -(\Delta + {}_-(\beta, \nabla \cdot)_+) \upharpoonright C_b^2.$$

The operator A associated with the Dirichlet form is called the *Dirichlet operator.*

In the following definitions we define different types of the uniqueness problem for the operator A.

Definition 1. *A linear set $\mathcal{D} \subset \mathcal{D}(A)$ is called a domain of strong uniqueness of A if there exists only one selfadjoint extension of $A \upharpoonright \mathcal{D}$ in L^2.*

We note that according to a theorem by J. von Neumann this is equivalent to the essential selfadjointness of $A \upharpoonright \mathcal{D}$.

Definition 2. *A linear set $\mathcal{D} \subset \mathcal{D}(A)$ is called a domain of Markov uniqueness of A if there exists only one extension of $A \upharpoonright \mathcal{D}$ which is a Markov generator in L^2.*

It is clear that if $\mathcal{D}$ is a domain of strong uniqueness of A it is also a domain of Markov uniqueness, but not vice versa. (There are counterexamples even if $\mathcal{H}_- = \mathcal{H}_0 = \mathcal{H}_+ = \mathbb{R}^1$, cf. [19].)

We also extend the strong uniqueness problem to the L^p-setting.

Definition 3. *Let $p \geqslant 1$. A linear set $\mathcal{D} \subset \mathcal{D}(A_p)$ is called a domain of L^p-strong uniqueness of A if all extensions B of $A \upharpoonright \mathcal{D}$ in L^p such that $-B$ is a generator of a C_0-semigroup in L^p coincide with A_p.*

According to a result by W.Arendt (cf. [35] Theorem A-II, 1.33, p.46) the latter is equivalent to the fact that $\mathcal{D}$ is a core of the operator A_p.

The following simple observation gives a link between the L^1-strong uniqueness and the Markov uniqueness.

Proposition 1. *Let A be a Markov generator. Let $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(A_1)$. Then if $\mathcal{D}$ is a domain of L^1-strong uniqueness for A, it is also a domain of Markov uniqueness.*

Proof Let $f \in \mathcal{D}$. Then by the definition of A_1 we have $e^{-tA_1} f = e^{-tA} f$ and

$$\frac{1}{t}(f - e^{-tA_1} f) = \frac{1}{t}(f - e^{-tA} f) = \frac{1}{t}\int_0^t e^{-tA} A f ds = \frac{1}{t}\int_0^t e^{-tA_1} A f ds.$$

Passing to the limit with respect to $t \downarrow 0$ we get $A_1 f = Af$. Now the result follows easily. □

Now we are ready to formulate the main results of the paper. The first result concerns the strong uniqueness problem in L^2. In [30] it was proved that if $\beta : \mathcal{H}_- \mapsto \mathcal{H}_0$ and $|\beta|_0 \in L^4(\mathcal{H}_-, d\nu)$ then $-\Delta - {}_-(\beta, \nabla \cdot)_+ \upharpoonright C_b^2$ is essentially selfadjoint. However, in the infinite dimensional case these conditions become quite restrictive in applications, in particular, because of the condition $\beta(x) \in \mathcal{H}_0$ for ν–a.e. $x \in \mathcal{H}_-$. The authors of [3] (using related analytic ideas, but also stochastic techniques in an essential way) derived another sufficient condition for essential selfadjointness, assuming the existence of the derivative of β and that it satisfies certain one-sided estimates. The following result is a strict generalization of [30] invoking, in addition, also a one sided estimate (but involving the $|\ |_+$–norm rather than the $|\ |_-$–norm as in [3]). The primary idea is to consider β as a sum of mappings $\alpha : \mathcal{H}_- \mapsto \mathcal{H}_0$ and $\delta : \mathcal{H}_- \mapsto \mathcal{H}_-$. The former satisfies the conditions of [30] whereas the derivative of the latter will satisfy a one-sided estimate. Technically this is achieved by means of an a–priori estimate (see Section 3). Though this result remains quite disjoint from those in [1], [2], [3] we are nevertheless able to apply it to new examples not covered by those papers. The most striking one is to show essential selfadjointness of the Dirichlet operator corresponding to the stochastic quantization of finite volume quantum fields. This is presented in Section 5 below, to which we refer for details and references.

Theorem 1. *Let $\beta = \alpha + \delta$, $|\alpha|_0 \in L^4(\mathcal{H}_-, \nu)$, $|\delta|_- \in L^2(\mathcal{H}_-, \nu)$. Suppose that there exist $\delta^m : \mathcal{H}_- \to \mathcal{H}_-$, $m \in \mathbb{N}$, such that*

(i) $\delta^m = \delta^m \circ P_{N_m}$ *for some* $N_m \in \mathbb{N}$ *and* $\delta_j^m \restriction \mathbb{R}^{N_m} \in C_b^1(\mathbb{R}^{N_m})$ *with globally Hölder continuous first order derivatives for all* $1 \leq j \leq N_m$ *(where* $\delta_j^m := {}_+(e_j, \delta^m)_-$*).*

(ii) $|\delta - \delta^m|_- \to 0$ *in* $L^2(\mathcal{H}_-, \nu)$ *as* $m \to \infty$.

(iii) *There exists* $c_+ \in \mathbb{R}$ *such that for all* $m \in \mathbb{N}$,

$$(\Lambda_{\delta^m}(x)y, y)_+ \leq c_+ \; |y|_+^2 \;\; \forall x, y \in \mathbb{R}^{N_m},$$

where $\Lambda_{\delta^m}(x) := (\nabla_i \delta_j^m(x))_{1 \leq i,j \leq N_m}$.

(iv) *If* $\alpha \neq 0$, *suppose, in addition, that* $\delta_j = \delta_j \circ P_j$ *with* $\delta_j \restriction \mathbb{R}^j \in C^1(\mathbb{R}^j)$ *for all* $j \in \mathbb{N}$ *and that there exists* $\varepsilon_0 \in (0,1)$, $c(\varepsilon_0) \in \mathbb{R}$ *such that for all* $N \in \mathbb{N}$

$$\begin{aligned}&\big\langle ((\nabla_i \delta_j \circ P_N)_{1 \leq i,j \leq N}\, w, w)_0 \big\rangle \\ &\leq (1 - \varepsilon_0) \sum_{i,j=1}^{N} \langle \nabla_j\, w_i \circ P_N, \nabla_j\, w_i \circ P_N \rangle + c(\varepsilon_0) \langle (w \circ P_N, w \circ P_N)_0 \rangle\end{aligned}$$

for all $w_1, \ldots, w_N \in C_b^1(\mathbb{R}^N)$ *and* $w := (w_1, \ldots, w_N)$.

Then $(-\Delta - {}_-(\beta, \nabla \cdot)_+ \restriction \mathcal{F}C_b^\infty)$ *is essentially self-adjoint in* $L^2(\mathcal{H}_-, \nu)$.

Remark 1. *Consider the following condition:*

(iii') $(\Lambda_{\delta^m}(x)y, y)_- \leqslant c_- |y|_-^2 \quad \forall x, y \in \mathbb{R}^{N_m}$.

To compare (iii) and (iii') one can rewrite them in the form

$$(iii) \quad (T \Lambda_{\delta^m}(x) T^{-1} z, z)_0 \leqslant c_+ |z|_0^2 \;\; \forall x, z \in \mathbb{R}^{N_m}.$$

and

$$(iii') \quad (T^{-1} \Lambda_{\delta^m}(x) T z, z)_0 \leqslant c_- |z|_0^2 \;\; \forall x, z \in \mathbb{R}^{R_N}.$$

Therefore (iii) and (iii') coincide if $\Lambda_{\delta^m} = \Lambda_{\delta^m}^*$.

The next theorem gives a criterium for the L^1-strong uniqueness of the operator A.

Theorem 2. *Let* $\beta = \alpha + \delta$, $|\alpha|_0, |\delta|_- \in L^2(\mathcal{H}_-, \nu)$. *Suppose that there exists a sequence of mappings* $(\delta^m)_{m \in \mathbb{N}}$, $\delta^m : \mathcal{H}_- \mapsto \mathcal{H}_-, m \in \mathbb{N}$ *such that*

(i) $(\delta^m)_{m \in \mathbb{N}}$ *satisfies condition (i) in Theorem 1.*

(ii) $|\delta - \delta^m|_- \longrightarrow 0$ *in* $L^1(\mathcal{H}_-, d\nu)$ *as* $m \to \infty$.

(iii) There exists a constant $c_+ \in \mathbb{R}$ *such that for all* $m \in \mathbb{N}$

$$(\Lambda_{\delta^m}(x)y, y)_+ \leqslant c_+ |y|_+^2 \;\; \forall x, y \in \mathbb{R}^N.$$

Then the operator $(\Delta + {}_-(\beta, \nabla \cdot)_+ \restriction \mathcal{F}C_b^\infty)$ *has a unique extension which generates a* C_0*-semigroup on* $L^1(\mathcal{H}_-, \nu)$.

Corollary 1. *Let the conditions of Theorem 2 be satisfied. Then* $\mathcal{F}C_b^\infty$ *is a domain of Markov uniqueness for the operator* A.

The proofs of Theorems 1 and 2 will be given in Section 4 after we derive a–priori estimates in Section 3. These estimates are the core of the method.

3. A–PRIORI ESTIMATES

The aim of this section is to obtain a–priori estimates for solutions of parabolic equations on $\mathbb{R}^d$. So here $\mathcal{H}_0, \mathcal{H}_+, \mathcal{H}_-$ will be just $\mathbb{R}^d$ endowed with the inner products $(y,z)_+ = \sum_{j=1}^d \mu_j^2 y_j z_j$, $(y,z)_- = \sum_{j=1}^d \mu_j^{-2} y_j z_j$ and the usual Euclidean product for $\mathcal{H}_0$. The measure ν above is correspondingly now a measure on $\mathbb{R}^d$.

Let $u(t,x) = u$ be the solution of the Cauchy problem

$$\left\{ \begin{array}{c} \frac{\partial}{\partial t} u = \Delta u + (b, \nabla u)_0 \\ u(\cdot, 0) = f(\cdot), \end{array} \right. \tag{2}$$

where $f \in C_0^\infty(\mathbb{R}^d)$, $b \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ with globally Hölder continuous first order derivatives. The operator $\mathcal{A} := \Delta + (b, \nabla \cdot)_0$ generates a C_0-semigroup on $C_\infty(\mathbb{R}^d)$ which can be extended to a C_0-semigroup on $L^p(\mathbb{R}^d)$ ($:= L^p(\mathbb{R}^d, dx)$, where dx denotes Lebesgue measure) and for the solution of (2) we have $u(t,\cdot) = e^{t\mathcal{A}} f(\cdot)$, $t > 0$, and $\|u\|_\infty \leqslant \|f\|_\infty$. Furthermore, we have $u(t,\cdot) \in C_b^2(\mathbb{R}^d) \cap C^3(\mathbb{R}^d)$ (even with globally Hölder continuous second order derivatives and locally Hölder continuous third order derivatives; cf [27, Theorems 9.2.3 and 8.12.1]).

Let us introduce the derivative of the mapping $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ as the linear operator Λ_b whose matrix is $(\Lambda_b)_{ij} = (\nabla_i b_j)$.

Proposition 2. *Let u be the solution to* (2) *with $f \in C_0^\infty(\mathbb{R}^d)$. Suppose that b is as above satisfying, in addition, that there exists $c_+ \in \mathbb{R}^+$ such that*

$$(\Lambda_b(x) y, y)_+ \leqslant c_+ \ (y,y)_+ \quad \forall x, y \in \mathbb{R}^d.$$

Then

$$\| |\nabla u(t,x)|_+ \|_\infty \leqslant e^{c_+ t} \| |\nabla f|_+ \|_\infty. \tag{3}$$

Proof. Let $w_i := \nabla_i u$ be the derivative of u in direction e_i. Denote the inner product in $L^2(\mathbb{R}^d)$ by $\langle \cdot, \cdot \rangle$ and the integral with respect to Lebesgue measure dx by $\langle \cdot \rangle$ (*only in this proof*). Differentiating equation (2) in the direction e_i we get

$$\frac{d}{dt} w_i - \Delta w_i - (b, \nabla w_i)_0 - (\Lambda_b w)_i = 0,$$

where $(\Lambda_b w)_i = \sum_{j=1}^d (\nabla_i b_j) w_j$. Multiplying both sides of the last equality by $\mu_i^2 w_i |w|_+^{p-2}$, $p > 4$, after integration with respect to dx and summation with respect to i from 1 to d we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \| |w|_+ \|_p^p + \sum_{i,j} \langle \nabla_j w_i, |w|_+^{p-2} \mu_i^2 \nabla_j w_i \rangle + \sum_{i,j} \langle w_i \mu_i^2 \nabla_j w_i, \nabla_j |w|_+^{p-2} \rangle \\ - \sum_{i,j} \langle (\nabla_i b_j) w_j, \mu_i^2 w_i |w|_+^{p-2} \rangle - \sum_i \langle (b, \nabla w_i)_0, \mu_i^2 w_i |w|_+^{p-2} \rangle = 0. \end{aligned}$$

The last term is equal to

$$-\frac{1}{2} \langle (b, \nabla |w|_+^2)_0 \ |w|_+^{2(p/2-1)} \rangle = -\frac{1}{p} \langle (b, \nabla |w|_+^p)_0 \rangle = \frac{1}{p} \langle |w|_+^p \ \mathrm{div}\, b \rangle \ .$$

Therefore, we obtain the equality

$$\frac{1}{p}\frac{d}{dt}\||w|_+\|_p^p + 4\frac{p-2}{p^2}\||\nabla(|w|_+^{p/2})|_0\|_2^2 + \sum_{j=1}^d \langle(\nabla w_j, \nabla w_j)_+, |w|_+^{p-2}\rangle$$

$$= \langle(\Lambda_b\, y, y)_+\rangle - \frac{1}{p}\langle|w|_+^p \mathrm{div}\ b\rangle, \tag{4}$$

where $y := w|w|_+^{p/2-1}$, $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^d)$. From (4) and the assumption it follows that

$$\frac{1}{p}\frac{d}{dt}\||w|_+\|_p^p \leqslant c_+\||w|_+\|_p^p + \frac{k}{p}\||w|_+\|_p^p, \quad \text{where} \quad k := \|div\ b\|_\infty$$

or

$$\||w(t)|_+\|_p \leqslant \||w(0)|_+\|_p\ e^{(c_+ + \frac{k}{p})t}.$$

Since $|w|_+(0) = |\nabla f|_+$, passing to the limit $p \to \infty$ we get $\||w|_+\|_\infty \leqslant \||\nabla f|_+\|_\infty\ e^{ct}$.
□

From now on as in Section 2 the norms $\|\ \|_p$ again denote the L^p–norms w.r.t ν.

Proposition 3. *Let $\beta = \alpha + \delta$, $b = \alpha^1 + \delta^1$ with $\alpha, \delta, \alpha^1, \delta^1 : \mathbb{R}^d \to \mathbb{R}^d$. Suppose that $|\alpha|_0 \in L^2(\mathbb{R}^d, \nu)$, $|\delta|_- \in L^1(\mathbb{R}^d, \nu)$ and that the condition of Proposition 2 is satisfied. Let u be the solution of* (2)*. Then*

$$\|u\|_2^2 + \int_0^t \||\nabla u|_0\|_2^2\ ds$$
$$\leqslant\ t\|f\|_\infty^2\||\alpha - \alpha^1|_0\|_2^2 + 2c_+^{-1}(e^{c+t} - 1)\|\,|\nabla f|_+\,\|_\infty\,\|f\|_\infty\,\||\delta - \delta^1|_-\|_1 + \|f\|_2^2.$$

Proof. Multiplying (2) by u and integrating w.r.t. ν we obtain after integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \||\nabla u|_0\|_2^2 = \langle(b - \beta, \nabla u)_0, u\rangle.$$

Estimating the r.h.s. as follows

$$|\langle(b-\beta, \nabla u)_0, u\rangle| \leqslant \frac{1}{2}\||\nabla u|_0\|_2^2 + \frac{1}{2}\|f\|_\infty^2\||\alpha - \alpha^1|_0\|_2^2 + \|f\|_\infty\||\delta - \delta^1|_-\|_1\,\||\nabla u|_+\|_\infty$$

after integration with respect to t and using Proposition 2 one completes the proof.
□

Proposition 4. *Let u be the solution to* (2)*. In addition to the conditions of Proposition 3 assume that $|\alpha|_0 \in L^4(\mathbb{R}^d, \nu)$, $|\delta|_- \in L^2(\mathbb{R}^d, \nu)$, $\delta \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, and that there exist $\varepsilon_0 \in (0,1)$, $c(\varepsilon_0) \in \mathbb{R}_+$ such that*

$$\langle(\Lambda_\delta w, w)_0\rangle \leqslant (1-\varepsilon_0)\sum_{j=1}^d \langle(\nabla_j w, \nabla_j w)_0\rangle + c(\varepsilon_0)\langle(w,w)_0\rangle \tag{5}$$

for all $w=(w_j)\in C_b^1(\mathbb{R}^d,\mathbb{R}^d)$. *Then there exists* $C(\varepsilon_0)\in\mathbb{R}_+$ *(depending only on* ε_0*) such that*

$$\int_0^t \||\nabla u(s)|_0\|_4^4 ds \leqslant C(\varepsilon_0)\{t\|f\|_\infty^4(\||\alpha|_0\|_4^4+\||\alpha^1|_0\|_4^4+\|\,|\alpha-\alpha^1|_0\|_2^2)$$
$$+\frac{1}{2c_+}(e^{2c_+t}-1)\|f\|_\infty^2\||\nabla f|_+\|_\infty^2\||\delta-\delta^1|_-\|_2^2$$
$$+2c_+^{-1}(e^{c_+t}-1)\|f\|_\infty^3\||\nabla f|_+\|_\infty\||\delta-\delta^1|_-\|_1+\|f\|_\infty^2\||\nabla f|_0\|_2^2+c\|f\|_2^2\ \|f\|_\infty^2\}. \tag{6}$$

Before proving Proposition 4 we prove several lemmas (all of them under the conditions of Proposition 4).

Lemma 1. *Let* $w:=\nabla u$. *Then*

$$\|\frac{du}{dt}\|_2^2+\frac{d}{dt}\||w|_0\|_2^2\leqslant\|(\beta-b,w)_0\|_2^2. \tag{7}$$

Proof. Using the equation and integrating by parts we have

$$\langle(b,w)_0,\frac{du}{dt}\rangle=\langle\frac{du}{dt}-div\,w,\frac{du}{dt}\rangle=\|\frac{du}{dt}\|_2^2+\langle(w,\frac{dw}{dt})_0\rangle+\langle(\beta,w)_0,\frac{du}{dt}\rangle.$$

Rewriting this in the form

$$\|\frac{du}{dt}\|_2^2+\frac{1}{2}\frac{d}{dt}\||w|_0\|_2^2=\langle(b-\beta,w)_0,\frac{du}{dt}\rangle\leq\frac{1}{2}\|(\beta-b,w)_0\|_2^2+\frac{1}{2}\|\frac{du}{dt}\|_2^2,$$

we get the result. □

Lemma 2. *Let* u *be the solution to (2),* $w=\nabla u$. *Then*

$$\frac{1}{2}\frac{d}{dt}\||w|_0\|_2^2+\sum_{i,j}\langle\nabla_iw_j,\nabla_iw_j\rangle=-\sum_i\langle(\alpha,\nabla w_i)_0,w_i\rangle+\langle(\Lambda_\delta w,w)_0\rangle$$
$$+\langle(\delta-b,w)_0,\frac{du}{dt}\rangle+\langle(\delta-\delta^1,w)_0,(\alpha-2\alpha^1,w)_0\rangle \tag{8}$$
$$-\langle(\alpha^1,w)_0,(\alpha,w)_0\rangle+\|(\alpha^1,w)_0\|_2^2+\|(\delta-\delta^1,w)_0\|_2^2.$$

Proof. Differentiating the equation in direction e_i we get

$$\frac{dw_i}{dt}=\Delta w_i+\nabla_i(\alpha^1,w)_0+(\delta^1,\nabla w_i)_0+(\nabla_i\delta^1)\cdot w.$$

Multiplying scalarly by w_i in $L^2(\mathbb{R}^d,\nu)$ and summing over i we have the equality

$$\frac{1}{2}\frac{d}{dt}\||w|_0\|_2^2+\sum_{i,j}\langle\nabla_iw_j,\nabla_iw_j\rangle+\sum_{i,j}\langle(\alpha_j+\delta_j)\nabla_jw_i,w_i\rangle+\langle(\alpha^1,w)_0,\sum_i(\nabla_i+\beta_i)w_i\rangle$$
$$-\sum_i\langle(\delta^1,\nabla w_i)_0,w_i\rangle-\langle(\Lambda_{\delta^1}w,w)_0\rangle=0.$$

Again using the equation we rewrite the last equality in the form

$$\frac{1}{2}\frac{d}{dt}\||w|_0\|_2^2+\sum_{i,j}\langle\nabla_i w_j,\nabla_i w_j\rangle+\sum_i\langle(\alpha,\nabla w_i)_0,w_i\rangle+\langle(\alpha^1,w)_0,(\alpha-\alpha^1,w)_0\rangle$$

$$+\langle(\alpha^1,w)_0,\frac{du}{dt}\rangle+\langle(\alpha^1,w)_0,(\delta-\delta^1,w)_0\rangle-\langle(\Lambda_{\delta^1}w,w)_0\rangle+\sum_i\langle(\delta-\delta^1,\nabla w_i)_0,w_i\rangle=0.$$

To finish the proof of the lemma one should observe that

$$\sum_i\langle(\delta-\delta^1,\nabla w_i)_0,w_i\rangle=\langle(\Lambda_{\delta^1}w,w)_0\rangle-\langle(\Lambda_\delta w,w)_0\rangle$$

$$-\langle(\delta-\delta^1,w)_0,\frac{du}{dt}\rangle-\langle(\delta-\delta^1,w)_0,(\alpha-\alpha^1,w)_0\rangle-\|(\delta-\delta^1,w)_0\|_2^2.\quad\square$$

Lemma 3. *Let u be the solution to (2), $w:=\nabla u$. Then*

$$\||w|_0\|_4^4\leqslant 16\|f\|_\infty^2(\frac{1}{2}\|(b-\beta,w)_0\|_2^2-\frac{1}{4}\frac{d}{dt}\||w|_0\|_2^2+\sum_{i,j}\|\nabla_i w_j\|_2^2).$$

Proof. Using equation (2) and integrating by parts we obtain

$$\||w|_0\|_4^4=\sum_{i,j}\langle w_i,w_iw_j^2\rangle=-\sum_{i,j}\langle u,w_j^2\nabla_i w_i\rangle-2\sum_{i,j}\langle u,w_iw_j\nabla_i w_j\rangle-\sum_{i,j}\langle u,\beta_iw_iw_j^2\rangle$$

$$=-\langle u,|w|_0^2\frac{du}{dt}\rangle-2\sum_{i,j}\langle u,w_iw_j\nabla_i w_j\rangle+\langle u,|w|_0^2(b-\beta,w)_0\rangle$$

$$\leqslant\|u\|_\infty\||w|_0\|_4^2\|\frac{du}{dt}\|_2+\|u\|_\infty\||w|_0\|_4^2\|(b-\beta,w)_0\|_2+2\|u\|_\infty\sum_{i,j}\|w_iw_j\|_2\|\nabla_i w_j\|_2$$

$$\leqslant\frac{1}{2}\||w|_0\|_4^4+\|f\|_\infty^2\|\frac{du}{dt}\|_2^2+\|f\|_\infty^2\|(b-\beta,w)_0\|_2^2$$

$$+2\|f\|_\infty(\sum_{i,j}\|w_iw_j\|_2^2)^{1/2}(\sum_{i,j}\|\nabla_i w_j\|_2^2)^{1/2}$$

$$\leqslant\frac{3}{4}\||w|_0\|_4^4+\|f\|_\infty^2(\|(b-\beta,w)_0\|_2^2-\frac{d}{dt}\||w|_0\|_2^2)$$

$$+\|f\|_\infty^2\|(b-\beta,w)_0\|_2^2+4\|f\|_\infty^2\sum_{i,j}\|\nabla_i w_j\|_2^2,$$

where we used that $ab\leq\frac{1}{4}a^2+b^2$ and Lemma 1 in the last step. Now the assertion follows. $\square$

Proof of Proposition 4. Let us first estimate the first term of the right hand side of (8):

$$\begin{aligned}|\sum_i \langle (\alpha, \nabla w_i)_0, w_i\rangle| &\leqslant \langle |\alpha|_0 |w|_0 (\sum_{i,j} |\nabla_i w_j|^2)^{1/2}\rangle \\ &\leqslant \frac{\varepsilon_0}{2}\sum_{i,j}\|\nabla_i w_j\|_2^2 + \frac{1}{2\varepsilon_0}\||\alpha|_0 \cdot |w|_0\|_2^2.\end{aligned}$$

The term in (8) containing $\frac{du}{dt}$ is estimated by Lemma 1 as follows

$$\begin{aligned}&|\langle (\delta - b, w)_0, \frac{du}{dt}\rangle| \leqslant \frac{1}{2}\|\frac{du}{dt}\|_2^2 + \frac{1}{2}\|(\delta - b, w)_0\|_2^2 \\ \leqslant\ &\frac{1}{2}\|(b-\beta, w)_0\|_2^2 - \frac{1}{2}\frac{d}{dt}\||w|_0\|_2^2 + \|(\delta - \delta^1, w)_0\|_2^2 + \|(\alpha^1, w)_0\|_2^2 .\end{aligned}$$

Therefore, using that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for estimating the rest of the right hand side of (8) we get by assumption (5) for any $\varepsilon, \varepsilon_1 > 0$ from (8) that

$$\begin{aligned}&\frac{d}{dt}\||w|_0\|_2^2 + \frac{\varepsilon_0}{2}(\sum_{i,j}\|\nabla_i w_j\|_2^2 + \frac{1}{2}\|(b-\beta, w)_0\|_2^2) \\ \leqslant\ &\frac{1}{2\varepsilon_0}\||\alpha|_0|w|_0\|_2^2 + \tilde{C}(\varepsilon_0)(\|(\alpha^1, w)_0\|_2^2 + \|(\alpha, w)_0\|_2^2 + \|(\delta - \delta^1, w)_0\|_2^2) + c(\varepsilon_0)\||w|_0\|_2^2 \\ \leqslant\ &\frac{\varepsilon}{2\varepsilon_0}\||w|_0\|_4^4 + \frac{1}{8\varepsilon_0\varepsilon}\||\alpha|_0\|_4^4 + \varepsilon_1\||w|_0\|_4^4 + \frac{\tilde{C}(\varepsilon_0)}{2\varepsilon_1}(\||\alpha^1|_0\|_4^4 + \||\alpha|_0\|_4^4) \\ &+\tilde{C}(\varepsilon_0)\|(\delta - \delta^1, w)_0\|_2^2 + c(\varepsilon_0)\||w|_0\|_2^2.\end{aligned}$$

where $\tilde{C}(\varepsilon_0) \in \mathbb{R}_+$ only depends on ε_0. Estimating the left hand side of the last inequality from below by Lemma 3 one obtains the desired result after integration with respect to t applying Propositions 2 and 3, and properly choosing ε and ε_1 (for instance, $\varepsilon := \frac{\varepsilon_0^2}{64\|f\|_\infty^2}$, $\varepsilon_1 = \frac{\varepsilon_0}{128\|f\|_\infty^2}$). □

4. Proofs of the uniqueness results

In this section we prove the results formulated in Section 2. Our strategy for the uniqueness in L^2 and in L^1 is the same. Namely, we take an arbitrary extension of the original operator defined on $\mathcal{F}C_b^\infty$, which generates a C_0-semigroup in L^2 (L^1 respectively). Then we construct the approximation sequence which converges to the generated semigroup in L^2 (L^1 respectively). This implies the uniqueness of the semigroup, and therefore the uniqueness of the extension. The same approach is extendable to all L^p-spaces (see Section 6 for more details).

Proof of Theorem 1. Let $f \in \mathcal{F}C_b^\infty$ and let $N \in \mathbb{N}$, $G \in C_b^\infty(\mathbb{R}^N)$ such that

$$f(x) = G(e_1(x), \dots, e_N(x)), \quad x \in \mathcal{H}_- .$$

For every $n \in \mathbb{N}$ there exist

$$\alpha_1^n, \dots, \alpha_{a_n}^n \in C_b^\infty(\mathbb{R}^{a_n})$$

such that for $\alpha^n \in C_b^\infty(\mathcal{H}_-, \mathcal{H}_-)$ defined by

$$\alpha^n(x) := \sum_{i=1}^{a_n} \alpha_i^n(e_1(x), \ldots, e_{a_n}(x))\ e_i, \quad x \in \mathcal{H}_- \ ,$$

we have

$$|\alpha - \alpha^n|_0 \longrightarrow 0 \text{ in } L^4(\mathcal{H}_-, \nu) \text{ as } n \to \infty.$$

We may assume that $a_n \geq N$ for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ define

$$d_{n,m} := \max(m, a_n, N_m)$$

where N_m is as in assumption (i) of the theorem. We now apply the results of Section 3 with b replaced by $b^{n,m} := P_{d_{n,m}} \circ (\alpha^n + \delta^m) \restriction \mathbb{R}^{d_{n,m}}$ and ν replaced by $\nu \circ P_{d_{n,m}}^{-1}$. For $k \in \mathbb{N}$ let $\chi_{n,m,k} \in C_0^\infty(\mathbb{R}^{d_{n,m}})$, such that for all $n, m \in \mathbb{N}$ the maps $\chi_{n,m,k}$, $\nabla\chi_{n,m,k}$, $k \in \mathbb{N}$, are uniformly bounded and such that $\chi_{n,m,k}(x) = 1$ provided $|x| \leq k$. Define $G_{n,m,k} := \chi_{n,m,k} \cdot G$, where G is considered as a function on $\mathbb{R}^{d_{n,m}}$. Then the solution of the Cauchy problem on $\mathbb{R}^{d_{n,m}}$

$$\begin{aligned}
\frac{\partial u_{n,m,k}}{\partial t} &= \Delta u_{n,m,k} + (b^{n,m}, \nabla u_{n,m,k})_0 \\
u_{n,m,k}(\ ,0) &= G_{n,m,k}
\end{aligned}$$

is given by a C_0–semigroup on $C_\infty(\mathbb{R}^{d_{n,m}})$, i.e.,

$$u_{n,m,k}(t, \cdot) = e^{-t\mathcal{A}_{n,m}} G_{n,m,k} \ , \quad t > 0,$$

where

$$\mathcal{A}_{n,m} = -\Delta - (b^{n,m}, \nabla\ \cdot)_0 \text{ on } C_0^\infty(\mathbb{R}^{d_{n,m}}).$$

Let now B with domain $D(B)$ be an arbitrary lower bounded self–adjoint extension of $(-\Delta - {}_-(\beta, \nabla\cdot)_+) \restriction \mathcal{F}C_b^\infty$ on $L^2(\mathcal{H}_-, \nu)$. It is an easy exercise to see that $\mathcal{F}C_b^2 \subset D(B)$ and that

$$B = -\Delta - {}_-(\beta, \nabla\cdot)_+ \text{ on } \mathcal{F}C_b^2,$$

in particular, $(e^{-t\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}} \in D(B)$, $t \geq 0$. Since $t \mapsto e^{-t\mathcal{A}_{n,m}} G_{n,m,k}$ is continuously differentiable from $\mathbb{R}_+$ to $C_\infty(\mathbb{R}^{d_{n,m}})$, so is $t \mapsto (e^{-t\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}}$ from $\mathbb{R}_+$ to $L^2(\mathcal{H}_-, \nu)$. Therefore,

$$\begin{aligned}
&-\frac{d}{ds}\left(e^{-(t-s)B}\left(e^{-s\mathcal{A}_{n,m}} G_{n,m,k}\right) \circ P_{d_{n,m}}\right) \\
= \ & e^{-(t-s)B}\left(\left(\mathcal{A}_{n,m}\ e^{-s\mathcal{A}_{n,m}} G_{n,m,k}\right) \circ P_{d_{n,m}}\right) \\
&-B\ e^{-(t-s)B}\left(\left(e^{-s\mathcal{A}_{n,m}} G_{n,m,k}\right) \circ P_{d_{n,m}}\right) \\
= \ & e^{-(t-s)B}\big(-\Delta - (b^{n,m} \circ P_{d_{n,m}}, \nabla\cdot)_0 - B\big)\left(e^{-s\mathcal{A}_{n,m}} G_{n,m,k}\right) \circ P_{d_{n,m}} \ .
\end{aligned}$$

Hence we have justified that the classical Duhamel formula applies in our case, since the above implies that for all $t > 0$ and $f_k := G_{n,m,k} \circ P_{d_{n,m}}$

$$
\begin{aligned}
& e^{-tB}\ f_k - (e^{-t\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}} \\
= & \int_0^t e^{-(t-s)B}\ {}_{-}(\beta - b^{n,m} \circ P_{d_{n,m}}, \nabla (e^{-s\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}})_{+}\ ds\ .
\end{aligned}
$$

Hence, if $\gamma \in \mathbb{R}$ such that $B \geq \gamma$, then

$$
\begin{aligned}
& \sup_{k\in\mathbb{N}} \| e^{-tB} f_k - (e^{-t\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}} \|_2 \\
\leq\ & e^{t\gamma} \sup_{k\in\mathbb{N}} \int_0^t \| {}_{-}\big((\alpha^n + \delta^m) \circ P_{d_{n,m}} - \beta, \nabla (e^{-s\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}} \big)_{+} \|_2\ ds \\
\leq\ & e^{t\gamma} \bigg[\|\, |\alpha^n - \alpha|_0 \|_4 \quad \sup_{k\in\mathbb{N}} \int_0^t \|\, |\nabla u_{n,m,k}(s, P_{d_{n,m}}(\cdot))|_0 \|_4\ ds \\
& \qquad + \|\, |\delta^m - \delta\|_{-} \|_2 \quad \sup_{k\in\mathbb{N}} \int_0^t \|\, |\nabla u_{n,m,k}(s, P_{d_{n,m}})|_{+} \|_\infty\ ds \bigg].
\end{aligned}
$$

Here we used the fact that, since $d_{n,m} \geq \max(a_n, N_m)$, both $\alpha^n \circ P_{d_{n,m}} = \alpha^n$ and $\delta^m \circ P_{d_{n,m}} = \delta^m$. We want to show that for all $t > 0$

$$
\overline{\lim_{n\to\infty}}\ \overline{\lim_{m\to\infty}} \sup_{k\in\mathbb{N}} \| e^{-tB} f_k - (e^{-t\mathcal{A}_{n,m}} G_{n,m,k}) \circ P_{d_{n,m}} \|_2 = 0. \tag{9}
$$

Applying Proposition 2 we see that as $m \to \infty$ the second term converges to zero by assumption (ii) for each fixed n. We note that the assumption in Proposition 2 is indeed satisfied with $b = b^{n,m}$ for fixed n and a constant c_+ independent of m, since $\alpha^n \restriction \mathbb{R}^{d_{n,m}}$ has bounded continuous derivatives and because of assumption (iii). If $\alpha = 0$, (9) follows. If $\alpha \neq 0$, we apply Proposition 4 to see that the first term converges to zero, too, if $n \to \infty$. We note that by assumption (iv) we can indeed apply Proposition 4, since the logarithmic derivative $\beta^{n,m}$ of $\nu \circ P^{-1}_{d_{n,m}}$ (as is easily checked) satisfies the equation

$$
\beta_j^{n,m} \circ P_{d_{n,m}} = E_\nu[\beta_j \mid \sigma(P_{d_{n,m}})] = E_\nu[\alpha_j \mid \sigma(P_{d_{n,m}})] + \delta_j
$$

for all $j \in \{1, \ldots, d_{n,m}\}$. Here $E_\nu[\,\cdot \mid \sigma(P_{d_{n,m}})]$ denotes the conditional expectation of ν given the σ–algebra $\sigma(P_{d_{n,m}})$ generated by $P_{d_{n,m}}$. Of course, the constant c_+ in the right hand side of inequality (6) depends on n and m. But since we first take $m \to \infty$ it disappears so when afterwards taking $n \to \infty$, the right hand side of (6) stays bounded.

Since $e^{-tB} f_k \longrightarrow e^{-tB} f$ in $L^2(\mathcal{H}_-, \nu)$ as $k \to \infty$, equality (9) implies that $e^{-tB} f$, $t > 0$, is independent of which extension B we took in the first place. Since $\mathcal{F}C_b^\infty$ is dense in $L^2(\mathcal{H}_-, \nu)$, it follows that $(e^{-tB})_{t\geq 0}$ and hence that B is uniquely determined. Thus, the theorem is proved. □

Proof of Theorem 2. The idea of the proof is similar to that of Theorem 1. We use the same approximating semigroups $e^{-t\mathcal{A}_{m,n}}$ as in the proof of Theorem 1.

Let B be an arbitrary extension of $(-\Delta - {}_{-}(\beta, \nabla \cdot)_{+} \restriction \mathcal{F}C_b^\infty)$ such that B is the generator of a C_0-semigroup in $L^1(\mathcal{H}_-, \nu)$.

Now let f, f_k, $G_{n,m,k}$ be as in the proof of Theorem 1. Then again $(e^{-s\mathcal{A}_{n,m}}G_{n,m,k})\circ P_{d_{n,m}} \in D(B)$ and using Duhamel's formula we obtain the estimate

$$\begin{aligned}
&\|e^{-tB}f_k - (e^{-t\mathcal{A}_{n,m}}G_{n,m,k}\circ P_{d_{n,m}})\|_1 \\
\leqslant\ & e^{t\gamma}\{\int_0^t \|(b^{n,m}\circ P_{d_{n,m}} - \beta, \nabla(e^{-s\mathcal{A}_{n,m}}G_{n,m,k})\circ P_{d_{n,m}})_0\|_1 ds\} \\
\leqslant\ & e^{t\gamma}\{\||\alpha^n - \alpha|_0\|_2 \sup_{k\in\mathbb{N}}\int_0^t \||\nabla u_{n,m,k}(s)|_0\|_2 ds + \||\delta - \delta^m|_-\|_1 \sup_{k\in\mathbb{N}}\int_0^t \||\nabla u_{n,m,k}(s)|_+\|_\infty ds\}.
\end{aligned}$$

Now we proceed as above, but using Proposition 3 instead of Proposition 4 to show that

$$\overline{\lim_n}\lim_m \sup_k \|e^{-tB}f_k - (e^{-t\mathcal{A}_{n,m}}G_{n,m,k})\circ P_{d_{n,m}}\|_1 = 0.$$

Therefore, the extension B is unique which implies the assertion of the theorem. □

5. Application to the stochastic quantization of field theory in finite volume

In this section we present our main application. After the programme of stochastic quantization of field theories was initiated by Parisi and Wu in [37] and after its implementation for Euclidean quantum fields with polynomial interaction in finite volume by Jona–Lasinio and Mitter in the pioneering work [24], there has been an enormous number of follow-up papers on this subject (see e.g. [4], [5], [6], [8], [10], [11], [12], [15], [18], [21],[23], [25], [26], [33], [34], [40], [42]. But the question whether the corresponding Dirichlet operator restricted to $\mathcal{F}C_b^\infty$ is essentially self–adjoint remained open. Below, we shall show that we can settle this problem positively as a simple application of Theorem 1 above. We note that Markov uniqueness in this case was already shown in [40, Section 7]. We use the same notation as in the latter paper, but for completeness we now recall the complete framework.

Let Λ be an open rectangle in $\mathbb{R}^2$. Let $(-\Delta+1)_N$ be the generator of the following quadratic form on $L^2(\Lambda, dx) : (u,v) \mapsto \int_\Lambda \langle\nabla u, \nabla v\rangle_{\mathbb{R}^2} dx + \int_\Lambda uv\ dx$ with $u, v \in \{g \in L^2(\Lambda, dx) \mid \nabla g \in L^2(\Lambda, dx)\}$ (where ∇ is in the sense of distribution). Let $\{e_n \mid n \in \mathbb{N}\} \subset C^\infty(\bar{\Lambda})$ be the (orthonormal) eigenbasis of $(-\Delta+1)_N$ and $\{\lambda_n \mid n \in \mathbb{N}\}$ $(\subset]0,\infty[)$ the corresponding eigenvalues (cf. [39, p. 226]), i.e., we consider Neumann boundary conditions. Define for $\alpha \in \mathbb{R}$

$$H_\alpha := \{u \in L^2(\Lambda, dx) \mid \sum_{i=1}^\infty \lambda_n^\alpha \langle u, e_n\rangle^2_{L^2(\Lambda,dx)} < \infty\} \tag{10}$$

equipped with the inner product

$$\langle u, v\rangle_{H_\alpha} := \sum_{n=1}^\infty \lambda_n^\alpha \langle u, e_n\rangle_{L^2(\Lambda,dx)}\ \langle v, e_n\rangle_{L^2(\Lambda,dx)}. \tag{11}$$

Clearly, we have that

$$H_\alpha \begin{cases} \text{completion of } C^\infty(\bar{\Lambda}) \text{ w.r.t } \| \ \|_{H_\alpha} \text{ if } \alpha \geq 0 \\ \text{completion of } C_0^\infty(\Lambda) \text{ w.r.t } \| \ \|_{H_\alpha} \text{ if } \alpha \leq 0 \end{cases} \tag{12}$$

(cf. [31, p. 79] for the latter).

Fix $\delta > 0$. Since $\sum_{i=1}^\infty \lambda_n^{-1-\delta} < \infty$, we have applying [47, Theorem 3.2] (i.e., the Gross–Minlos–Sazonov theorem) with $H := L^2(\Lambda, dx)$, $\| \cdot \| := \| \cdot \|_{H_{-\delta}}$, $A_1 := (-\Delta+1)_N^{-\delta/2}$ and $A_2 := (-\Delta+1)_N^{-1/2}$, that there exists a unique mean zero Gaussian probability measure μ on $\mathcal{H}_- := H_{-\delta}$ (called *free field* on Λ; see [36]) such that

$$\int_E l(z)^2 \ \mu(dz) = \|l\|_{H_{-1}}^2 \quad \text{for all } l \in \mathcal{H}_-' = H_\delta. \tag{13}$$

Clearly, supp $\mu = E$.

Remark 2. *In* (13) *we have realized the dual of $H_{-\delta}$ as H_δ using as usual the chain*

$$H_\alpha \subset H_0 = L^2(\Lambda, dx) \subset H_{-\alpha}, \quad \alpha \geq 0. \tag{14}$$

Let $h \in L^2(\Lambda, dx)$, $n \in \mathbb{N}$, and define $: z^n : (h)$ as follows (cf., e.g. [22, Sect. 8.5]): fix $n \in \mathbb{N}$ and let $H_n(t)$, $t \in \mathbb{R}$, be the n-th Hermite polynomial, i.e., $H_n(t) = \sum_{m=0}^{[n/2]}(-1)^m \alpha_{nm} t^{n-2m}$, with $\alpha_{mn} = n!/[(n-2m)!\, 2^m m!]$. Let $d \in C_0^\infty(\mathbb{R}^2)$, $d \geq 0$, $\int d(x) \ dx = 1$, and $d(x) = d(-x)$ for each $x \in \mathbb{R}^2$. Define for $k \in \mathbb{N}$, $d_{k,x}(y) := 2^{2k} d(2^k(x-y))$; $x, y \in \mathbb{R}^2$. Let $z_k(x) := {}_+(d_{k,x}, z)_-$, $z \in \mathcal{H}_-$, $x \in \Lambda$, and set

$$: z_k^n : \ (x) \ := c_k(x)^{n/2} H_n(c_k(x)^{-1/2} z_k(x)), \tag{15}$$

where $c_k(x) := \int z_k(x)^2 \ \mu(dz)$. Then it is known that $: z_k^n : \ (h) \ := \int \ : z_k^n : (x)\, h(x) \ dx \underset{k\to\infty}{\longrightarrow} \ : z^n : \ (h)$ both in every $L^p(\mathcal{H}_-, \mu)$, $p \geq 1$, and for μ–a.e. $z \in \mathcal{H}_-$ (cf., e.g., [38, Sect. 3] for the latter). The function $z \mapsto \limsup_{k\to\infty} : z_k^n : (h)$ is then a μ–version of $: z^n : \ (h)$. From now on $: z^n : \ (h)$ shall denote this particular version.

Now fix $N \in \mathbb{N}$, $a_n \in \mathbb{R}$, $0 \leq n \leq 2N$ with $a_{2N} < 0$ and define

$$V(z) := \sum_{n=0}^{2N} a_n : z^n : (1_\Lambda), \quad z \in E,$$

where 1_Λ denotes the indicator function of Λ. Let

$$\varphi := \exp(-\frac{1}{2}V). \tag{16}$$

Then $\varphi > 0$ μ–a.e. and $\varphi \in L^p(\mathcal{H}_-, \mu)$ for all $p \in [1, \infty[$ (cf. e.g. [44, Sect. 5.2] or [22, Sect. 8.6]). Set

$$\nu := \varphi^2 \cdot \mu.$$

We want to apply Theorem 1 to ν, $\mathcal{H}_- := H_{-\delta}$ (as above), $\mathcal{H}_0 := H_\alpha$, where

$$\alpha > \max(0, 1-\delta/2) \tag{17}$$

and $\mathcal{H}_+ := H_{\delta+2\alpha}$. Clearly, then since by (17), in particular, $\alpha > 1-\delta$, the embeddings

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$$

are Hilbert–Schmidt. We note that thus $\mathcal{H}_+ = (-\Delta+1)^\alpha(\mathcal{H}'_-)$ and that $\{\lambda_j^{-\alpha/2}e_j \mid j\in\mathbb{N}\}$, $\{\lambda_j^{\delta/2}e_j \mid j\in\mathbb{N}\}$ are orthonormal bases of $\mathcal{H}_0$, $\mathcal{H}_-$ respectively. Define $\beta := \alpha+\delta : \mathcal{H}_-\to\mathcal{H}_-$, where $\delta:\mathcal{H}_-\to\mathcal{H}_-$ and $\alpha:\mathcal{H}_-\to\mathcal{H}_0$ are for (ν–a.e.) $z\in\mathcal{H}_-$ defined by

$$\delta(z) := -\sum_{j=1}^\infty \lambda_j^{-\alpha-\frac{\delta}{2}+1}\ {}_{H_{-\delta}}(z,e_j)_{H_\delta}\ \lambda_j^{\delta/2}e_j$$

$$\alpha(z) := -\sum_{j=1}^\infty \lambda_j^{-\alpha/2}\ \sum_{n=1}^{2N} n\ a_n : z^{n-1} : (e_j)\ \lambda_j^{-\alpha/2}e_j\ .$$

We have that $|\delta|_-$, $|\alpha|_0^2 \in L^2(\mathcal{H}_-,\nu)$. Indeed,

$$\begin{aligned}\int |\delta|_-^2\ d\nu &= \int \sum_{j=1}^\infty \lambda_j^{-2\alpha-\delta+2}\ {}_{H_{-\delta}}(z,e_j)^2_{H_\delta}\ \varphi^2(z)\ \mu(dz)\\ &\underset{(13)}{\leqslant} \sum_{j=1}^\infty \lambda_j^{-2\alpha-\delta+2}\, 3^{1/4}\ \lambda_j^{-1}\left(\int\varphi^4\ d\mu\right)^{1/2}\\ &< \infty,\end{aligned}$$

since $2\alpha+\delta-1<\infty$ by (17), and

$$\begin{aligned}\left(\int|\alpha|_0^4\ d\nu\right)^{1/4} &= \left(\int\left[\sum_{j=1}^\infty\lambda_j^{-\alpha}\left(\sum_{n=1}^{2N} n\ a_n : z^{n-1} : (e_j)\right)^2\right]^2 \varphi^2(z)\ \mu(dz)\right)^{1/4}\\ &\le \sum_{n=1}^{2N} n\ a_n\ \left(\int\left[\sum_{j=1}^\infty \lambda_j^{-\alpha}\left(: z^{n-1} : (e_j)\right)^2\right]^2\ \varphi^2(z)\ \mu(dz)\right)^{1/4}\\ &\le \sum_{n=1}^{2N} n\ a_n\ \left(\int\left[\sum_{j=1}^\infty \lambda_j^{-\alpha}\left(: z^{n-1} : (e_j)\right)^2\right]^4\ \mu(dz)\right)^{1/8}\left(\int\varphi^4\ d\mu\right)^{1/8}\\ &< \infty\end{aligned}$$

since $\alpha>0$ by (17) (cf. the proof of [40, Theorem 7.5]). Furthermore, for all $j\in\mathbb{N}$

$$\begin{aligned} &{}_-(\beta(z),\lambda_j^{-\alpha/2}e_j)_+ = \left((-\Delta+1)_N^{-\delta/2}\beta, \lambda^{\delta/2+\alpha-\alpha/2}\ e_j\right)_{H_0}\\ = &-\lambda_j^{-\alpha-\delta/2+1}\ {}_{H_{-\delta}}(z,e_j)_{H_\delta}\ \lambda^{\delta/2+\alpha/2}\\ &-\lambda_j^{-\alpha/2-\delta/2}\sum_{n=1}^{2N} n\ a_n : z^{n-1} : (e_j)\ \lambda^{\delta/2}\\ = &-\lambda_j\ {}_{H_{-\delta}}(z,\lambda_j^{-\alpha/2}e_j)_{H_\delta} - \sum_{n=1}^{2N} n\ a_n : z^{n-1} : (\lambda_j^{-\alpha/2}e_j).\end{aligned}$$

It follows by [40, Proposition 7.2] that (1) holds. Defining for $m \in \mathbb{N}$

$$\delta^m(z) := -\sum_{j=1}^{m} \lambda_j^{-\alpha-\delta/2+1} \ {}_{H_{-\delta}}(z, e_j)_{H_\delta} \ \lambda_j^{\delta/2} e_j \ ,$$

we check immediately that conditions (i)–(iv) of Theorem 1 hold with $c_+ = 0 = c(\varepsilon_0)$ and $\varepsilon_0 = 1$, since $\lambda_j \leq 0 \ \forall j \in \mathbb{N}$. Hence the corresponding Dirichlet operator $(-\Delta - {}_{-}(\beta, \nabla\cdot)_+ \restriction \mathcal{F}C_b^\infty)$ is essentially self–adjoint in $L^2(\mathcal{H}_-, \nu)$.

For probabilistic consequences concerning uniqueness of the corresponding martingale problem (resp. the associated infinite–dimensional stochastic differential equation) we refer to [40] or the detailed discussion in [9] (see, in particular, [9, Theorem 3.5]).

Finally, we emphasize that the stronger results on uniqueness in L^p presented in the next section apply in the above case also.

6. Further results and final remarks

So far we have discussed the uniqueness problem in L^2 and in L^1. In both cases we used some a–priori estimates. In the case of L^1-uniqueness it was a relatively simple estimate (Proposition 3), whereas in case of L^2-uniqueness a much harder estimate was used (Proposition 4). The problem is naturally extended to the L^p setting, via Definition 3. A result on L^p-uniqueness can be proved along the same lines as in the proof of Theorem 1. The only difference is that one has to have a uniform estimate of ∇u_n in the corresponding L^{2p}-space, similar to that obtained in Proposition 4. Also the idea of obtaining such an estimate is the same as in Proposition 4, with one exception: one has to change the test function. Namely, the appropriate test function is $\nabla_i u_n(|\nabla u_n|^2 + \varepsilon)^{p-2}$, $\varepsilon > 0$, instead of $\nabla_i u_n$ used in the proof of Proposition 4 after differentiating the equation. We do not present this estimate and its proof here because it contains too many tedious technicalities. We refer to [29] where a similar estimate was obtained in a somewhat simpler situation. Our present result on L^p-uniqueness, a generalization of Theorem 1, reads as follows.

Theorem 3. *Let $\beta \in L^p$, $\beta = \alpha + \delta$, $|\alpha|_0 \in L^{2p}$, $|\delta|_- \in L^2$. Suppose that there exists a sequence of mappings $(\delta^m)_{m\in\mathbb{N}}$, $\delta^m : \mathcal{H}_- \mapsto \mathcal{H}_-, m \in \mathbb{N}$ such that all conditions of Theorem 1 hold. Then $(\Delta + (\beta, \nabla\cdot) \restriction \mathcal{F}C_b^\infty)$ has a unique extension which generates a C_0-semigroup on L^p for all $p \in (1 + \frac{1}{1+\sqrt{\varepsilon_0}}, 1 + \frac{1}{1-\sqrt{\varepsilon_0}})$.*

Remark 3. *The uniqueness result in [29] is now obtained as a particular case of Theorem 3: one has to take $\delta = 0$. In this case $\varepsilon_0 = 1$ and the interval of uniqueness becomes $3/2 < p < \infty$. We do not know at the moment whether the assumption on p is just a technical restriction or it reflects the essence of the problem.*

The analysis of the proof of Theorem 2 shows that it can also be extended to the L^p setting, for $p \in [1, 2)$. The corresponding result (i.e., Theorem 4 below) only complements Theorem 3 for the case $p \leqslant 1 + \frac{1}{1+\sqrt{\varepsilon_0}}$. Otherwise, it is, of course, contained in the latter result.

Theorem 4. *Let $1 \leqslant p < 2$, $\beta = \alpha + \delta$, $|\alpha|_0 \in L^{\frac{2p}{2-p}}$, $|\delta|_- \in L^2$. Suppose that there exists a sequence of mappings $(\delta^m)_{m\in\mathbb{N}}$, $\delta^m : \mathcal{H}_- \mapsto \mathcal{H}_-, m \in \mathbb{N}$ such that*
(i) $(\delta^m)_{m\in\mathbb{N}}$ satisfies conditions (i), (iii) of Theorem 2.
(ii) $|\delta - \delta^m|_- \longrightarrow 0$ in $L^p(\mathcal{H}_-, d\nu)$ as $m \to \infty$,
Then the operator $(\Delta + (\beta, \nabla\cdot)_0 \upharpoonright \mathcal{F}C_b^\infty)$ has a unique extension which generates a C_0-semigroup on L^p.

The proof is an obvious modification of the proof of Theorem 2.

Applications of these results have already been discussed in the previous section.

Finally, we mention that the method used in this paper can be extended to the case of Dirichlet forms with variable non-smooth coefficients. It is also important to investigate the L^p-uniqueness problem for the Dirichlet operator perturbed by a singular potential, especially with a form-bounded negative part (see [28] for abstract results in this direction). We intend to return to these problems in the future.

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