DOWN-UP ALGEBRAS

Georgia Benkart
Tom Roby

Abstract. The algebra generated by the down and up operators on a differential partially ordered set (poset) encodes essential enumerative and structural properties of the poset. Motivated by the algebras generated by the down and up operators on posets, we introduce here a family of infinite-dimensional associative algebras called down-up algebras. We show that down-up algebras exhibit many of the important features of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ including a Poincaré-Birkhoff-Witt type basis and a well-behaved representation theory. We investigate the structure and representations of down-up algebras and focus especially on Verma modules, highest weight representations, and category $\mathcal{O}$ modules for them. We calculate the exact expressions for all the weights, since that information has proven to be particularly useful in determining structural results about posets.

§1. Introduction

Combinatorial source of down-up algebras.

Assume P is a partially ordered set (poset), and let $\mathbf{C}P$ denote the complex vector space whose basis is the set P. For many posets there are two well-defined linear transformations on $\mathbf{C}P$ coming from the order relation on P,

$$d(y) = \sum_{x \prec y} x \qquad \text{and} \qquad u(y) = \sum_{y \prec z} z,$$

where $\prec$ denotes the covering relation in the poset. Thus, $d(y)$ is the sum of the elements which y covers, and $u(y)$ is the sum of the elements which cover

1991 *Mathematics Subject Classification.* 1991 Mathematics Subject Classifications: Primary 16S15, 16S30, 17B35, 17B10 Secondary 81R50, 06A07.

The first author gratefully acknowledges support from National Science Foundation Grant #DMS–9622447 and from the Mathematical Sciences Research Institute, Berkeley. Research at MSRI was supported in part by NSF grant 9701755.

The second author gratefully acknowledges support from National Science Foundation Grant #DMS–9353149 and the University of Wisconsin, Madison.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

y. (These are the "down" and "up" operators.) Young's lattice of all partitions of all nonnegative integers provides an important example of such a poset. If $\mu = (\mu_1, \mu_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$ are partitions (with parts arranged so $\mu_1 \geq \mu_2 \geq \cdots$ and $\nu_1 \geq \nu_2 \geq \cdots$), then $\mu \leq \nu$ if $\mu_i \leq \nu_i$ for each i. The partition ν covers μ if $\mu \leq \nu$ and $\sum_i \nu_i = 1 + \sum_i \mu_i$. In ([St1], [St2]), Stanley found that many of the interesting enumerative and structural properties of Young's lattice could be deduced from the following relation that the up and down operators satisfy,

$$du - ud = I,$$

where I is the identity transformation on $\mathbf{C}P$. Young's lattice of partitions is not the only example, and Stanley considered more general posets which satisfy the relation $du - ud = rI$ for some fixed positive integer r. Such posets afford a representation of the Weyl algebra generated by u and d/r. Since the Weyl algebra also can be realized as differential operators $y \mapsto d/dx$ and $x \mapsto x$ (multiplication by x) on $\mathbf{C}[x]$, Stanley referred to the posets satisfying $du - ud = rI$ as "r-differential" or simply "differential" when $r = 1$. Fomin [F] independently defined essentially the same class of posets for $r = 1$ calling them "Y-graphs", the terminology inspired by Young's lattice.

In his study of *uniform posets* [T], Terwilliger considered finite ranked posets P whose down and up operators satisfy the following relation

$$d_i d_{i+1} u_i = \alpha_i d_i u_{i-1} d_i + \beta_i u_{i-2} d_{i-1} d_i + \gamma_i d_i,$$

where d_i and u_i denote the restriction of d and u to the elements of rank i. (There is an analogous second relation,

$$d_{i+1} u_i u_{i-1} = \alpha_i u_{i-1} d_i u_{i-1} + \beta_i u_{i-1} u_{i-2} d_{i-1} + \gamma_i u_{i-1},$$

which holds automatically in this case because d_{i+1} and u_i are adjoint operators relative to a certain bilinear form.) The matrix algebra over the reals generated by these operators together with the idempotent projection maps onto each rank, (referred to as the "incidence algebra" in [T]), is finite-dimensional and in [T] its finite-dimensional simple representations are determined. Terwilliger also discusses many families of examples including the four presented below.

In many classical cases the constants in the above relations do not depend on the rank of the poset. A particular instance of this provides a q-analogue of the notion of differential poset. A partially ordered set whose down and up operators satisfy

$$\begin{aligned} d^2 u &= q(q+1)dud - q^3 ud^2 + rd \\ du^2 &= q(q+1)udu - q^3 u^2 d + ru \end{aligned} \tag{1.1}$$

where q and r are fixed complex numbers is said to be "(q,r)-differential". Examples of (q,r)-differential posets include the posets of alternating forms, quadratic forms, or Hermitian forms over a finite field, as well as Hemmeter's poset:

(1) $[Alt_q(n),\ (n \geq 2)]$ *The poset of alternating bilinear forms*

Assume H is an n-dimensional vector space over the field $GF(q)$ of q elements. Consider the set of pairs $P = \{(U,f) \mid U$ is a subspace of H and f is an alternating bilinear form on $U\}$ with the following ordering: $(U,f) \leq (V,g)$ if U is a subspace of V and $g|_U = f$. Then P is a (q,r)-differential poset with $r = -q^n(q+1)$.

(2) $[Quad_q(n),\ (n \geq 2)]$ *The poset of quadratic forms*

The poset is defined the same way as in the first example but with "quadratic form" replacing "alternating bilinear form". In this case P is $(q, -q^{n+1}(q+1))$-differential.

(3) $[Her_q(n),\ (n \geq 2)]$ *The poset of Hermitian forms*

In this example, H is taken to be an n-dimensional vector space over $GF(q^2)$, and the bilinear forms are "Hermitian". The result is a poset which is $(q^2, -q^{2n+1}(q^2+1))$-differential.

(4) $[Hem_q(n),\ (q \text{ odd},\ n \geq 2)]$ *Hemmeter's poset*

Suppose X^+ and X^- are two copies of the graph of the dual polar space $X = C_{n-1}(q)$. Let Y be the bipartite graph with vertex set $X^+ \cup X^-$ (forming the bipartition). Two vertices $x^+ \in X^+$ and $y^- \in X^-$ are adjacent in Y if and only if $x = y$ or x and y are adjacent in X. The graph Y is a distance regular graph recently discovered by Hemmeter. Now set $P = Y$ and fix $u_0 \in Y$. Declare $u \leq v$ if $\delta(u_0,u)+\delta(u,v) = \delta(u_0,v)$, where $\delta(u,v)$ is the distance between u and v. (The choice of u_0 is irrelevant since Y is vertex transitive.) Then like the poset $Alt_q(n)$, P is $(q, -q^n(q+1))$-differential.

Definition of down-up algebras.

In this paper we study certain infinite-dimensional associative algebras whose generators satisfy relations more general than (1.1). Although the original motivation for our investigations came from posets, we make no assumption about the existence of posets whose down and up operators satisfy our relations.

We say a unital associative algebra $A = A(\alpha,\beta,\gamma)$ over the complex numbers $\mathbf{C}$ with generators u, d and defining relations

(R1) $d^2u = \alpha dud + \beta ud^2 + \gamma d,$

(R2) $du^2 = \alpha udu + \beta u^2d + \gamma u,$

where α, β, γ are fixed but arbitrary elements of $\mathbf{C}$, is a **down-up algebra**.

It is easy to see that when $\gamma \neq 0$ the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$ by the map $d \mapsto d'$, $u \mapsto \gamma u'$. Therefore, it would suffice to treat just two cases $\gamma = 0, 1$. But rather than divide all our considerations into these cases, it is convenient to retain the notation γ throughout.

Examples of down-up algebras.

If d and u are the down and up operators of a (q, r)-differential poset, and B is the associative algebra they generate, then B satisfies (R1) and (R2) with $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$. Thus, B is a homomorphic image of the algebra $A(\alpha, \beta, \gamma)$ with these parameters. In an r-differential poset, the relation $du - ud = rI$ is satisfied. Multiplying on the left by d and on the right by d and adding the resulting equations, we get the relation $d^2u - ud^2 = 2rd$ of a $(-1, 2r)$-differential poset. Thus, the Weyl algebra is a homomorphic image (by the ideal generated by $du - ud - r1$) of the algebra $A(0, 1, 2r)$. More generally, the q-Weyl algebra is a homomorphic image of the algebra $A(0, q^2, q+1)$ (and also of the algebra $A(q-1, q, 2)$) by the ideal generated by $du - qud - 1$. The skew polynomial algebra $\mathbf{C}_q[d, u]$, or quantum plane (see [M]), is the associative algebra with generators d, u which satisfy the relation $du = qud$. Therefore, $\mathbf{C}_q[d, u]$ is a homomorphic image (by the ideal generated by $du - qud$) of the algebra $A(2q, -q^2, 0)$ (also of the algebra $A(aq, (1-a)q^2, 0)$ for any $a \in \mathbf{C}$).

Consider a 3-dimensional Lie algebra L over $\mathbf{C}$ having a basis $x, y, [x, y]$ such that $[x[x, y]] = \gamma x$ and $[[x, y], y] = \gamma y$. In the universal enveloping algebra $U(L)$ of L, the relations above become

$$\begin{aligned} x^2y - 2xyx + yx^2 &= \gamma x \\ xy^2 - 2yxy + y^2x &= \gamma y. \end{aligned}$$

Thus, $U(L)$ is a homomorphic image of the down-up algebra $A(2, -1, \gamma)$ via the mapping $\phi : A(2, -1, \gamma) \to U(L)$ with $\phi : d \mapsto x$, $\phi : u \mapsto y$. In fact, $U(L)$ is isomorphic to $A(2, -1, \gamma)$. The mapping $\psi : L \to A(2, -1, \gamma)$ with $\psi : x \mapsto d$, $\psi : y \mapsto u$, and $\psi : [x, y] \mapsto du - ud$ extends, by the universal property of $U(L)$, to an algebra homomorphism $\psi : U(L) \to A(2, -1, \gamma)$. The two homomorphisms ϕ and ψ are inverses of each other. Every 3-dimensional Lie algebra L over $\mathbf{C}$ with $L = [L, L]$ has such a basis (see [J, pp. 13-14]), so its enveloping algebra is a down-up algebra.

There are several noteworthy special cases. The Lie algebra $\mathfrak{sl}_2$ of 2×2 complex matrices of trace zero has a standard basis e, f, h, which satisfies $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. From this we see that $U(\mathfrak{sl}_2) \cong A(2, -1, -2)$. The Heisenberg Lie algebra $\mathfrak{H}$ has a basis x, y, z where $[x, y] = z$, and $[z, \mathfrak{H}] = 0$. Thus, $U(\mathfrak{H}) \cong A(2, -1, 0)$.

In [Sm], S.P. Smith investigated a class of associative algebras that have many of the properties of the enveloping algebra $U(\mathfrak{sl}_2)$ but also many new, interesting aspects. These algebras have a presentation by generators a, b, h and relations $[h, a] = a$, $[h, b] = -b$ and $ab - ba = f(h)$, where $f(h)$ is a polynomial in h. In the special situation that $\deg(f) \leq 1$, such an algebra is a homomorphic image of a down-up algebra $A(2, -1, \gamma)$ for some γ. Algebras related to Smith's appear in Hodges' work [H] on primitive factors of $U(\mathfrak{sl}_2)$ and type-A Kleinian singularities.

Woronowicz [Wo] introduced a family of algebras whose presentation is equivalent to having three generators x, y, z which are assumed to satisfy the relations

$$xz - \zeta^4 zx = (1 + \zeta^2)x \quad zy - \zeta^4 yz = (1 + \zeta^2)y \quad xy - \zeta^2 yx = \zeta z$$

for some scalar $\zeta \neq \pm 1, 0$. These algebras are down-up algebras with $\alpha = \zeta^2(1 + \zeta^2)$, $\beta = -\zeta^6$, and $\gamma = \zeta(1 + \zeta^2)$. Woronowicz's algebras fit into a more general construction studied by Jordan in [Jo].

The Lie superalgebra $S = \mathfrak{sl}(1, 1) = S_{\overline{0}} \oplus S_{\overline{1}}$ of 2×2 complex matrices, $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ with supertrace $y_1 - y_4 = 0$ and with multiplication given by the supercommutator $[x, y] = xy - (-1)^{ab} yx$ for $x \in S_{\overline{a}}$, $y \in S_{\overline{b}}$ has a presentation by generators e, f (which are odd and can be identified with the matrix units $e = e_{1,2}$, $f = e_{2,1}$) and relations $[e, [e, f]] = 0$, $[[e, f], f] = 0$, $[e, e] = 0$, $[f, f] = 0$. Its universal enveloping algebra $U(\mathfrak{sl}(1, 1))$ has generators e, f and relations $e^2 f - f e^2 = 0$, $ef^2 - f^2 e = 0$, $e^2 = 0$, $f^2 = 0$. Thus, $U(\mathfrak{sl}(1, 1))$ is a homomorphic image of the down-up algebra $A(0, 1, 0)$ by the ideal generated by the elements e^2 and f^2, which are central in $A(0, 1, 0)$.

Connections with Witten's deformation of $U(\mathfrak{sl}_2)$.

To provide an explanation of the existence of quantum groups, Witten ([W1], [W2]) introduced a 7-parameter deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$. Witten's deformation is a unital associative algebra over a field $\mathbf{K}$ (which is algebraically closed of characteristic zero and which could be assumed to be $\mathbf{C}$) and depends on a 7-tuple $\underline{\xi} = (\xi_1, \ldots, \xi_7)$ of elements of $\mathbf{K}$. It has a presentation by generators x, y, z and defining relations

$$xz - \xi_1 zx = \xi_2 x \qquad zy - \xi_3 yz = \xi_4 y$$
$$yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z.$$

We denote the resulting algebra by $\mathfrak{W}(\underline{\xi})$. It can be argued that the following holds:

Theorem 1.2. *([B, Thm. 2.6]) A Witten deformation algebra $\mathfrak{W}(\underline{\xi})$ with*

$$\xi_6 = 0, \quad \xi_5\xi_7 \neq 0, \ \xi_1 = \xi_3, \quad \text{and} \ \xi_2 = \xi_4 \tag{1.3}$$

is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with α, β, γ given by

$$\alpha = \frac{1 + \xi_1\xi_5}{\xi_5}, \qquad \beta = -\frac{\xi_1}{\xi_5}, \qquad \gamma = -\frac{\xi_2\xi_7}{\xi_5}. \tag{1.4}$$

Conversely, any down-up algebra $A(\alpha, \beta, \gamma)$ with not both α and β equal to 0 is isomorphic to a Witten deformation algebra $\mathfrak{W}(\underline{\xi})$ whose parameters satisfy (1.3).

A deformation algebra $\mathfrak{W}(\underline{\xi})$ has a filtration, and Le Bruyn ([L1], [L2]) investigated the algebras $\mathfrak{W}(\underline{\xi})$ whose associated graded algebras are Auslander regular. They determine a 3-parameter family of deformation algebras which are called *conformal $\mathfrak{sl}_2$ algebras* and whose defining relations are

$$xz - azx = x \qquad zy - ayz = y \qquad yx - cxy = bz^2 + z \tag{1.5}$$

When $c \neq 0$ and $b = 0$, the conformal $\mathfrak{sl}_2$ algebra with defining relations given by (1.5) is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = c^{-1}(1 + ac), \beta = -ac^{-1}$ and $\gamma = -c^{-1}$. If $b = c = 0$ and $a \neq 0$, then the conformal $\mathfrak{sl}_2$ algebra is isomorphic to the down-up algebra $A(\alpha, \beta, \gamma)$ with $\alpha = a^{-1}$, $\beta = 0$ and $\gamma = -a^{-1}$.

In a recent paper Kulkarni [K1] has shown that under certain assumptions on the parameters a Witten deformation algebra is isomorphic to a conformal $\mathfrak{sl}_2$ algebra or to a double skew polynomial extension. The precise statement of the result is

Theorem 1.6. *([K1, Thm. 3.0.3]) If $\xi_1\xi_3\xi_5\xi_2 \neq 0$ or $\xi_1\xi_3\xi_5\xi_4 \neq 0$, then $\mathfrak{W}(\underline{\xi})$ is isomorphic to one of the following algebras:*

(a) A conformal $\mathfrak{sl}_2$ algebra with generators x, y, z and relations given by (1.5) for some $a, b, c \in \mathbf{K}$.

(b) A double skew polynomial extension (that is, a skew polynomial extension of a skew polynomial ring) whose generators satisfy

(i) $xz - zx = x, \quad zy - yz = \zeta y, \quad yx - \eta xy = 0$ *or*

(ii) $xw = \theta wx, \quad wy = \kappa yw, \quad yx = \lambda xy$

for parameters $\zeta, \eta, \theta, \kappa, \lambda \in \mathbf{K}$.

Kulkarni [K1] studies the simple representations of the conformal sl_2 algebras and of the skew polynomial algebras in (b) by making essential use of the observation that the conformal $\mathfrak{sl}_2$ algebras of (1.5) can be realized as *hyperbolic rings.* Kulkarni applies results of Rosenberg [R] on left ideals of the left spectrum of hyperbolic rings to determine the maximal left ideals for the conformal sl_2 algebras. Further applications of results of [R] give the left spectrum of the double skew polynomial extensions in (b) of the theorem.

In this work we investigate the structure and representations of the down-up algebras $A(\alpha, \beta, \gamma)$. Remarkably down-up algebras have many of the important features of universal enveloping algebras including a Poincaré-Birkhoff-Witt basis theorem and a well-behaved representation theory. Adopting a Lie theoretic approach, we explicitly construct "highest weight" and "lowest weight" representations of the algebras $A(\alpha, \beta, \gamma)$ and determine when they are simple. We calculate the exact expressions for all the weights since that information has proven to be especially useful in determining structural results about posets. (In particular, for differential posets Stanley [St1] showed that the spectrum of ud is related to the eigenvalues of the adjacency matrix of certain finite graphs associated with the posets. This gave a new class of graphs whose spectra could be explicitly computed. Many interesting properties associated to random walks on the graph are related to its spectrum.)

In the spirit of the work of [BGG] we investigate two categories of modules ($\mathcal{O}$ and $\mathcal{O}'$) for down-up algebras whose simple objects include the highest weight simple modules. Corresponding to each complex number λ, there is a universal infinite-dimensional highest weight module $V(\lambda)$, the so-called Verma module. Each of the weight spaces of $V(\lambda)$ is 1-dimensional or infinite-dimensional depending on the choice of the parameters $\alpha, \beta, \gamma, \lambda$. (Roots of unity play a critical role in deciding which occurs.) Its weights are determined by the recurrence relation

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma, \qquad \lambda_0 = \lambda, \quad \lambda_{-1} = 0.$$

In particular, for the algebra $A(1, 1, 0)$, the weights of $V(\lambda)$ are given by the Fibonacci sequence $\lambda, \lambda, 2\lambda, 3\lambda, 5\lambda, 8\lambda, \dots$ for each λ. Besides the examples coming from (q, r)-differential posets and universal enveloping algebras, certain subalgebras of quantized enveloping algebras (quantum groups) are down-up algebras. The highest weight representations that we construct for these subalgebras are not the standard ones.

Our investigations arose from conversations one of the authors (TR) had with Paul Terwilliger. We would like to express our appreciation to him for many helpful suggestions over the course of this project.

§2. Representations of down-up algebras

As before, let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra over $\mathbf{C}$ with generators u, d and defining relations

(R1) $d^2u = \alpha dud + \beta ud^2 + \gamma d$,

(R2) $du^2 = \alpha udu + \beta u^2d + \gamma u$,

where α, β, γ are fixed but arbitrary elements of $\mathbf{C}$.

Highest weight modules.

Our first objective is to construct "highest weight" representations of down-up algebras. Involved in the construction is a certain sequence of elements of $\mathbf{C}$. Starting with $\lambda_{-1} = 0$ and an arbitrary $\lambda_0 = \lambda \in \mathbf{C}$, define λ_n for $n \geq 1$ inductively by the following recurrence relation:

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma. \tag{2.1}$$

Proposition 2.2. *Set $\lambda_{-1} = 0$. Let $\lambda_0 = \lambda \in \mathbf{C}$ be arbitrary, and define λ_n for $n \geq 1$ as in (2.1). Let $V(\lambda)$ be the $\mathbf{C}$-vector space having basis $\{v_n \mid n = 0, 1, 2, \dots\}$. Define*

$$\begin{aligned} d \cdot v_n &= \lambda_{n-1}v_{n-1}, \quad n \geq 1, \quad \textit{and} \quad d \cdot v_0 = 0 \\ u \cdot v_n &= v_{n+1}. \end{aligned} \tag{2.3}$$

Then this action extends to an $A(\alpha, \beta, \gamma)$-module action on $V(\lambda)$.

Proof. The free associative algebra $\mathbf{C}\langle u, d\rangle$ with 1 over $\mathbf{C}$ generated by u, d acts on $V(\lambda)$ by extending (2.3) making $V(\lambda)$ into an $\mathbf{C}\langle u, d\rangle$-module. To show there is an induced action of A, it suffices to verify that

$$\begin{aligned} (d^2u - \alpha dud - \beta ud^2 - \gamma d)v_n &= 0 \\ (du^2 - \alpha udu - \beta u^2d - \gamma u)v_n &= 0 \end{aligned}$$

for all $n \geq 0$. But this is easy to see since

$$(d^2u - \alpha dud - \beta ud^2 - \gamma d)v_n = \lambda_{n-1}(\lambda_n - \alpha\lambda_{n-1} - \beta\lambda_{n-2} - \gamma)v_{n-1},$$

and

$$(du^2 - \alpha udu - \beta u^2d - \gamma u)v_n = (\lambda_{n+1} - \alpha\lambda_n - \beta\lambda_{n-1} - \gamma)v_{n+1} = 0. \quad \square$$

Proposition 2.4.

(a) The $A(\alpha,\beta,\gamma)$-module $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for any n.

(b) If m is minimal with the property that $\lambda_m = 0$, then $M(\lambda) = span_{\mathbf{C}}\{v_j \mid j \geq m+1\}$ is a maximal submodule of $V(\lambda)$.

(c) Suppose N is a submodule of $V(\lambda)$ such that $N \subseteq span_{\mathbf{C}}\{v_j \mid j \geq 1\}$. Then $N \subseteq M(\lambda)$.

Proof. Assume N is a nonzero submodule of $V(\lambda)$, and suppose $x = \sum_{k=0}^{n} a_k v_k \in N$, where $a_n \neq 0$. Then

$$d^n \cdot x = \lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0 a_n v_0 \in N.$$

If the coefficient is nonzero, then $v_0 \in N$, and since v_0 generates $V(\lambda)$, we have $N = V(\lambda)$ in this case.

If $\lambda_r = 0$, then $\mathrm{span}_{\mathbf{C}}\{v_j \mid j \geq r+1\}$ is a proper submodule of $V(\lambda)$ by (2.3). Thus, $V(\lambda)$ is simple if and only if $\lambda_r \neq 0$ for any r.

Let m be the minimal value such that $\lambda_m = 0$. Then $M(\lambda) = \mathrm{span}_{\mathbf{C}}\{v_j \mid j \geq m+1\}$ is a submodule of $V(\lambda)$. Suppose N is a submodule of $V(\lambda)$ properly containing $M(\lambda)$. We may assume that some $y = \sum_{j=0}^{m} b_j v_j \neq 0$ belongs to N. If $n \leq m$ is maximal so $b_n \neq 0$, then $d^n \cdot y = \lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0 b_n v_0 \in N$. By the minimality of m this implies $v_0 \in N$ and hence that $N = V(\lambda)$. Thus, $M(\lambda)$ is a maximal submodule of $V(\lambda)$.

Now assume N is a submodule of $V(\lambda)$ contained in $V_1 = \mathrm{span}_{\mathbf{C}}\{v_j \mid j \geq 1\}$. The submodule $N + M(\lambda)$ is contained in V_1, so it is a proper submodule of $V(\lambda)$. By the maximality of $M(\lambda)$, we must have $N + M(\lambda) = M(\lambda)$, which implies $N \subseteq M(\lambda)$. $\square$

Definition 2.5. We will set $M(\lambda) = (0)$ if $V(\lambda)$ is simple and otherwise assume $M(\lambda)$ is defined by Proposition 2.4 (b) using the minimal m such that $\lambda_m = 0$. Note that when $M(\lambda) \neq (0)$, then $M(\lambda) \cong V(\lambda_{m+1})$.

Example 2.6. Observe for any down-up algebra $A = A(\alpha,\beta,\gamma)$ that $M(0) = \mathrm{span}_{\mathbf{C}}\{v_j \mid j \geq 1\}$, so that $V(0)/M(0)$ is one-dimensional. There is an algebra homomorphism $\epsilon : A \to \mathbf{C}$ with $\epsilon(u) = 0 = \epsilon(d)$ and $\epsilon(1) = 1$. This gives rise to a one-dimensional A-module, $\mathbf{C}v$, such that $a \cdot v = \epsilon(a)v$ for all $a \in A$, and $\mathbf{C}v \cong V(0)/M(0)$.

Definition 2.7. *A module V for $A = A(\alpha,\beta,\gamma)$ is said to be a* **highest weight module of weight** *λ if V has a vector y_0 such that $d \cdot y_0 = 0$, $du \cdot y_0 = \lambda y_0$, and $V = Ay_0$. The vector y_0 is a* **maximal vector** *or* **highest weight vector** *of V.*

The highest weight module $V(\lambda)$ is said to be the **Verma module** of highest weight λ. It is so named because it shares the same universal property as Verma modules for finite-dimensional semisimple complex Lie algebras:

Proposition 2.8. *If V is a highest weight module of weight λ of $A = A(\alpha, \beta, \gamma)$, then V is a homomorphic image of $V(\lambda)$.*

Proof. Let y_0 be a maximal vector of V of weight λ. Then $V = Ay_0$. The mapping $V(\lambda) \to V$ with $a \cdot v_0 \mapsto a \cdot y_0$ for all $a \in A$ is an A-module epimorphism. □

Suppose V is any highest weight A-module of weight λ. Starting with a nonzero maximal vector $y_0 \in V$ of weight λ, define y_j inductively by $y_j = u \cdot y_{j-1}$ for $j \geq 1$. Stop the inductive definition at the first place that a vector y_{m+1} is produced which is a linear combination of the previous vectors. In this case we claim that $V = \text{span}_{\mathbf{C}}\{y_j \mid j = 0, 1, \ldots, m\}$. If no such dependency relation occurs, then we claim that $V = \text{span}_{\mathbf{C}}\{y_j \mid j = 0, 1, \ldots\}$, where the vectors y_j are linearly independent. To see these two assertions, first we show that

$$du \cdot y_n = \lambda_n y_n \tag{2.9}$$

where $\lambda_0 = \lambda$ and the λ_n's satisfy the recurrence relation in (2.1). The proof is by induction on n with the inductive step being

$$\begin{aligned} du \cdot y_{n+1} = du^2 \cdot y_n &= \alpha udu \cdot y_n + \beta u^2 d \cdot y_n + \gamma u \cdot y_n \\ &= \alpha \lambda_n y_{n+1} + \beta u^2 du \cdot y_{n-1} + \gamma y_{n+1} \\ &= (\alpha \lambda_n + \beta \lambda_{n-1} + \gamma) y_{n+1} = \lambda_{n+1} y_{n+1}. \end{aligned}$$

Observe that this implies

$$d \cdot y_n = du \cdot y_{n-1} = \lambda_{n-1} y_{n-1} \tag{2.10}$$

for all $n \geq 1$. Since d and u generate A, the relation $y_{n+1} = u \cdot y_n$ together with (2.10) shows that $V = \text{span}_{\mathbf{C}}\{y_j \mid j = 0, 1, \ldots, m\}$ when a dependency occurs, and $V = \text{span}_{\mathbf{C}}\{y_j \mid j = 0, 1, \ldots\}$ otherwise. In both instances the y_j's determine a basis for V.

Weight modules.

If we multiply the relation $d^2u - \alpha dud - \beta ud^2 = \gamma d$ on the left by u and the relation $du^2 - \alpha udu - \beta u^2 d = \gamma u$ on the right by d and subtract the second from the first, the resulting equation is

$$0 = ud^2u - du^2d \qquad \text{or} \qquad (du)(ud) = (ud)(du). \tag{2.11}$$

Therefore, the elements du and ud commute in $A = A(\alpha, \beta, \gamma)$. For any basis element $v_n \in V(\lambda)$, we have $du \cdot v_n = \lambda_n v_n$ and $ud \cdot v_n = \lambda_{n-1} v_n$. Using that with

$n = 0$ and $\lambda \neq 0$, it is easy to see that du and ud are linearly independent. Let $\mathfrak{h} = \mathbf{C}du \oplus \mathbf{C}ud$.

We say an A-module V is a **weight module** if $V = \sum_{\nu \in \mathfrak{h}^*} V_\nu$, where $V_\nu = \{v \in V \mid h \cdot v = \nu(h)v \text{ for all } h \in \mathfrak{h}\}$, and the sum is over elements in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. (Necessarily this is a direct sum.) Any submodule of a weight module is a weight module. If $V_\nu \neq (0)$, then ν is a **weight** and V_ν is the corresponding **weight space**. Each weight ν is determined by the pair (ν', ν'') of complex numbers, $\nu' = \nu(du)$ and $\nu'' = \nu(ud)$, and often it is convenient to identify ν with (ν', ν''). In particular, highest weight modules are weight modules in this sense. The basis vector v_n of $V(\lambda)$ is a weight vector whose weight is given by the pair $(\lambda_n, \lambda_{n-1})$. We will explore next when two weights $(\lambda_k, \lambda_{k-1})$ and $(\lambda_\ell, \lambda_{\ell-1})$ can be equal. For that we will need to know the values of λ_n more precisely.

Proposition 2.12. *Assume $\lambda_{-1} = 0$, $\lambda_0 = \lambda$, and λ_n for $n \geq 1$ is given by the recurrence relation $\lambda_n - \alpha\lambda_{n-1} - \beta\lambda_{n-2} = \gamma$. Fix $t \in \mathbf{C}$ such that*

$$t^2 = \frac{\alpha^2 + 4\beta}{4}.$$

(i) If $\alpha^2 + 4\beta \neq 0$, then

$$\lambda_n = c_1 r_1^n + c_2 r_2^n + x_n,$$

where

$$r_1 = \frac{\alpha}{2} + t, \qquad r_2 = \frac{\alpha}{2} - t,$$

$$x_n = \begin{cases} (1 - \alpha - \beta)^{-1}\gamma & \textit{if} \quad \alpha + \beta \neq 1 \\ (2 - \alpha)^{-1}\gamma n & \textit{if} \quad \alpha + \beta = 1 \quad (\textit{necessarily} \quad \alpha \neq 2), \end{cases}$$

and $$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{r_2 - r_1} \begin{pmatrix} r_2 & -1 \\ -r_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda - x_0 \\ \alpha\lambda + \gamma - x_1 \end{pmatrix}.$$

(ii) If $\alpha^2 + 4\beta = 0$ and $\alpha \neq 0$, then

$$\lambda_n = c_1 s^n + c_2 n s^n + x_n \quad \textit{where}$$

$$s = \frac{\alpha}{2}$$

$$x_n = \begin{cases} (1 - \alpha - \beta)^{-1}\gamma & \textit{if} \quad \alpha + \beta \neq 1 \\ 2^{-1} n^2 \gamma & \textit{if} \quad \alpha + \beta = 1 \quad \textit{i.e.} \quad \textit{if} \quad \alpha = 2,\ \beta = -1, \end{cases}$$

and $$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2\alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda - x_0 \\ \alpha\lambda + \gamma - x_1 \end{pmatrix}.$$

(iii) If $\alpha^2 + 4\beta = 0$ and $\alpha = 0$, then $\beta = 0$ and $\lambda_n = \gamma$ for all $n \geq 1$.

Note in particular, that if α, β are real, then it is natural to take $t = (1/2)\sqrt{\alpha^2 + 4\beta}$.

Proof. These assertions can be shown using standard arguments for solving linear recurrence relations ([Br, Chap. 7]). $\square$

Theorem 2.13. *Assume for the algebra $A = A(\alpha, \beta, \gamma)$ that $\lambda_\ell = \lambda_k$ and $\lambda_{\ell-1} = \lambda_{k-1}$ for some $\ell > k$. Then*

(a) $A = A(\alpha, \beta, \gamma)$, where $\alpha^2 + 4\beta \neq 0$ and $\alpha + \beta \neq 1$, and either
(1) $\gamma = 0$ and $\lambda_n = 0$ for all n,
(2) $\gamma \neq 0$, $\beta = 0$, and $\lambda_n = (1-\alpha)^{-1}\gamma$ for all n,
(3)

$$\lambda_n = \Big(\lambda - \frac{\gamma}{1-\alpha-\beta}\Big)r^n + \frac{\gamma}{1-\alpha-\beta},$$

where $r = r_1 = (1/2)\alpha + t$ or $r = r_2 = (1/2)\alpha - t$ $(t^2 = (1/4)(\alpha^2 + 4\beta))$, and $r^{\ell-k} = 1$, or
(4) $r_1^{\ell-k} = 1 = r_2^{\ell-k}$ and

$$\lambda_n = c_1 r_1^n + c_2 r_2^n + \frac{\gamma}{1-\alpha-\beta},$$

where c_1 and c_2 are as in Proposition 2.12;

(b) $A = A(\alpha, 1-\alpha, 0)$, where $\alpha \neq 2$, and either
(1) $\lambda_n = 0$ for all n, or $\alpha = 1$ and $\lambda_n = \lambda \neq 0$ for all n, or
(2) $\lambda \neq 0$, $(\alpha - 1)^{\ell-k} = 1$, and

$$\lambda_n = \frac{\lambda}{2-\alpha}\Big(1 - (\alpha-1)^{n+1}\Big);$$

(c) $A = A(\alpha, -(1/4)\alpha^2, \gamma)$ where $\alpha \neq 0, 2$, and either
(1) $\gamma = 0$ and $\lambda_n = 0$ for all n, or
(2) $\gamma \neq 0$, $s = (1/2)\alpha$ satisfies $s^{\ell-k} = 1$, and

$$\lambda_n = \frac{2\gamma}{(2-\alpha)^2}\Big(-\alpha s^n + 2\Big);$$

(d) $A = A(2, -1, 0)$ and $\lambda_n = 0$ for all n;

(e) $A = A(0, 0, \gamma)$ and $\lambda_n = \gamma$ for all $n \geq 1$.

Proof. We will divide considerations into cases according to the solutions given for λ_n by Proposition 2.12.

Case (a): $\alpha + 4\beta \neq 0$ and $\alpha + \beta \neq 1$.

In this first case $r_1 \neq r_2$ and

$$\lambda_n = c_1 r_1^n + c_2 r_2^n + \frac{\gamma}{1-\alpha-\beta}.$$

From $\lambda_\ell = \lambda_k$ and $\lambda_{\ell-1} = \lambda_{k-1}$ we see

$$\begin{aligned} c_1 r_1^k (r_1^{\ell-k} - 1) &= -c_2 r_2^k (r_2^{\ell-k} - 1) \\ c_1 r_1^{k-1} (r_1^{\ell-k} - 1) &= -c_2 r_2^{k-1} (r_2^{\ell-k} - 1). \end{aligned}$$

Multiplying the second by r_2 and subtracting gives

$$c_1 r_1^{k-1} (r_1^{\ell-k} - 1)(r_1 - r_2) = 0.$$

This shows that $c_1 = 0, r_1 = 0$, or $r_1^{\ell-k} = 1$. Analogously, $c_2 = 0, r_2 = 0$, or $r_2^{\ell-k} = 1$.

Suppose initially that $0 = r_2 = (1/2)\alpha - t$. Then $r_1 = (1/2)\alpha + t = \alpha \neq 0$, and $t^2 = (1/4)\alpha^2$ implies $\beta = 0$. From Proposition 2.12 we see

$$\lambda_n = c_1 \alpha^n + \frac{\gamma}{1-\alpha} = \Big(\lambda - \frac{\gamma}{(1-\alpha)}\Big)\alpha^n + \frac{\gamma}{1-\alpha}. \tag{2.14}$$

When $\gamma = 0$, then $\lambda_n = \lambda\alpha^n$, and $\lambda_\ell = \lambda_k$ forces $\lambda = 0$ (Case (1)) or $\alpha^{\ell-k} = 1$ (Case (3) with $\gamma = 0$). When $\gamma \neq 0$, then either $\lambda_n = (1-\alpha)^{-1}\gamma$ for all n (Case (2)) or λ_n is as in (2.14) where $\alpha^{\lambda-k} = 1$ (Case (3) with $\beta = 0$).

When $r_1 = 0$ and $r_2 = \alpha$, the solutions are precisely the same. So we can assume that neither r_1 nor r_2 is 0.

Suppose next that $c_1 = 0$. Then $\lambda = \lambda_0 = c_2 + (1-\alpha-\beta)^{-1}\gamma$, which gives $\lambda_n = \Big(\lambda - (1-\alpha-\beta)^{-1}\gamma\Big) r_2^n + (1-\alpha-\beta)^{-1}\gamma$. The condition $c_1 = 0$ says that $r_1\lambda = (1-\alpha-\beta)^{-1}\gamma(\alpha+\beta-r_2)$. When $c_2 = 0$, then $\lambda = (1-\alpha-\beta)^{-1}\gamma$ and we have

$$\frac{r_1\gamma}{1-\alpha-\beta} = \frac{(\alpha+\beta-r_2)\gamma}{1-\alpha-\beta}.$$

If $\gamma = 0$, then $\lambda = 0$ and $\lambda_n = 0$ for all n (Case (1)). If $\gamma \neq 0$, then $\alpha = r_1 + r_2 = \alpha + \beta$ forces $\beta = 0$ and $\lambda_n = (1-\alpha)^{-1}\gamma$ (Case (2)). The only other option is that $c_2 \neq 0$ but $r_2^{\ell-k} = 1$. That gives Case (3). The situation when $c_2 = 0$ and $c_1 \neq 0$ also gives Case (3).

Finally, if r_1, r_2, c_1, c_2 are all nonzero, then $r_1^{\ell-k} = 1 = r_2^{\ell-k}$ and

$$\lambda_n = c_1 r_1^n + c_2 r_2^n + \frac{\gamma}{1-\alpha-\beta},$$

for all n (Case (4)).

Case (b): $\alpha + 4\beta \neq 0$ and $\alpha + \beta = 1$.

Under these hypotheses, $\beta = 1 - \alpha$ and $0 \neq \alpha^2 + 4(1-\alpha) = (2-\alpha)^2$. Then $t^2 = (1/4)(2-\alpha)^2$, and we may assume that $t = -(1/2)(2-\alpha)$. From that we have $r_1 = \alpha - 1$, $r_2 = 1$, and

$$\lambda_n = c_1(\alpha-1)^n + c_2 + \frac{\gamma n}{2-\alpha}. \tag{2.15}$$

When $\lambda_\ell = \lambda_k$ and $\lambda_{\ell-1} = \lambda_{k-1}$, then

$$\begin{aligned} c_1(\alpha-1)^\ell + c_2 + \frac{\gamma\ell}{2-\alpha} &= c_1(\alpha-1)^k + c_2 + \frac{\gamma k}{2-\alpha} \\ c_1(\alpha-1)^{\ell-1} + c_2 + \frac{\gamma(\ell-1)}{2-\alpha} &= c_1(\alpha-1)^{k-1} + c_2 + \frac{\gamma(k-1)}{2-\alpha} \end{aligned}$$

or

$$\begin{aligned} c_1(\alpha-1)^k\Big((\alpha-1)^{\ell-k} - 1\Big) &= \frac{\gamma(k-\ell)}{2-\alpha} \\ c_1(\alpha-1)^{k-1}\Big((\alpha-1)^{\ell-k} - 1\Big) &= \frac{\gamma(k-\ell)}{2-\alpha}. \end{aligned} \tag{2.16}$$

Therefore,

$$c_1(\alpha-1)^{k-1}(\alpha-2)\Big((\alpha-1)^{\ell-k} - 1\Big) = 0.$$

Since $\alpha \neq 2$, we see that the left-hand sides of (2.16) are 0, and $\gamma = 0$. Solving for c_1, c_2 from Proposition 2.12 we find that $c_1 = (2-\alpha)^{-1}(1-\alpha)\lambda$ and $c_2 = (2-\alpha)^{-1}\lambda$. Therefore,

$$\lambda_n = \frac{\lambda(1-\alpha)}{2-\alpha}(\alpha-1)^n + \frac{\lambda}{2-\alpha}1^n = \frac{\lambda}{2-\alpha}\Big(1 - (\alpha-1)^{n+1}\Big). \tag{2.17}$$

The condition $\lambda_\ell = \lambda_k$ implies that $\lambda = 0$, and hence all $\lambda_n = 0$ (Case (1)), or that $\alpha = 1$ and hence that $\lambda_n = \lambda$ for all n (Case (1)), or that $\lambda \neq 0$ and $(\alpha-1)^{\ell-k} = 1$ (Case (2)).

Case (c): $\alpha^2 + 4\beta = 0$, $\alpha + \beta \neq 1$ and $\alpha \neq 0$.

Here $\beta = -(1/4)\alpha^2$ so that

$$\lambda_n = c_1 s^n + c_2 n s^n + \frac{\gamma}{1-\alpha-\beta} = c_1 s^n + c_2 n s^n + \frac{4\gamma}{(2-\alpha)^2}. \tag{2.18}$$

It follows from $\lambda_\ell = \lambda_k$ and $\lambda_{\ell-1} = \lambda_{k-1}$ that

$$\begin{aligned} c_1 s^k \Big(s^{\ell-k} - 1\Big) &= -c_2 s^k \Big(\ell s^{\ell-k} - k\Big) \\ c_1 s^{k-1} \Big(s^{\ell-k} - 1\Big) &= -c_2 s^{k-1} \Big((\ell-1) s^{\ell-k} - (k-1)\Big) \end{aligned} \tag{2.19}$$

Since $s = (1/2)\alpha \neq 0$, we may use these equations to see that

$$c_2(s^{\ell-k} - 1) = 0 = c_1(s^{\ell-k} - 1).$$

Now if $s^{\ell-k} = 1$, then (2.19) shows that $c_2 = 0$, which says that $\lambda = (2-\alpha)^{-1}(2\gamma)$. Therefore, since $c_1 = \lambda - (2-\alpha)^{-2} 4\gamma$, we have

$$\lambda_n = \Big(\lambda - \frac{4\gamma}{(2-\alpha)^2}\Big) s^n + \frac{4\gamma}{(2-\alpha)^2} = \frac{2\gamma}{(2-\alpha)^2}\Big(-\alpha s^n + 2\Big), \tag{2.20}$$

where $s = (1/2)\alpha$ satisfies $s^{\ell-k} = 1$. This gives the solutions in (c). Now if $s^{\ell-k} \neq 1$, then $c_1 = 0 = c_2$ and $\lambda_n = (2-\alpha)^{-2}4\gamma$ for all n. Those relations imply $(2-\alpha)^{-1}2\gamma = (2-\alpha)^{-2}4\gamma$, which gives $\gamma = 0$ or $\alpha = 0$. Since $\alpha \neq 0$ by assumption, $\gamma = 0$ and $\lambda_n = 0$ for all n.

Case (d): $\alpha^2 + 4\beta = 0$ *and* $\alpha = 2$ *and* $\beta = -1$.

The values of λ_n are given by $\lambda_n = c_1 s^n + c_2 n s^n + (1/2)\gamma n^2$, which reduces to $\lambda_n = c_1 + c_2 n + (1/2)\gamma n^2$ because $s = \alpha/2 = 1$. Solving for c_1 and c_2 using Proposition 2.12, we see that $c_1 = \lambda$ and $c_2 = \lambda + \gamma/2$. Therefore,

$$\begin{aligned} \lambda_n &= c_1 + c_2 n + \frac{\gamma n^2}{2} = \lambda + (\lambda + \frac{\gamma}{2})n + \frac{\gamma n^2}{2} \\ &= (\frac{\gamma}{2} n + \lambda)(n+1). \end{aligned} \tag{2.21}$$

Then

$$\lambda_\ell = \lambda + (\lambda + \frac{\gamma}{2})\ell + \frac{\gamma \ell^2}{2} = \lambda + (\lambda + \frac{\gamma}{2})k + \frac{\gamma k^2}{2} = \lambda_k$$

implies that

$$(\frac{\gamma}{2})(\ell^2 - k^2) + (\lambda + \frac{\gamma}{2})(\ell - k) = 0.$$

Similarly, from $\lambda_{\ell-1} = \lambda_{k-1}$ we obtain

$$(\frac{\gamma}{2})((\ell-1)^2 - (k-1)^2) + (\lambda + \frac{\gamma}{2})(\ell - k) = 0,$$

and those equations combine to give

$$\gamma(\ell - k) = 0.$$

It must be $\gamma = 0$ and $\lambda_n = \lambda(n+1)$. But then $\lambda_\ell = \lambda(\ell+1) = \lambda(k+1) = \lambda_k$ shows that $\lambda = 0$. We conclude that for the algebra $A(2,-1,\gamma)$, the only time two weights $(\lambda_\ell, \lambda_{\ell-1})$ and $(\lambda_k, \lambda_{k-1})$ can be equal is if $\gamma = 0$ and $\lambda_n = 0$ for all n.

Case (e): $\alpha = \beta = 0$

In this case $\lambda_n = \gamma$ for all $n \geq 1$. $\square$

Corollary 2.22. *In $V(\lambda)$ each weight space is either one-dimensional or infinite-dimensional. If an infinite-dimensional weight space occurs, there are only finitely many weights. The cases when $V(\lambda)$ has an infinite-dimensional weight space are the ones listed in Theorem 2.13.*

Submodules of $V(\lambda)$.

Proposition 2.23. *If each weight space of $V(\lambda)$ is one-dimensional (i.e. if we are not in one of the cases described by Theorem 2.13), then the proper submodules of $V(\lambda)$ have the form $N = span_{\mathbf{C}}\{v_j \mid j \geq n\}$ for some $n > 0$ with $\lambda_{n-1} = 0$, and hence they are contained in $M(\lambda)$.*

Proof. Suppose N is a proper submodule of $V(\lambda)$. There exists some weight $\nu = (\lambda_n, \lambda_{n-1})$ such that $N_\nu \neq (0)$, and we may suppose that n is the least value with that property. Then $v_n \in N$, and n is the smallest value for which that is true. Applying u we see that $v_j \in N$ for all $j \geq n$. $\square$

To describe the submodules of $V(\lambda)$ when its weights are given by Theorem 2.13, it is helpful to observe that if N is a proper submodule of any $V(\lambda)$, then

$$I_N = \{f \in \mathbf{C}[x] \mid f(u) \cdot v_0 \in N\} \tag{2.24}$$

is an ideal of $\mathbf{C}[x]$. There exists a monic polynomial $g(x)$ which generates that ideal. Then $\dim_{\mathbf{C}} V(\lambda)/N = \deg(g) = k$, and $v_i + N$ for $i = 0, 1, \ldots, k-1$ is a basis for $V(\lambda)/N$.

The situations described by Theorem 2.13 can be grouped into four general types (see 2.17 and 2.20):

(2.25)
(i) $\gamma = 0$ and $\lambda_n = 0$ for all n.
(ii) $\lambda_n = c$ for all n where $c \in \mathbf{C}$ satisfies $c - \alpha c - \beta c = \gamma$.

(iii) $\lambda_n = (\lambda - c)r^n + c$ where $c \in \mathbf{C}$ and r is a root of $x^2 - \alpha x - \beta = 0$. There is a least integer $p \in \mathbf{Z}_{>0}$ such that $r^p = 1$.

(iv) $\lambda_n = c_1 r_1^n + c_2 r_2^n + c$ where $r_1 \neq r_2$ are roots of the equation $x^2 - \alpha x - \beta = 0$, and there is some smallest value $p \in \mathbf{Z}_{>0}$ so that $r_1^p = 1 = r_2^p$.

(i) Assume first that $A = A(\alpha, \beta, 0)$ and consider the module $V(0)$. Since $\lambda_n = 0$ for all n, for any $g \in \mathbf{C}[x]$,

$$N^{(g)} = \mathrm{span}_{\mathbf{C}}\{f(u)g(u)v_0 \mid f \in \mathbf{C}[x]\} \tag{2.26}$$

is a submodule of $V(0)$, which is proper if $\deg(g) \geq 1$. For each $\xi \in \mathbf{C}$, the submodule $N^{(x-\xi)}$ is maximal, and every maximal submodule has this form. In particular, $M(\lambda) = N^{(x)}$. The quotient module $L(0, \xi) \overset{\text{def}}{=} V(0)/N^{(x-\xi)}$ is spanned by $v_0 + N^{(x-\xi)}$ where $d \cdot (v_0 + N^{(x-\xi)}) = 0$ and $u \cdot (v_0 + N^{(x-\xi)}) = \xi v_0 + N^{(x-\xi)}$.

(ii) Next suppose that $A = A(\alpha, \beta, \gamma)$, and the module is $V(c)$ where $0 \neq c \in \mathbf{C}$ is such that $c - \alpha c - \beta c = \gamma$. Then $\lambda_n = c \neq 0$ for all n. By Proposition 2.4, $V(c)$ is simple.

(iii) and (iv) These two cases behave similarly, and so we will treat them simultaneously. Assume for $A = A(\alpha, \beta, \gamma)$ that the module $V(\lambda)$ has weights given by

$$\lambda_n = (\lambda - c)r^n + c \text{ where } c \in \mathbf{C},\ \lambda \neq c, \text{ and } r^p = 1, \text{ or}$$
$$\lambda_n = c_1 r_1^n + c_2 r_2^n + c \text{ where } c \in \mathbf{C},\ c_1, c_2 \neq 0, \text{ and } r_1^p = 1 = r_2^p.$$

In each instance, we will assume that p is chosen to be minimal. Then $\lambda_{n+p} = \lambda_n$ for all $n \geq 0$, and $V(\lambda) = \sum_{i=0}^{p} V(\lambda)_{\nu_i}$, where $\nu_i = (\lambda_i, \lambda_{i-1})$. Now $\nu_p = \nu_0$ if and only if $\lambda_{p-1} = 0$, which happens, for example, if $\beta \neq 0$.

Suppose first that $\nu_p \neq \nu_0$, so that $V(\lambda)$ has $p+1$ distinct weights. If N is a submodule of $V(\lambda)$ and $N_{\nu_0} \neq (0)$, then $N = V(\lambda)$. In particular, any proper submodule N is contained in $M(\lambda)$ by Proposition 2.4, so that $M(\lambda)$ is the unique maximal submodule. We may assume that $N_{\nu_j} \neq (0)$ for some $j = 1, \ldots, p$. Then $(0) \neq u^{p-j+k} N_{\nu_j} \subseteq N_{\nu_k}$ so that all $N_{\nu_k} \neq (0)$. Therefore, for $k \neq p$,

$$\{f \in \mathbf{C}[x] \mid u^k f(u^p) \cdot v_0 \in N_{\nu_k}\} \tag{2.27}$$

is a nonzero ideal of $\mathbf{C}[x]$, and we may assume that g_k is its monic generator. Similarly, $\{f \in \mathbf{C}[x] \mid f(u^p) \cdot v_0 \in N_{\nu_p}\}$ is a nonzero ideal, whose monic generator we denote by g_p. Then $u^k g_k(u^p) \cdot v_0 \in N_{\nu_k}$ implies $u(u^k g_k(u^p) \cdot v_0) \in N_{\nu_{k+1}}$. It follows that

$$g_{p-1} \mid g_{p-2} \mid \cdots \mid g_2 \mid g_1 \mid g_p \mid x g_{p-1}.$$

Either $g_1 = g_2 = \cdots = g_p = g$ and $N = N^{(g(x^p))}$ (see (2.26) for the notation), or there is some value m such that $g_{p-1} = \cdots = g_{m+1} = g$ and $g_m = \cdots = g_1 = g_p = xg$. Since $d(u^{m+1}g(u^p)\cdot v_0) = \lambda_m u^m g(u^p)\cdot v_0$, it must be that $\lambda_m = 0$ in this case. It is easy to see that $\lambda_k \neq \lambda_\ell$ for $1 \leq k < \ell \leq p$. Therefore m is the minimal value such that $\lambda_m = 0$. In this situation, $N = N^{(x^{m+1}g(x^p))}$. As a special case, we have $M(\lambda) = N^{(x^{m+1})}$.

In the case that $\nu_p = \nu_0$, the module $V(\lambda)$ has p distinct weights. Assume N is a proper submodule of $V(\lambda)$. Arguing as above, we have that $N_{\nu_k} \neq (0)$ for each $k = 0, 1, \ldots, p-1$. Letting g_k be the monic generator for the ideal in (2.27), we have

$$g_{p-1} \mid g_{p-2} \mid \cdots \mid g_2 \mid g_1 \mid g_0 \mid xg_{p-1}.$$

Again there are two possibilities. Either $g_0 = g_1 = \cdots = g_{p-1} = g$ and $N = N^{(g(x^p))}$, or else there is some value m such that $g_{p-1} = \cdots = g_{m+1} = g$ and $g_m = \cdots = g_1 = g_0 = xg$. Here $N = N^{(x^{m+1}g(x^p))}$, and $M(\lambda) = N^{(x^{m+1})}$.

Corollary 2.28. *If $\gamma = 0 = \lambda$, then $V(\lambda)$ has infinitely many maximal proper submodules, each of the form $N^{(x-\xi)} = \mathrm{span}_{\mathbf{C}}\{f(u)(u-\xi)v_0 \mid f \in \mathbf{C}[x]\} = \mathrm{span}_{\mathbf{C}}\{v_n - \xi v_{n-1} \mid n = 1, 2, \ldots\}$ for some $\xi \in \mathbf{C}$, and infinitely many one-dimensional simple modules $L(0,\xi) = V(0)/N^{(x-\xi)}$. In all other cases, $M(\lambda)$ is the unique maximal submodule of $V(\lambda)$, and there is a unique simple highest weight module of weight λ, $L(\lambda) \stackrel{\text{def}}{=} V(\lambda)/M(\lambda)$, up to isomorphism.*

Enveloping algebra examples.

Recall that the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is isomorphic to the algebra $A(2,-1,-2)$, and the universal enveloping algebra $U(\mathfrak{H})$ of the Heisenberg Lie algebra $\mathfrak{H}$ is isomorphic to $A(2,-1,0)$. For all the algebras $A(2,-1,\gamma)$, we have computed the values of λ_n in the proof of Case (d) of Theorem 2.13, and from (2.21) we see that

$$\lambda_n = (\frac{\gamma}{2}n + \lambda)(n+1).$$

In the $\mathfrak{sl}_2$-case, the operator $h = du - ud$ is used rather than du. The eigenvalues of h are $\lambda_n - \lambda_{n-1} = \lambda + n\gamma = \lambda - 2n$, $n = 0, 1, \ldots,$ (as is customary in the representation theory of $\mathfrak{sl}_2$), and $V(\lambda)$ is simple if and only if $\lambda \notin \mathbf{Z}_{\geq 0}$. The analogous computation in the Heisenberg Lie algebra shows that the central element $z = du - ud$ has constant eigenvalue $\lambda_n = \lambda$.

Quantum examples.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra (for example, a finite-dimensional simple Lie algebra) over $\mathbf{C}$ corresponding to the Cartan matrix $\mathfrak{A} = (a_{i,j})_{i,j=1}^n$. Then there are relatively prime integers ℓ_i so that the matrix $(\ell_i a_{i,j})$ is symmetric. Let $\mathbf{C}(q)$ be the field of rational functions in the indeterminate q over $\mathbf{C}$. Assume

$$q_i = q^{\ell_i}, \qquad \text{and} \qquad [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$$

for all $m \in \mathbf{Z}_{\geq 0}$. When $m \geq 1$, let

$$[m]_i! = \prod_{j=1}^{m} [j]_i.$$

Set $[0]_i! = 1$ and define

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i![m-n]_i!}.$$

Then the quantized enveloping algebra (quantum group) $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is the unital associative algebra over $\mathbf{C}(q)$ with generators E_i, F_i, K_i, K_i^{-1} $(i = 1, \ldots, n)$ subject to the relations

(Q1) $K_i K_i^{-1} = K_i^{-1} K_i, \qquad K_i K_j = K_j K_i$

(Q2) $K_i E_j K_i^{-1} = q_i^{a_{i,j}} E_j$

(Q3) $K_i F_j K_i^{-1} = q_i^{-a_{i,j}} F_j$

(Q4) $E_i F_j - F_j E_i = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

(Q5) $\displaystyle\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_i E_i^{1-a_{i,j}-k} E_j E_i^k = 0 \qquad \text{for} \quad i \neq j$

(Q6) $\displaystyle\sum_{k=0}^{1-a_{i,j}} (-1)^k \begin{bmatrix} 1-a_{i,j} \\ k \end{bmatrix}_i F_i^{1-a_{i,j}-k} F_j F_i^k = 0 \qquad \text{for} \quad i \neq j.$

Suppose $a_{i,j} = -1 = a_{j,i}$ for some $i \neq j$, and consider the subalgebra $U_{i,j}$ generated by E_i, E_j. In this special case, the quantum Serre relation (Q5) reduces to

$$E_i^2 E_j - [2]_i E_i E_j E_i + E_j E_i^2 = 0 \qquad \text{and}$$
$$E_j^2 E_i - [2]_j E_j E_i E_j + E_i E_j^2 = 0.$$

Since $-\ell_i = \ell_i a_{i,j} = \ell_j a_{j,i} = -\ell_j$, the coefficients $[2]_i$ and $[2]_j$ are equal. The algebra $U_{i,j}$ (with q is specialized to a complex number which is not a root of unity) is isomorphic to $A([2]_i, -1, 0)$ by the mapping $E_i \mapsto d$, $E_j \mapsto u$. The same result is true if the corresponding F's are used in place of the E's. In particular, when $\mathfrak{g} = \mathfrak{sl}_3$, the algebra $U_{i,j}$ is just the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by the E's.

Here we compute the values of λ_n in the case of the algebra $A([2]_i, -1, 0)$. For convenience of notation we write

$$p = q_i.$$

Then $\alpha = [2]_i = \dfrac{p^2 - p^{-2}}{p - p^{-1}} = p + p^{-1}$, $\beta = -1$ and $\gamma = 0$ so that

$$\alpha^2 + 4\beta = p^2 + 2 + p^{-2} - 4 = (p - p^{-1})^2.$$

Thus

$$\begin{aligned} r_1 &= \frac{p + p^{-1} + p - p^{-1}}{2} = p \\ r_2 &= \frac{p + p^{-1} - (p - p^{-1})}{2} = p^{-1}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \frac{1}{p^{-1} - p} \begin{pmatrix} p^{-1} & -1 \\ -p & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ (p + p^{-1})\lambda \end{pmatrix} \\ &= \frac{1}{p^{-1} - p} \begin{pmatrix} -p\lambda \\ p^{-1}\lambda \end{pmatrix} = \frac{1}{p - p^{-1}} \begin{pmatrix} p\lambda \\ -p^{-1}\lambda \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_n &= \frac{p\lambda}{p - p^{-1}} p^n - \frac{p^{-1}\lambda}{p - p^{-1}} p^{-n} \\ &= \Big(\frac{p^{n+1} - p^{-(n+1)}}{p - p^{-1}} \Big) \lambda \\ &= [n+1]_i \lambda. \end{aligned}$$

(q, r)-differential poset examples.

For an algebra arising from a (q, r)-differential poset the case of interest is the algebra $A = A(\alpha, \beta, \gamma)$ with defining relations

$$\begin{aligned} d^2 u &= q(q+1) dud - q^3 ud^2 + rd, \\ du^2 &= q(q+1) udu - q^3 u^2 d + ru. \end{aligned}$$

Here r is an arbitrary element of the field. For this algebra $\alpha^2+4\beta=q^2(q-1)^2$, so we may assume that $t=(1/2)q(q-1)$ and $r_1=1/2(\alpha+q(q-1))=q^2$ and $r_2=1/2(\alpha-q(q-1))=q$. (We will suppose throughout that $q\neq 0,\pm 1$). Then

$$1-\alpha-\beta=1-q(q+1)+q^3=q^2(q-1)-(q-1)=(q-1)(q^2-1)$$

so that

$$x_n=(1-\alpha-\beta)^{-1}\gamma=\frac{r}{(q-1)(q^2-1)}$$

for all n. We call that common value x. Recall that

$$\begin{pmatrix} c_1\\ c_2\end{pmatrix}=\frac{1}{r_2-r_1}\begin{pmatrix} r_2 & -1\\ -r_1 & 1\end{pmatrix}\begin{pmatrix}\lambda-x\\ \alpha\lambda+r-x\end{pmatrix}.$$

Now

$$\frac{1}{r_2-r_1}=\frac{1}{q(1-q)},$$

and putting that into the above equation, we get after a bit of calculation

$$\begin{aligned} c_1&=\frac{1}{q(1-q)}\Big(q(\lambda-x)-(\alpha\lambda+r-x)\Big)\\ &=\frac{-q}{1-q}\Big(\lambda+\frac{r}{q^2-1}\Big)\\ c_2&=\frac{1}{q(1-q)}\Big(-q^2(\lambda-x)+(\alpha\lambda+r-x)\Big)\\ &=\frac{1}{1-q}\Big(\lambda+\frac{r}{q-1}\Big). \end{aligned}$$

Thus the eigenvalues of du for this algebra are

$$\begin{aligned}\lambda_n&=\frac{-q}{1-q}\Big(\lambda+\frac{r}{q^2-1}\Big)q^{2n}+\frac{1}{1-q}\Big(\lambda+\frac{r}{q-1}\Big)q^n+\frac{r}{(q-1)(q^2-1)}\\ &=\frac{q^{n+1}-1}{q-1}\Big(q^n\lambda+\frac{q^n-1}{q^2-1}r\Big).\end{aligned}$$

Fibonacci examples.

For the algebra $A(1,1,0)$, the solutions to the associated linear recurrence $\lambda_n=\lambda_{n-1}+\lambda_{n-2}$, $\lambda_0=\lambda$, $\lambda_{-1}=0$, (hence the eigenvalues of du on $V(\lambda)$) are given by the Fibonacci sequence $\lambda_0=\lambda$, $\lambda_1=\lambda$, $\lambda_2=2\lambda$, $\lambda_3=3\lambda$, $\lambda_4=5\lambda$, ... In this case, the equations in Proposition 2.12 reduce to

$$\lambda_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where

$$c_1 = \frac{\lambda\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)$$
$$c_2 = -\frac{\lambda\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)$$

or

$$\lambda_n = \lambda\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

Lowest weight modules.

To create lowest weight modules for $A = A(\alpha,\beta,\gamma)$ we reverse the roles of d and u. Thus, we assume there is a vector w_0 such that $u \cdot w_0 = 0$, $ud \cdot w_0 = \kappa w_0$ and $W = Aw_0$. When $\beta \neq 0$, the eigenvalues of du are given by the sequence which has $\kappa_{-1} = 0$, $\kappa_0 = \kappa$, an arbitrary complex number, and

$$\begin{aligned} &\beta\kappa_n + \alpha\kappa_{n-1} - \kappa_{n-2} = -\gamma, \quad \text{or equivalently} \\ &\kappa_n = \beta^{-1}\big(-\alpha\kappa_{n-1} + \kappa_{n-2} - \gamma\big) \end{aligned} \tag{2.29}$$

for all $n \geq 1$.

Proposition 2.30. *Let $W(\kappa)$ be the $\mathbf{C}$-vector space having basis $\{w_n \mid n = 0,1,2,\dots\}$.*

(a) Assume $\beta \neq 0$, $\kappa_{-1} = 0$, and $\kappa_0 = \kappa$, an arbitrary element of $\mathbf{C}$. Suppose κ_n for $n \geq 1$ is as in (2.29), and define

$$\begin{aligned} &u \cdot w_n = \kappa_{n-1} w_{n-1}, \quad n \geq 1, \quad \textit{and} \quad u \cdot w_0 = 0 \\ &d \cdot w_n = w_{n+1}. \end{aligned} \tag{2.31}$$

Then this action gives $W(\kappa)$ the structure of a lowest weight $A(\alpha,\beta,\gamma)$-module.

(b) When $\beta = 0$ and $\alpha \neq 0$, let κ_n be defined by

$$\kappa_n = -\gamma \sum_{j=1}^{n+1} \alpha^{-j}$$

for all $n \geq 0$. Then $W(-\gamma\alpha^{-1})$ with the action given by (2.31) is a lowest weight $A(\alpha,\beta,\gamma)$-module.

(c) *When $\gamma \neq 0$, the only lowest weight $A(0,0,\gamma)$-module is the 1-dimensional module $W = \mathbf{C}w_0$ with $d \cdot w_0 = 0 = u \cdot w_0$. When $\gamma = 0$, set $\kappa_n = 0$ for all n. Then $W(0)$ with the action given by (2.31) is a lowest weight $A(0,0,0)$-module.*

The proof of this result is identical to that of Proposition 2.2, and the following proposition can be shown using standard recurrence relation techniques.

Proposition 2.32. *Assume $\beta \neq 0$ and fix $t \in \mathbf{C}$ such that*

$$t^2 = \frac{\alpha^2 + 4\beta}{4\beta}.$$

Let $\kappa_{-1} = 0$, $\kappa_0 = \kappa \in \mathbf{C}$, and for $n \geq 1$, let κ_n be given by the recurrence relation $\beta\kappa_n + \alpha\kappa_{n-1} - \kappa_{n-2} = -\gamma$.

(i) If $\alpha^2 + 4\beta \neq 0$, then

$$\kappa_n = c_1 r_1^n + c_2 r_2^n + x_n,$$

where

$$r_1 = -\frac{\alpha}{2\beta} + t, \qquad r_2 = -\frac{\alpha}{2\beta} - t,$$
$$x_n = \begin{cases} (1-\alpha-\beta)^{-1}\gamma & \text{if } \alpha+\beta \neq 1 \\ (\alpha-2)^{-1}\gamma n & \text{if } \alpha+\beta = 1 \text{ and } \alpha \neq 2, \end{cases}$$
$$\text{and} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{r_2 - r_1}\begin{pmatrix} r_2 & -1 \\ -r_1 & 1 \end{pmatrix}\begin{pmatrix} \lambda - x_0 \\ \alpha\lambda + \gamma - x_1 \end{pmatrix}.$$

(ii) If $\alpha^2 + 4\beta = 0$, then $\alpha \neq 0$ and

$$\kappa_n = c_1 s^n + c_2 n s^n + x_n$$

where

$$s = \frac{\alpha}{2\beta}$$
$$x_n = \begin{cases} (1-\alpha-\beta)^{-1}\gamma & \text{if } \alpha+\beta \neq 1 \\ -\frac{\gamma}{2}n^2 & \text{if } \alpha+\beta = 1 \text{ i.e. if } \alpha \neq 2 \text{ and } \beta = -1, \end{cases}$$
$$\text{and} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2\beta}{\alpha} & \frac{2\beta}{\alpha} \end{pmatrix}\begin{pmatrix} \kappa - x_0 \\ -\frac{(\gamma+\alpha\kappa)}{\beta} - x_1 \end{pmatrix}.$$

Doubly-infinite modules.

Proposition 2.33. *Assume $\beta \neq 0$ and let κ, λ be arbitrary but fixed complex numbers. Suppose $\lambda_0 = \lambda$, $\lambda_1 = \alpha\lambda + \beta\kappa + \gamma$, and for $n \geq 2$, λ_n is defined by the recurrence relation $\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma$. Set $\lambda_{-1} = \kappa$ and define λ_{n-2} for all $n \leq 0$ by the equation $\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma$. The complex vector space $V(\kappa, \lambda)$ with basis $\{v_n \mid n \in \mathbf{Z}\}$ and with*

$$\begin{aligned} d \cdot v_n &= \lambda_{n-1} v_{n-1} \\ u \cdot v_n &= v_{n+1} \end{aligned} \tag{2.34}$$

is an $A(\alpha, \beta, \gamma)$-module.

Proof. As in our previous arguments, we may assume there is an action of the free associative algebra generated by d and u on $V(\kappa, \lambda)$ given by (2.34). Then

$$\begin{aligned} \Big(d^2u - \alpha dud - \beta ud^2 - \gamma d\Big)v_m &= \lambda_{m-1}\Big(\lambda_m - \alpha\lambda_{m-1} - \beta\lambda_{m-2} - \gamma\Big)v_{m-1} = 0 \\ \Big(du^2 - \alpha udu - \beta u^2d - \gamma u\Big)v_m &= \Big(\lambda_{m+1} - \alpha\lambda_m - \beta\lambda_{m-1} - \gamma\Big)v_m = 0, \end{aligned}$$

so that there is an induced action by $A(\alpha, \beta, \gamma)$. $\square$

If $\lambda_m = 0$ for any m, then the span of the vectors $\{v_n \mid n \geq m+1\}$ is a submodule of $V(\kappa, \lambda)$.

Dual modules.

Suppose V is any module for $A = A(\alpha, \beta, \gamma)$ and let $\{v_a \mid a \in I\}$ be a basis for V. Assume $v_a^* \in \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ is defined by $v_a^*(v_b) = \delta_{a,b}$ for all $a, b \in I$. Let V^* be the finite dual of V (i.e. the finite linear combinations of the elements v_a^*). We will show that V^* carries the structure of an A-module.

The antiautomorphism η

Define a mapping $\eta : A \to A$ by first specifying $\eta(u) = d$ and $\eta(d) = u$ and extending this to an antiautomorphism of the free associative algebra generated by u and d. Then

$$\eta(d^2u) = du^2 = \alpha udu + \beta u^2d + \gamma u = \eta(\alpha dud + \beta ud^2 + \gamma d),$$

and a similar relation holds for du^2. Therefore η induces an antiautomorphism of order 2 on A (which we also denote by η).

Now define

$$(x \cdot v^*)(w) = v^*(\eta(x) \cdot w)$$

for all $x \in A$, $w \in V$, and $v^* \in V^*$. It is easy to see that this makes V^* an A-module.

Suppose now that $V = V(\lambda)$ is a highest weight A-module with basis $\{v_n \mid n = 0, 1, \dots\}$. Then

$$\begin{aligned}(u \cdot v_\ell^*)(v_n) &= v_\ell^*(d \cdot v_n) = \lambda_{n-1} v_\ell^*(v_{n-1}) \\ &= \begin{cases} 0 & \text{if } n \neq \ell + 1 \\ \lambda_\ell & \text{if } n = \ell + 1. \end{cases}\end{aligned}$$

Therefore $u \cdot v_\ell^* = \lambda_\ell v_{\ell+1}^*$. Similarly, $d \cdot v_\ell^* = v_{\ell-1}^*$. If $\lambda_n = 0$ for some n, then $V(\lambda)^*$ is not generated by v_0^*. But whenever $\lambda_n \neq 0$ for all n, then $V(\lambda)^*$ is a highest weight module. In fact, in that case $V(\lambda)^* \cong V(\lambda)$ under the mapping which sends $\lambda_{n-1} \cdots \lambda_1 \lambda_0 v_n^*$ to v_n. We summarize what we have shown for highest weight modules and the analogous results for lowest weight and doubly-infinite modules in the following proposition.

Proposition 2.35. *If Y is a module for $A(\alpha, \beta, \gamma)$, let Y^* denote the finite dual of Y.*

(a) If $V(\lambda)$ is a highest weight module as in Proposition 2.2, then $V(\lambda)^*$ is an $A(\alpha, \beta, \gamma)$-module (which is a highest weight module isomorphic to $V(\lambda)$ whenever $\lambda_n \neq 0$ for all n) under the action

$$\begin{aligned} u \cdot v_n^* &= \lambda_n v_{n+1}^* \\ d \cdot v_n^* &= v_{n-1}^*. \end{aligned}$$

(b) If $W(\kappa)$ is a lowest weight module as in Proposition 2.30, then $W(\kappa)^*$ is an $A(\alpha, \beta, \gamma)$-module (which is a lowest weight module isomorphic to $W(\kappa)$ whenever $\kappa_n \neq 0$ for all n) under the action

$$\begin{aligned} u \cdot w_n^* &= w_{n-1}^* \\ d \cdot v_n^* &= \kappa_n w_{n+1}^*. \end{aligned}$$

(c) If $V(\kappa, \lambda)$ is a doubly-infinite module as in Proposition 2.35, then $V(\kappa, \lambda)^*$ is a doubly-infinite module under the action

$$\begin{aligned} u \cdot v_n^* &= \lambda_n v_{n+1}^* \\ d \cdot v_n^* &= v_{n-1}^*. \end{aligned}$$

§3. Structural results on down-up algebras

Theorem 3.1. *(Poincaré-Birkhoff-Witt Theorem) Assume* $A = A(\alpha, \beta, \gamma)$ *is a down-up algebra over* $\mathbf{C}$. *Then* $\{u^i(du)^j d^k \mid i, j, k = 0, 1, \dots\}$ *is a basis of* A.

Proof. It is clear from the defining relators $f_1 = d(du) - \alpha(du)d - \beta ud^2 - \gamma d$ and $f_2 = (du)u - \alpha u(du) - \beta u^2 d - \gamma u$ that the monomials $u^i(du)^j d^k$, $i, j, k = 0, 1, \dots$ are all the irreducible words in A and so span A. We show that they are linearly independent by applying the Diamond Lemma (see [Be]).

We can order the words in the free algebra $\mathbf{C}\langle d, u\rangle$ by their total degree in d and u, and then lexicographically with $d > u$. Two words in $\mathbf{C}\langle d, u\rangle$ *admit a composition* if an end of one is a beginning of the other. We need to argue that whenever the largest words $w_1 = d^2u$ and $w_2 = du^2$ in the relators f_1 and f_2 admit a composition say, $w_1 = ab$ and $w_2 = bc$ or $w_2 = a'b'$ and $w_1 = b'c'$, then

$$\begin{aligned} f_1 c - a f_2 &= a_1 t_1 f_1 x_1 + a_2 t_2 f_2 x_2 \qquad \text{or} \\ f_2 c' - a' f_1 &= b_1 y_1 f_1 z_1 + b_2 y_2 f_2 z_2, \end{aligned}$$

where $a_i, b_i \in \mathbf{C}$, t_i, x_i, y_i, z_i are words in d, u (possibly empty), $t_i f_i x_i < abc$, and $y_i f_i z_i < a'b'c'$ for $i = 1, 2$. There is exactly one admissible composition, $d^2u^2 = d(du)u$, of the largest words in the relators, and

$$\begin{aligned} (d^2u - \alpha dud - \beta ud^2 - \gamma d)u &- d(du^2 - \alpha udu - \beta u^2 d - \gamma u) \\ &= \beta ud^2u - \beta du^2 d = 0. \end{aligned}$$

Therefore by the Diamond Lemma, the irreducible words form a basis of A. $\square$

Corollary 3.2. *The Gelfand-Kirillov dimension of any down-up algebra* $A(\alpha, \beta, \gamma)$ *is 3.*

Proof. Let $A^{(n)} = \mathrm{span}_{\mathbf{C}}\{u^i(du)^j d^k \mid i + 2j + k \le n\}$. The spaces $A^{(n)}$ afford a filtration $(0) \subset A^{(0)} \subset A^{(1)} \subset \cdots \subset \cup_n A^{(n)} = A(\alpha, \beta, \gamma)$ of the down-up algebra, and $A^{(m)}A^{(n)} \subseteq A^{(m+n)}$ since the defining relations replace the words d^2u and du^2 by words of the same or lower total degree. The number of monomials $u^i(du)^j d^k$ with $i + 2j + k = \ell$ is $(m+1)(m+1)$ if $\ell = 2m$ and is $(m+1)(m+2)$ if $\ell = 2m+1$. Thus, $\dim A^{(n)}$ is a polynomial in n with positive coefficients of degree 3, and the Gelfand-Kirillov dimension is given by

$$\begin{aligned} \mathrm{GKdim}\big(A(\alpha, \beta, \gamma)\big) &= \limsup_{n\to\infty} \log_n(\dim A^{(n)}) \\ &= \lim_{n\to\infty} \frac{\ln\left(\dim A^{(n)}\right)}{\ln n} \\ &= 3. \quad \square \end{aligned}$$

Recall that the **Jacobson radical** of a ring R is the intersection of all the annihilators of the simple R-modules. It has many other characterizations, but we will not need them here. A ring whose Jacobson radical is zero is said to be **semiprimitive**.

Theorem 3.3. *If $A(\alpha,\beta,\gamma)$ has infinitely many simple Verma modules $V(\lambda)$, then the intersection of the annihilators of the simple Verma modules is zero. Therefore, such a down-up algebra $A(\alpha,\beta,\gamma)$ is semiprimitive.*

Proof. Suppose the element $a=\sum_{i,j,k} a_{i,j,k}u^i(du)^j d^k$ annihilates all the simple modules $V(\lambda)$. Assume there is a least value ℓ so that $a_{i,j,\ell}\neq 0$ for some i,j. Let $V(\lambda)$ be a simple highest weight module for $A(\alpha,\beta,\gamma)$ of weight λ. Applying a to the vector v_ℓ in $V(\lambda)$ we get

$$\begin{aligned}0=a\cdot v_\ell &= (\lambda_{\ell-1}\cdots\lambda_1\lambda_0)\sum_{i,j} a_{i,j,\ell}u^i(du)^j\cdot v_0\\ &= (\lambda_{\ell-1}\cdots\lambda_1\lambda_0)\sum_{i,j} a_{i,j,\ell}\lambda_0^j u^i\cdot v_0\\ &= (\lambda_{\ell-1}\cdots\lambda_1\lambda_0)\sum_{i,j} a_{i,j,\ell}\lambda_0^j v_i.\end{aligned}$$

Recall that a necessary and sufficient condition for $V(\lambda)$ to be simple is that none of the λ_n is zero. The elements v_i are linearly independent in $V(\lambda)$, so combining those results, we see for each i that

$$0=\sum_j a_{i,j,\ell}\lambda_0^j=\sum_j a_{i,j,\ell}\lambda^j.$$

Because there are infinitely many values of λ that give simple highest weight modules and only finitely many λ that can be the roots of a nonzero polynomial, we see that the coefficients $a_{i,j,\ell}$ must all be zero. Since the Jacobson radical is the intersection of the annihilators of all the simple A-modules, it too must be zero. $\square$

Let us examine in detail when a down-up algebra $A(\alpha,\beta,\gamma)$ has infinitely many simple Verma modules. Since the Verma module $V(\lambda)$ is simple if and only if $\lambda_n\neq 0$ for all $n\geq 1$, the analysis requires the expressions for λ_n in Proposition 2.12.

Case 1. $\alpha^2+4\beta\neq 0$. Here $\lambda_n=c_1r_1^n+c_2r_2^n+x_n$, (see Proposition 2.12 (i)), which can be expressed as a polynomial in λ:

$$\lambda_n=\frac{1}{r_2-r_1}\Big((r_2-\alpha)r_1^n+(-r_1+\alpha)r_2^n\Big)\lambda+b_n,$$

where $r_2-\alpha=1/2\alpha-t-\alpha=-r_1$ and $-r_1+\alpha=-1/2\alpha-t+\alpha=r_2$. Thus the coefficient of λ is $(r_2-r_1)^{-1}(r_2^{n+1}-r_1^{n+1})$. If this is 0, then $r_1^{n+1}=r_2^{n+1}$, which implies $(\alpha+\tau)^{n+1}=(\alpha-\tau)^{n+1}$ where $\tau=2t$ and $\tau^2=\alpha^2+4\beta$. Since $\tau^2=\alpha^2+4\beta\neq$

0, $\alpha+\tau$ and $\alpha-\tau$ are unequal, and we may suppose one of them — say $\alpha+\tau \neq 0$. Then $1 = \Big((\alpha+\tau)^{-1}(\alpha-\tau)\Big)^{n+1}$, so in fact both are nonzero, and the same holds true if $\alpha-\tau \neq 0$. We may suppose $(\alpha+\tau)^{-1}(\alpha-\tau) = e^{i\theta}$ where $\theta = 2\pi k/(n+1)$ for some $1 \leq k \leq n$. Then $\alpha-\tau = (\alpha+\tau)e^{i\theta}$ gives $\alpha(e^{i\theta}-1) = -\tau(e^{i\theta}+1)$. Squaring that relation shows that $\alpha^2 = -\beta(e^{i\theta}+e^{-i\theta})^2 = -4\beta\cos^2(\theta/2)$. Consequently, λ_n is a degree 1 polynomial in λ except when $\alpha^2 = -4\beta\cos^2(\theta/2)$ where $e^{i\theta}$ is an $(n+1)$st root of unity.

Case 2. $\alpha^2+4\beta = 0$ and $\alpha \neq 0$. In this case $\lambda_n = c_1 s^n + c_2 n s^n + x_n = \big((n+1)s^n\big)\lambda + b_n$, where $s = \alpha/2$. Thus, λ_n is a degree 1 polynomial in λ for all n.

Case 3. $\alpha^2+4\beta = 0$ and $\alpha = 0$ (so $\beta = 0$). Here $\lambda_n = \gamma$ for all $n \geq 1$.

Proposition 3.4. *If*

(a) $\alpha^2+4\beta \neq 0$ *and* $\alpha^2 \neq -4\beta\cos^2(\theta/2)$ *for some root of unity* $e^{i\theta}$,

(b) $\alpha^2+4\beta = 0$ *and* $\alpha \neq 0$, *or*

(c) $\alpha = 0 = \beta$ *and* $\gamma \neq 0$,

then $A(\alpha,\beta,\gamma)$ *has infinitely many simple Verma modules* $V(\lambda)$.

Proof. In the cases described in (a) and (b), each λ_n can be regarded as a line $\lambda_n = m_n\lambda + b_n$ with nonzero slope m_n. Each such line intersects the line $y = 0$ at one value of λ. Avoiding all those countably many values by choosing λ appropriately, we get infinitely many simple Verma modules $V(\lambda)$. □

Proposition 3.5. *A down-up algebra* $A = A(\alpha,\beta,\gamma)$ *is* $\mathbf{Z}$*-graded,* $A = \oplus_n A_n$, *where* $A_n = span_{\mathbf{C}}\{u^i(du)^j d^k \mid i,j,k = 0,1,\ldots,$ *and* $i-k = n\}$, *and* A_0 *is a commutative subalgebra.*

Proof. The free associative algebra over $\mathbf{C}$ generated by d and u can graded by assigning $\deg(d) = -1$ and $\deg(u) = 1$ and extending this by using $\deg(ab) = \deg(a)+\deg(b)$. The relations $d^2u = \alpha dud + \beta ud^2 + \gamma d$ and $du^2 = \alpha udu + \beta u^2 d + \gamma u$ are homogeneous, so the down-up algebra inherits the grading.

To prove that A_0 is commutative, we first show that $u^i(du)^j d^i$ commutes with du and ud for all i,j. This is true if $i+j \leq 1$ by (2.11). Suppose we know the result for all $i+j < N$, and let $i+j = N \geq 2$. Consider $(u^i(du)^j d^i)(ud)$. Now if $i = 0$ this equals $(ud)(u^i(du)^j d^i)$ since ud and du commute. If $i = 1$, then $u(du)^j d = (ud)^{j+1}$, which commutes with ud. So we may assume that $i \geq 2$. Then

$$\begin{aligned}(u^i(du)^j d^i)(ud) &= u^i(du)^j d^{i-2}(\alpha dud + \beta ud^2 + \gamma d)d \\ &= \alpha u^2(u^{i-2}(du)^j d^{i-2})(du)d^2 + \\ &\qquad \beta u^2(u^{i-2}(du)^j d^{i-2})(ud)d^2 + \gamma u^i(du)^j d^i \qquad \text{while}\end{aligned}$$

$$\begin{aligned}(ud)(u^i(du)^jd^i) &= u(\alpha udu+\beta u^2d+\gamma u)(u^{i-2}(du)^jd^i)\\ &= \alpha u^2(du)(u^{i-2}(du)^jd^{i-2})d^2+\\ &\qquad \beta u^2(ud)(u^{i-2}(du)^jd^{i-2})d^2+\gamma u^i(du)^jd^i.\end{aligned}$$

These two expressions are equal by our induction hypothesis. Analogously, we consider $(u^i(du)^jd^i)(du)$, where we can suppose that $i \geq 1$. Then

$$\begin{aligned}(u^i(du)^jd^i)(du) &= u^i(du)^jd^{i-1}(\alpha dud+\beta ud^2+\gamma d)\\ &= \alpha(u^i(du)^jd^i)(ud)+\\ &\qquad \beta u(u^{i-1}(du)^jd^{i-1})(ud)d+\gamma u^i(du)^jd^i \qquad \text{which equals}\end{aligned}$$

$$\begin{aligned}(du)(u^i(du)^jd^i) &= (\alpha udu+\beta u^2d+\gamma u)(u^{i-1}(du)^jd^i)\\ &= \alpha(ud)(u^i(du)^jd^i)+\\ &\qquad \beta u(ud)(u^{i-1}(du)^jd^{i-1})d+\gamma u^i(du)^jd^i\end{aligned}$$

by what we have already established.

The proof of the proposition will be complete once we show that $u^i(du)^jd^i$ commutes with $u^k(du)^\ell d^k$ for all i, j, k, ℓ. This is true if $i+j \leq 1$ or if i or k is ≤ 1 by our previous considerations, so we assume it is true when $i+j+k+\ell < N$ and prove it for $i+j+k+\ell = N$. We suppose $i, k \geq 2$. Now

$$\begin{aligned}(u^i(du)^jd^i)(u^k(du)^\ell d^k) &= u^i(du)^jd^{i-2}(\alpha dud+\beta ud^2+\gamma d)u^{k-1}(du)^\ell d^k\\ &= \alpha u(u^{i-1}(du)^jd^{i-1})(ud)(u^{k-1}(du)^\ell d^{k-1})d+\\ &\qquad \beta u^2(u^{i-2}(du)^jd^{i-2})(ud)(du)(u^{k-2}(du)^\ell d^{k-2})d^2+\\ &\qquad u(u^{i-1}(du)^jd^{i-1})(u^{k-1}(du)^\ell d^{k-1})d,\end{aligned}$$

which can be seen to equal the same expression with the roles of i, j and k, ℓ reversed by induction. □

The center of a down-up algebra.

Let $Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}$ denote the center of a down-up algebra $A = A(\alpha, \beta, \gamma)$.

Proposition 3.6. *Suppose $A = \sum_{j\in\mathbf{Z}} A_j$ is the $\mathbf{Z}$-grading of $A = A(\alpha, \beta, \gamma)$ as in Proposition 3.5, and assume that A has infinitely many simple Verma modules. Then:*

(a) The center $Z(A)$ of A is contained in $A_0 = span_{\mathbf{C}}\{u^i(du)^jd^i \mid i, j = 0, 1, \ldots\}$.

(b) If $z \in Z(A)$ and $V(\lambda)$ is any Verma module of A, then z acts as a scalar, say $\chi_\lambda(z)$, on $V(\lambda)$. The mapping $\chi_\lambda : Z(A) \to \mathbf{C}$ is an algebra homomorphism.

(c) A scalar $\pi \in \mathbf{C}$ is **linked** *to λ if $\pi = \lambda_{n+1}$ for some $n \geq 0$ with $\lambda_n = 0$, where the sequence $\lambda_1, \lambda_2, \ldots$ is constructed using the recurrence relation in (2.1) starting with $\lambda_{-1} = 0$ and $\lambda_0 = \lambda$. If π is linked to λ, then $\chi_\pi = \chi_\lambda$.*

Proof. Consider the sequence $\lambda = \lambda_0, \lambda_1, \lambda_2, \ldots$ from (2.1), and let $\Lambda(\lambda) = \{n \in \mathbf{Z}_{\geq 0} \mid \lambda_n = 0\}$. Suppose $v_0, v_1, \ldots$ is the canonical basis for $V(\lambda)$ as in Proposition 2.2, and let $z \in Z(A)$. From $dz \cdot v_0 = zd \cdot v_0 = 0$ we see that $z \cdot v_0 = \chi_\lambda(z)v_0 + \sum_{n \in \Lambda(\lambda), n>0} a_n v_n$ for some scalars a_n and $\chi_\lambda(z)$. Therefore, when $\lambda \neq 0$ and $V(\lambda)$ is simple (i.e. when $\Lambda(\lambda) = \emptyset$) then $z \cdot v_0 = \chi_\lambda(z)v_0$. But $za \cdot v_0 = az \cdot v_0 = \chi_\lambda(z)a \cdot v_0$ for all $a \in A$, and since $V(\lambda) = Av_0$, this shows that z acts as the scalar $\chi_\lambda(z)$ on such Verma modules. Now write z as a sum of its homogeneous components, $z = \sum_j z_j$, $z \in A_j$. Then since $A_j v_n \subseteq \mathbf{C}v_{j+n}$ for all j, n, and since $zv_n = \chi_\lambda(z)v_n$ for all n when $V(\lambda)$ is simple, it must be that z_j for $j \neq 0$ lies in the annihilator of $V(\lambda)$. But the intersection of the annihilators of the simple $V(\lambda)$ modules is zero by Theorem 3.3. Thus, $z = z_0 \in A_0$, as claimed in (a). Once we know that, then it follows for any Verma module $V(\lambda)$ that $z \cdot v_0 = \chi_\lambda(z)v_0$ for some scalar $\chi_\lambda(z) \in \mathbf{C}$. As before using $V(\lambda) = Av_0$, we see z acts as $\chi_\lambda(z)$ on all of $V(\lambda)$. As $\chi_\lambda(zz')v_0 = zz' \cdot v_0 = z \cdot (z'v_0) = \chi_\lambda(z)\chi_\lambda(z')v_0$, it is clear χ_λ is an algebra homomorphism. Part (c) is apparent also, since when $\lambda_n = 0$, then $\text{span}_{\mathbf{C}}\{v_j \mid j \geq n+1\}$ is a submodule of $V(\lambda)$ isomorphic to $V(\lambda_{n+1})$. □

§4. Category $\mathcal{O}$ modules

The representation theory of finite-dimensional complex semisimple Lie algebras centers on a certain important collection of weight modules, the so-called category $\mathcal{O}$ modules, whose study was initiated by Bernstein, Gelfand, and Gelfand [BGG]. In this section we define an analogous category of modules for each down-up algebra $A = A(\alpha, \beta, \gamma)$. We begin with a general result about weight vectors and then specialize to the modules in $\mathcal{O}$ and derive some results about them.

Proposition 4.1. *Suppose M is an A-module and $m \in M$ is a vector of weight $\nu = (\nu', \nu'')$.*

(a) Then $u \cdot m$ is a vector of weight $\mu = (\mu', \mu'')$ for $\mu' = \alpha\nu' + \beta\nu'' + \gamma$ and $\mu'' = \nu'$.

(b) If $\beta \neq 0$, then $d \cdot m$ is a vector of weight $\delta = (\delta', \delta'')$ where $\delta' = \nu''$ and $\delta'' = \beta^{-1}(\nu' - \alpha\nu'' - \gamma)$.

Proof. The assertions are evident from the following calculations:

(4.2)

$$
\begin{aligned}
(du)u\cdot m &= \alpha u(du)\cdot m + \beta u(ud)\cdot m + \gamma u\cdot m = (\alpha\nu' + \beta\nu'' + \gamma)u\cdot m\\
(ud)u\cdot m &= u(du)\cdot m = \nu' u\cdot m\\
(du)d\cdot m &= d(ud)\cdot m = \nu'' d\cdot m\\
(ud)d\cdot m &= \beta^{-1}\big(d(du)\cdot m - \alpha d(ud)\cdot m - \gamma d\cdot m\big) = \beta^{-1}\big(\nu' - \alpha\nu'' - \gamma\big)d\cdot m. \quad \square
\end{aligned}
$$

(4.3) The category $\mathcal{O}$ consists of all A-modules M satisfying the following conditions:

(a) M is a weight module relative to $\mathfrak{h} = \mathrm{span}_{\mathbf{C}}\{du, ud\}$, i.e. $M = \bigoplus_\nu M_\nu$ where $M_\nu = \{m \in M \mid h\cdot m = \nu(h)m \text{ for all } h \in \mathfrak{h}\}$;

(b) d acts locally nilpotently on M, so that for each $m \in M$, $d^n \cdot m = 0$ for some n.

(c) M is a finitely generated A-module.

The category $\mathcal{O}$ is closed under taking submodules and quotients. It contains all the Verma modules $V(\lambda)$ and their simple quotient modules $L(\lambda) = V(\lambda)/M(\lambda)$ (and in the case that $\gamma = 0$ and $\lambda = 0$, the one-dimensional quotients $L(0,\xi) \stackrel{\text{def}}{=} V(0)/N^{(x-\xi)}$ (see Corollary 2.28)). Modules in the category $\mathcal{O}$ enjoy some of the same basic properties as the category $\mathcal{O}$ modules for complex semisimple Lie algebras and for the algebras studied by Smith [Sm].

(4.4) Suppose $\omega = (\omega', \omega'')$, where $\omega', \omega'' \in \mathbf{C}$. *Let $\Omega(\omega)$ denote the set weights defined inductively starting with $\omega_0 = (\omega_0', \omega_0'')$, where $\omega_0' = \omega'$ and $\omega_0'' = \omega''$, and proceeding for $n \geq 1$ by setting $\omega_n = (\omega_n', \omega_n'')$ where*

$$
\begin{aligned}
\omega_n' &= \alpha\omega_{n-1}' + \beta\omega_{n-1}'' + \gamma\\
\omega_n'' &= \omega_{n-1}'.
\end{aligned}
$$

Note that $\omega_n' = \alpha\omega_{n-1}' + \beta\omega_{n-2}' + \gamma$ for $n \geq 2$. When $\omega = (\lambda, 0)$ then $\omega_n = (\lambda_n, \lambda_{n-1})$, where λ_n is as in Proposition 2.1. Thus $\Omega(\omega)$ is just the set of weights of the Verma module $V(\lambda)$ in the particular instance that $\omega = (\lambda, 0)$.

Lemma 4.5. *Suppose $\beta \neq 0$. Assume M is an object in the category $\mathcal{O}$, and let $\Omega(M) = \{\nu \in \mathfrak{h}^* \mid M_\nu \neq (0)\}$ be the set of weights of M. Then there exists a finite set of weights $\omega^1, \ldots, \omega^r$ so that $\Omega(M) \subseteq \cup_i \Omega(\omega^i)$.*

Proof. Because M is finitely generated, there exists a finite set of weight vectors $m_1, \ldots, m_\ell$ generating M. Then $\sum_j \mathbf{C}[d]m_j$ is finite-dimensional, and moreover it is $\mathfrak{h}$-invariant for $\mathfrak{h} = \mathrm{span}_{\mathbf{C}}\{du, ud\}$ by Proposition 4.1. So we can choose a basis $y_1, \dots, y_r$ for it such that y_i has some weight ω^i for each i. Then by Theorem 3.1,

$$M = \sum_j Am_j = \sum_j \mathbf{C}[u]\mathbf{C}[du]\mathbf{C}[d]m_j = \sum_i \mathbf{C}[u]\mathbf{C}[du]y_i = \sum_i \mathbf{C}[u]y_i.$$

The vectors in $\mathbf{C}[u]y_i$ have weights in $\Omega(\omega^i)$ by Proposition 4.1, so the relation $\Omega(M) \subseteq \cup_i \Omega(\omega^i)$ is apparent. $\square$

Lemma 4.6. *Suppose $\beta \neq 0$. If M is a simple object in the category $\mathcal{O}$, then $M \cong L(\lambda)$ for some λ, or $\gamma = 0$ and $M \cong V(0)/N^{(x-\xi)} = L(0,\xi)$ for some $\xi \in \mathbf{C}$ (as in Corollary 2.28).*

Proof. It follows from the local nilpotence of d and from Proposition 4.1 that M contains some highest weight vector v_0. The simplicity of M forces $M = Av_0$ so that M is a highest weight module. As highest weight modules are quotients of Verma modules, M is as asserted. $\square$

§5. Category $\mathcal{O}'$ modules

In this section we introduce and investigate a more general category of modules for the down-up algebra $A = A(\alpha, \beta, \gamma)$. **We require throughout that $\beta \neq 0$.**

(5.1) The category $\mathcal{O}'$ consists of all A-modules M satisfying the following conditions:

(a) M is a weight module relative to $\mathfrak{h} = \mathrm{span}_{\mathbf{C}}\{du, ud\}$
(b) $\mathbf{C}[d]m$ is finite-dimensional for each $m \in M$.
(c) M is a finitely generated A-module.

The modules in $\mathcal{O}$ clearly belong to $\mathcal{O}'$, but we will show later that $\mathcal{O}'$ is larger than $\mathcal{O}$ by determining the simple modules in $\mathcal{O}'$. Two mappings on weights play a prominent role in these investigations.

Recall (Proposition 4.1) that if a vector m in an A-module has weight $\nu = (\nu', \nu'')$ relative to $\mathfrak{h}$, then $d \cdot m$ has weight

$$\begin{aligned} &\delta(\nu) = (\delta(\nu)', \delta(\nu)'') \qquad \text{where} \\ &\delta(\nu)' = \nu'' \qquad \text{and} \qquad \delta(\nu)'' = \beta^{-1}(\nu' - \alpha\nu'' - \gamma), \end{aligned} \tag{5.2}$$

and $u \cdot m$ has weight

$$\mu(\nu) = (\mu(\nu)', \mu(\nu)'') \qquad \text{where}$$
$$\mu(\nu)' = \alpha\nu' + \beta\nu'' + \gamma \qquad \text{and} \qquad \mu(\nu)'' = \nu'. \tag{5.3}$$

An easy direct computation shows that $\delta(\mu(\nu)) = \nu$ and $\mu(\delta(\nu)) = \nu$, (or this can be readily seen from the fact that $(du) \cdot m = \nu' m$ and $(ud) \cdot m = \nu'' m$). Observe that the set of weights $\Omega(\omega)$ in (4.4) alternately can be described as

$$\Omega(\omega) = \{\mu^k(\omega) \mid k = 0, 1, \dots\}. \tag{5.4}$$

The set of weights of the lowest weight module $W(\kappa)$ is just $\{\delta^k(\omega) \mid k = 0, 1, \dots\}$, where $\omega = (0, \kappa)$.

Theorem 5.5. *Suppose F is a set of weights such that $\delta(\omega), \mu(\omega) \in F$ whenever $\omega \in F$. Suppose $\rho \in \mathbf{C}$ is nonzero, and let $N(F, \rho)$ be the $\mathbf{C}$-vector space with basis $\{v_\omega \mid \omega \in F\}$.*

(a) Define

$$d \cdot v_\omega = \rho v_{\delta(\omega)}$$
$$u \cdot v_\omega = \rho^{-1} \mu(\omega)'' v_{\mu(\omega)}. \tag{5.6}$$

Then this action extends to give an $A(\alpha, \beta, \gamma)$-module action on $N(F, \rho)$.

(b) If F is generated by any weight $\nu = (\nu', \nu'') \in F$ under the action of δ and μ, and if $\nu' \neq 0$ for any $\nu \in F$, then $N(F, \rho)$ is a simple $A(\alpha, \beta, \gamma)$-module.

Proof. Let $N(F, \rho)$ be as in the statement of the theorem, and define an action of the free associative algebra $\mathbf{C}\langle d, u \rangle$ on $N(F, \rho)$ using (5.6). Then

$$(du) \cdot v_\omega = \rho^{-1} \mu(\omega)'' d \cdot v_{\mu(\omega)} = \mu(\omega)'' v_\omega = \omega' v_\omega$$
$$(ud) \cdot v_\omega = \rho u \cdot v_{\delta(\omega)} = \mu(\delta(\omega))'' v_\omega = \omega'' v_\omega. \tag{5.7}$$

Utilizing (5.7) together with (5.2) and (5.3), it is easy to see that

$$\Big(d^2u - \alpha dud - \beta ud^2 - \gamma d\Big) \cdot v_\omega = \rho\Big(\omega' - \alpha\omega'' - \beta\delta(\omega)'' - \gamma\Big) v_{\delta(\omega)} = 0$$
$$\Big(du^2 - \alpha udu - \beta u^2 d - \gamma u\Big) \cdot v_\omega = \rho^{-1}\mu(\omega)''\Big(\mu(\omega)' - \alpha\omega' - \beta\omega'' - \gamma\Big) v_{\mu(\omega)} = 0.$$

Thus, there is an induced action making $N(F, \rho)$ into an $A(\alpha, \beta, \gamma)$-module.

Now suppose F is generated by any weight $\nu \in F$ under the maps δ and μ defined in (5.2) and (5.3). It follows from (5.7) that $N(F,\rho)$ is a weight module. Let M be a nonzero submodule of $N(F,\rho)$, and assume $v = \sum_\omega a_\omega v_\omega$ is a nonzero vector in M. Since any submodule of a weight module is a weight module, $v_\omega \in M$ if $a_\omega \neq 0$. Now whenever $v_\omega \in M$ for some ω, then $d \cdot v_\omega = \rho v_{\delta(\omega)}$ and $u \cdot v_\omega = \rho^{-1}\mu(\omega)'' v_{\mu(\omega)}$ both belong to M. Since $\mu(\omega)'' = \omega' \neq 0$ for any $\omega \in F$ by assumption, $v_{\mu(\omega)} \in F$. Because F is generated by ω under δ and μ, M contains v_τ for all $\tau \in F$. As a consequence, $M = N(F,\rho)$ and $N(F,\rho)$ is a simple $A(\alpha,\beta,\gamma)$-module. $\square$

Suppose now that M is a simple module in the category $\mathcal{O}'$. If M contains a highest weight vector, then M is as in Lemma 4.6. So we may assume henceforth that M has no highest weight vectors. Let m be a nonzero weight vector in M and consider the finite-dimensional space $\mathbf{C}[d]m$. If $d^k \cdot m = 0$ for some k, then $d^{k-1} \cdot m$ is a highest weight vector in M contrary to assumption. It must be that d acts nonnilpotently on $\mathbf{C}[d]m$. Let v be an eigenvector of d in $\mathbf{C}[d]m$ corresponding to nonzero eigenvalue, say $d \cdot v = \rho v$, $\rho \neq 0$. Write $v = \sum_\omega v_\omega$ where v_ω belongs to the weight space M_ω.

Since $d \cdot v = \rho v$, it must be that $d \cdot v$ has the same number of weight components as v. Equating corresponding components on both sides of the equation gives

$$d \cdot v_\omega = \rho v_{\delta(\omega)}. \tag{5.8}$$

Thus if S is the $\mathbf{C}$-span of the weight components v_ω of v, then $dS = S$. Now $\rho u \cdot v = (ud) \cdot v = \sum_\omega \omega'' v_\omega$, which implies that

$$u \cdot v_\omega = \rho^{-1}\mu(\omega)'' v_{\mu(\omega)}. \tag{5.9}$$

Thus, S is an $A(\alpha,\beta,\gamma)$-submodule of M. By simplicity, $S = M$.

Since there are only finitely many nonzero weight component summands, $\delta^\ell(\omega) = \delta^k(\omega)$ for some $\ell > k$. Applying μ^k to both sides, we see $\delta^{(\ell-k)}(\omega) = \omega$. Thus, there is some smallest value p so that $\delta^p(\omega) = \omega$ and $v_\omega, v_{\delta(\omega)}, \ldots, v_{\delta^{p-1}(\omega)}$ are all nonzero. Their $\mathbf{C}$-span S' is invariant under d. Using (5.9) and the fact that $\mu(\delta^i(\omega)) = \delta^{i-1}(\omega)$ for $i \geq 1$ and $\mu(\omega) = \mu(\delta^p(\omega)) = \delta^{p-1}(\omega)$, we see that $uS' \subseteq S'$ also. Thus, $M = S'$ by simplicity. The set $F = \{\omega, \delta(\omega), \ldots, \delta^{p-1}(\omega)\}$ is δ and μ-invariant. Any weight in it generates the whole set under δ and μ. Now suppose ν is a weight of F and $\nu' = \mu(\nu)'' = 0$. Setting $v_n = \rho^n v_{\delta^n(\nu)}$ for $n = 0, 1, \ldots, p-1$, we have $u \cdot v_0 = 0$, $d \cdot v_n = v_{n+1}$ for $0 \leq n \leq p-1$, and $d \cdot v_n = \rho^p v_0$. In this case M is a finite-dimensional lowest weight module – a quotient of $W(\kappa)$ for $\kappa = \nu''$. The sole remaining possibility is that $\nu' \neq 0$ for any $\nu \in F$. In this situation $M = N(F,\rho)$, (and $N(F,\rho)$ is in fact a simple module - compare Theorem 5.5 (b)).

Let us summarize what we have shown:

Theorem 5.10. *Assume M is a simple module in the category $\mathcal{O}'$. Then there are three possibilities:*

(a) M is a highest weight module, that is, M is isomorphic to $L(\lambda)$ for some λ or to $L(0,\xi)$ for some $\xi \in \mathbf{C}$ (when $\gamma = 0$).

(b) M is a finite-dimensional lowest weight module with weights $\nu, \delta(\nu), \ldots, \delta^{p-1}(\nu)$ such that $\delta^p(\nu) = \nu$.

(c) M is isomorphic to $N(F,\rho)$ for some $\rho \neq 0$ and some finite set $F = \{\nu, \delta(\nu), \ldots, \delta^{p-1}(\nu)\}$ such that $\delta^p(\nu) = \nu$.

We conclude this section by analyzing the special case that the set F is cyclically generated by $\omega = (\lambda, \kappa)$ under δ and μ, and the weights in

$$F = \{\ldots, \mu^2(\omega), \mu^1(\omega), \omega, \delta(\omega), \delta^2(\omega), \ldots\}$$

are all distinct. Observe that if we define λ_n by first setting $\lambda_0 = \lambda$ and $\lambda_{-1} = \kappa$ and proceeding by

$$\begin{aligned} \lambda_n &= \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma \qquad \text{for } n \geq 1 \\ \lambda_{-n} &= \beta^{-1}(\lambda_{-n+2} - \alpha\lambda_{-n+1} - \gamma) \qquad \text{for } n \geq 2, \end{aligned}$$

then

$$\begin{aligned} (\lambda_n, \lambda_{n-1}) &= \mu^n(\omega) \qquad \text{for } n \geq 0 \\ (\lambda_{-n}, \lambda_{-n-1}) &= \delta^n(\omega) \qquad \text{for } n \geq 1. \end{aligned}$$

These are just the weights of the doubly-infinite module $V(\kappa, \lambda)$ in Proposition 2.33. It is easy to see using Proposition 2.35 that there is an isomorphism from the dual module $V(\kappa,\lambda)^*$ to $N(F,1)$ taking $v_n^* \mapsto v_{\mu^n(\omega)}$ for $n \geq 0$ and $v_{-n}^* \mapsto v_{\delta^n(\omega)}$ for $n \geq 1$. Theorem 5.5 shows that if $\nu' = \lambda_n \neq 0$ for any $\nu = (\lambda_n, \lambda_{n-1})$, $n \in \mathbf{Z}$, then $N(F,1) \cong V(\kappa,\lambda)^*$ is simple. This forces $V(\kappa,\lambda)$ to be simple as well, for the annihilator in $V(\kappa,\lambda)^*$ of any proper submodule of $V(\kappa,\lambda)$ would be a proper submodule of $V(\kappa,\lambda)^*$. Consequently, an analogue of the result in Proposition 2.4 holds for the doubly-infinite module $V(\kappa,\lambda)$:

Proposition 5.11. *Suppose $\lambda_0 = \lambda$ and $\lambda_{-1} = \kappa$ and define the sequence λ_n as in Proposition 2.33. If $\lambda_n \neq 0$ for any n, then the doubly-infinite module $V(\kappa,\lambda)$ is simple.*

§6. One-dimensional $A(\alpha,\beta,\gamma)$-modules and isomorphisms

Consider a one-dimensional space $X = \mathbf{C}x$ with

$$d \cdot x = ax \qquad \text{and} \qquad u \cdot x = bx.$$

Then this extends to a module for $A = A(\alpha,\beta,\gamma)$ if and only if

$$\begin{aligned} 0 &= (d^2u - \alpha dud - \beta ud^2 - \gamma d)\cdot x = a\big((1-\alpha-\beta)ab - \gamma\big)x \\ 0 &= (du^2 - \alpha udu - \beta u^2 d - \gamma u)\cdot x = b\big((1-\alpha-\beta)ab - \gamma\big)x. \end{aligned}$$

If $a = 0 = b$, then $X = V(0)/M(0)$. We may suppose that not both a, b are 0, so that X is an A-module if and only if $(1-\alpha-\beta)ab = \gamma$ holds. When $\gamma = 0$ and $\alpha+\beta = 1$, then any choice of a and b gives X the structure of an A-module, so the one-dimensional A-modules are parametrized by $\mathbf{C}^2$ in this instance. When $\gamma = 0$ and $\alpha+\beta \neq 1$, then either $a = 0$ or $b = 0$ must hold. The one-dimensional A-modules are parametrized by $\{(a,0) \mid a \in \mathbf{C}\} \cup \{(0,b) \mid b \in \mathbf{C}\}$ in this case. When $\gamma \neq 0$, we may suppose that $\gamma = 1$. ¿From $(1-\alpha-\beta)ab = \gamma = 1$, we see that $b = a^{-1}(1-\alpha-\beta)^{-1}$. Thus, the one-dimensional A-modules are parametrized by $\mathbf{C}$ for $\gamma \neq 0$ and $\alpha+\beta \neq 1$, and by $a = 0 = b$ when $\gamma \neq 0$ and $\alpha+\beta = 1$.

Now $(du)\cdot x = abx = (ud)\cdot x$, so that x has weight $\nu = (ab, ab)$ for all one-dimensional A-modules X. When $\beta \neq 0$, then $\delta(\nu) = (ab, \beta^{-1}(ab - \alpha ab - 1)$, which equals (ab, ab) when $\gamma = 1$. Thus, in the case $\beta \neq 0, \gamma \neq 0$ the one-dimensional module X is isomorphic to $N(F, a)$ where $F = \{\nu = (ab, ab)\}$ and $\delta(\nu) = \nu$ (compare Theorem 5.10). To summarize we have:

Theorem 6.1. *The one-dimensional modules for the down-up algebra $A(\alpha,\beta,\gamma)$ are parametrized by*

(a) $\mathbf{C}^2$ *when* $\gamma = 0$ *and* $\alpha+\beta = 1$*;*
(b) $\{(a,0) \mid a \in \mathbf{C}\} \cup \{(0,b) \mid b \in \mathbf{C}\}$ *when* $\gamma = 0$ *and* $\alpha+\beta \neq 1$*;*
(c) $\mathbf{C}$ *when* $\gamma \neq 0$ *and* $\alpha+\beta \neq 1$;
(d) $(0,0)$ *when* $\gamma \neq 0$ *and* $\alpha+\beta = 1$.

Corollary 6.2. *If the down-up algebras $A(\alpha,\beta,\gamma)$ and $A(\alpha',\beta',\gamma')$ are isomorphic, then either both $\alpha+\beta$ and $\alpha'+\beta'$ are equal to 1 or both are different from 1, and either both γ and γ' are 0 or both are different from 0.*

Proof. The result is a simple consequence of Theorem 6.1 and the fact that if $\phi : A(\alpha,\beta,\gamma) \to A(\alpha',\beta',\gamma')$ is an isomorphism, then any one-dimensional module $X = \mathbf{C}x$ for $A(\alpha',\beta',\gamma')$ becomes one for $A(\alpha,\beta,\gamma)$ by defining $t\cdot x = \phi(t)\cdot x$ for all $t \in A(\alpha,\beta,\gamma)$. $\square$

Open Problems. In conclusion we mention some noteworthy open questions concerning down-up algebras which are not addressed here. Problems (a)-(f) were posed in [B], and since the original version of this paper appeared, several of these problems have been solved.

(a) *Determine when the algebra $A(\alpha, \beta, \gamma)$ is Noetherian.* Kirkman, Musson, and Passman [KMP] have recently proven that $A(\alpha, \beta, \gamma)$ is left and right Noetherian if and only if $\beta \neq 0$ if and only if du and ud are algebraically independent.

(b) *Find conditions on α, β, γ for $A(\alpha, \beta, \gamma)$ to be a domain.* When $\beta = 0$, then $d(du - \alpha ud - \gamma 1) = 0$ so that $A(\alpha, \beta, \gamma)$ has zero divisors for any choice of $\alpha, \gamma \in \mathbf{C}$. It has been shown in [KMP] that $A(\alpha, \beta, \gamma)$ is a domain if and only if $\beta \neq 0$ (i.e if and only if the conditions in (a) hold.) It has been proved independently by Kulkarni [K2] (using methods from hyperbolic rings) that $\beta \neq 0$ is a necessary and sufficient condition for $A(\alpha, \beta, \gamma)$ to be a domain.

(c) *Compute the center of $A(\alpha, \beta, \gamma)$.* This has been solved in a very recent preprint by Zhao [Z] and announced by Kulkarni [K2].

(d) *Determine when $A(\alpha, \beta, \gamma)$ is a Hopf algebra.* The mapping $S(d) = -d$, $S(u) = -u$, which is an algebra antiautomorphism (in fact, the antipode) in the enveloping algebra case, is an algebra antiautomorphism of $A(\alpha, \beta, \gamma)$ if and only if $\beta = -1$ or $\beta = 1$ and $\alpha = 0 = \gamma$.

(e) *Relate Kulkarni's presentation of the maximal left ideals in conformal $\mathfrak{sl}_2$ algebras to the simple modules in category $\mathcal{O}$ and category $\mathcal{O}'$.* The approach in [K1] using noncommutative algebraic geometry is quite different from the one adopted here and so is the description of the simple modules. Our primary focus here was on determining explicit information about the representations of down-up algebras that could be used in the study of posets in the spirit of the work of Stanley [St1] and Terwilliger [T].

(f) *Study the homogenization $A[t]$ of the down-up algebra $A = A(\alpha, \beta, \gamma)$, which is the graded algebra generated by d, u, t subject to the relations*

$$d^2u = \alpha dud + \beta ud^2 + \gamma dt^2, \quad du^2 = \alpha udu + \beta u^2 d + \gamma ut^2,$$
$$dt = td, \quad ut = tu.$$

Homogenized $\mathfrak{sl}_2$ is a positively graded Noetherian domain and a maximal order, which is Auslander-regular of dimension 4 and satisfies the Cohen-Macaulay property. Le Bruyn and Smith [LS] have determined the point, line, and plane modules of homogenized $\mathfrak{sl}_2$ and shown the line modules are homogenizations of the Verma modules.

(g) *Investigate indecomposable and projective modules for down-up algebras.*

(h) *Determine when two down-up algebras are isomorphic.* Some information on this problem is derived in Section 6 as a consequence of the determination of the 1-dimensional modules, but we do not address the general problem here.

(i) *Investigate prime and primitive ideals of down-up algebras.*

References

[B] G. Benkart, *Down-up algebras and Witten's deformations of the universal enveloping algebra of* $\mathfrak{sl}_2$, *Contemp. Math. AMS* (to appear).

[Be] G.M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), 178–218.

[BGG] J. Bernstein, I.M. Gelfand, and S.I. Gelfand, *A category of* $\mathfrak{g}$*-modules*, Func. Anal. Appl. **10** (1976), 87–92.

[Br] R.A. Brualdi, *Introductory Combinatorics*, Second Edition, North Holland, New York, 1992.

[F] S.V. Fomin, *Duality of graded graphs*, J. Alg. Comb. **3** (1994), 357–404.

[H] T.J. Hodges, *Noncommutative deformations of type-A Kleinian singularities*, J. Algebra **161** (1993), 271–290..

[J] N. Jacobson, *Lie Algebras*, Wiley Interscience, New York, 1962.

[Jo] D.A. Jordan, *Finite-dimensional simple modules over certain iterated skew polynomial rings*, J. Pure Appl. Algebra **98** (1995), 45–55.

[K1] R. Kulkarni, *Irreducible representations of Witten's deformations of* $U(sl_2)$ (to appear).

[K2] R. Kulkarni, *Down-up algebras and their representations* (to appear).

[KMP] E. Kirkman, I. Musson, and D. Passman, *private communication.*

[L1] L. Le Bruyn, *Two remarks on Witten's quantum enveloping algebra*, Comm. Algebra **22** (1994), 865–876.

[L2] L. Le Bruyn, *Conformal* $\mathfrak{sl}_2$ *enveloping algebras*, Comm. Algebra **23** (1995), 1325–1362.

[LS] L. Le Bruyn and S.P. Smith, *Homogenized* $\mathfrak{sl}_2$, Proc. Amer. Math. Soc. **118** (1993), 725–730.

[M] Yu. Manin, *Some remarks on Koszul algebras and quantum groups*, Ann. Inst. Fourier **37** (1987), 191–205.

[R] A.L. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Kluwer, 1995.

[Sm] S.P. Smith, *A class of algebras similar to the enveloping algebra of* $sl(2)$, Trans. Amer. Math. Soc. **322** (1990), 285–314.

[St1] R.P. Stanley, *Differential posets*, J. of Amer. Math. Soc. **4** (1988), 919–961.

[St2] R.P. Stanley, *Variations on differential posets*, 145-165; *Invariant Theory and Tableaux*, D. Stanton, ed., IMA Series, vol. 19, Springer-Verlag, New York, 1990.

[T] P. Terwilliger, *The incidence algebra of a uniform poset*, 193-212; *Coding Theory and Design Theory, Part I*, D. Ray-Chaudhuri, ed., IMA Series, vol. 20, Springer-Verlag, New York, 1990.

[W1] E. Witten, *Gauge theories, vertex models, and quantum groups*, Nuclear Phys. B **330** (1990), 285–346.

[W2] E. Witten, *Quantization of Chern-Simons gauge theory with complex gauge group*, Comm. Math. Phys. **137** (1991), 29–66.

[Wo] S.L. Woronowicz, *Twisted SU(2) group. An example of non-commutative differential calculus*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **23** (1987), 117–181.

[Z] K. Zhao, *Centers of down-up algebras* (to appear).

Georgia Benkart, Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706-1388
benkart@math.wisc.edu

Tom Roby, Department of Mathematics and Computer Science, California State University, Hayward, California 94542-3092
troby@mcs.csuhayward.edu