ISOPERIMETRIC FLOW AND CONVEXITY OF H-GRAPHS

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To Leon Simon who taught me De Giorgi-Nash-Moser Theory, and Craig Evans who alerted me to the fact that the time had come to use it.

ABSTRACT. In this paper we consider a "flow" of nonparametric solutions of the volume constrained Plateau problem with respect to a convex planar curve. Existence and regularity is obtained from standard elliptic theory, and convexity results for small volumes are obtained as an immediate consequence. Finally, the regularity is applied to show a strong stability condition (Theorem 8) for all volumes considered. This condition, in turn, allows us to adapt an argument of Cabré and Chanillo [CC97] which yields that any solution enclosing a non-zero volume has a unique nondegenerate critical point.

CONTENTS

INTRODUCTION

Let Ω be a smooth $(C^{2,\alpha})$ bounded, strictly convex domain in $\mathbb{R}^2$. We consider classical solutions $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of the constant mean curvature boundary value problem:

$$(*) \qquad \begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 2H & \text{on } \Omega, \\ u\Big|_{\partial\Omega} \equiv 0 \end{cases}$$

where H is a given constant. The word "flow" in the title refers simply to the fact that we consider the solutions of $(*)$ as a continuous family $u = u(x; H)$ for $x \in \bar{\Omega}$

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and H in a suitable interval (which we can think of as a time interval). We remark also that each solution (for fixed H) provides the least area graph which, along with the domain Ω, encloses a certain prescribed volume. See §3 below for details.

In the following §1 we delineate existence and regularity results for the family $u(x; H)$. These results follow from standard theorems in elliptic pde for which we cite extensively [GT83]. The utility of the results in §1 is demonstrated throughout the rest of the paper.

In §2 we characterize (in terms of Poisson's equation) two aspects of the convexity properties of solutions that enclose a sufficiently small volume. It follows from this characterization, for example, that if Ω has boundary an ellipse, then for $|H| \neq 0$ small enough, solutions of $(*)$ are convex. It is also shown for all ($C^{2,\alpha}$) convex domains that small enough solutions (in the sense just mentioned) have convex level curves. More precisely, we show that small solutions are $1/2$ (power) convex (see §2 for details).

We are unable at present to derive any convexity property for larger solutions. In §3, however, we establish a strong stability condition. The condition is "strong" in the sense that it holds for arbitrary variations, not just those that preserve the volume with respect to which the solution is known to be a minimizer of area. The stability condition is then used in §4 to show that each non-zero solution has a unique critical point. In this regard, we follow an argument of Cabré and Chanillo [CC97] which they applied to semilinear equations. In particular, each solution has a unique global extremum (maximum or minimum), a corollary that can also be deduced from the convexity of the level curves for small solutions (§2). It also follows from the method of Cabré and Chanillo that the critical point is nondegenerate, i.e., solutions are strictly convex in some neighborhood of the critical point.

Most of the arguments below (usually in an ill-formed state) and many ill-fated versions of them have been inflicted on my friends and colleagues. I thank them for their patience and helpful comments. Among them are Claire Chan, Mikhail Feldman, Melinda McCuan, Robert Osserman, and Tatiana Toro. I should specifically like to thank Robert Finn for Example 1, Henry Wente for suggesting the "form" of Theorem 6, and Brian White for making Remark 6. Finally, I am indebted to David Hoffman for introducing me to "bubble problems."

1. Solvability and regularity

The results of this section apply more generally to problems

$$(**) \qquad \begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 2H \quad \text{on } \Omega, \\ u\big|_{\partial\Omega} \equiv \phi \end{cases}$$

where Ω is a $C^{2,\alpha}$ domain in $\mathbb{R}^n$ (not necessarily convex) and either (i) $\phi \equiv 0$, or (ii) ϕ (a function defined on $\partial\Omega$) extends to a $C^{2,\alpha}$ function on $\bar{\Omega}$ and the mean curvature of $\partial\Omega$ is everywhere positive. The proofs, for the most part, apply to both cases (i) and (ii) though certain details not directly related to our main results are presented in Appendix A.

We begin by determining a suitable interval on which to consider the mean curvature H.

Theorem 1 (solvability)**.** *There is a unique value $H_{max} > 0$ depending on Ω such that the following hold.*

i. *If $|H| \leq H_{max}$, $(*)$ has a unique solution $u = u(x; H)$.*
ii. *If $|H| > H_{max}$, $(*)$ has no solution.*

Furthermore, if $|H| < H_{max}$, $u \in C^{2,\alpha}(\bar{\Omega})$, but

$$\sup_{x\in\Omega} |Du(x; H_{max})| = \infty.$$

Remark 1. *If κ denotes the minimum mean curvature of $\partial\Omega$ and $2|H| \leq \kappa$, then solvability follows from Theorem 16.11* [GT83, pg. 409]*. For the particular boundary condition in $(*)$ solvability will, in general, persist for $2|H| > \kappa$, and we wish to include these solutions in our discussion.*

Remark 2. *Any results numbered 6.2-16.11 without specific reference, refer to* [GT83].

Remark 3. *The mean curvature operator that appears in $(*)$ will be denoted by $\mathcal{M}$. Furthermore, given $p \in \mathbb{R}^n$ we define the vector $A = p/\sqrt{1+|p|^2}$ and write $\mathcal{M}$ in its "pure divergence form" (1) which expands to an alternative "non-divergence quasilinear form" (2).*

$$\mathcal{M}u = \sum_i D_i A^i(Du) \tag{1}$$

$$= \sum_{i,j} \frac{\partial A^i}{\partial p_j}(Du) D_i D_j u. \tag{2}$$

The coefficients in the last expression will be denoted by $A_{ij} = A_{ij}(Du)$, and one easily checks that the coefficient matrix (A_{ij}) is positive definite, i.e., the operator is elliptic (and uniformly elliptic if $|Du|$ remains bounded).

We may also consider $\mathcal{M}$ in "non-divergence linear form"

$$\mathcal{M}u = \sum_{i,j} a_{ij} D_i D_j u, \tag{3}$$

by simply setting $a_{ij}(x) = A_{ij}(Du(x))$.

Given a solution $u \in C^2(\bar{\Omega})$ of $(*)$, it follows that $u \in C^\omega(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$. See Corollary 16.7 [GT83, pg. 407; see also pp. 109–111].

Remark 4. *When convenient, we indicate the dependence of the problem* $(*)$ *on the mean curvature by a subscript:* $(*)_H$.

Proof of Theorem 1. We set

$$H_{\max} \equiv \sup\{H : (*) \text{ is (uniquely) solvable in } C^2(\bar{\Omega})\}.$$

Let us assume for the moment that $H_{\max} > 0$. Note that in our case of primary interest (convex planar domains) this condition of "nondegeneracy" follows from Remark 1. Uniqueness of a given solution u is immediate from the comparison principle, Theorem 10.1 [GT83, pg. 263]; see also Theorem 10.2.

Following the usual Leray-Schauder approach (Theorem 13.8 [GT83, pg. 331]), solvability follows if there is a constant M such that the apriori bound

$$|u|_{C^1(\Omega)} = \sup_\Omega |u| + \sup_\Omega |Du| < M \tag{4}$$

holds for any $C^{2,\alpha}(\bar{\Omega})$ solution of $(*)_{\sigma H}$. Note that the constant M is required to be independent of u and $\sigma > 0$.

To obtain a such a bound we assume that $|H| < H_{\max}$ and take $\tilde{H} \in (|H|, H_{\max})$ such that $(*)_{\tilde{H}}$ has a solution $\tilde{u} \in C^2(\bar{\Omega})$. ($\tilde{H}$ exists by the definition of $H_{\max}$.) By the comparison principle, any solution of $(*)_{\sigma H}$ satisfies $|u| \le |\tilde{u}|$ and thus,

$$|Du|\Big|_{\partial\Omega} \le |D\tilde{u}|\Big|_{\partial\Omega} \le |\tilde{u}|_{C^1(\Omega)} \equiv \tilde{M} < \infty. \tag{5}$$

On the other hand, we can differentiate the expression (3) to obtain a linear elliptic equation satisfied by $v = D_k u$. In fact,

$$\mathcal{L}v \equiv \sum_{i,j} a_{ij} D_i D_j v + \sum_l b_l D_l v = 0 \tag{6}$$

where

$$b_l(x) = \sum_{i,j} \frac{\partial^2 A^i}{\partial p_j \partial p_l}(Du) D_i D_j u.$$

By the weak maximum principle, Theorem 3.1 [GT83, pg. 32],

$$\sup_\Omega |D_k u| = \sup_{\partial\Omega} |D_k u|.$$

Consequently, we have from (5)

$$\begin{aligned}\sup_\Omega |Du| &\le \sqrt{n} \max_k \sup_\Omega |D_k u| \\ &= \sqrt{n} \max_k \sup_{\partial\Omega} |D_k u| \\ &\le \sqrt{n}\tilde{M}. \end{aligned} \tag{7}$$

We have therefore established the apriori bound (4) with $M = (1+\sqrt{n})\tilde{M}$, and solvability follows for $|H| < H_{\max}$.

Since our primary results concern convex planar domains and the case $|H| < H_{\max}$, we postpone the remainder of the proof of Theorem 1 (see Appendix A) and use only the assertions established above. Concerning the extremal solution $u(x; H_{\max})$, we note that the nondegeneracy condition, $H_{\max} > 0$, and the gradient blow-up condition follow in general from a "short time existence" result, Theorem 11, that is of independent interest. □

This is a convenient time to point out two other immediate consequences of the comparison principle (Theorem 10.1).

Corollary 1 (monotonicity and symmetry)**.** *If $-H_{\max} < H \le \tilde{H} < H_{\max}$, then*

$$u(x; H) \ge u(x, \tilde{H})$$

for all $x \in \Omega$ with equality only if $H = \tilde{H}$. There also holds

$$u(x, -H) \equiv -u(x, H).$$

The uniform estimate (4) in the proof of Theorem 1 allows us to concentrate on certain questions of uniformity in H for higher derivatives of u and to effectively ignore dependencies on ellipticity constants and bounds for the top order coefficients (usually denoted by λ and Λ respectively in [GT83]). This observation is recorded for reference in the following

Corollary 2. *If $0 < \tilde{H} < H_{\max}$, then we have a uniform bound*

$$|u|_{C^{1,\tilde{\alpha}}(\Omega)} \le \tilde{M} \tag{8}$$

depending only on $\tilde{H}$. In addition, the coefficients in (2) and (3) are uniformly elliptic and bounded for all $x \in \bar{\Omega}$ and $|H| < \tilde{H}$, i.e., there is some $\tilde{M}$ and some $\tilde{\lambda} > 0$, both independent of H, such that

$$|a_{ij}|_{C^{0,\tilde{\alpha}}(\Omega)} \le \tilde{\Lambda}, \text{ and} \tag{9}$$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \tilde{\lambda}|\xi|^2 \tag{10}$$

for all $\xi \in \mathbb{R}^n$ and $x \in \bar{\Omega}$.

Proof. The ellipticity constant λ for (a_{ij}) is $\inf_\Omega (1+|Du|^2)^{-3/2}$. Thus, by (4) $\lambda \ge (1+n\tilde{M}^2)^{-3/2} \equiv \tilde{\lambda} > 0$, and (10) holds.

Since $a_{ij} = A_{ij}(Du)$ and A is smooth, $|a_{ij}|_{C^0(\Omega)} \le |A|_{C^2(G)}$ where $G = \{Du \in \mathbb{R}^n : x \in \Omega\}$. Thus, we have a bound, $|a_{ij}|_{C^0(\Omega)} \le \tilde{\Lambda}_0$.

Given $\tilde{\lambda}$, $\tilde{\Lambda}_0$ and M from (4), Theorem 13.2 [GT83, pg. 323] implies[1] that for some $\tilde{\alpha} = \tilde{\alpha}(\tilde{\lambda}, \tilde{\Lambda}_0, \Omega)$

$$[Du]_{C^{\tilde{\alpha}}(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \frac{|Du(x) - Du(y)|}{|x-y|} \leq C$$

where $C = C(\tilde{\lambda}, \tilde{\Lambda}_0, M, \Omega, \phi)$. Thus, (8) holds. Notice that we returned to the "pure divergence form" $\mathcal{M}u = \operatorname{div} A(Du) = 2H$ in order to apply Theorem 13.2.

Extending slightly our estimate for a_{ij}, see Lemma 9 Appendix B, we see that $[a_{ij}]_{C^{\tilde{\alpha}}(\Omega)} \leq |A_{ij}|_{C^1(G)}[Du]_{C^{\tilde{\alpha}}(\Omega)} \leq |A|_{C^3(G)}\tilde{M}$. Thus, (9) follows from (8). ■

Remark 5. *The regularity assertion of Remark 3, the solvability theorem above (n.b. Theorem 13.8), and the bounds given in Corollary 2 all depend crucially on the C^α gradient bound of Ladyzhenskaya and Ural'tseva (Theorem 13.2) which follows from the De Giorgi-Nash-Moser Theory of Chapter 8 (or alternatively—in two dimensions—from earlier results of Morrey). This dependence also presents itself as the main difficulty when one tries to prove the continuity theorem below by applying the Schauder estimates from Chapters 6 and 8.*

Theorem 2 (continuity)**.** *If $|H_0|, |H| \leq \tilde{H} < H_{\max}$, then the following estimates hold.*

(i) *If Ω is a $C^{k,\alpha}$ domain for some $k \geq 1$ and $\alpha > 0$,*

$$|u(\,\cdot\,; H) - u(\,\cdot\,; H_0)|_{C^{k,\alpha_0}(\Omega)} \leq C_0|H - H_0|$$

where C_0 and $\alpha_0 > 0$ are independent of H.

(ii) *If $\Omega' \subset\subset \Omega$, then*

$$|u(\,\cdot\,; H) - u(\,\cdot\,; H_0)|_{C^k(\Omega')} \leq C_0'|H - H_0|$$

where C_0' depends on k and Ω' but not H.

Proof. Let us consider H_0 and $\tilde{H} < H_{\max}$ fixed and write $u_0 = u(\,\cdot\,, H_0)$, so that our primary focus becomes dependence on H. Accordingly, we will denote various constants that are independent of H with a subscript 0, and by C_0 in particular. As mentioned above, constants that can be taken to depend only on $\tilde{H}$ will be, for the most part, ignored. Various other constants will be denoted by C.

[1] Strictly speaking, Gilbarg and Trudinger state the theorem with a dependence on $K \equiv |u|_{C^1(\Omega)}$. The dependence however essentially arises from the De Giorgi-Nash estimates (Theorems 8.22 and 8.29 [GT83, pp. 200–205]) when they are applied to the equation

$$\sum_{ij} D_i(a_{ij}D_jw) = 0$$

where $w = D_ku$. In this instance, only a bound for $|a_{ij}|_{C^0(\Omega)}$ is required, not the explicit value K.

We begin by observing that the difference $v = u - u_0 = u(\,\cdot\,; H) - u(\,\cdot\,; H_0)$ satisfies a linear elliptic equation. If $\delta = H - H_0$,

$$\begin{aligned} 2\delta &= \operatorname{div} A(Du) - \operatorname{div} A(Du_0) \\ &= \sum_i D_i \left\{ A^i(tDu + (1-t)Du_0)\Big|_{t=0}^{1} \right\} \\ &= \sum_i D_i(\alpha_{ij}(x)\, D_j v) \end{aligned}$$

where

$$\alpha_{ij} = \alpha_{ij}^{\delta}(x) = \int_0^1 \frac{\partial A^i}{\partial p_j}(tDu + (1-t)Du_0)\, dt. \tag{11}$$

The uniform ellipticity of Corollary 2 is easily seen to hold for (α_{ij}). Thus, we see that v satisfies the uniformly elliptic divergence structure boundary value problem

$$\begin{cases} \sum_{i,j} D_i(\alpha_{ij} D_j v) = 2\delta & \text{on } \Omega, \\ v\big|_{\partial\Omega} \equiv 0, \end{cases} \tag{12}$$

and the estimates of the theorem follow, at least formally, from slight extensions of the "weak" Schauder estimates Theorems 8.33 and 8.32 [GT83, pg. 210]. For reference we give the statements as they apply to a general divergence form linear boundary value problem.

Let $v \in C^{k,\alpha}(\Omega)$ (where Ω is a bounded domain and $k = 1$ or 2 or $3\dots$) be a (weak) solution of the linear boundary value problem

$$\begin{cases} Lv = g + \sum_j D_j f_j & \text{on } \Omega, \\ v\big|_{\partial\Omega} \equiv \phi, \end{cases}$$

where $Lv = \sum_{i,j} D_i(\alpha_{ij}(x) D_j v + \beta_i(x) v) + \sum_j c_j(x) D_j v + d(x) v$ and the coefficients satisfy

$$|\alpha_{ij}|_{C^{k-1,\alpha}(\Omega)},\ |\beta_i|_{C^{k-1,\alpha}(\Omega)},\ |c_j|_{C^{k-1}(\Omega)},\ |d|_{C^{k-1}(\Omega)} \le \Lambda_k.$$

Theorem 8.32′. *If $\Omega' \subset\subset \Omega$, then*

$$|v|_{C^{k,\alpha}(\Omega')} \le C(|v|_{C^0(\Omega)} + |g|_{C^{k-1}(\Omega)} + |f|_{C^{k-1,\alpha}(\Omega)})$$

where $C = C(n, \lambda, \Lambda_k, \Omega', \Omega)$ (λ being the ellipticity constant for (α_{ij})) and $|f|_{C^{k-1,\alpha}(\Omega)} = \sum |f_j|_{C^{k-1,\alpha}(\Omega)}$.

Theorem 8.33′. *If Ω is a $C^{k,\alpha}$ domain and $v \in C^{k,\alpha}(\bar{\Omega})$, then*

$$|v|_{C^{k,\alpha}(\Omega)} \le C(|v|_{C^0(\Omega)} + |\phi|_{C^{k,\alpha}(\Omega)} + |g|_{C^{k-1}(\Omega)} + |f|_{C^{k-1,\alpha}(\Omega)})$$

where $C = C(n, \lambda, \Lambda_k, \Omega)$.

Applying Theorem 8.33′, for example, we have: *If* $|\alpha_{ij}|_{C^{k-1,\alpha}(\Omega)} \le \Lambda$, *then*

$$|v|_{C^{k,\alpha}(\Omega)} \le C(|v|_{C^0(\Omega)} + 2|\delta|)$$

where $C = C(n, \lambda, \Lambda, \Omega)$.

The only dependence in C on H is through Λ, so we see that the following two lemmas together establish statement (i) of the theorem.

Lemma 1. *There are constants Λ_0 and α_0 (independent of H) such that*

$$|\alpha_{ij}|_{C^{k-1,\alpha_0}(\Omega)} \le \Lambda_0.$$

Lemma 2. *There is some $C_0 > 0$ such that*

$$|u(\,\cdot\,; H) - u(\,\cdot\,; H_0)|_{C^0(\Omega)} \le C_0|\delta|.$$

Lemma 2 follows immediately from Theorem 8.16 [GT83, pg. 191]. In fact, we have

$$|v|_{C^0(\Omega)} \le (C/\lambda)(2|\delta|)|\Omega|^{2/q}$$

for any $q > n$ where $C = C_0(n, q, \Omega)$ is independent of H.

Proof of Lemma 1. We see from the definition of α_{ij} in (11) that for any $\alpha > 0$,

$$|\alpha_{ij}|_{C^{k-1,\alpha}(\Omega)} \le \sup_{0\le t\le 1} |\tilde{\alpha}_{ij}|_{C^{k-1,\alpha}(\Omega)}$$

where $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ij}(x,t) \equiv A_{ij}(tDu + (1-t)Du_0)$. By Lemma 9 (see Appendix B) $|\tilde{\alpha}_{ij}|_{C^{k-1,\alpha}(\Omega)}$ can be bounded in terms of B_1 and B_2 where

$$|A_{ij}|_{C^k(G)} \le B_1, \text{ and}$$

$$|tDu + (1-t)Du_0|_{C^{k-1,\alpha}(\Omega)} \le B_2.$$

In this instance $G = \{tDu + (1-t)Du_0 \in \mathbb{R}^n : x \in \Omega\}$. As in the estimates in Corollary 2, A_{ij} is smooth and G is bounded independently of H, so B_1 can be taken independently of H.

To find B_2 independently of H it suffices to bound $|u|_{C^{k,\alpha}(\Omega)}$. We proceed by induction.

The initial case $k = 1$ is obtained from (8) by taking $\alpha_0 = \tilde{\alpha}$.

For $k \ge 2$, we take α_0 to be the minimum of $\tilde{\alpha}$ and the Hölder exponent of $\partial\Omega$ and assume inductively that

$$|u|_{C^{k-1,\alpha_0}(\Omega)} \le M_0,$$

and (as a consequence)

$$|\alpha_{ij}|_{C^{k-2,\alpha_0}(\Omega)} \le \Lambda_0$$

for some Λ_0 independent of H. The latter assumption puts us in a position to apply an extension of the "classical" Schauder global estimate, Theorem 6.6 n.b., Problem 6.2, which again we state for convenience.

Let $v \in C^{k,\alpha}(\Omega)$ be a (classical) solution of the linear boundary value problem

$$\begin{cases} Lv = f & \text{on } \Omega, \\ v\big|_{\partial\Omega} \equiv \phi, \end{cases}$$

where $Lv = \sum_{i,j} a_{ij}(x) D_i D_j v + \sum_i b_i(x) D_i v + c(x) v$ and the coefficients satisfy

$$|a_{ij}|_{C^{k-2,\alpha}(\Omega)},\ |b_i|_{C^{k-2,\alpha}(\Omega)},\ |c|_{C^{k-2,\alpha}(\Omega)} \leq \Lambda_k.$$

Theorem 6.6′. *If Ω is a $C^{k,\alpha}$ domain and $v \in C^{k,\alpha}(\bar{\Omega})$, then*

$$|v|_{C^{k,\alpha}(\Omega)} \leq C(|v|_{C^0(\Omega)} + |\phi|_{C^{k,\alpha}(\Omega)} + |f|_{C^{k-2,\alpha}(\Omega)})$$

where $C = C(n, \lambda, \Lambda_k, \Omega)$.

When applied to the equation in $(*)$ Theorem 6.6′ yields

$$\begin{aligned} |u|_{C^{k,\alpha_0}(\Omega)} &\leq C_0(|u|_{C^0(\Omega)} + 2|H|) \\ &\leq C_0 \quad \text{independent of } H. \end{aligned}$$

The induction is concluded with the use of Lemma 9 which implies a bound for $|\alpha_{ij}|_{C^{k-1,\alpha_0}(\Omega)}$. This establishes Lemma 1 and Theorem 2 part (i).

If we replace Ω by $\Omega' \subset\subset \Omega$ in the proof of Lemma 1 and use

Theorem 6.2′. *Let $v \in C^{k,\alpha}(\Omega)$ be a solution of the linear boundary value problem described just above. Then*

$$|v|_{C^{k,\alpha}(\Omega')} \leq C(|v|_{C^0(\Omega)} + |f|_{C^{k-2,\alpha}(\Omega)})$$

where $C = C(n, \lambda, \Lambda_k, \Omega', \Omega)$.

then the same reasoning yields an estimate

$$|u|_{C^{k,\alpha_0}(\Omega')} \leq C_0(k), \tag{13}$$

which in turn gives by Lemma 9

Lemma 1′. For any k, there is a constant Λ_0' such that

$$|\alpha_{ij}|_{C^k(\Omega')} \leq \Lambda_0'.$$

Having made this observation, Theorem 2 part (ii) follows from Theorem 8.32′. ■

The main theorem of this section is the following:

Theorem 3 (regularity). $u \in C^\infty(\Omega \times (-H_{\max}, H_{\max}))$.

Proof. By Theorem 2 statement (i), if $\tilde{H} < H_{\max}$ and $|H| + |\delta| \le \tilde{H}$, then

$$\Delta_H^\delta u \equiv \frac{u(\,\cdot\,; H+\delta) - u(\,\cdot\,; H)}{\delta} = \frac{u_1 - u_0}{\delta}$$

satisfies

$$|\Delta_H^\delta u|_{C^{2,\alpha}(\bar{\Omega})} \le C_0,$$

i.e., $\{\Delta_H^\delta u\}_{|H|+|\delta|\le\tilde{H}}$ is bounded in $C^{2,\alpha}(\bar{\Omega})$. By Lemma 6.36, this set is therefore precompact in $C^2(\bar{\Omega})$. It follows that there is a function $\dot{u} \in C^2(\bar{\Omega})$ such that $\lim_{\delta\to 0} |\dot{u} - \Delta_H^\delta u|_{C^2(\bar{\Omega})} = 0$, and

$$\begin{cases} \sum_{ij} D_i\left(a_{ij} D_j \dot{u}\right) = 2 \quad \text{on } \Omega, \\ \dot{u}\big|_{\partial\Omega} \equiv 0. \end{cases} \tag{14}$$

Technically, $\Delta_H^\delta u$ satisfies a boundary value problem (divide equation (12) by δ), and by taking the limit of subsequences $\delta_j \to 0$ we arrive at (14). The existence of the limit as $\delta \to 0$ then follows from the uniqueness of solutions to (14).

It also follows from (14) that $\dot{u} \in C^\infty(\Omega) \cap C^{k,\alpha}(\bar{\Omega})$ (if Ω is a $C^{k,\alpha}$ domain).

Remark 6. *Note that the velocity $\dot{u}$ of our flow is determined by a linear boundary value problem—rather than by a local expression as for example in the heat equation.*

In our derivation of (14) we assumed that Ω was at least $C^{2,\alpha}$. So as not to require $\partial\Omega$ to be inordinately smooth, we work locally from now on.

We next consider the continuity of $\dot{u}$ as a function of H. Extending the notation above, we write $\dot{v} = \dot{u} - \dot{u}_0 \equiv \dot{u}(\,\cdot\,; H) - \dot{u}(\,\cdot\,; H_0)$. Note that $\dot{v}$ satisfies a linear boundary value problem

$$\begin{cases} \sum_{ij} D_i\left(a_{ij}^0 D_j \dot{v}\right) = f \quad \text{on } \Omega, \\ \dot{v}\big|_{\partial\Omega} \equiv 0 \end{cases} \tag{15}$$

where $a_{ij}^0 = A_{ij}(Du_0)$ and $f = -\sum_{i,j} D_i[(a_{ij} - a_{ij}^0) D_j \dot{u}]$.

Letting Ω' be a smooth domain compactly contained in Ω (n.b. Problem 6.9) and $\Omega'' \subset\subset \Omega'$, we have from Theorem 8.32$'$

$$|\dot{v}|_{C^{k,\alpha}(\Omega'')} \le C(|\dot{v}|_{C^0(\Omega')} + \sum_{i,j} |f_{ij}|_{C^{k-1,\alpha}(\Omega')})$$

where $C = C(n, \lambda, \Lambda, \Omega'', \Omega')$ and $f_{ij} = (a_{ij} - a_{ij}^0) D_j \dot{u}$. In this case, C is independent of H. Thus, we have an estimate

$$|\dot{v}|_{C^k(\Omega'')} \le C_0''|H - H_0| = C_0''|\delta| \tag{16}$$

(for arbitrary k as in Theorem 2) as long as the following lemmas hold.

Lemma 3. *There is some C_0 and some $\alpha_0 > 0$ such that*

$$|f_{ij}|_{C^{k-1,\alpha_0}(\Omega')} \leq C_0 |\delta|.$$

Lemma 4. *There is some C_0 such that*

$$|\dot{v}|_{C^0(\Omega')} \leq C_0 |\delta|.$$

As before Theorem 8.16 implies

$$|\dot{v}|_{C^0(\Omega')} \leq C(q) \|f\|_{L^{q/2}}$$

for any $q > n$. Therefore, Lemma 4 follows from Lemma 3.

Proof of Lemma 3. Since $|a_{ij} - a^0_{ij}|_{C^l(\Omega')} \leq C_0|\delta|$ for any fixed l (by Theorem 2 part (ii)), it is sufficient to show that for some $\alpha_0 > 0$

$$|D\dot{u}|_{C^{k-1,\alpha_0}(\Omega')}$$

is *bounded* independently of H. Such a bound follows from Theorem 8.32$'$ when applied to the equation in (14). One must check that the coefficients a_{ij} are bounded in $C^{k-1,\alpha_0}(\Omega')$ and that $|\dot{u}|_{C^0(\Omega)}$ can also be bounded (independently of H). The first bound follows from Lemma 9 and the bound on $|u|_{C^{k,\alpha_0}(\Omega')}$ given in (13). The latter bound follows from Theorem 8.16. This completes the proof of Lemma 3.

From the estimate (16) it follows that

$$\ddot{u} = \frac{\partial^2 u}{\partial H^2}$$

exists and is well defined in $C^\infty(\Omega)$—satisfying the boundary value problem

$$\begin{cases} \sum_{ij} D_i\left(a_{ij} D_j \ddot{u}\right) = f_2 \quad \text{on } \Omega, \\ \ddot{v}\Big|_{\partial\Omega} \equiv 0 \end{cases}$$

where $f_2 = -\sum_{i,j} \dot{a}_{ij} D_j \dot{u}$ and

$$\dot{a}_{ij} = \sum_k \frac{\partial^2 A}{\partial p_j \partial p_k}(Du)\ D_k \dot{u}. \tag{17}$$

Since we have infinitely many derivatives to go (in proving Theorem 3), let us assume for $l = 1, 2, \ldots, m$, that $u^{(l)} = \partial^l u / \partial H^l \in C^\infty(\Omega)$ satisfies

$$\begin{cases} \sum_{i,j} D_i\left(a_{ij} D_j u^{(l)}\right) = f_l \quad \text{on } \Omega, \\ \\ u^{(l)}\Big|_{\partial\Omega} \equiv 0 \end{cases} \tag{18}$$

where $f_1 = 2H$, $f_2 = 2$, and

$$f_{l+1} = f_{l+1}(x; H) = \dot{f}_l - \sum_{i,j} \dot{a}_{ij} D_j u^{(l)}, \quad 2 < l \leq m-2 \tag{19}$$

with $\dot{a}_{ij}$ given by (17).

Under these assumptions, we reason as follows:

Theorem 4. *For $|H|, |H_0| \leq \tilde{H} < H_{\max}$ and $\Omega'' \subset\subset \Omega$, $v^{(l)} \equiv u^{(l)} - u_0^{(l)} = u^{(l)}(\,\cdot\,; H) - u^{(l)}(\,\cdot\,; H_0)$ satisfies an estimate*

$$|v^{(l)}|_{C^k(\Omega'')} \leq C_0 |\delta| \tag{20}$$

where C_0 is independent of H and $\delta = H - H_0$.

Corollary 3. *There exists $u^{(m+1)} \in C^\infty(\Omega)$ satisfying (18) with $m+1$ in place of l.*

Note that Theorem 3 clearly follows from Corollary 3, and Theorem 4 is a direct generalization of Theorem 2.

Proof of Theorem 4. According to (18), $v^{(m)}$ satisfies

$$\begin{cases} \sum_{i,j} D_i \left(a_{ij}^0 D_j v^{(m)} \right) = f_l - f_l^0 - \sum_{i,j} D_i[(a_{ij} - a_{ij}^0) D_j u^{(m)}] \quad \text{on } \Omega, \\ v^{(m)}\Big|_{\partial\Omega} \equiv 0 \end{cases} \tag{21}$$

where $f_m^0 = f_m(x; H_0)$ and the other functions have been defined above, n.b. (15). We take, as before, $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, and Lemma 8.32$'$ implies for any k and α

$$|v^{(m)}|_{C^{k,\alpha}(\Omega'')} \leq C_0 (|v^{(m)}|_{C^0(\Omega')} + |f_m - f_m^0|_{C^{k-1}(\Omega')} + \sum_{i,j} |f_{ij}|_{C^{k-1,\alpha}(\Omega')})$$

where $f_{ij} = (a_{ij} - a_{ij}^0) D_j u^{(m)}$.

Thus, in general, we have three terms to estimate:

Lemma 5. $|f_{ij}|_{C^{k-1,\alpha_0}(\Omega')} \leq C_0 |\delta|$, *(for some α_0).*

Lemma 6. $|f_m - f_m^0|_{C^{k-1}(\Omega')} \leq C_0 |\delta|$.

Lemma 7. $|v^{(m)}|_{C^0(\Omega')} \leq C_0 |\delta|$.

Lemmas 5 and 7 follow from the reasoning that gave us Lemmas 3 and 4, provided we produce a

Proof of Lemma 6. From the definition

$$|f_m - f_m^0|_{C^{k-1}(\Omega')} \leq |\dot{f}_{m-1} - \dot{f}_{m-1}^0|_{C^{k-1}(\Omega')} + \sum_{i,j} |\dot{a}_{ij} D_j u^{(m-1)} - \dot{a}_{ij}^0 D_j u_0^{(m-1)}|_{C^{k-1}(\Omega')}.$$

The first term can be handled by induction using the assertion of the lemma itself. If we consider one of the terms in the sum we have

$$|\dot{a}_{ij} D_j u^{(m-1)} - \dot{a}^0_{ij} D_j u_0^{(m-1)}|_{C^{k-1}(\Omega')}$$
$$\leq C_0(|\dot{a}_{ij} - \dot{a}^0_{ij}|_{C^{k-1}(\Omega')} |D_j u^{(m-1)}|_{C^{k-1}(\Omega')} + |\dot{a}^0_{ij}|_{C^{k-1}(\Omega')} |D_j v^{(m-1)}|_{C^{k-1}(\Omega')}).$$

The first product on the right can be handled by the reasoning in the proof of Lemma 3. The second we can estimate by incorporating (20) in our induction hypothesis.

This completes the proof of Lemma 6 and Theorem 4.

Proof of Corollary 3. Since k is arbitrary in Theorem 4, any sequence $\delta_j \to 0$ provides a (sub)sequence of difference quotients $\Delta^\delta u^{(m)} \to w \in C^2(\Omega'')$. Dividing (21) by δ we also have

$$\sum_{i,j} D_i(a^0_{ij} D_j \Delta^\delta u^{(m)}) = \Delta^\delta f_m - \sum_{i,j} D_i(\Delta^\delta a_{ij} D_j u^{(m)}).$$

On the other hand, it follows inductively from (19) that f_m is a linear combination of terms $a^{(k)}_{ij} D_j u^{(l)}$ where $a^{(k)}_{ij} = \partial^k a_{ij}/\partial H^k$ and $k, l \geq 1$, $k + l \leq m$. Consequently, $\dot{f}_m = \lim_{\delta\to 0} \Delta^\delta f_m$ is well defined, and the limit w mentioned above satisfies

$$\begin{cases} \sum_{i,j} D_i\left(a_{ij} D_j w\right) = \dot{f}_m - \sum_{i,j} D_i(\dot{a}_{ij} D_j u^{(m)}) & \text{on } \Omega, \\ w\big|_{\partial\Omega} \equiv 0 & \end{cases}$$

Since the solutions of this boundary value problem are unique, the limit of *every* such subsequence must be

$$w = \lim_{\delta\to 0} \Delta^\delta u^{(m)} = u^{(m+1)}.$$

This completes the proof of Corollary 3 and, hence, of Theorem 3. ■

We conclude this section with the following observation.

Theorem 5 (relation of volume and mean curvature)**.** *Let*

$$W = W(H) \equiv \int_\Omega u(x; H). \tag{22}$$

There is a unique value $V_{\max} > 0$ such that $W : [-H_{\max}, H_{\max}] \to [-V_{\max}, V_{\max}]$ is a smooth strictly decreasing function.

Proof.

$$\dot{W} = \int_\Omega \dot{u}.$$

Recall from (14) that $\dot{u}$ satisfies

$$\begin{cases} \sum_{ij} D_i\left(a_{ij} D_j \dot{u}\right) = 2 \quad \text{on } \Omega, \\ \dot{u}\big|_{\partial\Omega} \equiv 0. \end{cases}$$

By the maximum principle, any solution of this problem (for any H) is negative. ■

2. Convexity for small volumes

The regularity result of the previous section allows us to linearize the problem $(*)$ around the zero solution and determine (in terms of Poisson's equation) the signs of expressions involving relatively high derivatives of $u(x;H)$ in both x and H. If these expressions are chosen appropriately as below, we obtain information about the convexity of solutions with $|H|$ small.

In our introductory remarks we were somewhat carefree with the term *convexity*. This is essentially justified by the symmetry (Corollary 1) of the H-graphs under consideration. Nevertheless, it will be convenient from now on to distinguish between *concave* functions ($D^2u \leq 0$) and *convex* functions ($D^2u \geq 0$). See [Mor66], Lemma 1.8.1, for equivalent definitions.

It will also be convenient for us to detect convexity (or concavity) by considering a *single number*. A simple way to do this in two dimensions is the following. Let $v \in C^2(\Omega)$. We say that v is *strictly second order convex* (alt. concave) if D^2v is positive definite (alt. negative definite) on Ω. Define an auxiliary function G_v on Ω by

$$G_v = v_{xx}v_{yy} - v_{xy}^2$$

where we have used the classical "x, y" notation to denote the second partials. We then have

Lemma 8. *If $\inf_\Omega v < \inf_{\partial\Omega} v$ and $G_v > 0$ on Ω, v is strictly second order convex. Similarly, $\sup_\Omega v > \sup_{\partial\Omega} v$ (and $G_v > 0$) implies v is strictly second order concave.*

Proof. Since $D^2v(x)$ is a real symmetric matrix, there is an orthogonal matrix M and a diagonal matrix Λ (both of which depend smoothly on x) such that $MD^2vM^{-1} = \Lambda$. If the diagonal elements of Λ are λ_1 and λ_2, then $G_v = \det D^2v = \lambda_1\lambda_2$, and the convexity form $D^2ve_\theta \cdot e_\theta = \lambda_1\xi_1^2 + \lambda_2\xi_2^2$ where $\xi = Me_\theta$ and $e_\theta = (\cos\theta, \sin\theta)$. Thus, if $G_v > 0$, then neither λ_1 nor λ_2 can vanish.

On the other hand, if $\inf_\Omega v < \inf_{\partial\Omega} v$ then there is a large lower-hemispherical graph $h \leq v$ such that at one or more points $h(x_0) = v(x_0)$. Therefore, the λ_i must be positive (and v strictly second order convex). ■

Kawohl [Kaw84] gives essentially the same reasoning as in the above proof under the additional assumption that v be subharmonic.

The assumptions in Lemma 8 are natural for the applications we have in mind, but a more general discussion may be found in Appendix C. At present, our primary objective was to justify the following terminology.

We say that v is *uniformly* second order convex (alt. concave) if v is convex (alt. concave) and for some $\lambda > 0$ we have $G_v \geq \lambda$ on Ω.

The two "small volume" results of this section are obtained by the following basic line of reasoning.

If $u = u(x; H)$ is the solution discussed in §1, then the convexity properties of u are the same as those of u/H. To be precise, if $w = u/H$ and $G_w > 0$, then $G_u > 0$. On the other hand the scaled function w is the difference quotient

$$\Delta^H u = \frac{u(x;H) - u(x;0)}{H} \tag{23}$$

which, according to the proof of Theorem 3 converges in $C^2(\bar{\Omega})$ to $\dot{u}$. Since G_w is a second order operator in x we have

$$|G_w - G_{\dot{u}}|_{C^0(\bar{\Omega})} \to 0 \quad \text{as } |H| \to 0.$$

On the other hand, at $H = 0$ we have from (14) that $\dot{u}$ is a solution of the Saint Venant torsion problem

$$\begin{cases} \Delta \dot{u} = 2 & \text{on } \Omega, \\ \dot{u}\big|_{\partial\Omega} \equiv 0. \end{cases}$$

Combining these observations, we have proved

Theorem 6 (1 convexity)**.** *Let Ω be a strictly convex domain in the sense that the curvature κ of $\partial\Omega$ is everywhere positive. Consider the problem*

$$\begin{cases} \Delta v = 2 & \textit{on } \Omega, \\ v\big|_{\partial\Omega} \equiv 0. \end{cases} \tag{24}$$

If v is uniformly second order convex, then there is some $\epsilon > 0$ such that $u(x;H)$ is strictly second order convex for $0 < H < \epsilon$. In particular, if Ω is an ellipse, "small bubbles" are convex.

If on the other hand, v has a point of strict non-convexity, i.e., the Gauss curvature of graph(v) is negative at some point, then arbitrarily small bubbles u are likewise non-convex. This may be observed for smooth convex domains whose boundaries converge to a square.

Although solutions $\dot{u}$ of (24) are not convex in general, they are "1/2 power convex." That is to say, $v = (-\dot{u})^{1/2}$ is strictly second order concave. The crucial step in proving this fact (showing that G_v is subharmonic) was carried out by Makar-Limonov [ML71] though the strict second order convexity was actually noted later in [Kaw84].

We note further that this condition is *uniform*. In order to see this, we introduce another auxiliary function $L_v \equiv v_y^2 v_{xx} - 2v_x v_y v_{xy} + v_x^2 v_{yy}$ which essentially measures the convexity of the level curves; see Appendix C. Given any positive function ϕ defined on Ω it is easy to see that

$$G_\psi = \frac{2\phi G_\phi - L_\phi}{8\phi^2} \tag{25}$$

where $\psi = \sqrt{\phi}$. In our case,

$$G_v = \frac{L_{\dot u} - 2\dot u G_{\dot u}}{8\dot u^2}.$$

Now in some closed neighborhood $\mathcal{N}$ of $\partial\Omega$ we may assume $|D\dot u| \geq \delta > 0$ and (consequently) that the curvature of the level curves $L_{\dot u}/|D\dot u|^3 \geq \kappa/2 > 0$. Taking a smaller neighborhood if necessary we may also assume that $|\dot u| < 1$ and $|\dot u G_{\dot u}| < \kappa\delta^3/8$. Thus, on $\mathcal{N}\backslash\partial\Omega$

$$\begin{aligned} G_v &= \frac{1}{8\dot u^2}\left(\frac{L_{\dot u}}{|D\dot u|^3}|D\dot u|^3 - 2\dot u G_{\dot u}\right) \\ &\geq \frac{1}{8}\left(\frac{\kappa}{2}\delta^3 - \frac{\kappa\delta^3}{4}\right) \\ &= \frac{\kappa\delta^3}{32} > 0. \end{aligned}$$

Finally, on $\Omega' = \Omega\backslash\mathcal{N}$ (by smoothness and the observation of Kawohl) there is some $\lambda' > 0$ such that $G_v \geq \lambda'$. Letting $\lambda = \min\{\kappa\delta^3/32, \lambda'\}$ we have established uniformity.

Our basic line of reasoning now yields

Theorem 7. *Given a strictly convex ($\kappa > 0$) domain Ω, there is some $\epsilon = \epsilon(\Omega) > 0$ such that $u = u(x; H)$ is $1/2$ concave for $-\epsilon < H < 0$.*

Proof. Again we scale up. For the function $w = \sqrt{-u/H}$ we have from (25)

$$G_w = \frac{1}{8(\Delta u)^2}(L_{\Delta u} - 2\Delta u G_{\Delta u})$$

where $\Delta u = \Delta^H u$ is the difference quotient given in (23). Since $\Delta u \to \dot u$ in $C^2(\bar\Omega)$ as $H \to 0$ we conclude (essentially from the discussion above) that $L_{\Delta u} - 2\Delta u G_{\Delta u} \geq \mu/2 > 0$ for $|H|$ small enough where. Thus, $G_w > 0$ on Ω for $|H|$ small. ∎

Were we able to extend the reasoning of Makar-Limonov to the linear problem (14)—and show any degree of strict power convexity—the methods of this section would apply to show the convexity of the level curves for solutions of ($*$). Another related (and perhaps more tractable) approach will be described at the end of the paper.

For now, we concentrate on showing that the "level curves" are in fact smooth simple closed curves.

3. Stability

For this section, let $u = u(x; H)$ be a positive solution $(-H_{\max} < H < 0)$ of $(**)$ on a smooth domain in $\mathbb{R}^n$. We first observe that graph(u) has minimal area among smooth graphs that enclose the same volume. More precisely, if $v \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$, $v\big|_{\partial\Omega} \equiv \phi$, and

$$V(v) \equiv \int_\Omega v = V(u),$$

then

$$\begin{aligned} A(v) &\equiv \int_\Omega \sqrt{1 + |Dv|^2} \\ &= A(u) + \int_0^1 \frac{d}{dt}\left[\int_\Omega \sqrt{1 + |(1-t)Du + tDv|^2}\right] dt \qquad (26) \\ &= A(u) + \int_0^1 \left[\int_\Omega \frac{[(1-t)Du + tDv]\cdot(Dv - Du)}{\sqrt{1 + |(1-t)Du + tDv|^2}}\right] dt \qquad (27) \\ &= A(u) + \left[\int_\Omega \frac{Du \cdot (Dv - Du)}{\sqrt{1 + |Du|^2}}\right] \qquad (28) \\ &\qquad + \int_0^1 \left[\int_\Omega \frac{(1 + |Dv_*|^2)|Dh|^2 - (Dv_* \cdot Dh)^2}{(1 + |Dv_*|^2)^{3/2}}\right] dt \end{aligned}$$

where we have expanded the integrand in (27) by Taylor's formula at $t = 0$; $v_* = (1 - t_*)u + t_* v$ for some $t_* \in (0, 1)$, and $h = v - u$. The numerator in the third term of (28),

$$|Dh|^2 + |Dv_*|^2|Dh|^2 - (Dv_* \cdot Dh)^2,$$

is nonnegative by Schwarz' inequality, and integrating the second term by parts yields

$$\int_\Omega \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right)(u - v) = 2H \int_\Omega (u - v) = 0.$$

Hence, $A(v) \geq A(u)$. (In other words, the area integrand in (26) is a convex function of t and has a critical point, hence a minimum, at $t = 0$.)

Thus, u is a *stable critical point for A with respect to volume preserving variations*:

Corollary 4. *If $v = v(x;\epsilon)$ is a smooth volume preserving variation of u, (i.e., $v \in C^\infty(\Omega \times I) \cap C^0(\bar{\Omega} \times I)$ for some interval $I = (-\epsilon, \epsilon)$ and satisfies $V(v) \equiv V(u)$, $v \equiv \phi$ on $\partial\Omega$, and $v(x;0) \equiv u$), then*

$$\delta A(v) \equiv \frac{d}{d\epsilon} A(v)\Big|_{\epsilon=0} = 0, \quad \textit{and} \tag{29}$$

$$\delta^2 A(v) \equiv \frac{d^2}{d\epsilon^2} A(v)\Big|_{\epsilon=0} \geq 0. \tag{30}$$

We say that u is a *critical point for A with respect to volume preserving variations* if (29) holds and a *(semi)stable critical point with respect to volume preserving variations* if (29) and (30) hold.[2]

We define an alternative functional

$$J(v) = A(v) + 2HV(v).$$

It is easy to see that the assertion of statement (29) in Corollary 4 is equivalent to the following condition.

(29)′ $\delta J(v) = 0$ for *any* (not necessarily volume preserving) variation v.

Following the advice of Bolza [Bol09] and Barbosa and do Carmo [BdC84] we note that a similar equivalence does not hold in general for the second variation. To be precise, if $\mathcal{D}$ is a domain in a *parametric* surface of constant mean curvature, then we have

Proposition 1 ([BdC84]; see also [Wen66]). *$\mathcal{D}$ is stable with respect to volume preserving variations if and only if $\delta^2 J(\vec{v}) \geq 0$ for all smooth compactly supported parametric variations $\vec{v}$ satisfying $\delta V(\vec{v}) = 0$.*

The reasoning of Cabré and Chanillo in the next section, however, essentially requires such an equivalence to hold for non-parametric solutions.

Definition 1. *$u = u(x;H)$ is said to be* overstable *(or more accurately oversemistable) if $\delta^2 J(v) \geq 0$ for* all *compactly supported variations v.*

Theorem 8. *Let $u_0 = u(x;H_0)$ where $|H_0| < H_{\max}$. Then u_0 is overstable.*

Remark 7. *Several proofs may be given of Theorem 8. Probably the simplest—pointed out to me by C. Chan and H. Wente—is obtained by repeating the calculation leading to Corollary 4 with J in place of A and an arbitrary variation v in place of the volume preserving one. One then has $J(v) \geq J(u)$. The result also follows—as*

[2]Other authors have typically considered "parametric" variations, but since any smooth variation of a graph is locally "non-parametric" for small ϵ, we lose no generality for the surfaces under consideration.

pointed out by R. Schoen—from the discussion in [FCS80] *by noting that* $g = N_3$ *(the vertical component of the normal to* $\mathrm{graph}(u)$*) is a positive solution to the equation* $\Delta g + \|B\|^2 g = 0$ *where* Δ *denotes the intrinsic Laplacian on* $\mathrm{graph}(u)$ *and* $\|B\|^2$ *the sum of the squares of the principal curvatures of the graph. The proof presented below demonstrates that* $u(x; H)$ *provides, in some sense, the flow which optimally changes volume. More precisely, given any variation* v*, there is a variation* w *consisting only of members of* $\{u(x; H)\}$ *such that (31) holds.*

Proof or Theorem 8. Let $v = v(\,\cdot\,; \epsilon)$ be a smooth compactly supported variation.
Since $V(v) \to V(u)$ as $\epsilon \to 0$, we may define

$$H(\epsilon) = W^{-1}(V(v))$$

where W is given by (22) in Theorem 5.

Setting $w = u(\,\cdot\,; H(\epsilon))$ we obtain another variation. Since $V(v) = V(w)$ and $A(w) \le A(v)$ we see that $J(w) \le J(v)$ with equality at $\epsilon = 0$. Therefore,

$$\delta^2 J(w) \le \delta^2 J(v). \tag{31}$$

On the other hand, we can compute $\delta^2 J(w)$ explicitly.

$$\begin{aligned} \frac{d}{d\epsilon} J(w) &= \frac{d}{d\epsilon} \int_\Omega \left[\sqrt{1 + |Du(x; H(\epsilon))|^2} + 2Hu(x; H(\epsilon)) \right] \\ &= H'(\epsilon) \int_\Omega [2H - 2H(\epsilon)] \dot{u}(x; H(\epsilon)) \end{aligned}$$

where $\dot{u} = \partial u / \partial H$ (which is well defined by Theorem 3) and we have integrated by parts. Differentiating again and setting $\epsilon = 0$,

$$\delta^2 J(w) = -2H'(0)^2 \int_\Omega \dot{u}(x; H).$$

Since $\dot{u}$ is a solution of (14) (see the proof of Theorem 5), the integral on the right is strictly negative, and $\delta^2 J(v) \ge \delta^2 J(w) \ge 0$. ■

Remark 8. *The condition of overstability has been considered by various authors including Gulliver* [Gul73]*, Mori* [Mor83]*, and Ruchert* [Ruc79]*. Ruchert obtains the condition*

$$\int_{\mathcal{G}} \frac{1}{2} \|B\|^2 < 2\pi \tag{32}$$

for overstability where $\mathcal{G} = graph(u)$ *and* $\|B\|^2$ *is the sum of the squares of the principal curvatures. Finn has pointed out that this condition in inadequate to show the overstability of* H*-graphs as follows.*

Example 1. *Rewriting the integral in (32) we have*

$$\begin{aligned}\int_{\mathcal{G}} \frac{1}{2}\|B\|^2 &= 2\int_{\mathcal{G}} H^2 - \int_{\mathcal{G}} K \\ &\geq H^2 A(u).\end{aligned}$$

Let N_r be a nodoid (i.e., an inflectionless, rotationally symmetric surface of non-zero constant mean curvature) with $H = 1$ and maximum distance from its axis of rotation r. If we assume the axis to be the z-axis and consider $\mathcal{G}_r = \{(x, y, z) \in N_r : x \geq r - 1/8\}$ we obtain graphs with $H^2 A_r = A_r \to +\infty$ as $r \to \infty$. It is clear that (32) fails for these graphs. Note: $\mathcal{G}_r$ approximates a portion of a circular torus with axis the z-axis and dimensions $(r - 1/2) \times 1/2$.

Remark 9. *It should also be noted that while $u(x; H)$ minimizes area among graphs that enclose the same volume, it has not been proved that $u(x; H)$ is the classical Douglas-Rado-Wente* [Wen71] *solution of the volume constrained Plateau problem.*

Before we proceed, let us recall the formulation of "overstability" in terms of eigenvalues. An elementary computation taking $v = u + \epsilon\phi$ gives

$$\delta^2 J(v) = \delta^2 A(v) = \langle -\mathcal{L}\phi, \phi\rangle$$

where $\mathcal{L}$ is the linearization of $\mathcal{M}$ at u given in (6) and the inner product is taken in L^2. Thus, the first eigenvalue of $\mathcal{L}$ on Ω, $\lambda_1(\mathcal{L}, \Omega) \geq 0$. From the variational characterization of eigenvalues ($\lambda_1 = \inf_{|\phi|_{L^2}=1}\langle -\mathcal{L}\phi, \phi\rangle$) and the regularity of eigenfunctions the following corollary follows at once.

Corollary 5. *If $\phi \neq \Omega' \subset\subset \Omega$, then $\lambda_1(\mathcal{L}, \Omega') > \lambda_1(\mathcal{L}, \Omega) \geq 0$.*

4. Uniqueness of critical points

Here we apply arguments of Cabré and Chanillo [CC97] to show the two theorems stated below. The reasoning applies to strictly convex ($\kappa > 0$) domains Ω in $\mathbb{R}^2$.

Theorem 9. *If $0 < |H| < H_{\max}$, then for each direction $e_\theta = (\cos\theta, \sin\theta)$ the solution $u = u(x; H)$ satisfies*

(33)
$$\begin{cases} \text{(i) } N_\theta \equiv \{x \in \bar{\Omega} : u_{e_\theta} = Du(x) \cdot e_\theta = 0\} & \textit{is a smooth embedded curve in } \bar{\Omega}. \\ \text{(ii) } M_\theta \equiv \{x \in N_\theta : Du_{e_\theta} = 0\} = \phi. & \end{cases}$$

Theorem 10. *If $u \in C^2(\bar{\Omega})$ is any positive function satisfying* (i), (ii), *and*

$$\begin{cases} u\big|_{\partial\Omega} = 0 \\ |Du\big|_{\partial\Omega}| > 0, \end{cases}$$

then u has a unique critical point in Ω.

Detailed proofs of both theorems may be found in [CC97], but for completeness and to give a more detailed exposition of certain points we include an outline of the reasoning.

Proof of Theorem 9. Notice that N_θ is a smooth embedded curve locally near any point $x_0 \in N_\theta \backslash M_\theta$. Because of this, (i) essentially follows from (ii). In order to verify (ii) we consider two cases.

If $x_0 \in M_\theta \cap \partial\Omega$, then x_0 must be one of the two points p_1, p_2 on $\partial\Omega$ where in inward normal n to $\partial\Omega$ is orthogonal to e_θ: $n \cdot e_\theta = 0$. (In any case, these two points are in N_θ.) Calculating the normal curvature κ_σ of $\mathcal{G} = \text{graph}(u)$ with respect to $N = (Du, -1)/\sqrt{1+|Du|^2}$ along $\partial\Omega$ we have

$$\kappa_\sigma = \kappa n \cdot N = \frac{\kappa n \cdot Du}{\sqrt{1+|Du|^2}} = \frac{\kappa}{\sqrt{1+|Du|^2}} \frac{\partial u}{\partial n} > 0. \tag{34}$$

On the other hand, since $x_0 \in N_\theta$ an alternative expression is given by

$$\kappa_\sigma = \frac{\partial^2 u}{\partial e_\theta^2}(0,0,1) \cdot N = -\frac{u_{e_\theta e_\theta}}{\sqrt{1+|Du|^2}}.$$

Equating the two expressions we find

$$Du_{e_\theta} \cdot e_\theta = u_{e_\theta e_\theta} = -\kappa \frac{\partial u}{\partial n} < 0. \tag{35}$$

Evidently, $Du_{e_\theta} \neq 0$ and the first case is complete.

We have only used the equation in $(*)$ when we asserted, by the Hopf boundary point lemma, the inequality in (34). The statement (35) also implies that N_θ is *transverse* to $\partial\Omega$ at $x_0 = p_1, p_2$.

The second possibility is that $x_0 \in M_\theta \cap \Omega$. This assumption may be slightly refined as follows. From the first case there is a closed neighborhood $\mathcal{N}$ of $\partial\Omega$ for which $\mathcal{N} \cap N_\theta$ consists precisely of connected portions Γ_1 and Γ_2 of the smooth curves (in N_θ) near p_1 and p_2. We can also assume that $\mathcal{N} \backslash (\Gamma_1 \cup \Gamma_2)$ has exactly two connected components C_+ and C_- with $u_{e_\theta} > 0$ on C_+ and $u_{e_\theta} < 0$ on C_-.

Accordingly, we assume $x_0 \in M_\theta \cap (\Omega \backslash \mathcal{N})$. According to Hartman [Har58, pg. 381 (iv)], since $u_{e_\theta}(x_0) = 0$, $Du_{e_\theta}(x_0) = 0$, and $\mathcal{L}u_{e_\theta} = 0$, a small disk B centered at x_0 consists of $4k$ disjoint regions in $\Omega \backslash N_\theta$ ($k \geq 1$) along with $4k$ arcs in N_θ connecting x_0 to ∂B which are smooth and (with the exception of x_0) disjoint. We may furthermore order the regions consecutively (say clockwise) so that the first three are R_+, R_- and R'_+ with u_{e_θ} alternating in sign on the regions as indicated. Each region must belong to a connected component of $\Omega \backslash N_\theta$. If every such component C extends to $\partial\Omega$, then each such C must be path connected to C_+ or to C_-. Assuming this, and connecting R_+ and R'_+ to C_+ by paths in $\{x : u_{e_\theta}(x) > 0\}$, we see that it is impossible to connect R_- to C_- by a path in $\{x : u_{e_\theta}(x) < 0\}$. Consequently, some component $C = \Omega'$ of $\Omega \backslash N_\theta$ is compactly contained in Ω. Furthermore, since $\partial\Omega' \subset N_\theta$, u_{e_θ} is a nontrivial eigenfunction for $\mathcal{L}$:

$$\begin{cases} \mathcal{L}u_{e_\theta} = 0 & \text{on } \Omega', \\ u\big|_{\partial\Omega'} \equiv 0. \end{cases}$$

This implies that $0 \geq \lambda_1(\mathcal{L}, \Omega')$ and contradicts Corollary 5. The contradiction establishes Theorem 9. ■

Proof of Theorem 10. The basic assertion in this proof is that there is a natural flow $\xi : \bar{\Omega} \times \mathbb{R} \to \bar{\Omega}$ that "rotates" the nodal sets N_θ. That is, we primarily want ξ to satisfy the condition

$$\xi(N_\theta, \tau) = N_{\theta+\tau}. \tag{36}$$

This flow also fixes the set of critical points $K \equiv \{x \in \Omega : Du(x) = 0\} = \cap N_\theta$. In terms of an autonomous system of ode's

$$\begin{cases} \dot{\xi} = \vec{F}(\xi), \\ \xi(x, 0) = x, \end{cases}$$

the condition on the critical points becomes

$$\vec{F}(x) \equiv 0, \quad x \in K.$$

At points x away from the critical set K, (36) imposes a useful necessary condition on $\vec{F}$ as follows. Let $\bar{\theta} = \bar{\theta}(x)$ be defined by

$$e_{\bar{\theta}} = (\cos\bar{\theta}, \sin\bar{\theta}) = (u_y, -u_x)/|Du|.$$

Notice that $x \in N_{\bar{\theta}} \backslash N_\theta$ for $\theta \neq \bar{\theta}$. Consequently, as long as $\xi(x, \tau) \notin K$ we conclude from (36) that

$$\bar{\theta}(\xi(x, \tau)) = \bar{\theta}(x) + \tau.$$

Differentiating with respect to τ,

$$D\bar{\theta} \cdot \vec{F} \equiv 1. \tag{37}$$

On the other hand, we can compute $D\bar{\theta}$ explicitly:

$$D\bar{\theta} = \frac{1}{|Du|^2} D^2u \cdot (-u_y, u_x) = -\frac{1}{|Du|} D^2u \cdot e_{\bar{\theta}}.$$

Looking then at (37), there is an obvious choice for $\vec{F}$:

$$\vec{F}_0 = \frac{D\bar{\theta}}{|D\bar{\theta}|^2} = -|Du| \frac{D^2u \cdot e_{\bar{\theta}}}{|D^2u \cdot e_{\bar{\theta}}|^2}.$$

Note that the condition (ii) $M_\theta = \phi$ implies that $|D^2u \cdot e_{\bar{\theta}}| = |Du_{e_\theta}| \geq \lambda > 0$ uniformly for points $x \in K$. By continuity, a similar bound holds in a neighborhood of K, and one obtains the estimate $|\vec{F}_0| \le C|Du|$ for some constant C. From this it follows that $\vec{F}_0$ extends to a Lipschitz vector field on $\bar{\Omega}$ that vanishes on K.

Unfortunately, there is no reason be believe that the resulting flow leaves $\bar{\Omega}$ invariant, or equivalently that $\vec{F}_0$ is proportional to e_θ on $\partial\Omega$. There are many other choices for $\vec{F}$ however. In fact, if $\vec{V}$ is (almost) *any* vector field, then

$$\vec{F} = \frac{\vec{V}}{D\bar{\theta} \cdot \vec{V}} \tag{38}$$

will satisfy (37) and imply the main condition (36). $\vec{F}_0$ is obtained by (extending to K) the particular choice $\vec{V}_0 = D\bar{\theta}$. Another choice, at least near $\partial\Omega$, is given by

$$\vec{V}_1 = e_{\bar{\theta}}$$

which—if it can be extended—will ensure invariance of the domain. Note first of all that the formula (38) is valid (i.e., finite valued) near $\partial\Omega$. In fact,

$$D\bar{\theta} \cdot e_{\bar{\theta}} = -\frac{1}{|D\bar{\theta}|} Du_{e_{\bar{\theta}}} \cdot e_{\bar{\theta}} > 0 \quad \text{by (35).} \tag{39}$$

By taking a partition of unity:

$$\begin{cases} \phi_0, \phi_1 & \text{smooth, nonnegative on } \Omega,\ \sum \phi_j \equiv 1, \\ \phi_0 \equiv 1 & \text{on a large convex domain } \Omega' \subset\subset \Omega,\ K \subset \Omega', \\ \phi_1 \equiv 1 & \text{on a small neighborhood } \mathcal{N} \subset\subset \bar{\Omega}\backslash\Omega',\ \partial\Omega \subset \mathcal{N}, \end{cases}$$

and considering $\vec{V} = \sum \phi_j \vec{V}_j$ we get the advantages of both $\vec{V}_0$ and $\vec{V}_1$. (Notice that the sign of the inner product in (39) agrees with $D\bar{\theta} \cdot D\bar{\theta} > 0$.)

The resulting flow ξ satisfies all the requirements outlined at the beginning of the proof, and $N_\theta \ni x \mapsto \xi(x, \pi) \in N_{\theta+\pi} = N_\theta$ is a homeomorphism that reverses the endpoints p_1 and $p_2 \in \partial\Omega$ of N_θ. Such a map has a unique fixed point, and this establishes Theorem 10. ■

Under the conditions established by Theorems 9 and 10—in particular, that $Du(x; H)$ vanishes at a unique point $x_0 \in \Omega$—it follows from Theorem 12 Appendix C that the convexity of the level curves is equivalent to the condition $HL_u \ge 0$. This observation along with the nondegeneracy of the critical point at $x_0 = x_0(H)$ (see Lemma 11) suggests the following strategy for proving the convexity of the level curves.

Let

$$\begin{aligned} H_0 &= \max\{H : HL_u \ge 0\} \\ &= \max\{H : L_u \ge 0\}. \end{aligned}$$

Note that $H_0 > 0$ by Theorem 7. If $H_0 < H_{\max}$, then $u_0 = u(\,\cdot\,; H_0)$ satisfies for some neighborhood Ω' of $x_0 = x_0(H_0)$

$$\begin{cases} L_{u_0} > 0 & \text{on } \Omega' \backslash \{x_0\}. \\ L_{u_0} > 0 & \text{on } \partial\Omega. \\ L_{u_0} = 0 & \text{at some point } x_1 \in \Omega\backslash\Omega'. \end{cases}$$

Under these conditions it is natural to try to show $L_{u_0}/|Du_0|^3$ or (more likely) $L_{u_0}/|Du_0|^2$ is a supersolution in $\Omega\backslash\{x_0\}$ of some homogeneous elliptic equation. Convexity of the level curves for all $0 < |H| < H_{\max}$ would follow in either case.

Appendix A. Short time existence

We now return to the proof of Theorem 1. We must address the extremal case $H = H_{\max}$.

First of all note that $H_{\max} < \infty$. In fact, if $u = u(x; H)$ is any solution to $(*)$ then by integrating the equation we have

$$2H|\Omega| = \int_\Omega \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \int_{\partial\Omega} \nu \cdot N$$

where ν is the outward pointing unit normal to Ω and N is the normal to graph(u). The integral on the right is clearly bounded in absolute value by $|\partial\Omega|$.

From Corollary 1 it is natural to define

$$u(x; \pm H_{\max}) \equiv \pm \lim_{H \nearrow H_{\max}} u(x, H). \tag{40}$$

Serrin [Ser69] gives a more general bound for $|u|_{C^0(\Omega)}$ than that described in §1. To be precise he shows $|u|_{C^0(\Omega)} = \sup_\Omega |u| \leq 1/(\sigma|H|)$ for $|H| \neq 0$ and the alternative bound

$$|u|_{C^0(\Omega)} \leq 1/(\sigma|H|) - \sqrt{1/(\sigma|H|)^2 - a^2}$$

if Ω happens to be contained in a disk of radius $a \leq 1/(\sigma|H|)$. Since any domain is contained in a disk of radius $C(n)\text{diam}(\Omega)$ for some constant $C(n) < 1$, we have the absolute bound

$$|u|_{C^0(\Omega)} \leq C(n)\text{diam}(\Omega). \tag{41}$$

From (41) and the monotonicity it is clear that (40) gives a well defined finite pointwise limit satisfying the boundary condition of $(*)$. In order to show that the equation is satisfied we restrict to a smooth domain $\Omega' \subset\subset \Omega$ (n.b. Problem 6.9) and apply Corollary 16.7 [GT83, pg. 407]. In our case, $\{u(\,\cdot\,; H)\}_{|H|<H_{\max}}$ is bounded in $C^k(\bar{\Omega}')$ for any k, and by Lemma 6.36 there is a subsequence $u(\,\cdot\,; H_j)$ with $H_j \nearrow H_{\max}$ converging to $u(\,\cdot\,; H_{\max})$ in $C^k(\bar{\Omega}')$. Passing to a limit in the equation, we see that $(*)_{H_{\max}}$ is satisfied by the limit function $u(\,\cdot\,; H_{\max})$.

If the gradient blow-up condition

$$\sup_{x\in\Omega} |Du(x;H_{\max})| = \infty$$

were to fail, then additional regularity follows from Lemma 6.18—see Remark 3 equation (3)—and we have $u(\,\cdot\,;H_{\max}) \in C^{2,\alpha}(\bar{\Omega})$. Thus, we arrive at a contradiction (of the definition of $H_{\max}$) from

Theorem 11 (short time existence)**.** *If $u_0 = u(x;H_0) \in C^2(\bar{\Omega})$ solves $(*)_{H_0}$, then there is some $\delta > 0$ such that $(*)_H$ is solvable in $C^2(\bar{\Omega})$ for $|H - H_0| < \delta$.*

Proof. Recall that, using Corollary 1 and the comparison principle, it is enough to assume $H_0 > 0$ and find an *apriori* gradient bound for solutions $u(x;H)$ for H in some interval $H_0 < H < H_0 + \delta$. A common way to obtain such a bound is to produce a barrier, i.e., we want to find a fixed value $H > H_0$ and a fixed subsolution $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ for the problem $(*)_H$. We produce such a subsolution as follows.

Consider the linearization of $\mathcal{M}$ at u_0

$$\mathcal{L}w = \sum_{ij} \left[A_{ij}(Du_0)D_iD_jw + \sum_k \frac{\partial^2 A_i}{\partial p_j \partial p_k}(Du_0)D_iD_ju_0D_kw \right].$$

We define $w = w(x;H)$ as the solution to the linear boundary value problem

$$\begin{cases} \mathcal{L}w & = 2H \quad \text{on } \Omega, \\ w\big|_{\partial\Omega} & \equiv \phi. \end{cases} \tag{42}$$

Note that

$$w(x;H_0) \equiv u_0. \tag{43}$$

Furthermore, $w \in C^\infty(\Omega\times\mathbb{R})$. See the proof of Theorem 3. In particular, $\dot{w} = \partial w/\partial H$ satisfies

$$\sum_{ij} \left[A_{ij}(Du_0)D_iD_j\dot{w} + \sum_k \frac{\partial^2 A_i}{\partial p_j \partial p_k}(Du_0)D_iD_ju_0D_k\dot{w} \right] = 2. \tag{44}$$

It remains to show that for some $H > H_0$, $w(x;H)$ is a barrier, i.e., that $\mathcal{M}w \geq 2H > 2H_0$. We see immediately that such a constant H exists by differentiating $\mathcal{M}w$ with respect to H and setting $H = H_0$; using (43) and (44), the value is 2. ■

Applying Theorem 11 to the zero solution $u(x;0)$ we obtain the nondegeneracy $H_{\max} > 0$, and the gradient blow-up condition follows as outlined above. This completes the proof of Theorem 1.

Remark 10. *We have assumed no convexity in Theorem 11, though the Wiener condition is required for the linear problem (42) to be solvable with regular boundary values. This, however, is accomplished for us by a simple regularity assumption on $\partial\Omega$*

(see the discussion preceding Theorem 6.13). The regularity assertion $w \in C^\infty(\Omega \times \mathbb{R})$ is completely analogous to Theorem 3 and follows from the Schauder estimates (without any additional work).

Appendix B. Hölder inequalities

Lemma 9. *Let $g : \Omega \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$. If $|V|_{C^{k+1}(G)} \leq C_1$ where $G = g(\Omega)$ and $|g|_{C^{k,\alpha}(\Omega)} \leq C_2$, then*

$$|V \circ g|_{C^{k,\alpha}(\Omega)} \leq B$$

where $B = B(C_1, C_2, k, n)$.

To prove Lemma 9 we use induction starting from $k = 0$ and

Lemma 10. *If $u, v : \Omega \to \mathbb{R}$, then for any multiindex β with $|\beta| = m$,*

$$[D^\beta(uv)]_{C^\alpha(\Omega)} \leq 2^{m+1}|u|_{C^{m,\alpha}(\Omega)}|v|_{C^{m,\alpha}(\Omega)}.$$

Appendix C. Hadamard type theorems

In what follows $v \in C^2(\bar{\Omega})$ is a positive function with zero boundary values on the planar C^2 domain Ω.

For lack of a proof, we begin with a conjecture.

Conjecture 1. (i) *$\mathcal{G} \equiv graph(v)$ is concave if and only if*

$$G_v \equiv v_{xx}v_{yy} - v_{xy}^2 \geq 0 \quad \text{for } x \in \Omega.$$

(ii) *v is $-\infty$ concave, i.e., graph(v) has convex level sets $\{x \in \Omega : v(x) > c\}$, if and only if*

$$L_v \equiv v_y^2 v_{xx} - 2v_x v_y v_{xy} + v_x^2 v_{yy} \leq 0 \quad \text{for } x \in \Omega.$$

Remark 11. *The Gauss curvature of $\mathcal{G}$ is given by $G_v/(1+|Dv|^2)^{3/2}$. The curvature of the level curves (at least where $|Dv| \neq 0$) is given by $-L_v/|Dv|^3$.*

The reasoning in the proof of Lemma 8 proves most of statement (i). Recall that λ_1 and λ_2 denoted the eigenvalues of D^2u. If v is concave then $\lambda_1, \lambda_2 \leq 0$, and we clearly have that $G_v \geq 0$. If the condition $\lambda_1, \lambda_2 < 0$ holds throughout Ω, we say that v is *strictly second order concave*. In such a case it is clear that $G_v > 0$, and the converse is also true.

The outstanding case of statement (i), that $G_v \geq 0$ on Ω implies v is concave, is related to the borderline case of Hadamard's Theorem for Ovaloids. See [Hop89], pp. 119-122, esp. Remark 1.5 and [dC76], pg. 387, Remark 3. We note, however, that none of the proofs of Hadamard's Theorem given in the above references apply in a straightforward way to yield what we find by the simple reasoning above. The essential difficulty is that the image of the Gauss map need not be simply connected.

For the same reason, the argument of Chern and Lashoff [CL58] (also for compact surfaces) is unlikely to settle the conjecture easily.

Statement (ii) is, presumably, even more difficult. We have only the following restricted version which is used in our remarks at the end of §4.

Theorem 12. *If v has a unique critical point, then statement* (ii) *holds.*

Proof. From our assumption of a unique critical point, it follows that each of the level sets $\Omega_c = \{x \in \Omega : v(x) > c\}$ for $0 \leq c < \max v$ is bounded by a smooth simple closed curve. Since the inward normal to this curve is given by $n = Dv/|Dv|$, it is easy to check that the curvature with respect to n is given by $-L_v/|Dv|^3$. Thus, we need only show a version of Hadamard's theorem for simple closed planar curves. A proof for this may be found in [dC76]; see Proposition 1 pg. 397. ■

Our reasoning at the end of § 4 also uses the following simple observation.

Lemma 11. *If $x_0 \in \Omega$ is the unique nondegenerate critical point for v, i.e., $D^2v(x_0) < 0$, then in some neighborhood Ω' of x_0, $L_v < 0$.*

Proof. From the nondegeneracy of the critical point, we may assume $Dv \neq 0$ and $D^2v < 0$ in Ω'. Since $L_v = w^T D^2 v w$ where $w = (v_y, -v_x) \neq 0$, our assertion follows at once. ■

References

[BdC84] J.L. Barbosa and M.P. do Carmo. Stability of hypersurfaces with constant mean curvature. *Math. Z.*, 185:339–353, 1984.

[Bol09] Oscar Bolza. *Vorlesungen Über Variationsrechnung.* Teubner, Leipzig, 1909.

[CC97] Xavier Cabré and Sagun Chanillo. Stable solutions of semilinear elliptic problems in convex domains. *preprint*, 1997.

[CL58] S.S. Chern and R.K. Lashof. On the total curvature of immersed manifolds. *Michigan Math. J.*, 5:5–12, 1958.

[dC76] Manfredo P. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. in Pure Appl. Math.*, 33:199–211, 1980.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, second edition, 1983.

[Gul73] Robert Gulliver. Regularity of minimizing surfaces of prescribed mean curvature. *Ann. of Math.*, 97(2):275–305, 1973.

[Har58] Philip Hartman. On elliptic partial differential equations and uniqueness theorems for closed surfaces. *J. Math. Mech.*, 7:377–392, 1958.

[Hop89] Heinz Hopf. *Differential Geometry in the Large.* Number 1000 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1989.

[Kaw84] Bernhard Kawohl. When are superharmonic functions concave? Applications to the St. Venant torsion problem and to the fundamental mode of the clamped membrane. *Z. Angew. Math. Mech.*, 64:364–366, 1984.

[ML71] L.G. Makar-Limonov. Solution of Dirichlet's problem for the equation $\Delta u = -1$ on a convex region. *Math. Notes Acad. Sci. USSR*, 9:52–53, 1971.

[Mor66] Charles B. Morrey. *Multiple Integrals in the Calculus of Variations*. Springer, Berlin, 1966.

[Mor83] Hiroshi Mori. Stable complete constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. *Trans. Amer. Math. Soc.*, 278(2):671–687, 1983.

[Ruc79] Hartmut Ruchert. Ein Eindeutigkeitssatz für Flächen konstanter mittlerer Krümmung. *Arch. Math. (Basel)*, 33:91–104, 1979.

[Ser69] James Serrin. On surfaces of constant mean curvature which span a given space curve. *Math. Z.*, 112:77–88, 1969.

[Wen66] Henry Wente. Dissertation. *Harvard*, 1966.

[Wen71] H.C. Wente. A general existence theorem for surfaces of constant mean curvature. *Math. Z.*, 120:277–288, 1971.

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