COMBINATORICS OF TOPOLOGICAL POSETS: HOMOTOPY COMPLEMENTATION FORMULAS

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ABSTRACT. We show that the well known *homotopy complementation formula* of Björner and Walker admits several closely related generalizations on different classes of topological posets (lattices). The utility of this technique is demonstrated on some classes of topological posets including the Grassmannian and configuration posets, $\widetilde{\mathcal{G}}_n(R)$ and $\exp_n(X)$ which were introduced and studied by V. Vassiliev. Among other applications we present a reasonably complete description, in terms of more standard spaces, of homology types of configuration posets $\exp_n(S^m)$ which leads to a negative answer to a question of Vassilev raised at the workshop "Geometric Combinatorics" (MSRI, February 1997).

1. INTRODUCTION

One of the objectives of this paper is to initiate the study of topological (continuous) posets and their order complexes from the point of view of Geometric Combinatorics. Recall that finite or more generally locally finite partially ordered sets (posets) already occupy one of privileged positions in this field. The well-known identity (Philip Hall)

$$\mu(P) = \tilde{\chi}(\Delta(P))$$

in spite of its simplicity, often serves as a good initial example illustrating both the combinatorial and the geometric (topological) nature of these objects. As usual, $\mu(P)$ is the Möbius number of the poset P, $\Delta(P)$ is equally well known *order complex* of all chains in P and $\tilde{\chi}(K)$ is the reduced Euler characteristic of the space K.

The order complex construction is of fundamental importance for geometric combinatorics. Recall some of the highlights. As demonstrated by the Goresky-MacPherson formula and its subsequent generalizations, [16, 25, 47], the homology (homotopy) type of interesting spaces associated to an affine arrangement $\mathcal{A}$ in R^n can be described in terms of order complexes of the cones in the associated intersection lattice $L(\mathcal{A})$. The problem of finding combinatorial formulas for Pontryagin classes can be reduced, [14], to the study of combinatorial models of Grassmannians which in turn are defined as the order complexes of posets of classes of oriented matroids. Order complexes, as combinatorial models of spaces associated to loop spaces, appear in [3]. The homotopy colimit construction, viewed as a natural generalization of the order complex construction, has been applied in [45, 47] on diagrams of spaces over posets

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which led to new results about affine, projective and other arrangements and to new insights about toric varieties, deleted joins etc.

Our final example brings us closer to the immediate objectives of this paper. Victor Vassiliev used systematically and skilfully his "geometric resolution" method, [37, 38, 39], together with a variety of other techniques, to obtain far reaching results in several mathematical fields. Some of these applications, especially the theory of Vassiliev knot invariants and invariants of ornaments, reveal that the geometric resolution is in some cases very close to or generalizes the order complex construction. One of surprising Vassiliev's discoveries, [37, 39], is that the order complex construction $\Delta(\mathcal{P})$, properly interpreted, generalized and applied to interesting **topological posets** $\mathcal{P}$ leads to interesting and elegant, geometric observations.

In this paper we focus out attention on topological posets and their order complexes, their homotopy and homology types as combinatorial objects interesting in their own right. We raise a general problem of determining which combinatorial results about (discrete) posets have interesting continuous analogs. The emphasis is of course on results which belong to the field of geometric combinatorics, more precisely the results which do not necessarily have a proper analogue (generalization) if the posets $\mathcal{P}$ is replaced by a more general topological category $\mathcal{C}$.

Our central result in this direction is that the well known *homotopy complementation formula* of Björner and Walker, [4, 6], which deals with the order complex of a finite lattice, admits several closely related generalizations on different classes of topological posets (lattices).

These generalized formulas lead to explicit "computations" of homotopy (homology) types of order complexes of natural topological posets, including posets associated to Grassmannians and configuration spaces $F(X, n)$. As an illustration we show how some recent results of Vassiliev mentioned above can be obtained by this method. Another application is a reasonably complete description, in terms of more standard spaces, of homology types of configuration posets $\exp_n(S^m)$ of spheres, known previously in the case of S^1. As a consequence we obtain a negative answer to a question of Vassiliev, [40], who asked whether the order complex $\Delta(\exp_n(X))$ is homeomorphic to the join $X^{*(n)} = X * \ldots * X$.

The paper is organized as follows. In section 2 we give a brief review of some of the central results of the paper emphasizing the link with the usual "homotopy complementation formula" of Björner and Walker. In section 3 several different classes of topological posets are introduced and analyzed. Most of these classes naturally appeared in the course of the proof of homotopy complementation formulas which occupy section 4. Applications to Grassmannian and configuration posets and elementary proofs are presented in section 5.

2. Main results

Homotopy complementation formula of A. Björner and J.W. Walker ([6], see also theorem 4.1) is a versatile tool for computing homotopy types of order complexes $\Delta(P)$ of finite posets. For example let Π_n be the lattice of all partitions of the set $\{1, \ldots, n\}$, ordered by the refinement relation, and $\widetilde{\Pi}_n$ the poset obtained by deleting the maximum and the minimum elements, $(12\dots n)$ and $(1)(2)\dots(n)$. Assume $n \geq 3$. Then the homotopy complementation formula implies the following homotopy recurrence relation, [6]

$$\Delta(\widetilde{\Pi}_n) \simeq \bigvee_{i=2}^{n} \Sigma(\Delta(\widetilde{\Pi}^i_{n-1})) \tag{1}$$

where $\widetilde{\Pi}^i_{n-1}$ is a lattice isomorphic to $\widetilde{\Pi}_{n-1}$ and Σ is the suspension operator. ¿From here, it is easily deduced by induction that the homotopy type of the lattice Π_n, i.e. the homotopy type of the order complex $\Delta(\widetilde{\Pi}_n)$, is the wedge of $(n-1)!$ copies of the sphere S^{n-3}.

Our first objective is to prove an analogous homotopy complementation formula for topological posets (lattices) which yields the following homotopy recurrence relations. In these examples, $\widetilde{\mathcal{G}}_n(R), \widetilde{\mathcal{G}}^{\pm}_n(R), \exp_n(X)$ are the Grassmannian and configuration topological posets defined in section 3 and $\widetilde{\mathcal{B}}_n := \mathcal{B}_n \setminus \{\emptyset\}$ is the the face poset of an n-simplex, i.e. the (truncated) Boolean lattice on $\{0, 1, \dots, n\}$.

$$\Delta(\widetilde{\mathcal{G}}_n(R)) \simeq S^{n-1} \wedge \Sigma(\Delta(\widetilde{\mathcal{G}}_{n-1}(R))) \tag{2}$$

$$\Delta(\widetilde{\mathcal{G}}^{\pm}_n(R)) \simeq (S^{n-1} \vee S^{n-1}) \wedge \Sigma(\Delta(\widetilde{\mathcal{G}}^{\pm}_{n-1}(R))) \tag{3}$$

$$\Delta(\exp_n(S^1)) \simeq S^n \wedge (\Delta(\widetilde{\mathcal{B}}_{n-1})/\partial\Delta(\widetilde{\mathcal{B}}_{n-1})) \simeq S^{2n-1} \tag{4}$$

These formulas should not be viewed as isolated examples, rather they illustrate a general and simple proof scheme which is potentially applicable in many different situations. Very often, see section 4, these recurrence formulas have the form

$$\Delta(\mathcal{P}_n) \simeq P_n \wedge \Sigma(\Delta(\mathcal{P}_{n-1})) \tag{5}$$

where P_n is a pointed "parameter" space. Specially if P_n is a finite set we recover the usual wedge form as in the example (1). Examples (2) and (4) lead to new proofs of results of Vassiliev, [37] and [39], which was an initial motivation for a general complementation formula for topological posets. The example (4) is a special case of a general formula of the form

$$\Delta(\exp_n(X)) \simeq \mathrm{Thom}_n(X \setminus \{x_0\}) \tag{6}$$

where $\mathrm{Thom}_n(Y)$ is the one-point compactification of a vector bundle over a configuration space $B(Y, n)$ of n-element subsets of Y. Motivated by (4), Vassiliev asked, [40], if an analogous formula

$$\Delta(\exp_n(X)) \simeq X^{*(n)} = X * \dots * X \tag{7}$$

holds for arbitrary topological spaces, specially if the order complex of $\exp_n(S^2)$ is homotopic to S^{3n-1}. We show that the answer to this question is in general negative and that formula 6 implies that the conjecture (6) is false already in the case $X = S^2$. More importantly, we are able to give a sufficiently complete description of homology types of order complexes $\Delta(\exp_n(S^m))$ in terms of homologies of some standard spaces.

3. Topological posets

3.1. **Motivating examples.** A topological poset $(\mathcal{P}, \leq, \tau)$ is a poset $(\mathcal{P}, \leq)$ and a Hausdorff topology τ on the set $\mathcal{P}$ such that the order relation $R_{\leq} := \{(p, q) \in \mathcal{P} \times \mathcal{P} \mid p \leq q\}$ is a closed subspace of $\mathcal{P} \times \mathcal{P}$.

A morphism $f : (\mathcal{P}_1, \leq_1, \tau_1) \to (\mathcal{P}_2, \leq_2, \tau_2)$ in the category $TPos$ of topological posets is a continuous, monotone map $f : \mathcal{P}_1 \to \mathcal{P}_2$.

Before we introduce special classes of topological posets and begin their analysis, let us review some motivating examples.

Example 3.1. *(Grassmannian posets,* [37]*)* Suppose that K is one of the classical (skew) fields R, C or Q. The Grassmannian poset $\mathcal{G}_n(K) = (G(K^n), \subseteq)$, is the disjoint sum

$$G(K^n) := \coprod_{i=0}^{n} G_i(K^n)$$

where $G_i(K^n)$ is the manifold of all i-dimensional linear subspaces of K^n. The order in this poset is by inclusion, $U \leq V$ iff $U \subseteq V$. Every Grassmannian poset $\mathcal{G}_n(K)$ is a lattice with the minimum element $\hat{0} = \{0\}$, maximum element $\hat{1} = K^n$ and the rank function $r : G(K^n) \to N$ defined by $r(V) := \dim(V)$. The poset $\widetilde{\mathcal{G}}_n(K) := \mathcal{G}_n(K) \setminus \{\hat{0}, \hat{1}\}$ is called the truncated Grassmannian poset.

Example 3.2. *(Subspace posets)* For a given space X, let $\exp(X)$ be the topological space of all closed subspaces equipped with the Vietoris topology, [11]. The inclusion relation $\subseteq$ turns this space into a topological poset. This poset is not very interesting itself, at least from the point of view of combinatorics. Its major role is to serve as a source of interesting subposets. For example circle, polygon, disc etc. posets from [1] are examples of subposets of $\exp(R^d)$.

Example 3.3. *(Configuration posets)* Let $\exp_n(X) := \{A \in \exp(X) \mid |A| \leq n\}$ be the subposet of $\exp(X)$ consisting of all finite, nonempty, closed subsets of X of cardinality less or equal to n, $n \in N$. The space $\exp_n(X)$ was under the name n-th symmetric product of X introduced by Borsuk and Ulam, [7]. Note that $\exp_n(X)$ is related to but not the same as the orbit space X^n/S_n where S_n is the symmetric group. The space $\exp_n(X)$ is viewed as a topological subposet of $\exp(X)$ and it has a natural rank function $p : \exp_n(X) \to N$ defined by $p(A) = |A|$. Let $B(X, k) = p^{-1}(k)$ be the space of all k-element subsets in $\exp_n(X)$. If $F(X, k)$ is the usual configuration space of all ordered collections $(x_1, \ldots, x_k) \in X^k$, $x_i \neq x_j$ for $i \neq j$, then $B(X, k)$ is the space of all unordered, k-element configurations, $B(X, k) \cong F(X, k)/S_k$.

Example 3.4. *(Semialgebraic posets)* Let M be a semialgebraic set in R^n and $\leq$ an order relation on M such that $R_{\leq} = \{(x, y) \in M \times M \mid x \leq y\}$ is a semialgebraic subset of $M \times M$. These posets will be generally called *semialgebraic posets* although in this generality they often intersect with other classes of posets. An example of a semialgebraic poset is $R^{n+1} \cong R^n \oplus R$ with the quadratic form $q(x, t) = x_1^2 + \ldots + x_n^2 - t^2$ and the order relation defined by $(x, t) \leq (y, s)$ iff $q(x - y, t - s) \leq 0$. Other examples of semialgebraic posets which have interesting applications in combinatorics can be found in [1].

Example 3.5. *(Diagram posets)* Diagrams of spaces, specially diagrams of spaces over finite posets entered combinatorics in papers [45, 47]. Recall that a diagram $\mathcal{D}$ over a (discrete) poset $(P, \leq)$ is a (contravariant) functor $\mathcal{D} : P^{op} \to Top$ where $(P, \leq)$ is viewed as a small category such that $p \to q$ iff $p \leq q$. In a more informal language a diagram $\mathcal{D}$ consists of a family of spaces $\{D_p\}_{p\in P}$ and a family of "connecting" maps $\{d_{pq} : D_q \to D_p\}_{p\leq q}$ such that $d_{pp} = 1_{D_p}$ and $d_{pq} \circ d_{qr} = d_{pr}$ for $p \leq q \leq r$. Every diagram $\mathcal{D}$ gives rise to a topological poset $(\widetilde{\mathcal{D}}, \preccurlyeq)$ on the space $\widetilde{\mathcal{D}} := \coprod_{p\in P} \mathcal{D}(p)$ where for $x \in \mathcal{D}(p)$ and $y \in \mathcal{D}(q)$

$$x \preccurlyeq y \Longleftrightarrow (p \leq q) \quad \text{and} \quad f_{pq}(y) = x\,.$$

3.2. **The order complex of a topological poset.** The order complex of a discrete poset P is the simplicial complex $\Delta(P)$ of all chains in P. If $(\mathcal{P}, \leq, \tau)$, or simply $\mathcal{P}$, is a topological poset then there exists a space $\Delta(\mathcal{P})$ naturally associated to $\mathcal{P}$ which can serve as an 'ordered complex' of $\mathcal{P}$. If the poset $\mathcal{P}$ is interpreted as a small, topological category $\widetilde{\mathcal{P}}$ where $\mathrm{ob}(\widetilde{\mathcal{P}}) = \mathcal{P}$ and $\mathrm{mor}(\widetilde{\mathcal{P}}) = \{(x, y) \in \mathcal{P}^2 \mid x \leq y\}$, then the space $\Delta(\mathcal{P})$ is naturally homeomorphic to the classifying space $B(\widetilde{\mathcal{P}})$ of $\widetilde{\mathcal{P}}$, [30]. The reader who is not familiar with basics of this theory may find it convenient to review first the definition 3.12 which gives an alternative and more elementary way of defining the order complex for a narrow but important class of topological M-posets.

Definition 3.6. *Given a topological poset $\mathcal{P}$, let $N_* = \{N_n(\mathcal{P})\}_{n=0}^{\infty}$ be the associated simplicial space where $N_n(\mathcal{P})$ is the space of all (not necessarily strictly) increasing, finite chains $x_0 \leq x_1 \leq \ldots \leq x_n$ in $\mathcal{P}$, topologized as a subspace of $\mathcal{P}^{n+1}$. The order complex $\Delta(\mathcal{P})$ of $\mathcal{P}$ is by definition the geometric realization,* [30, 21, 22, 31]*, of the simplicial space $N_*(\mathcal{P})$. More explicitly, the space $\Delta(\mathcal{P})$ is described as the union (colimit) of the inductively defined sequence of spaces $F_p = F_p(\Delta(\mathcal{P}))$, where $F_0 \cong \mathcal{P}$ and F_p is constructed from F_{p-1} and N_* by the following push-out diagram*

$$\begin{array}{ccc} (N_p \times \dot{\Delta}^p) \cup (\delta N_p \times \Delta^p) & \longrightarrow & F_{p-1} \\ \downarrow & & \downarrow \\ N_p \times \Delta^p & \longrightarrow & F_p \end{array}$$

The space δN_p of all degenerated p-simplices is in the case of posets the space of all non-strictly increasing p-chains in $\mathcal{P}$.

Informally speaking, the space $\Delta(\mathcal{P})$ should be the union, as in the case of discrete posets, of all simplices spanned by all finite, *strictly increasing* chains in $\mathcal{P}$. This is indeed the case, since nondegenerated simplices in the simplicial space $N_*(\mathcal{P})$ are in one–to–one correspondence with strictly increasing finite chains in $\mathcal{P}$. In other words $\Delta(\mathcal{P})$ is isomorphic as a set, with the usual order complex of $\mathcal{P}$ seen as a discrete poset. On the other hand, the topology on $\mathcal{P}$ enters the definition of $\Delta(\mathcal{P})$ in an essential way and the following simple example should make the difference perfectly clear.

Example: Let $(R^2, \leq)$ be the topological poset where R^2 has the usual topology and $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. Let $A = \{(x, y) \mid x + y = 1,\ x, y \geq 0\}$ and let $\mathcal{P} = \{(0, 0)\} \cup A$ be a topological subposet of R^2. Then the topological order complex $\Delta(\mathcal{P})$ is homeomorphic to the triangle with vertices $(0,0),(1,0),(0,1)$ with the topology induced from R^2 while the discrete order complex is the cone with vertex $(0,0)$ over the (discrete) set A.

The following elementary proposition, connecting the order complex of a *diagram poset* $\widetilde{\mathcal{D}}$ (example 3.5) with the *homotopy colimit* ([18, 47, 45]) of the diagram $\mathcal{D}$, serves as an initial example for the naturality of the order complex construction.

Proposition 3.7. *If $\mathcal{D} : Q \to Top$ is a diagram of spaces over a finite poset Q and $\widetilde{\mathcal{D}}$ the associated topological poset (example 3.5), then*

$$\Delta(\widetilde{\mathcal{D}}) \cong \mathbf{hocolim}_Q\, \mathcal{D}.$$

3.3. **Special classes of topological posets.** In this section we introduce some special classes of topological posets. Our goal is to define classes which are broad enough to include interesting posets and sufficiently narrow to allow special technical arguments. The first two classes, the classes of *A-posets* and *B-posets*, are characterized as topological posets satisfying some cofibration conditions. The other two classes of *C-posets* and *M-posets* are much more special but have many useful properties. Their detailed analysis is given in section 3.4.

Definition 3.8. *(A-posets) A topological poset $\mathcal{P}$ is an A-poset if the associated simplicial space $N_*(\mathcal{P})$ has the property that $(N_n(\mathcal{P}), \delta N_n(\mathcal{P}))$ is a cofibration pair for each $n \in N$ where $\delta N_n(\mathcal{P})$ is the space of degenerated n-simplices. A simplicial space with this or other closely related property is often in the literature referred to as* cofibrant*,* good *or* proper*,* [30, 21, 22, 31].

Definition 3.9. *(B-posets) A topological poset $\mathcal{P}$ is a B-poset or an* intervally cofibrant *poset if for each two closed intervals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{P}$, if $\mathcal{I}_1 \subset \mathcal{I}_2$ then the inclusion $\Delta(\mathcal{I}_1) \hookrightarrow \Delta(\mathcal{I}_2)$ is a cofibration. A closed interval in $\mathcal{P}$ is by convention any set of the form $[a,b]_{\mathcal{P}}, \mathcal{P}_{\geq a}, \mathcal{P}_{\leq b}$ including the poset $\mathcal{P}$ itself.*

Remark 3.10. The inclusion of CW-complexes is always a cofibration. It is well known that a finite family $\mathcal{F}$ of (semi)algebraic sets in R^n admits a compatible triangulation, [17, 20]. These two facts together are often sufficient, especially in the context of M-posets defined bellow, to conclude that a given topological poset is an A or B-poset. Alternatively for the same purpose one can use well known general properties of cofibrations, [19, 34, 21, 9].

The class of M-posets introduced in the following definition includes Grassmannian and Diagram posets but the class of configuration posets is out of its scope.

Definition 3.11. *Let $\mathcal{P}$ be a topological poset. A pair (Q, μ) where Q is a finite poset and $\mu : \mathcal{P} \to Q$ is a monotone map, is called a* mirror *of $\mathcal{P}$ if*

1. *for all $p, q \in \mathcal{P}$, if $p < q$ then $\mu(p) < \mu(q)$,*
2. *for all $q \in Q$, $\mu^{-1}(q)$ is a nonempty closed subspace of $\mathcal{P}$.*

In this case $\mathcal{P}$ is called an M-poset over Q and $\mu : \mathcal{P} \to Q$ is an associated mirror map or shortly an M-map from $\mathcal{P}$ to Q.

By definition 3.11, every strictly increasing chain $x_1 < \ldots < x_k$ in $\mathcal{P}$ is by an M-map sent to a strictly increasing chain in Q of the same length. This useful property of M-posets is crucial in the following definition and the subsequent proposition 3.13.

Definition 3.12. *Assume that $\mathcal{P}$ is an M-poset with $\mu : \mathcal{P} \to Q$ as an associated mirror map. Let $\mathcal{P}^{*(Q)} := \mathcal{J}_{i\in Q}\, \mathcal{P}_i$ be the join of the family $\{\mathcal{P}_i\}_{i\in Q}$ of spaces where $\mathcal{P}_i \cong \mathcal{P}$ for all i. Each element $a \in \mathcal{P}^{*(Q)}$ has the form $x = \Sigma_{i\in Q}\, \lambda_i a_i$ where $a_i \in \mathcal{P}_i$, $\lambda_i \geq 0$ and $\Sigma_{i\in Q}\lambda_i = 1$. Let $\mathrm{supp}(a) = \{i \in Q \mid \lambda_i > 0\}$. Define the μ-order complex $\Delta_\mu(\mathcal{P})$ of $\mathcal{P}$ as the subspace of $\mathcal{P}^{*(Q)}$ where*

$$a \in \Delta_\mu(\mathcal{P}) \Longleftrightarrow \mathrm{supp}(a) \ \text{ is a chain and if } i < j \text{ in } \mathrm{supp}(a) \text{ then } a_i < a_j \text{ in } \mathcal{P}$$

Proposition 3.13. *Suppose $\mathcal{P}$ is an M-poset and $\mu : \mathcal{P} \to Q$ an associated M-map. Then the spaces $\Delta_\mu(\mathcal{P})$ and $\Delta(\mathcal{P})$ are homeomorphic.*

Proof: The proof is based on the fact that for each M-poset $\mathcal{P}$, the space $N_n(\mathcal{P}) \setminus \delta N_n(\mathcal{P})$ of all nondegenerated simplices in $\mathcal{P}$ is a closed subspace of $N_n(\mathcal{P})$ for each $n \in N$. □

As already noted above, there are two situations when the definition of $\Delta_\mu(\mathcal{P})$ seems to be specially convenient.

1. $\mathcal{P}$ is a topological poset with a finite rank function $\rho : \mathcal{P} \to [n]$, $[n] = \{0, 1, \ldots, n\}$ as a mirror function. An example of such a poset is the Grassmannian poset $\mathcal{G}_n(K) = (G(K^n), \subseteq)$.
2. $\mathcal{D} : Q \to Top$ is a diagram of spaces. If $(\widetilde{\mathcal{D}}, \preccurlyeq)$ is the associated topological poset then the obvious monotone map $\mu : \widetilde{\mathcal{D}} \to Q$ is a mirror map.

The first condition in the definition of a mirror poset, definition 3.11, looks quite restrictive. If this condition is deleted, we obtain a much broader class of topological posets which still preserve some of the favorable properties of M-posets.

Definition 3.14. *A topological poset $(\mathcal{P}, \leq)$ is called a C-poset, or more precisely a C-poset over a finite poset Q, if there exists a monotone map $\alpha : \mathcal{P} \to Q$, called a C-map, such that $\alpha^{-1}(q)$ is a nonempty closed subspace of $\mathcal{P}$ for each $q \in Q$.*

3.4. **Mirrors and diagrams.** The main result of this section, proposition 3.17, is that there exists a functor from the category of M-posets over a fixed finite set Q to the category of diagrams over another poset, closely related to Q. This is a useful observation because the replacement of a topological poset (topological category) with a *non–discrete* set of objects, by a diagram over a *discrete* poset, allows us to use a variety of results which are not available for general topological categories, [18, 47, 45].

Recall (example 3.5) that every diagram of spaces $\mathcal{D} : Q \to Top$ over a finite poset Q yields a topological poset $\widetilde{\mathcal{D}}$. We ask when the converse is true.

Question: Given a topological poset $\mathcal{P}$ and a mirror map $\mu : \mathcal{P} \to Q$, how far is $\mathcal{P}$ from being a topological poset $\widetilde{\mathcal{D}}$ associated to a diagram $\mathcal{D} : Q \to Top$ over Q. More generally, when is it possible to define a diagram $\mathcal{E} : R \to Top$ over a finite poset

R, constructed naturally from Q, such that the order complexes $\Delta(\mathcal{P})$ and $\Delta(\widetilde{\mathcal{E}})$ are naturally homeomorphic.

Here are two examples that help us predict the answer.

Example: Let $X = X_0 * \ldots * X_n$ be the join of a sequence $X_0, \ldots, X_n$ of spaces. Then $X \cong \Delta(\mathcal{P})$ where $\mathcal{P} := \coprod_{i=0}^{n} X_i$ and for $x \in X_i$ and $y \in X_j$, $x \lneqq y$ if and only if $i < j$. The obvious rank function $r : \mathcal{P} \to [n]$, $[n] := \{0, 1, \ldots, n\}$, is a mirror map. Let $R := \Delta([n])$ be the order complex of $[n]$. Then $R \cong \mathcal{B}([n]) \setminus \{\emptyset\}$ is the face poset of an n-simplex. Let $\mathcal{E} : R^{op} \to Top$ be the diagram defined by $\mathcal{E}(A) := \prod_{i \in A} X_i$, $A \in \mathcal{B}([n]) \setminus \{\emptyset\}$, with obvious projections as the diagram maps. Then

$$\Delta(\mathcal{P}) = \mathbf{hocolim}_Q \mathcal{E} = \Delta(\widetilde{\mathcal{E}}).$$

Example 3.15. Let $\mathcal{P} = \widetilde{\mathcal{G}}_n(R)$ be the truncated Grassmannian poset defined in the example 3.1. Then the rank function $r : \widetilde{\mathcal{G}}_n(R) \to \langle n \rangle$ can be seen as a mirror map from $\mathcal{P}$ to the poset $\langle n \rangle := \{1, \ldots, n\}$. Let $R = \Delta(\langle n \rangle)$ be the order complex of $\langle n \rangle$ and assume that R is ordered by inclusion. Let $\mathcal{E} : R^{op} \to Top$ be the diagram over R defined by $\mathcal{E}(I) := G_I(R^n)$ where $I = \{i_1 < i_2 < \ldots < i_m\}$ is a nonempty subset (sequence) of $\langle n \rangle$ and $G_I(R^n) = G_{i_1, \ldots, i_m}(R^n)$ is the flag manifold of all linear flags $F_1 \subset \ldots \subset F_m$ in R^n where $\dim(F_k) = i_k$ for all $k = 1, \ldots, m$. The diagram map $e(J, I) : G_I(R^n) \to G_J(R^n)$, for each pair of subsets $J \subset I \subset \langle n \rangle$, is the obvious projection (restriction) map. Then,

$$\Delta(\widetilde{\mathcal{G}}_n(R)) \cong \mathbf{hocolim}_R \mathcal{E} \cong \Delta(\widetilde{\mathcal{E}}).$$

Definition 3.16. *Suppose that $\mathcal{P}$ is an M-poset and $\mu : \mathcal{P} \to Q$ an associated mirror map. Let $R = \Delta(Q)$ be the order complex of Q, seen as a poset of faces of $\Delta(Q)$ ordered by inclusion. There exists a naturally defined diagram of spaces $\mathcal{E} : R^{op} \to Top$ over R defined as follows. Let $I = \{q_1 < q_2 < \ldots < q_m\}$ be a chain in Q representing an element of R. Define $\mathcal{C}(I) := \prod_{k=1}^{m} \mu^{-1}(q_k)$ and let $\mathcal{E}(I)$ be the, possibly empty, subspace of $\mathcal{C}(I)$ defined by*

$$x = (x_1, \ldots, x_m) \in \mathcal{E}(I) \quad \Longleftrightarrow \quad x_1 < x_2 < \ldots < x_m \quad \text{in} \quad \mathcal{P}.$$

For a subchain J of I let $e(J, I) : \mathcal{E}(I) \to \mathcal{E}(J)$ be the projection (restriction) map which restricts $x = (x_1, \ldots, x_m)$ to the subchain J.

Proposition 3.17. *Let $\mathcal{P}$ be an M-poset with a mirror map $\mu : \mathcal{P} \to Q$. Let $R = \Delta(Q)$ be the ordered complex of Q, viewed as the face poset ordered by the inclusion and let $\mathcal{E} : R^{op} \to Top$ be the diagram from definition 3.16. Then,*

$$\Delta(\mathcal{P}) \cong \mathbf{hocolim}_R \mathcal{E} \cong \Delta(\widetilde{\mathcal{E}}).$$

Proof: From the assumption that $\mu^{-1}(q)$ is closed for all $q \in Q$, we deduce that, in the spirit of proposition 3.13, the space $\Delta(\mathcal{P})$ can be identified as a subspace of the join $\mathcal{J}_{q \in Q}\, \mu^{-1}(q)$. The rest of the proof follows by inspection. $\square$

As an application of proposition 3.17, we show how one can get the information about the fundamental group $\pi_1(\Delta(\mathcal{P}))$ of the order complex of an M-poset $\mathcal{P}$ from the associated diagram $\mathcal{E}$.

Proposition 3.18. *Let $\mathcal{P}$ be an M-poset with a mirror map $\mu : \mathcal{P} \to Q$. Let $R := \Delta(Q)$ be the poset and $\mathcal{E} : R^{op} \to Top$ the associated diagram of spaces from definition 3.16 and proposition 3.17. Then*

$$\pi_1(\Delta(\mathcal{P})) \cong \mathbf{colim}_R\ \pi_1(\mathcal{E})$$

where $\pi_1(\mathcal{E}) : R^{op} \to Group$ is the diagram of groups, $\pi_1(\mathcal{E})(I) := \pi_1(\mathcal{E}(I))$, $I \in R$.

Proof: The result follows from proposition 3.17 and the Seifert-van Kampen theorem which for diagrams over posets, cf. Brown [10] p. 206, reads as follows

$$\pi_1(\mathbf{hocolim}_R\, \mathcal{E}) \cong \mathbf{colim}_R\, \pi_1(\mathcal{E}).$$

We continue this section with a discussion of C-posets, their comparison with M-posets and some useful results which will be used in section 4.

Proposition 3.19. *Suppose that $(\mathcal{P}, \leq)$ is a C-poset over Q and let $\alpha : \mathcal{P} \to Q$ be an associated C-map. Suppose that the order complex $\Delta(D_q)$ is compact for each subposet $D_q := \alpha^{-1}(q) \subset \mathcal{P}$, $q \in Q$. Then there exists an M-poset $(\mathcal{P}_{\mathrm{M}}, \leq_{\mathrm{M}})$, where $\mathcal{P}_{\mathrm{M}} := \coprod_{q\in Q} \Delta(D_q)$, and an M-map $\mu : \mathcal{P}_{\mathrm{M}} \to Q$ such that*

$$\Delta(\mathcal{P}_{\mathrm{M}}) \cong \Delta(\mathcal{P}).$$

Proof: The assumption about $(\mathcal{P}, \leq)$ being a C-poset over Q can be restated as the fact that there exists a decomposition $\mathcal{P} = \coprod_{q\in Q} D_q$ of $\mathcal{P}$ into closed, convex subposets D_q such that for $x \in D_q, y \in D_{q'}$, if $x \leq y$ then $q \leq q'$. The compactness assumption on $\Delta(D_q)$, $q \in Q$, implies that D_q are compact subspaces of the compact space $\mathcal{P}$. By the *saturation* $(\mathcal{E}, \leq_e)$ of $(\mathcal{P}, \leq)$, we mean a new order relation $\leq_e$ on the space $\mathcal{E} := \mathcal{P} = \coprod_{q\in Q} D_q$ where, for given $a \in D_q$ and $b \in D_{q'}$,

$$a \leq_e b \iff (q = q' \text{ and } a \leq b) \text{ or } q < q'.$$

Obviously, $\Delta(\mathcal{E}) \cong \mathcal{J}_{q\in Q}\, \Delta(D_q)$ and $\Delta(\mathcal{P})$ can be identified as a closed (compact) subspace of $\Delta(\mathcal{E})$. Define $(\mathcal{E}_{\mathrm{E}}, \leq_{\mathrm{E}})$ as the topological poset with $\mathcal{E}_{\mathrm{E}} := \coprod_{q\in Q} \Delta(D_q)$ and for $x \in \Delta(D_q), y \in \Delta(D_{q'})$, $x \leq_{\mathrm{E}} y$ iff $x = y$ or $q < q'$. Obviously

$$\Delta(\mathcal{E}_{\mathrm{E}}) \cong \Delta(\mathcal{E}) \cong \mathcal{J}_{q\in Q}\, \Delta(D_q).$$

Let $(\mathcal{P}_{\mathrm{M}}, \leq_{\mathrm{M}})$ be a subposet of $(\mathcal{E}_{\mathrm{E}}, \leq_{\mathrm{E}})$ where $\mathcal{P}_{\mathrm{M}} := \mathcal{E}_{\mathrm{E}} = \coprod_{q\in Q} \Delta(D_q)$ and the order relation is defined as follows. Recall that for $x \in D_q$, $\mathrm{supp}(x)$ is the unique minimal chain C in D_q such that $x \in \Delta(C) \subset \Delta(D_q)$. Then for $x \in D_q$ and $y \in D_{q'}$,

$$x \leq_{\mathrm{M}} y \iff (q = q' \wedge x = y) \bigvee (q < q' \wedge \text{ supp}(x) \cup \text{supp}(y) \text{ is a chain in } \mathcal{P}).$$

Both $(\mathcal{E}_{\mathrm{E}}, \leq_{\mathrm{E}})$ and $(\mathcal{P}_{\mathrm{M}}, \leq_{\mathrm{M}})$ are M-posets over Q where for example the M-map $\mu : \mathcal{P}_{\mathrm{M}} \to Q$ is determined by $\mu(\Delta(D_q)) = \{q\}$. The desired isomorphism

$$\Delta(\mathcal{P}_{\mathrm{M}}) \cong \Delta(\mathcal{P})$$

is now transparent since both spaces are identified with the same subspace of

$$\Delta(\mathcal{E}_{\mathrm{E}}) \cong \Delta(\mathcal{E}) \cong \mathcal{J}_{q\in Q}\, \Delta(D_q).$$

An important class of C-posets arises by the so called "Grothendieck construction", [35, 18].

Definition 3.20. *Let $\mathcal{D} : Q^{op} \to TPos$ be a diagram of topological posets over a finite poset Q. In other words $\mathcal{D}$ consists of a collection $\{(D_q, \leq_q)\}_{q\in Q}$ of topological posets indexed by $q \in Q$ and a collection of continuous, monotone maps $\{d_{qq'}\}_{q\leq q'}$, $d_{qq'} : D_{q'} \to D_q$, such that $d_{qq} = \mathrm{id}_{D_q}$ and $d_{qq'} \circ d_{q'q''} = d_{qq''}$ for $q \leq q' \leq q''$. The Grothendieck construction applied on $\mathcal{D}$ yields a new topological poset $(\mathcal{C}, \leq_c)$ where $\mathcal{C} := \coprod_{q\in Q} D_q \times \{q\}$ and $(x, q) \leq_c (y, q')$ iff $x \leq_q d_{qq'}(y)$.*

The following proposition is well known in the case of discrete categories (posets), [35].

Proposition 3.21. *Suppose $\mathcal{D} : Q^{op} \to TPos$ is a diagram of compact topological posets over a finite poset Q. Let $(\mathcal{C}, \leq_c)$ be the topological poset obtained from $\mathcal{D}$ by the Grothendieck construction, definition 3.20. Let $X_{\mathcal{D}} : Q^{op} \to Top$ be the diagram of spaces defined by*

$$X_{\mathcal{D}}(q) = X_q := \Delta(\mathcal{D}(q)) \ \text{ and } \ x_{qq'} = \Delta(d_{qq'}) : X_{q'} \to X_q.$$

Then there is a homotopy equivalence

$$\mathbf{hocolim}_Q\, X_{\mathcal{D}} \simeq \Delta(\mathcal{C}).$$

Proof: There is another diagram of spaces $Y : Q^{op} \to Top$ over Q defined by $Y_q := \Delta(\mathcal{C}_{\geq q})$ where $\mathcal{C}_{\geq q} := \coprod_{r\geq q} D_r \subset \mathcal{C}$. The inclusion $Y_{q'} \hookrightarrow Y_q,\ q \leq q'$, is a cofibration so by the projection lemma in [45], $\Delta(\mathcal{C}) \cong \mathbf{colim}_Q\, Y$. There is a map of diagrams $\alpha : X_{\mathcal{D}} \to Y$ where $\alpha_q : X_q \to Y_q$ is the inclusion $X_q = \Delta(D_q) \hookrightarrow \Delta(\mathcal{C}_{\geq q}) = Y_q$. The map α_q is a homotopy equivalence which is deduced from proposition 2.1 in [30], see also section 3.3 of [45]. By the homotopy lemma ([45]) which is better known as the May-Tornehave-Segal theorem, [31, 22, 18], the map $\bar{\alpha} : \mathbf{hocolim}_Q\, X_{\mathcal{D}} \to \mathbf{hocolim}_Q\, Y$ is also a homotopy equivalence. Finally,

$$\Delta(\mathcal{C}) \cong \mathbf{colim}_Q\, Y \simeq \mathbf{hocolim}_Q\, X_{\mathcal{D}}.$$

The proposition 3.24 is needed in the proof of theorem 4.6. The proof of this proposition is based based on proposition 3.22 which, together with its corollary 3.23 may have some independent interest.

Proposition 3.22. *Suppose that $\mathcal{E} : R^{op} \to Top$ is a diagram of spaces over a finite poset R. Let $\Sigma^n(X)$ be the n-fold iterated suspension of the space X and $\Sigma^n(\mathcal{E})$ the diagram over R defined by $\Sigma^n(\mathcal{E})(r) := \Sigma^n(\mathcal{E}(r))$ for each $r \in R$. Let $\mathcal{D} : \widetilde{\mathcal{B}}_n^{op} \to Top$ be a diagram over the truncated Boolean lattice $\widetilde{\mathcal{B}}_n := \mathcal{B}_n \setminus \{\emptyset\}$ on $[n] = \{0, 1, \dots, n\}$ defined by*

$$\mathcal{D}(I) := \begin{cases} \Delta(R), & \text{for } I \neq [n] \\ \mathbf{hocolim}_R\, \mathcal{E}, & \text{for } I = [n] \end{cases}$$

where $d_{I,J} : \mathcal{D}(J) \to \mathcal{D}(I)$ is defined by $d_{I,J} = id_{\Delta(R)}$ for $J \neq [n]$ and $d_{I,[n]}$ is the canonical projection map. Then,

$$\mathbf{hocolim}_R\, \Sigma^n(\mathcal{E}) \cong \mathbf{hocolim}_{\widetilde{\mathcal{B}}_n}\, \mathcal{D}.$$

Proof: Let $\mathcal{F} : R \times \widetilde{\mathcal{B}}_n \to Top$ be the diagram over the product poset defined by

$$\mathcal{F}(r, I) := \begin{cases} \mathcal{E}(r), & \text{if } I = [n] \\ *, & \text{otherwise} \end{cases}$$

The Segal's homotopy push down construction, [31, 45, 18], applied on $\mathcal{F}$ and two projections $\pi_1 : R \times \widetilde{\mathcal{B}}_n \to R$ and $\pi_2 : R \times \widetilde{\mathcal{B}}_n \to \widetilde{\mathcal{B}}_n$, yields two diagrams $\mathcal{F}_1 := (\pi_1)_*(\mathcal{F})$ and $\mathcal{F}_2 := (\pi_2)_*(\mathcal{F})$, over R and $\widetilde{\mathcal{B}}_n$. In light of the homotopy pushdown theorem, [18, 45], it is sufficient to show that these two diagrams are locally homotopy equivalent to diagrams $\Sigma^n(\mathcal{E})$ and $\mathcal{D}$ respectively. More precisely, we will show that there exist maps of diagrams $\alpha : \mathcal{F}_1 \to \Sigma^n(\mathcal{E})$ and $\beta : \mathcal{F}_2 \to \mathcal{D}$ such that the maps $\alpha(r) : \mathcal{F}_1(r) \to \Sigma^n(\mathcal{E})(r)$ and $\beta(I) : \mathcal{F}_2(I) \to \mathcal{D}(I)$ are homotopy equivalences. By definition

$$\mathcal{F}_1(r) = \textbf{hocolim}\ \mathcal{F}|\pi^{-1}(R_{\geq r})$$

On the other hand the restriction diagram $\mathcal{F}|(\{r\} \times \widetilde{\mathcal{B}}_n)$ is a "retract" of $\mathcal{F}|\pi^{-1}(R_{\geq r})$. The general principle, [31, 18] and section 3.3 of [45], about homotopies arising from natural transformations, permits us to conclude that

$$\Sigma^n \mathcal{E}(r) \cong \textbf{hocolim}\ \mathcal{F}|(\{r\} \times \widetilde{\mathcal{B}}_n) \simeq \mathcal{F}_1(r) = (\pi_1)_*(\mathcal{F})(r)$$

Similarly,

$$\mathcal{F}_2([n]) = (\pi_2)_*(\mathcal{F})([n]) = \textbf{hocolim}\ \mathcal{F}|\pi_2^{-1}([n]) \cong \textbf{hocolim}_R\, \mathcal{E}$$

and for $I \neq [n]$, $I \in \widetilde{\mathcal{B}}_n$,

$$\mathcal{F}_2(I) = (\pi_2)_*(\mathcal{F})(I) \simeq \textbf{hocolim}\ \mathcal{F}|(R \times \{I\}) \cong \Delta(R)$$

which completes the proof. □

The following corollary, which is needed in the proof of proposition 3.24, follows from the proposition 3.22 by a single application of the *homotopy* lemma, [45].

Corollary 3.23. *If the order complex $\Delta(R)$ is contractible than the operations Σ^n and* **hocolim** *commute up to homotopy,*

$$\textbf{hocolim}_R\, \Sigma^n \mathcal{E} \simeq \Sigma^n(\textbf{hocolim}_R\, \mathcal{E}).$$

Proposition 3.24. *Suppose $\mathcal{D} : Q^{op} \to TPos$ is a diagram of compact topological posets over the linearly ordered set $Q = \langle n \rangle := \{1, \dots, n\}$. Suppose $D_i := \mathcal{D}(i)$ is stably or Σ-contractible for each $i = 1, \dots, n$ which means that the iterated suspension $\Sigma^k(\Delta(D_i))$ is contractible for some $k \geq 0$. Let $(\mathcal{C}, \leq_c)$ be the poset from the definition 3.20 and $\alpha : \mathcal{C} \to Q$ the monotone map which turns $\mathcal{C}$ into a C-poset over Q. Then each fiber of the map $\bar{\alpha} = \Delta(\alpha) : \Delta(\mathcal{C}) \to \Delta(\langle n \rangle)$ is Σ-contractible, specially cohomologically trivial.*

Proof: The lemma is proved by induction on $n \in N$. Note that $\Delta(\mathcal{C})$ is Σ-contractible by propositions 3.21 and 3.22. Assume, as the induction hypothesis, that the fiber $\bar{\alpha}^{-1}(x)$ is Σ-contractible for each element $x \in \partial\Delta(\langle n \rangle)$, where $\partial\Delta(\langle n \rangle)$ is the boundary of the simplex $\Delta(\langle n \rangle) \cong \Delta^{n-1}$. Suppose that x belongs to the interior of $\Delta(\langle n \rangle)$.

Let $\mathcal{C}_M$ be the M-poset associated to the C-poset $\mathcal{C}$ constructed in the proposition 3.19. The poset $\Delta(\langle n\rangle)$, ordered by the reversed inclusion, is in different notation the truncated Boolean lattice $\widetilde{\mathcal{B}}_{n-1} = \mathcal{B}_{n-1} \setminus \{\emptyset\}$ i.e. the face poset of an $(n-1)$-dimensional simplex. Then, cf. definition 3.16 and proposition 3.17, there exists a diagram of spaces $\mathcal{E} : \widetilde{\mathcal{B}}_n \to Top$ associated to the M-poset $\mathcal{C}$ such that $\mathbf{hocolim}\,\mathcal{E} \cong \Delta(\mathcal{C})$ and such that $\mathcal{E}(I)$, $I = \{i_1 < \ldots < i_k\}$, can be identified as the fiber F_I of the map $\bar{\alpha} = \Delta(\alpha)$ over the barycenter of the face I. By applying the Σ^m operation on the diagram $\mathcal{E}$, where m is big enough, and using the corollary 3.23, we obtain

$$* \simeq \mathbf{hocolim}_{\widetilde{\mathcal{B}}_n}\, \Sigma^m(\mathcal{E}) \simeq S^{n-1} * \Sigma^m(F_{\langle n\rangle}) \cong \Sigma^{m+n}(F_{\langle n\rangle})$$

which means that the fiber $F_{\langle n\rangle}$ is indeed Σ-contractible. □

4. Homotopy complementation formulas

Theorem 4.1. (Homotopy Complementation Theorem, Björner and Walker 1983) *Let L be a bounded lattice and $z \in \widetilde{L} := L - \{\hat{0}, \hat{1}\}$. Let $\mathrm{Co}(z) := \{x \in \widetilde{L} \mid x \wedge z = \hat{0}, x \vee z = \hat{1}\}$. Then*

1. *The poset $\widetilde{L} \setminus \mathrm{Co}(z)$ is contractible i.e. the order complex $\Delta(\widetilde{L} \setminus \mathrm{Co}(z))$ is contractible.*
2. *If $\mathrm{Co}(z)$ is an antichain, then*

$$\Delta(\widetilde{L}) \simeq \bigvee_{y \in \mathrm{Co}(z)} \Sigma(\Delta(\widetilde{L}_{<y}) * \Delta(\widetilde{L}_{>y})). \tag{8}$$

The proof of this theorem consists of two parts. In the first part it is shown that the poset $\widetilde{L}_1 := \widetilde{L} \setminus \mathrm{Co}(z)$ is contractible which implies that $\Delta(\widetilde{L}) \simeq \Delta(\widetilde{L})/\Delta(\widetilde{L}_1)$. In the second part it is shown that the homotopy type of any space of the form $\Delta(P)/\Delta(P \setminus C)$ where P is a finite poset and C an antichain, specially in the case $P = \widetilde{L}, C = \mathrm{Co}(z)$, has the desired wedge decomposition.

Our objective is to extend this theorem to the case of topological posets. This goal is achieved in three steps, each step consisting of an appropriate answer, or several answers to one of the following questions.

Q1.: Let $\mathcal{P}$ be topological poset and $C \subset \mathcal{P}$ its subset, specially an antichain. When is $\mathcal{P} \setminus C$ contractible?

Q2.: If $\mathcal{R} := \mathcal{P} \setminus C$ is contractible, when can we conclude that $\Delta(\mathcal{P})$ and the quotient space $\Delta(\mathcal{P})/\Delta(\mathcal{R})$ have the same homotopy type?

Q3.: Is there a decomposition formula, analogous to (8), for the homotopy type of the space $\Delta(\mathcal{P})/\Delta(\mathcal{R})$?

4.1. **The first question.**

Proposition 4.2. *Suppose that $(\mathcal{R}, \leq)$ is a topological poset and let $z \in \mathcal{R}$ such that the least upper bound $x \vee z$ exists for each $x \in \mathcal{R}$. Assume that the map $\phi_z : \mathcal{R} \to \mathcal{R}$, $\phi_z(x) := x \vee z$ is continuous. Then the order complex $\Delta(\mathcal{R})$ of $\mathcal{R}$ is contractible.*

Proof: By proposition 2.1 from [30], if $f, g : \mathcal{R} \to \mathcal{R}$ a two morphisms in $TPos$ such that $f(x) \leq g(x)$ for each $x \in \mathcal{R}$, then the induced maps $\Delta(f), \Delta(g) : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$ are homotopic. It follows that all three maps id, ϕ_z and $c : \mathcal{R} \to \mathcal{R}$, $c(x) := z$, induce homotopic maps of order complexes, hence $\Delta(\mathcal{R})$ must be contractible. □

Remark 4.3. The homotopy $H : \Delta(\mathcal{R}) \times I \to \Delta(\mathcal{R})$ of two maps $\Delta(f), \Delta(g) : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$, where $f(x) \leq g(x)$ for each $x \in \mathcal{R}$, is defined as follows. Define $F : \mathcal{R} \times \{0 < 1\} \to \mathcal{R}$ by $F(x, 0) := f(x)$ and $F(x, 1) := g(x)$. Then,

$$\Delta(\mathcal{R} \times \{0 < 1\}) \cong \Delta(\mathcal{R}) \times \Delta(\{0 < 1\}) \cong \Delta(\mathcal{R}) \times I$$

and $H := \Delta(F) : \Delta(\mathcal{R}) \times I \to \Delta(\mathcal{R})$. More explicit description of H, at least in the case of M-posets (definition 3.11) is following. Let $C = \{x_0 < \ldots < x_p\}$ be a chain in $\mathcal{R}$ and let $x \in \Delta(C) \subset \Delta(\mathcal{R})$ which, definition 3.12, has the form $x = \Sigma_i\ \lambda_i x_i$. Then H can be described as a "linear" homotopy between $\Delta(f)$ and $\Delta(g)$ where

$$H(x, t) = (1 - t)\Delta(f)(x) + t\Delta(g)(x) = (1 - t)\Sigma_i\ \lambda_i f(x_i) + t\Sigma_i\ \lambda_i g(x_i).$$

The continuity of the map $\phi_z : \mathcal{R} \to \mathcal{R}$ was used in essential way in the proof of proposition 4.2. In some cases it is still possible to reach the desired conclusion assuming that ϕ_z is only partially continuous. We illustrate the relevant ideas in the special case of the Grassmannian poset $\mathcal{G}_n(R)$.

Proposition 4.4. *Let $\widetilde{\mathcal{G}}_n(R)$ be the truncated Grassmannian poset, example 3.1. Given $z \in G_1(R^n)$, let $\mathrm{Co}(z) := \{l \in G_{n-1}(R^n) \mid z \not\subseteq l\}$ be the space of all complements of z in $\mathcal{G}_n(R)$. Define $\mathcal{R} := \widetilde{\mathcal{G}}_n(R) \setminus \mathrm{Co}(z)$. Then the order complex $\Delta(\mathcal{R})$ is contractible.*

Proof: The map $\phi_z : \mathcal{R} \to \mathcal{R}$, $\phi_z(x) := x \vee z$, is no longer continuous so it has to be modified. Let $\mathcal{Z} := \{l \in \mathcal{R} \mid z \subseteq l\}$. Let us observe first that the inclusion $\Delta(\mathcal{Z}) \hookrightarrow \Delta(\mathcal{R})$ is a cofibration, cf. remark 3.10. We deduce ([9]) that there exists a continuous function $\alpha : \Delta(\mathcal{R}) \to I$ such that $\alpha^{-1}(0) = \Delta(\mathcal{Z})$ and $\mathcal{U} := \alpha^{-1}([0, 1))$ deforms to $\Delta(\mathcal{Z})$ through $\Delta(\mathcal{R})$, with $\Delta(\mathcal{Z})$ fixed. The function ϕ_z is modified with the aid of the function α. Using the notation of the remark 4.3, let $\psi : \Delta(\mathcal{R}) \to \Delta(\mathcal{R})$ be defined as follows. For $x_i \in C$, let $\psi(x_i) := \alpha(x_i)(x_i \vee z) + [1 - \alpha(x_i)]x_i$. For a general $x \in \Delta(C)$, ψ is defined by linear extension,

$$\psi(x) = \psi(\Sigma_i \lambda_i x_i) = \Sigma_i \lambda_i \psi(x_i).$$

The continuity of this map follows from the fact that ϕ_z is continuous away from $\mathcal{Z}$ and the modification takes care of points close or inside $\mathcal{Z}$. Note that ψ does not arise as a map of the form $\Delta(h)$ for some $TPos$-morphism $h : \mathcal{R} \to \mathcal{R}$. On the other hand $\psi(x)$ is always on the "interval" connecting $\Delta(\phi_z)(x)$ and $x \in \Delta(\mathcal{R})$, hence a continuous "linear" homotopy $H : \Delta(\mathcal{R}) \times I \to \Delta(\mathcal{R})$ is still well defined and continuous. Note that ψ maps $\Delta(\mathcal{R})$ into the neighborhood $\mathcal{U}$ which itself can be deformed into $\Delta(\mathcal{Z})$. Finally, since the last space is contractible, ψ is null-homotopic, so is the identity map and $\Delta(\mathcal{R})$ is contractible. □

In the following proposition we record for the future reference, a more general statement which is proved along the lines of the proof of proposition 4.4.

Proposition 4.5. *Suppose that $\mathcal{R}$ is a compact topological poset and let $z \in \mathcal{R}$ be a minimal (maximal) element such that the least upper bound $\phi_z(x) := x \vee z$ (the least lower bound $x \wedge z$) exists for each $x \in \mathcal{R}$. Suppose that the function $\phi_z : \mathcal{R} \setminus \mathcal{R}_{\geq z} \to \mathcal{R}$ is continuous and assume that the inclusion map $\Delta(\mathcal{R}_{\geq z}) \to \Delta(\mathcal{R})$ is a cofibration. In the dual case the upper cone $\mathcal{R}_{\geq z}$ is replaced by the upper cone $\mathcal{R}_{\leq z}$. Then the order complex $\Delta(\mathcal{R})$ is contractible.*

The following theorem is in some sense our most complete answer to the first question (**Q1**). Its proof reveals that this question is reducible to a form of the Quillen's *fiber theorem*, [27, 5, 45], for topological posets. Quillen's result is known to hold, [18], for diagrams of spaces over topological categories with the discrete set of objects. Unfortunately it doesn't seem to be available yet in the case of general topological categories, specially in the case of topological posets. This is the reason why we use instead the Vietoris-Begle mapping theorem, [33].

Theorem 4.6. *Let $(\mathcal{P}, \leq)$ be a compact topological B and M-poset, definitions 3.9 and 3.11, such that the poset $\hat{\mathcal{P}} := \mathcal{P} \cup \{\hat{0}, \hat{1}\}$ with added minimum and maximum elements $\hat{0}$ and $\hat{1}$ is a lattice. Let $z \in \mathcal{P}$ and let $\mathrm{Co}(z)$ be the set of all complements of z in $\hat{\mathcal{P}}$, $\mathrm{Co}(z) := \{x \in \mathcal{P} \mid x \wedge z = \hat{0} \text{ and } x \vee z = \hat{1}\}$. Then the order complex $\Delta(\mathcal{R})$ of the poset $\mathcal{R} := \mathcal{P} \setminus \mathrm{Co}(z)$ is cohomologically trivial, i.e. has the cohomology of a point. More generally, for the conclusion of the theorem it is not necessary to assume that $\hat{\mathcal{P}}$ is a lattice. It suffices to assume that $z \vee x$ and $z \wedge x$ exist in $\hat{\mathcal{P}}$ for all $x \in \mathcal{P}$.*

Proof: Let $e : \mathcal{R} \to \mathcal{R}, e(x) = x$, be the identity map and $c : \mathcal{R} \to \mathcal{R}, c(x) = z$ a constant map. In order to show that $\Delta(\mathcal{R})$ is cohomologically trivial, it suffices to prove that the maps $\bar{e} = \Delta(e)$ and $\bar{c} = \Delta(c)$ from $\Delta(\mathcal{R})$ to $\Delta(\mathcal{R})$ induce the same homomorphism $e^* = c^* : \check{H}^*(\Delta(\mathcal{R})) \to \check{H}^*(\Delta(\mathcal{R}))$ of the associated Alexander cohomology groups.

Let $\mathcal{R} \times \mathcal{R}$ be the product poset and let $\mathcal{F} := \{(x, y) \in \mathcal{R} \times \mathcal{R} \mid x \in \mathcal{R} \text{ and } (x \leq y \text{ or } y \leq z)\}$. Let $\pi_1, \pi_2 : \mathcal{F} \to \mathcal{R}$ be the obvious projection maps, $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, restricted on $\mathcal{F}$. Define $d, j : \mathcal{R} \to \mathcal{F}$ by $d(x) = (x, x)$ and $j(x) = (x, z), x \in \mathcal{R}$. All maps π_1, π_2, d, j are monotone and continuous so they induce the corresponding maps, $\bar{\pi}_1, \bar{\pi}_2, \bar{j}, \bar{d}$ of ordered complexes. Obviously,

$$\pi_1 \circ d = \pi_1 \circ j = e,\ e = \pi_2 \circ d,\ c = \pi_2 \circ j. \tag{9}$$

The equality $e^* = c^* : \check{H}^*(\Delta(\mathcal{R})) \to \check{H}^*(\Delta(\mathcal{R}))$ follows immediately if we prove that the homomorphism $\pi_1^* : \check{H}^*(\Delta(\mathcal{R})) \to \check{H}^*(\Delta(\mathcal{F}))$, associated to the projection $\pi_1 : \mathcal{F} \to \mathcal{R}$, is an isomorphism. Indeed, $d^* = j^*$ follows from $d^* \circ \pi_1^* = j^* \circ \pi_1^*$ and $e^* = d^* \circ \pi_2^* = j^* \circ \pi_2^* = c^*$. We shall demonstrate that π_1^* is an isomorphism by showing that $\bar{\pi}_1 : \Delta(\mathcal{F}) \to \Delta(\mathcal{R})$ satisfies all conditions of the Vietoris-Begle mapping theorem, [33]. The complex $\Delta(\mathcal{P})$ is compact since $\mathcal{P}$ is compact and $\mathcal{P}$ is an M-poset. Hence, the map $\bar{\pi}_1$ is closed as a continuous map of compact spaces. It is surjective since the map $\pi_1 : \mathcal{F} \to \mathcal{R}$ is surjective. It remains to be proved that the fiber $\bar{\pi}_1^{-1}(w)$ is cohomologically trivial for each $w \in \Delta(\mathcal{R})$. We actually prove that the fiber $\bar{\pi}^{-1}(w)$ is Σ-contractible (cf. proposition 3.24).

Let $w = \lambda_1 x_1 + \ldots + \lambda_n x_n$ where $x_1 < x_2 < \ldots < x_n$ is a strictly increasing chain C in $\mathcal{R}$ and $\lambda_i > 0$ for all $i = 1, \ldots, n$. Let $\mathcal{E} := \pi_1^{-1}(C)$ be the subposet of $\mathcal{F}$ "over" the chain C. Let us show that $\mathcal{E}$, together with the monotone map $\bar{\pi}_1 : \mathcal{E} \to C$, satisfies conditions of proposition 3.24. Let $x \in \{x_1, \ldots, x_n\}$. We are supposed to show that $B(x) = \bar{\pi}_1^{-1}(x) = \{(x,t) \mid x \leq t \text{ or } t \leq z\}$ is contractible. By assumption either $z \wedge x$ or $z \vee x$ exists in $\mathcal{R}$. Suppose $a := z \wedge x$ exists, the other case is treated analogously. Let $U(x) := \{t \in B(x) \mid t \leq a \text{ or } a \leq t\}$, $V(x) := \{t \in B(x) \mid t \leq z\}$ and $W(x) := U(x) \cap V(x)$. Obviously $\Delta(B(x)) = \Delta(U(x)) \cup \Delta(V(x))$ and $\Delta(U(x)) \cap \Delta(V(x)) = \Delta(W(x))$. Here we use in an essential way the fact that a is the greatest lower bound of x and z so every chain in $B(x)$ which intersects $U(x) \setminus V(x)$ is contained in $U(x)$. All spaces $\Delta(U(x)), \Delta(V(x)), \Delta(W(x))$ are contractible and the inclusion maps between them are cofibrations since $\mathcal{P}$ is a B-poset. By the gluing lemma, [10, 45], $B(x)$ is also contractible. Finally, an application of proposition 3.24 yields that the fiber $\bar{\pi}_1^{-1}(a)$ is Σ-contractible. This proves that π_1^* is an isomorphism and the theorem follows. $\square$

4.2. **The second and the third question.** Assuming that $\Delta(\mathcal{R})$ is contractible, the spaces $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P})/\Delta(\mathcal{R})$ have the same homotopy type if the inclusion map $\Delta(\mathcal{R}) \hookrightarrow \Delta(\mathcal{P})$ is a cofibration. Some of the methods how this condition can be checked are discussed in the remark 3.10. A general impression is that the answer to the question **Q2** is positive in all cases of interest.

We focus our attention now to the third question. Assume that the following assumptions hold until the end of this section.

A1.: $\Delta(\mathcal{P})$ is a compact,
A2.: $\mathcal{R} := \mathcal{P} \setminus C$ where $C \subset \mathcal{P}$ is an open antichain.

It follows from **A2** that $\Delta(\mathcal{R})$ is a closed subspace of $\Delta(\mathcal{P})$. By **A1**, the space $Y := \Delta(\mathcal{P}) \setminus \Delta(\mathcal{R})$ is locally compact so the question **Q3** reduces to the analysis of the one-point compactification $\bar{Y} := Y \cup \{\infty\}$ of Y. Let us define auxiliary posets

$$E_{\mathcal{P},C} := \{(c,x) \in \mathcal{P}^2 \mid c \in C \text{ and } (x \geq c \text{ or } x \leq c)\}$$

$$\dot{E}_{\mathcal{P},C} := \{(c,x) \in \mathcal{P}^2 \mid c \in C \text{ and } (x > c \text{ or } x < c)\}.$$

The order complexes $\mathcal{E}_{\mathcal{P},C} := \Delta(E_{\mathcal{P},C})$ and $\dot{\mathcal{E}}_{\mathcal{P},C} := \Delta(\dot{E}_{\mathcal{P},C})$ of these two posets can be naively seen as the disc and the sphere bundle associated to a "vector bundle" over C, with $s : C \to \mathcal{E}_{\mathcal{P},C},\ s(c) = (c,c)$, as the zero section.

Definition 4.7. *Let us define the "Thom-space" of the "disc"-bundle $p : \mathcal{E}_{\mathcal{P},C} \to C$ as the one-point compactification*

$$\mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) := (\mathcal{E}_{\mathcal{P},C} \setminus \dot{\mathcal{E}}_{\mathcal{P},C}) \cup \{\infty\}.$$

Proposition 4.8. *Under assumptions* **A1** *and* **A2**, *the second projection map $\pi_2 : \Delta(\mathcal{P})^2 \to \Delta(\mathcal{P})$ induces a homeomorphism*

$$\mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) \xrightarrow{\cong} \Delta(\mathcal{P})/\Delta(\mathcal{R})\ .$$

Proof: The projection π_2 induces a continuous map $\alpha : (\mathcal{E}_{\mathcal{P},C} \setminus \dot{\mathcal{E}}_{\mathcal{P},C}) \to \Delta(\mathcal{P}) \setminus \Delta(\mathcal{R})$ and a map $\overline{\alpha} : \mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) \to (\Delta(\mathcal{P}) \setminus \Delta(\mathcal{R})) \cup \{\infty\}$. Note that $\Delta(\mathcal{R})$ is compact which implies that the letter space is homeomorphic to $\Delta(\mathcal{P})/\Delta(\mathcal{R})$. The map α is 1–1 because C is an antichain. It remains to be checked that the inverse function α^{-1} is also continuous. This will follow from the continuity of of the function $\beta = p \circ \alpha^{-1} : \Delta(\mathcal{P}) \setminus \Delta(\mathcal{R}) \to C$. Note that if $x = \lambda_1 x_1 + \ldots + \lambda_n x_n \in \Delta(\mathcal{P}) \setminus \Delta(\mathcal{R}), \lambda_i > 0$, then $\beta(x) = y$ where y is the unique element in the intersection of the chain $Z := \{x_1 < \ldots < x_n\}$ and antichain C. Let $U \subset C$ be a neighborhood of y in C. It follows from **A2** that $\mathcal{P} \setminus U$ is a closed subposet of $\mathcal{P}$, hence $\Delta(\mathcal{P} \setminus U)$ is a closed subspace of $\Delta(\mathcal{P})$. If $V := \Delta(\mathcal{P}) \setminus \Delta(\mathcal{P} \setminus U)$, then V is a neighborhood of x and $\beta(V) \subset U$. This means that β is continuous which completes the proof of the proposition. □

In the most interesting cases, $p : \mathcal{E}_{\mathcal{P},C} \to C$ is indeed homeomorphic to the disc bundle of an actual vector bundle over C. In this case the "Thom-space" from definition 4.7 can be described as the *twisted smash product* in the sense of the following definition.

Definition 4.9. *Suppose that $\mathcal{E}$ is a fibre bundle pair over B with total pair $(E, \dot{E})$, fiber pair $(F, \dot{F})$ and a projection $p : E \to B$, cf.* [33], *p.256. For example $\mathcal{E}$ can be the disc and sphere bundle pair $(D(V), S(V))$ associated to a vector bundle $V \xrightarrow{p} B$. Suppose $\dot{B} \subset B$ is a closed subspace. Then the twisted smash product*

$$(B, \dot{B}) \rtimes_{\mathcal{E}} (F, \dot{F})$$

of pairs $(B, \dot{B})$ and $(F, \dot{F})$ relative $\mathcal{E}$, is the quotient space $E/(\dot{E} \cup p^{-1}(\dot{B})$.

Corollary 4.10. *Suppose that the compact bundle pair $(\mathcal{E}_{\mathcal{P},C}, \dot{\mathcal{E}}_{\mathcal{P},C}) \to C$ from definition 4.7 is isomorphic to the disc and sphere bundle pair $(D(V), S(V)) \to C$ associated to some m-dimensional vector bundle over C. Assume that V can be extended to a vector bundle W over a compact space $\widetilde{C} \supset C$. Then,*

$$\Delta(\mathcal{P})/\Delta(\mathcal{R}) \cong \mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) \cong (\widetilde{C}, C) \rtimes_{\mathcal{W}} (D^m, S^{m-1})$$

where $\mathcal{W} = (D(W), S(W))$ is the disc and sphere bundle pair associated to W.

Corollary 4.11. *If the bundle V in the corollary 4.10 is trivial, then W can be taken to be the trivial bundle over $\bar{C} := C \cup \infty$ in which case*

$$\Delta(\mathcal{P})/\Delta(\mathcal{R}) \cong \mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) \cong (\bar{C}, \{\infty\}) \wedge (D^m, S^{m-1})$$

where " $\wedge$ " is the usual smash product of pointed spaces.

For the sake of completeness we formulate one more corollary of proposition 4.8 which shows explicitly the connection of this result with the equation (8).

Corollary 4.12. *Suppose that the topological poset $\mathcal{P}$ satisfies, besides* **A1** *and* **A2**, *the condition that the antichain C is finite. Then,*

$$\Delta(\mathcal{P})/\Delta(\mathcal{R}) \cong \mathrm{Thom}(\mathcal{E}_{\mathcal{P},C}) \cong \bigvee_{c \in C} \Sigma(\Delta(\mathcal{P}_{<c}) * \Delta(\mathcal{P}_{>c}))$$

5. Applications

5.1. Grassmannian posets.

Theorem 5.1. *(Vassilev, [37, 38]) Let $K = R, C, Q$ be one of the classical fields and let $G_k(K^n)$ be the Grassmann manifold of all linear, k-dimensional subspaces of K^n. Let $\mathcal{G}_n(K) = (G(K^n), \subseteq)$ be the associated Grassmannian poset where $G(K^n)$ is a disjoint sum of Grassmannians*

$$G(K^n) := \coprod_{i=0}^{n} G_i(K^n)$$

Let $\widetilde{\mathcal{G}}_n(K) = (\widetilde{G}(K^n), \subseteq)$ be the truncated Grassmannian poset where $\widetilde{G}(K^n) := \coprod_{i=1}^{n-1} G_i(K^n)$. Then,

$$\Delta(\widetilde{\mathcal{G}}_n(K)) \cong S^{\binom{n}{2}d+n-2} \quad \text{where} \quad d = \dim_R(K).$$

Proof: We want to illustrate the technique developed in this paper so we concentrate on the proof that the order complex $\Delta(\widetilde{\mathcal{G}}_n(K))$ has the same homotopy type as the sphere of dimension $\binom{n}{2}d+n-2$. The proof that $\Delta(\widetilde{\mathcal{G}}_n(K))$ is actually a sphere can be completed along the lines of [37], see also the section 5.3 for a direct, elementary proof. To simplify the notation we prove the result in the case $K = R$, in other two cases the proof is completely analogous. Let $Z \in G_1(R^n)$ and let $C := \mathrm{Co}(Z)$ be the space of all complements of Z in $\mathcal{G}_n(R)$. Obviously, $\mathrm{Co}(Z) = \{L \in G_{n-1}(R^n) \mid Z \nsubseteq L\}$. It is easy to see that the space C is homeomorphic to R^{n-1} so the one-point compactification $\bar{C}$ of C is homeomorphic to S^{n-1}. Let $\mathcal{G}'_n(R) = \widetilde{\mathcal{G}}_n(R) \setminus \mathrm{Co}(Z)$. Then by proposition 4.4 and corollary 4.11,

$$\Delta(\widetilde{\mathcal{G}}_n(R)) \simeq \Delta(\widetilde{\mathcal{G}}_n(R))/\Delta(\mathcal{G}'_n(R)) \simeq (\bar{C}, \infty) \wedge \Sigma(\bar{\mathcal{G}}_{n-1}(R)) \simeq S^{n-1} \wedge \Sigma(\widetilde{\mathcal{G}}_{n-1}(R))$$

Obviously $\Delta(\widetilde{\mathcal{G}}_2(R)) \simeq S^1$ so an induction based on the homotopy recurrence relation above yields the desired formula

$$\Delta(\widetilde{\mathcal{G}}_n(R)) \simeq S^{\binom{n}{2}+n-2}\ .$$

The idea of the proof of theorem 5.1 is quite general and can be applied in many other situations. Here is another example where we compute the homotopy type of an oriented Grassmannian poset.

Theorem 5.2. *Let $G_k^{\pm}(R^n)$ be the Grassmann manifold of all linear, oriented, k-dimensional subspaces of R^n. Let $\mathcal{G}_n^{\pm}(R) = (G^{\pm}(R^n), \subseteq)$ be the associated Grassmannian poset and $\widetilde{\mathcal{G}}_n^{\pm}(R) = (\widetilde{G}^{\pm}(R^n), \subseteq)$ the associated truncated poset where $\widetilde{G}^{\pm}(R^n) := \coprod_{i=1}^{n-1} G_i^{\pm}(R^n)$ and $G^{\pm}(R^n) := \widetilde{G}^{\pm}(R^n) \cup G_0^{\pm}(R^n) \cup G_n^{\pm}(R^n)$. Then the homotopy type of the truncated poset is equal to the wedge of 2^{n-2} spheres of dimension $\binom{n}{2}+n-2$,*

$$\Delta(\widetilde{\mathcal{G}}_n^{\pm}(R)) \simeq \bigvee_{j=1}^{2^{n-2}} S_j^{\binom{n}{2}+n-2}.$$

Proof: The proof is similar to the proof of theorem 5.1. Choose $Z \in G_1^{\pm}(R^n)$ and denote by $C = \mathrm{Co}(Z)$ the space of all complements of Z in $\mathcal{G}_n^{\pm}(R)$. This space is seen to be a disjoint union of 'positive' and 'negative' hyperplanes

$$\mathrm{Co}(Z) = \{L \in G_{n-1}^{\pm}(R^n) \mid Z \not\subseteq L\} \cong R^{n-1} \sqcup R^{n-1}.$$

¿From here we deduce the recurrence formula (3) in section 2 and the desired formula follows by induction. □

Let $I := \{i_\nu\}_{\nu=1}^k \subseteq [n]$ be a subset of $[n] = \{0, 1, \dots, n\}$. Define

$$\mathcal{G}_I(K) := \coprod_{\nu=1}^{k} G_{i_\nu}(K^n)$$

as the "rank selected" subposet of $\mathcal{G}_n(K)$. A natural problem is to express the homotopy type of $\Delta(\mathcal{G}_I(K))$ in terms of "standard" spaces and constructions. Here is an example.

Proposition 5.3. *Let $I = \langle k \rangle = \{1, \dots, k\}$. Let $\mathcal{G}_I(R) = \coprod_{\nu=1}^k G_\nu(R^n)$ be the rank selected subposets of $\mathcal{G}_n(R)$ associated to I. Then, cf. definition 4.10,*

$$\Delta(\mathcal{G}_I(R)) \cong (G_k(R^n), G_{k-1}(R^{n-1})) \rtimes_{\mathcal{W}} (D^m, S^{m-1})$$

where $m = \binom{k}{2} + k - 1$ and $\mathcal{W}$ is the disc bundle which arises if each fibre L in the tautological k-plane bundle over $G_k(R^n)$ is replaced by the cone over $\Delta(\widetilde{\mathcal{G}}_k(R)) \cong D^m$.

Remark 5.4. *In the case of general rank selected posets $\mathcal{P} = \mathcal{G}_I(K)$, given $Z \in \mathcal{P}$, the space $\mathrm{Co}(Z)$ of complements of Z is not necessarily an antichain. The answer to the question* **Q3** *from section 4 gets more complicated, compare* [5]*, and may require new ideas.*

5.2. **Configuration posets.** Configuration posets $\exp_n(X)$ and configuration spaces $F(X, n)$ and $B(X, n)$, of labeled and unlabeled points in the space X respectively, were defined in example 3.3. The following construction has been introduced by Vassiliev under the name *geometric resolution* of configuration spaces $\exp_n(M)$ and $B(M, r)$. Suppose that a manifold or more generally a finite CW-complex M is generically embedded in the space R^N of very large dimension N. Let $\mathrm{Conv}_r(M)$ be the union of all (closed) $(r-1)$-dimensional simplices with vertices in the embedded space M. The genericity of the embedding means that two simplices spanned by different sets of vertices must have disjoint interiors. It is easy to observe that $\mathrm{Conv}_n(M) \cong \Delta(\exp_n(M))$, more precisely the order complex $\Delta(\exp_n(M))$ can be seen as the barycentric subdivision of $\mathrm{Conv}_n(M)$. This is precisely the context in which order complexes of continuous posets arose in Vassiliev's work. For example he proved in [39] the following theorem.

Theorem 5.5. *The space $\mathrm{Conv}_n(S^1) \cong \Delta(\exp_n(S^1))$ is homeomorphic to S^{2n-1}.*

Vassiliev actually proved a little more by showing that spaces above are PL-homeomorphic. Motivated by this example, he asked in his lecture at the workshop

"Geometric Combinatorics", MSRI, February 1997, whether a similar formula holds for other spaces (manifolds) M. In other words is it always true that

$$\Delta(\exp_n(X)) \cong X * \ldots * X = X^{*(n)}$$

where $X^{*(n)}$ denotes the n-fold join of the space X. Here we show, theorem 5.8, corollary 5.9 and proposition 5.10, that the homology type of the space $\Delta(\exp_n(M))$ is very closely related to the homology type of the space $B(M,n)$. As a consequence we obtain a proof of theorem 5.5 and show that already in the case $X = S^2, n \geq 2$ the answer to the question raised by Vassiliev is negative.

We start with a proof of $\Delta(\exp_n(S^1)) \cong S^{2n-1}$. Vassilev gave in [39] an elegant and short proof of this result. Another elementary proof is given in section 5.3 while the following proof illustrates in the first place some of the tools developed in this paper.

Proof of theorem 5.5: It is not difficult to check that $\Delta(\exp_n(S^1))$ is a PL-manifold so, in light of the fact that the Poincaré conjecture holds for PL–manifolds of dimension $n \geq 5$ (the case $n = 2$ is established by a direct argument), we concentrate on the proof that $\Delta(\exp_n(S^1))$ has the correct homotopy type. Let $x_0 \in S^1 = B(S^1, 1) \subset \exp_n(S^1)$. The set $\mathrm{Co}(\{x_0\})$ of all complements of $\{x_0\}$ in the lattice $\exp_n(S^1) \cup \{\hat{0}, \hat{1}\}$ is $B(S^1 \setminus \{x_0\}, n)$. The space $B(S^1 \setminus \{x_0\}, n) \cong B(R^1, n)$ is an open, n-dimensional convex set so its one–point compactification is homeomorphic to S^n. Finally, by the results of sections 4.1 and 4.2,

$$\Delta(\exp_n(S^1)) \simeq S^n \wedge \Delta(\widetilde{\mathcal{B}}_{n-1}) \simeq S^{2n-1}$$

and the result follows. □

The idea of the proof of theorem 5.5 can be obviously used for more general configuration posets.

Definition 5.6. *The configuration space $F(Y,n)$ of all labeled n-element subsets of Y is a free S_n-space. Let V be the standard representation of S_n i.e. the $(n-1)$-dimensional representation obtained by subtracting the trivial 1-dimensional representation of S_n from the permutation representation. Let ξ be the vector bundle*

$$R^{n-1} \longrightarrow F(Y,n) \times_{S_n} V \longrightarrow B(Y,n)$$

and let us define its Thom-space as its one-point compactification

$$\mathrm{Thom}_n(Y) := (F(Y,n) \times_{S_n} V) \cup \{\infty\}.$$

Definition 5.7. *Let $\mathcal{P} = \exp_n(X)$ be the n^{th} configuration poset of X and $x_0 \in X$. Let $C := \mathrm{Co}(\{x_0\})$ be the space of all complements of $\{x_0\}$ in the lattice $\hat{\mathcal{P}} = \mathcal{P} \cup \{\hat{0}, \hat{1}\}$ and $\mathcal{R} := \mathcal{P} \setminus C$. The pointed space (X, x_0) is called* admissible *if the inclusion $\Delta(\mathcal{R}) \hookrightarrow \Delta(\mathcal{P})$ is a cofibration, cf. section 4.2. For example every pair (M,p) is admissible if M is a smooth, compact manifold and $p \in M$.*

Theorem 5.8. *Suppose that (X, x_0) is a compact, admissible space. Then*

$$\Delta(\exp_n(X)) \simeq \mathrm{Thom}_n(X \setminus \{x_0\})$$

Proof: The compactness of $\Delta(\exp_n(X))$ follows from the compactness of X. The space $C = \mathrm{Co}(\{x_0\}) \cong B(X \setminus \{x_0\}, n)$ is an open antichain in $\Delta(\exp_n(X))$ so both conditions $A1$ and $A2$ of proposition 4.8 are satisfied. The bundles $\mathcal{E}_{\mathcal{P},C}$ and $\dot{\mathcal{E}}_{\mathcal{P},C}$ from definition 4.7 are identified as the disc and the sphere bundle associated to the bundle from the definition 5.6. The result is therefore a consequence of proposition 4.8. □

Corollary 5.9. *The order complex $\Delta(\exp_n(S^m))$, has the homotopy type of the space*

$$\mathrm{Thom}_n(R^m) = (F(R^m, n) \rtimes_{S_n} V) \cup \{\infty\}$$

By the Poincaré-Lefschetz duality, taking the (co)homology with Z_2-coefficients,

$$\widetilde{H}_p(\Delta(\exp_n(S^m))) \cong H^{(m+1)n-p-1}(B(R^m, n)).$$

Proposition 5.10. $\Delta(\exp_n(S^2))$ *does not have the homotopy type of a sphere for $n \geq 2$.*

Proof: By corollary 5.9

$$H_p(\Delta(\exp_n(S^2))) \simeq H^{3n-p-1}(B(R^2, n)).$$

The cohomology of the configuration space $B(R^2, n)$ with Z_2 coefficients has been computed by D.B. Fuchs, [12, 38]. He proved that the dimension of $H^k(B(R^2, n), Z_2)$ over Z_2 is equal to the number of representations of the form $n = 2^{\alpha_1} + \ldots + 2^{\alpha_{n-k}}$, $\alpha_i \geq 0$, where two representations that differ only by the order of summation are considered to be equal. ¿From here we deduce that $H^0(B(R^2, n), Z_2) \cong H^1(B(R^2, n), Z_2) \cong Z_2$, hence the reduced homology of $\Delta(\exp_n(S^2))$ is nontrivial in two dimensions i.e. this space cannot be homotopic to a sphere. □

The following simple corollary reveals the connection of theorem 5.8 and its consequences with the interesting problem of n-neighborly submanifolds of R^N, [41]. Note that in this statement, the embedding of S^2 is not necessarily smooth, let alone stably n-supported in the sense of [41]. This suggests that a more general problem of finding n-neighborly embeddings of more general spaces, say simplicial complexes, may be interesting.

Corollary 5.11. *Suppose that the sphere S^2 is topologically embedded in R^N in such a way that for any collection Σ of r different points on the sphere, there exists a hyperplane in R^N such that $\Sigma \subset H$ and $S^2 \setminus \Sigma$ is in an open halfspace determined by H. Then $N \geq 3n$.*

Proof: Every embedding of the sphere S^2 in R^N which is n-neighborly in the sense above, leads to an embedding of the order complex $\exp_n(S^2)$ into the same space. By the proof of the corollary 5.10 we know that $H_{3n-1}(\exp_n(S^2)) \neq 0$, hence $3n \leq N$.

5.3. **Elementary proofs.** In this section we present direct, elementary proofs of two motivating results of Vassiliev, theorems 5.1 and 5.5 of this paper. I am obliged to W. Thurston for the idea of the first, to G. Kalai for the idea of the second proof and to B. Shapiro for the information related to convex curves in R^n.

Second proof of theorem 5.1: Let $\mathrm{Symm}(n)$ be the vector space of all symmetric, $n \times n$ matrices with real entries. The dimension of this space is $\binom{n}{2} + n$. Given a matrix $A := [a_{ij}]$ let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the ordered sequence of its eigenvalues and let $u_1, u_2, \ldots, u_n$ be a sequence of linearly independent vectors such that u_i is an eigenvector associated to the eigenvalue λ_i. Let $L_1 \subset L_2 \subset \ldots \subset L_n$ be the flag associated to this sequence, $L_i := \mathrm{span}\{u_j \mid j = 1, \ldots, i\}$. Note that this flag is not well defined if some of the eigenvectors coincide. On the other hand if $M := \mathrm{Symm}(n) \setminus \{\alpha I \mid \alpha \in R\}$ is the space of all matrices which are not multiples of the unit matrix, then the map

$$\Phi : M \longrightarrow G_1(R^n) * \ldots * G_{n-1}(R^n)$$

defined by

$$A := [a_{ij}] \mapsto \frac{\lambda_2 - \lambda_1}{\lambda_n - \lambda_1} L_1 + \frac{\lambda_3 - \lambda_2}{\lambda_n - \lambda_1} L_2 + \ldots + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n - \lambda_1} L_{n-1}$$

is well defined. This map is not one–to–one. More precisely, $\Phi(A) = \Phi(B)$ iff there exist $\alpha > 0$ and $\beta \in R$ such that $B = \alpha A + \beta I$ or in other words Φ is constant on orbits of the group $R^+ \times R$ which acts on M by the formula $(\alpha, \beta)A := \alpha A + \beta I$. We conclude that the image of this map, $\mathrm{Image}(\Phi) = \Delta(\widetilde{\mathcal{G}}_n(R))$, is diffeomorphic to the orbit manifold $M/(R^+ \times R)$. The orbits are easily identified as 'vertical' two dimensional halfplanes with the origin removed. Hence, if H is the codimension one linear subspace of $\mathrm{Symm}(n)$ orthogonal to I, then the $(\binom{n}{2} + n - 2)$-dimensional unit sphere $S(H)$ in H intersects each orbit in exactly one point. This completes the proof.

Second proof of theorem 5.5: This proof relies on some characteristic properties of so called *closed convex curves* in R^{2n}, [32, 29]. We outline the main idea of the proof leaving details to the reader. A simple, closed curve $\gamma : S^1 \to R^{2n}$ is called *convex* if the total multiplicity of its intersection with any affine hyperplane does not exceed $2n$. An example is the Carathéodory curve

$$\Gamma : t \mapsto (\sin t, \cos t, \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)).$$

Let $C := \mathrm{conv}(\mathrm{Im}(\Gamma))$ be the convex hull of this curve and let $D := \partial(C) \cong S^{2n-1}$. All we have to show is that every point in D can be expressed uniquely as a convex combination of the form $\alpha_1 x_1 + \ldots + \alpha_n x_n$ where $x_1, x_2, \ldots, x_n$ are distinct points in $\mathrm{Im}(\Gamma)$ and conversely that every such convex combination determines a point in D. Both facts follow from the following observation of Schoenberg, [32], who proved that the convex hull of every closed convex curve γ is the intersection of a family of closed halfspaces determined to support hyperplanes which are tangent to γ at n distinct points. Note that, if C is seen as a continuous analogue of the *cyclic* polytope, then this result of Schoenberg can be related to the well known Gale's evenness condition, [24, 46]. Indeed, hyperplanes tangent to γ at n distinct points are spanned by n pairs of "infinitesimally closed" points on this curve.

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