On a Singular Limit Problem for Nonlinear Maxwell's Equations

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Abstract: In this paper we study the following nonlinear Maxwell's equations $\varepsilon \mathbf{E}_t + \sigma(x, |\mathbf{E}|)\mathbf{E} = \nabla \times \mathbf{H} + \mathbf{F}$, $\mathbf{H}_t + \nabla \times \mathbf{E} = 0$, where $\sigma(x, s)$ is a monotone graph of s. It is shown that the system has a unique weak solution. Moreover, the limit of the solution as $\varepsilon \to 0$ converges to the solution of quasi-stationary Maxwell's equations.

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1. Introduction

Let Ω be a bounded domain in R^3 and $Q_T = \Omega \times (0, T]$ for any fixed $T > 0$. Let $\mathbf{E}$ and $\mathbf{H}$ be the electric and magnetic fields, respectively, in Ω (here and thereafter a bold letter represents a vector in R^3). Let σ be the electric conductivity in the field, which is assumed to be a function of x and $|\mathbf{E}|$. Consider the following Maxwell's equations (see Landau-Lifschitz [13]):

$$\varepsilon \mathbf{E}_t + \sigma(x, |\mathbf{E}|)\mathbf{E} = \nabla \times \mathbf{H} + \mathbf{F}, \qquad (x,t) \in Q_T, \tag{1.1}$$

$$\mathbf{H}_t + \nabla \times \mathbf{E} = 0, \qquad (x,t) \in Q_T, \tag{1.2}$$

$$\nabla \times \mathbf{E} = 0, \qquad (x,t) \in \partial\Omega \times (0,T], \tag{1.3}$$

$$\mathbf{E}(x,0) = \mathbf{E}_0(x), \mathbf{H}(x,0) = \mathbf{H}_0(x), \qquad x \in \Omega, \tag{1.4}$$

where ε is the dielectric parameter and other physical parameters are normalized.

In some applications ([4, 11]), the electric conductivity, σ, strongly depends on the electric field $|\mathbf{E}|$, hence the electric current density. Particularly, the electric conductivity may act like a switch-like function in some electromagnetic fields. On the other hand, for many types of micron devices and other industrial problems (such as microwave heating [11, 8, 17]) etc.) the experiment shows that the displacement current, $\varepsilon \mathbf{E}_t$, is often negligible since it is small in comparison of the eddy current, $\mathbf{J} = \sigma \mathbf{E}$. This motivates us to investigate the nonlinear problem (1.1)-(1.4) and the singular limit problem as $\varepsilon \to 0$. It is shown that there exists a unique global solution to (1.1)-(1.4). Moreover, the limit of the solution converges to the solution of the quasi-stationary system (i.e., the system (1.1)-(1.4) with $\varepsilon = 0$ in (1.1)). This limit solution provides new existence result for the quasi-stationary system. Indeed, when $\varepsilon = 0$, the system (1.1) becomes

$$\sigma(x, |\mathbf{E}|)\mathbf{E} = \nabla \times \mathbf{H} + \mathbf{F}, \tag{1.5}$$

Thus, one can solve Eq. (1.5) for $|\mathbf{E}|$ in terms of $|\nabla \times \mathbf{H}|$ and known data,

$$|\mathbf{E}| = g(x, |\nabla \times \mathbf{H}|),$$

where $g(x, s)$ is the inverse function of $\sigma(x, s)s$.

It follows from (1.2) that $\mathbf{H}$ satisfies

$$\mathbf{H}_t + \nabla \times [\rho(x, |\nabla \times \mathbf{H}|)\nabla \times \mathbf{H}] = 0, \tag{1.6}$$

where

$$\rho(x, |\nabla \times \mathbf{H}||) = \frac{1}{\sigma(x, |\mathbf{E}|)} = \frac{1}{\sigma(x, g(x, |\nabla \times \mathbf{H}|))}$$

represents the electric resistivity in the field.

The research on Maxwell's equations is of great interesting because of the important applications in plasma physics, semiconductor-superconductor modeling and other industrial problems ([8, 9, 13, 17] etc.). The study on the system (1.1)-(1.4) as well as the quasi-stationary form (1.6) received considerable attention recently. In [12], the authors established the well-posedness for a quasi-stationary system, where a constitutive relation between the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ is assumed to be nonlinear. In [15], the author studied the regularity of weak solution to a linear system of (1.6) with minimal requirement on coefficients. There is a special interest when

$$\rho(|\nabla \times \mathbf{H}|) = |\nabla \times \mathbf{H}|^{p-2}, \qquad p > 2.$$

On one hand, if $\mathbf{H}$ is restricted in one direction (scalar field) then the evolution system (1.6) becomes the p-Laplacian which has been studied extensively (see [5] and the references therein). On the other hand, in a recent work [16] (also see [1, 3] for the scalar case), it is shown that the limit of the solution to (1.6) as $p \to \infty$ is the unique solution to Bean's critical-state model in the superconductivity theory ([2]). Thus, for large p the system (1.6) provides a good approximation to Bean's model. More recently, the author of [10] studied the similar problem to this paper in a domain with a bounded complement in R^3. The conditions on σ in [10] is quite different from ours here. Like many nonlinear problems, the major difficulty is how to pass the weak limit of an approximate solution for a nonlinear function $\sigma(x, s)$. This is done by employing a monotonicity argument ([6]). The monotonicity of $\sigma(x, s)$ in s is essential in the proof.

In §2, we use the finite element method to establish the well-posedness of the system (1.1)-(1.4) for fixed $\varepsilon > 0$. In §3, we show that the singular limit of the solution to (1.1)-(1.4) has a unique limit. Moreover, the limit solution solves the quasi-stationary Maxwell's equations. Some examples are also discussed in this section.

2. Existence and Uniqueness for fixed $\varepsilon > 0$

Introduce some standard spaces (see [4, 7]).

$$\begin{aligned}
&H(div, \Omega) = \{\mathbf{U} \in L^2(\Omega)^3, div\mathbf{U} \in L^2(\Omega)\};\\
&H(curl, \Omega) = \{\mathbf{U} \in L^2(\Omega)^3, curl\mathbf{U} \in L^2(\Omega)^3\};\\
&H(div0, \Omega) = \{\mathbf{U} \in H(div, \Omega) : div\mathbf{U} = 0 \text{ in } \Omega\},\\
&H_0(curl, \Omega) = \{\mathbf{U} \in H(curl, \Omega) : \mathbf{N} \times \mathbf{U} = 0 \,\text{on}\, \partial\Omega\},
\end{aligned}$$

where $\mathbf{N}$ is the exterior unit normal on $\partial\Omega$.

Note that the trace of a function in $H(curl, \Omega)$ is well defined (see [4] for example).

We shall assume the following conditions on $\sigma(x, s)$ and data $\mathbf{E}_0(x), \mathbf{H}_0(x)$ and $\mathbf{F}(x, t)$.

H(2.1): Let $\sigma(x, s)$ be measurable in $\Omega \times [0, \infty)$ and monotone increasing in s. Moreover,

$$\begin{aligned}
&\int_0^{s^2} \sigma(x, \sqrt{s})dx \geq a_0 s^{p+2} - a_1, \text{ if } s \text{ is sufficiently large},\\
&0 \leq \sigma(x, s) \leq b_0(1 + s^p), s \in [0, \infty), \text{for some } p \geq 0,
\end{aligned}$$

where the constants $a_0 > 0, a_1 \geq 0$ and $b_0 \geq 0$.

H(2.2): Assume that $\mathbf{H}_0 \in H(curl, \Omega) \bigcap H(div0, \Omega), \mathbf{F} \in H^1(0, T; H^1(\Omega))$.

Definition 2.1: *A pair of vector fields $(\mathbf{E}(x, t), \mathbf{H}(x, t))$ is said to be a weak solution of the problem (1.1)-(1.4), if*

$$\mathbf{E} \in L^2(0, T; H_0(curl, \Omega)) \bigcap L^{p+2}(0, T; \Omega), \mathbf{H} \in L^2(0, T; H(div0, \Omega)) \bigcap H(0, T, L^2(\Omega))$$

which satisfy the following integral identities:

$$\int\int_{Q_T} \left[-\varepsilon \mathbf{E}\cdot \mathbf{\Phi}_t + \sigma(x,|\mathbf{E}|)\mathbf{E}\cdot\mathbf{\Phi}\right] dxdt$$

$$= \int\int_{Q_T} \mathbf{H}\cdot(\nabla\times\mathbf{\Phi})dxdt + \varepsilon\int_\Omega \mathbf{E}_0\cdot\mathbf{\Phi}(x,0)dx, \tag{2.1}$$

$$\int\int_{Q_T} \left[-\mathbf{H}\cdot\mathbf{\Psi}_t + \mathbf{E}\cdot(\nabla\times\mathbf{\Psi})\right] dxdt = \int_\Omega \left[\mathbf{H}_0(x)\cdot\mathbf{\Psi}(x,0)\right] dx \tag{2.2}$$

for all test functions $\mathbf{\Phi}\in H^1(0,T;H_0(curl,\Omega))$, $\mathbf{\Psi}\in H^1(0,T;H(curl,\Omega))$ *with* $\mathbf{\Phi}(x,T)=\mathbf{\Psi}(x,T)=0.$

First of all, we derive some energy estimates. A special attention is paid on how various constants depend on ε since we will study the singular limit problem in section 3.

Lemma 2.1: *Under the assumptions H(2.1)-H(2.2) there exist constants* C_1, C_2 *and* C_3 *such that*

$$\sup_{[0,T]}\int_\Omega \left[\varepsilon|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{E}|^{p+2}\right] dx + \int_0^T\int_\Omega \left[\varepsilon|\mathbf{E}_t|^2 + |\mathbf{H}_t|^2\right] dxdt$$

$$\leq C_1\int_\Omega \left[|\mathbf{E}_0|^2 + |\mathbf{H}_0|^2 + |\nabla\times\mathbf{H}_0|^2\right] dx + C_2\int\int_{Q_T}[|\mathbf{F}|^2 + |\mathbf{F}_t|^2]dxdt + C_3,$$

where C_1, C_2 *and* C_3 *depend only on known data.*

Proof: Note that for any vector fields $\mathbf{A},\mathbf{B}\in H(curl,\Omega)$ with either $\mathbf{A}$ or $\mathbf{B}$ in $H_0(curl,\Omega)$, the following identity holds:

$$\int_\Omega \mathbf{A}\cdot(\nabla\times\mathbf{B})dx = \int_\Omega \mathbf{B}\cdot(\nabla\times\mathbf{A})dx.$$

Taking inner product to the system (1.1) and (1.2) by $\mathbf{E}$ and $\mathbf{H}$, respectively, we add up the resulting equations to obtain

$$\sup_{[0,T]}\int_\Omega \left[\varepsilon|\mathbf{E}|^2 + |\mathbf{H}|^2\right] dx + \int\int_{Q_T}\sigma(x,|\mathbf{E}|)|\mathbf{E}|^2 dxdt$$

$$\leq C\int_\Omega \left[|\mathbf{E}_0|^2 + \mathbf{H}_0|^2\right] dx + \int\int_{Q_T}[|\mathbf{E}\cdot\mathbf{F}|]dxdt \tag{2.3}$$

where the constant C depends only on known data, but not on ε.

We first assume that $\sigma(x,s)$ is differentiable with respect to s. Then we formally differentiate Eq.(1.1) and Eq.(1.2) with respect to t to obtain

$$\begin{aligned}&\varepsilon\mathbf{E}_{tt}+\sigma(x,|\mathbf{E}|)\mathbf{E}_t+\sigma_s(x,|\mathbf{E}|)(|\mathbf{E}|)_t\mathbf{E}=\nabla\times\mathbf{H}_t+\mathbf{F}_t,\\&\mathbf{H}_{tt}+\nabla\times\mathbf{E}_t=0.\end{aligned}$$

It is clear that

$$\int\int_{Q_T}(\nabla\times\mathbf{E}_t)\cdot\mathbf{H}_t dxdt=\int\int_{Q_T}(\nabla\times\mathbf{H}_t)\cdot\mathbf{E}_t dxdt.$$

We take the inner product by $\mathbf{E}_t$ for the first equation and by $\mathbf{H}_t$ for the second equation and add up the resulting equations to obtain:

$$\begin{aligned}&\sup_{[0,T]}\int_\Omega\left[\varepsilon|\mathbf{E}_t|^2+|\mathbf{H}_t|^2\right]dxdt+\int\int_{Q_T}\left[\sigma|\mathbf{E}_t|^2+\sigma_s(|\mathbf{E}|)_t\mathbf{E}\cdot\mathbf{E}_t\right]dxdt\\&\le C,\end{aligned}$$

where C depends only on known data.

Note that $\sigma_s\ge 0$, we see that

$$\begin{aligned}&\int\int_{Q_T}\sigma_s(|\mathbf{E}|)_t\mathbf{E}\cdot\mathbf{E}_t dxdt\\&=\int\int_{Q_T}\sigma_s(|\mathbf{E}|)_t\frac{d}{dt}\frac{|\mathbf{E}|^2}{2}dxdt\\&=\int\int_{Q_T}\sigma_s|\mathbf{E}|[|\mathbf{E}|_t]^2dxdt\ge 0.\end{aligned}$$

It follows that

$$\sup_{[0,T]}\int_\Omega\left[\varepsilon|\mathbf{E}_t|^2+|\mathbf{H}_t|^2\right]dxdt+\int\int_{Q_T}[\sigma|\mathbf{E}_t|^2dxdt\le C.$$

Since the above estimate does not depend on the differentiability of σ with respect to s, therefore the above estimate holds as long as σ is monotone increasing with respect to s.

Now we take the inner product by $\mathbf{E}_t$ to (1.1) and by $\mathbf{H}_t$ to (1.2) and then add up the resulting equations to obtain

$$\int\int_{Q_T}\left[\varepsilon|\mathbf{E}_t|^2+|\mathbf{H}_t|^2\right]dxdt+\frac{1}{2}\int_\Omega\int_0^{|\mathbf{E}(x,T)|^2}\sigma(x,\sqrt{s})dsdx$$
$$\leq\int\int_{Q_T}\mathbf{E}_t\cdot\mathbf{F}dxdt+C\int_\Omega\int_0^{|\mathbf{E}_0(x)|^2}\sigma(x,\sqrt{s})dsdx+\int_\Omega|\nabla\times\mathbf{H}_0|^2dx+C.$$

Now

$$\int\int_{Q_T}\mathbf{E}_t\cdot\mathbf{F}dxdt$$
$$=\int_\Omega\left[\mathbf{E}(x,T)\cdot\mathbf{F}(x,T)-\mathbf{E}_0(x)\cdot\mathbf{F}(x,0)\right]dx-\int\int_{Q_T}\mathbf{E}\cdot\mathbf{F}_tdxdt$$
$$\leq\int_\Omega\left[\frac{a_0}{4}\mathbf{E}(x,T)|^{p+2}+\frac{16}{a_0}|\mathbf{F}_t|^{\frac{p+2}{p+1}}\right]dxdt.$$

On the other hand, by the assumption H(2.1) we may assume that the growth condition of $\sigma(x,s)$ on s holds for all $s\geq M_0$, i.e.,

$$\int_0^{s^2}\sigma(x,\sqrt{s})dx\geq a_0s^{p+2}-a_1,\ \text{if } s\geq M_0,$$

where M_0 is a fixed constant.

It follows that

$$\int_\Omega\int_0^{|\mathbf{E}(x,T)|^2}\sigma(x,s)dsdx$$
$$\geq a_0\int_{\Omega\bigcap\{x:|\mathbf{E}(x,T)|\geq M_0\}}|\mathbf{E}(x,T)|^{p+2}dx-C. \tag{2.4}$$

Combining (2.3)-(2.4) yields

$$\sup_{0\leq t\leq T}\int_\Omega[\varepsilon|\mathbf{E}|^2+|\mathbf{H}|^2]dx+\sup_{0\leq t\leq T}\int_\Omega|\mathbf{E}(x,t)|^{p+2}dx+\int_0^T\int_\Omega[\varepsilon|\mathbf{E}_t|^2+|\mathbf{H}_t|^2]dxdt$$
$$\leq C\int\int_{Q_T}[|\mathbf{F}|^2+|\mathbf{F}_t|^2]dxdt+\int_\Omega[|\mathbf{E}_0|^2+|\mathbf{H}_0|^2+|\nabla\times\mathbf{H}_0|^2]dx+C.$$

Q.E.D.

Theorem 2.2: *Under the assumptions H(2.1)-H(2.2) the problem (1.1)-(1.4) has a unique weak solution. Moreover,*

$$curl\mathbf{E} \in L^2(Q_T), \mathbf{E}_t \in L^2(Q_T)$$

and

$$\mathbf{H}_t \in L^2(Q_T), \nabla \times \mathbf{H} \in L^{\frac{p+2}{p+1}}(Q_T).$$

Proof: The proof is based on the finite element method (see [14] for parabolic equations). The monotonicity of $\sigma(x,s)$ on s plays an important role. We shall first deal with the case where $\sigma(x,s)$ is continuous on s. For convenience, we rewrite the system (1.1)-(1.4) to the following form:

$$\varepsilon \mathbf{W}_{tt} + \sigma(x, |\mathbf{W}_t|)\mathbf{W}_t = \nabla \times [\mathbf{H}_0 - \nabla \times \mathbf{W}] + \mathbf{F}, \qquad (x,t) \in Q_T, \quad (2.5)$$

$$\mathbf{N} \times (\mathbf{W}_t) = 0, \qquad x \in \partial\Omega, 0 \leq t \leq T, \quad (2.6)$$

$$\mathbf{W}(x,0) = 0, \mathbf{W}_t(x,0) = \mathbf{E}_0(x), \qquad x \in \Omega, \quad (2.7)$$

where $\mathbf{W}$ is defined as follows:

$$\mathbf{W}(x,t) = \int_0^t \mathbf{E}(x,\tau)d\tau.$$

It is clear that if $\mathbf{W}$ is a solution of the system (2.5)-(2.7) then a pair of functions defined by

$$\mathbf{E}(x,t) = \mathbf{W}_t(x,t), \mathbf{H}(x,t) = \mathbf{H}_0(x) - \nabla \times \mathbf{W}(x,t)$$

will be a weak solution of (1.1)-(1.4). Let $\{\mathbf{e}_k\} = \{e_k^{(1)}, e_k^{(2)}, e_k^{(3)}\}$ be a smooth basis of $H_0(curl, \Omega)$ and orthonormal in $L^2(\Omega)^3$, i.e.

$$< \mathbf{e}_i, \mathbf{e}_j >= \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Now we expand the known data as follows:

$$\mathbf{H}_0(x) = \sum_{k=1}^{\infty} diag[\mathbf{a}_k \circ \mathbf{e}_k],$$

$$\mathbf{E}_0(x) = \sum_{k=1}^{\infty} diag[\mathbf{b}_k \circ \mathbf{e}_k],$$
$$\mathbf{F}(x,t) = \sum_{k=1}^{\infty} diag[\mathbf{g}_k(t) \circ \mathbf{e}_k],$$

where $\mathbf{a}_k, \mathbf{b}_k$ and $\mathbf{g}_k$ are 3×1 matrices, the symbol $\circ$ is the matrix product and $diag[]$ represents the diagonal vector of a matrix.

Let

$$\mathbf{W}_n(x,t) = \sum_{k=1}^{n} diag[\mathbf{c}_n^{(k)} \circ \mathbf{e}_k],$$

where $\mathbf{c}_n^{(k)}(t)$ is a 3×1 vector which is determined by the following ordinary differential system:

$$\varepsilon \frac{d^2}{dt^2}\mathbf{c}_n^{(k)} + \sigma(x, |\mathbf{W}_{nt}|)\frac{d}{dt}\mathbf{c}_n^{(k)} = A_k(\mathbf{W}_n^{(k)}, \mathbf{e}_k) + B_k(t), \tag{2.8}$$
$$\mathbf{c}_n^{(k)})(0) = 0, \tag{2.9}$$
$$\frac{d}{dt}\mathbf{c}_n^{(k)}(0) = \mathbf{b}_k, \tag{2.10}$$

where

$$\mathbf{W}_n^{(k)} = diag[\mathbf{c}_n^{(k)} \circ \mathbf{e}_k],$$
$$A_k(\mathbf{W}_n^{(k)}, \mathbf{e}_k) = \int_\Omega diag\{[\nabla \times \mathbf{W}_n^{(k)}] \circ [\nabla \times \mathbf{e}_k]\}dx,$$
$$B_k(t) = \int_\Omega diag\{(\nabla \times \mathbf{H}_0 + \mathbf{F}) \circ \mathbf{e}_k\}dx, \qquad k = 1, 2, \cdots n.$$

Now we define the approximate solution $(\mathbf{E}_n, \mathbf{H}_n)$ as follows:

$$\mathbf{E}_n(x,t) = \mathbf{W}_{nt}(x,t), \mathbf{H}_n(x,t) = \mathbf{H}_{0n} - \nabla \times \mathbf{W}_n(x,t),$$

where

$$\mathbf{H}_{0n}(x) = \sum_{k=1}^{n} diag[\mathbf{a}_k \circ \mathbf{e}_k].$$

Equivalently, then $(\mathbf{E}_n, \mathbf{H}_n)$ satisfies the following system in the weak sense:

$$\varepsilon \mathbf{E}_{nt} + \sigma(x, |\mathbf{E}_n|)\mathbf{E}_n = \nabla \times \mathbf{H}_n, \qquad (x,t) \in Q_T, \tag{2.11}$$
$$\mathbf{H}_{nt} + \nabla \times \mathbf{E}_n = 0, \qquad (x,t) \in Q_T. \tag{2.12}$$

Similar to Lemma 2.1, one can easily derive the following energy estimates :

$$\sup_{[0,T]}\int_\Omega\left[\varepsilon|\mathbf{E}_n|^2+|\mathbf{H}_n|^2+|\mathbf{E}_n|^{p+2}\right]dx+\int_0^T\int_\Omega\left[\varepsilon|\mathbf{E}_{nt}|^2+|\mathbf{H}_{nt}|^2\right]dxdt$$
$$\leq C_1\int_\Omega\left[|\mathbf{E}_{0n}|^2+|\mathbf{H}_{0n}|^2+|\nabla\times\mathbf{H}_{0n}|^2\right]dx+C_2\int\int_{Q_T}[|\mathbf{F}_n|^2+|\mathbf{F}_{nt}|^2]dxdt+C_3,$$

where C_1, C_2 and C_3 are independent of n and ε.

By the weak compactness property, we can extract a subsequence (still denoted by $(\mathbf{E}_n, \mathbf{H}_n)$) such that

$$\begin{aligned}&\mathbf{E}_n\to\mathbf{E},\mathbf{E}_{nt}\to\mathbf{E}_t,\mathbf{H}_{nt}\to\mathbf{H}_t,\text{ weakly in } L^2(Q_T),\\&\mathbf{H}_n\to\mathbf{H},\ \text{weakly in } L^2(0,T;W^{1,\frac{p+2}{p+1}}(\Omega)),\\&\mathbf{H}_n\to\mathbf{H},\ \text{a.e. in } Q_T.\end{aligned}$$

Moreover,

$$\mathbf{E}_n\to\mathbf{E}\qquad\text{weakly in } L^{p+2}(Q_T).$$

Next we claim that the sequence $\mathbf{E}_n$ converges to $\mathbf{E}$ strongly in $L^2(Q_T)^3$. To prove the claim we only need to show that $\{\mathbf{E}_n\}$ is a Cauchy sequence in $L^2(Q_T)^3$. Let

$$\mathbf{E}_n^*(x,t)=\mathbf{E}_n(x,t)-\mathbf{E}_m(x,t),\mathbf{H}_n^*(x,t)=\mathbf{H}_n(x,t)-\mathbf{H}_m(x,t).$$

By energy estimates, we see

$$\begin{aligned}&\sup_{0\leq t\leq T}\int_\Omega[|\mathbf{E}_n^*|^2+|\mathbf{H}_n^*|^2]dx+\\&\int\int_{Q_T}\{[\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n-\sigma(x,|\mathbf{E}_m|)\mathbf{E}_m]\cdot[\mathbf{E}_n-\mathbf{E}_m]\}\,dxdt\\&\leq C\int_\Omega[|\mathbf{E}_{0n}-\mathbf{E}_{0m}|^2+|\mathbf{H}_{0n}-\mathbf{H}_{0m}|^2]dx+C\int\int_{Q_T}[|\mathbf{F}_n-\mathbf{F}_m|^2]dxdt,\end{aligned}$$

where C is a constant independent of n and m.

Note that $\sigma(x,s)$ is monotonic increasing in s, then

$$\begin{aligned}&[\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n-\sigma(x,|\mathbf{E}_m|)\mathbf{E}_m]\cdot[\mathbf{E}_n-\mathbf{E}_m]\\&\geq\frac{\sigma(x,|\mathbf{E}_n|)-\sigma(x,|\mathbf{E}_m|)][|\mathbf{E}_n|^2-|\mathbf{E}_m|^2]}{2}\geq 0.\end{aligned}$$

It follows that

$$\sup_{0\le t\le T}\int_\Omega[|\mathbf{E}_n^*|^2+|\mathbf{H}_n^*|^2dx$$
$$\le C\int_\Omega[|\mathbf{E}_{0n}-\mathbf{E}_{0m}|^2+|\mathbf{H}_{0n}-\mathbf{H}_{0m}|^2]dx+C\int\int_{Q_T}|\mathbf{F}_n-\mathbf{F}_m|^2dxdt.$$

This implies that $\mathbf{E}_n,\mathbf{H}_n$ are Cauchy sequences since both $\mathbf{E}_{0n},\mathbf{H}_{0n}$ and $\mathbf{F}_n$ are Cauchy sequences in $L^2(Q_T)^3$. Hence,

$$\mathbf{E}_n,\mathbf{H}_n\to\mathbf{E},\mathbf{H}\,\text{strongly in } L^2(Q_T).$$

After taking a subsequence if necessary, we see that

$$\mathbf{E}_n\to\mathbf{E},\qquad a.e.\,in\,Q_T.$$

To show the existence of a weak solution to (1.1)-(1.4), we only need to show

$$\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n\to\sigma(x,|\mathbf{E}|)\mathbf{E}\qquad \text{in } L^1(Q_T).$$

As $\sigma(x,s)$ is continuous on s and $\mathbf{E}_n$ converges to $\mathbf{E}$ almost everywhere in Q_T, we know

$$\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n\to\sigma(x,|\mathbf{E}|)\mathbf{E}\qquad a.e.\text{ in } Q_T.$$

We now show that $\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n$ is equip-integrable in Q_T. We adopt a technique used for scalar elliptic and parabolic equations. Let A be any measurable subset of Q_T. For any large $m>0$,

$$\int\int_A\sigma(x,|\mathbf{E}_n)|\mathbf{E}_n|dxdt$$
$$\le\int\int_{A\bigcap\{|\mathbf{E}_n|\le m\}}\sigma(x,|\mathbf{E}_n|)|\mathbf{E}_n|dxdt+\int\int_{A\bigcap\{|\mathbf{E}_n|\ge m\}}\sigma(x,|\mathbf{E}_n|)|\mathbf{E}_n|dxdt$$
$$\equiv I_1+I_2.$$

The assumption on $\sigma(x,s)$ yields

$$I_1\le C\int\int_{A\bigcap\{|\mathbf{E}_n|\le m\}}[1+|\mathbf{E}_n|^p|\mathbf{E}_n|]dxdt,$$

which can be arbitrarily small if $|A|$ is small since $\mathbf{E}_n \in L^{p+2}(Q_T)$.

On the other hand,

$$I_2 \leq \frac{1}{m}\int\int_{A\bigcap\{|\mathbf{E}_n|\geq m\}} \sigma(x,|\mathbf{E}_n|)|\mathbf{E}_n|^2 dxdt \leq \frac{C}{m},$$

which is also small if m is sufficiently large.

This concludes that $\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n$ is equip-integrable in Q_T.

It follows by Vitali's theorem that

$$\sigma(x,|\mathbf{E}_n|)\mathbf{E}_n \to \sigma(x,|\mathbf{E}|)\mathbf{E} \qquad \text{in } L^1(Q_T).$$

Finally, we show that $(\mathbf{E},\mathbf{H})$ is a weak solution of (1.1)-(1.4). By multiplying Eq.(2.11) and Eq.(2.12) by test functions $\mathbf{\Psi}$ and $\mathbf{\Phi}$, respectively, and then taking integration over Q_T, after some routine calculations and taking the limit, we see that $(\mathbf{E},\mathbf{H})$ is a weak solution to the system (1.1)-(1.4).

Now we consider the case where $\sigma(x,s)$ is discontinuous on s at some points. Without loss of generality, we may assume that $\sigma(x,s)$ has a jump only at one point $s=1$. In this case $\sigma(x,s)$ is not uniquely defined at $s=1$. We shall understand the value of $\sigma(x,1)$ in the following sense:

$$\sigma(x,1) \in [\sigma(x,1-),\sigma(x,1+)],$$

where $\sigma(x,1\pm)$ represents the right or left limit as $s\to 1$.

By the standard approximation, we can construct a smooth approximation sequence $\sigma_m(x,s)$ such that

$$\begin{aligned} &(i) \qquad \sigma_m(x,s) \text{ is monotonic increasing for all } s\geq 0,\\ &(ii) \qquad \sigma_m(x,s)=\sigma(x,s), \text{ if } |s-1|\geq \tfrac{1}{m}. \end{aligned}$$

Let $(\mathbf{E}_m,\mathbf{H}_m)$ be a solution of (1.1)-(1.4) in which $\sigma(x,s)$ is replaced by $\sigma_m(x,s)$. By the same energy estimate we see that there exists a measurable function $\beta(x,t)\in L^{\frac{p+2}{p+1}}(Q_T)$ such that

$$\sigma_m(x,|\mathbf{E}_m|) \to \beta(x,t), \text{weakly in } L^{\frac{p+2}{p+1}}(Q_T).$$

Define

$$A_m = \{(x,t) : 1 - \frac{1}{m} \leq |\mathbf{E}| \leq 1 + \frac{1}{m}\},$$
$$A = \{(x,t) : |\mathbf{E}(x,t)| = 1\}.$$

Since $\sigma(x,s)$ is continuous except at $s = 1$, we see

$$\beta(x,t) = \sigma(x, |\mathbf{E}|)\mathbf{E}, \text{if } (x,t) \in Q_T \backslash A.$$

Now it is clear that

$$A = \bigcap_{m=1}^{\infty} A_m.$$

Recall that $\sigma_m(x,s) = \sigma(x,s)$ if $|s-1| \geq \frac{1}{m}$. It follows that for all $(x,t) \in A_m$

$$\sigma(x, 1 - \frac{1}{m}) \leq \sigma_m(x, |\mathbf{E}|) \leq \sigma(x, 1 + \frac{1}{m}).$$

Consequently, as $m \to 0$,

$$\sigma(x, 1-) \leq \beta(x,t) \leq \sigma(x, 1+), (x,t) \in A.$$

Thus, $(\mathbf{E}, \mathbf{H})$ is a weak solution of (1.1)-(1.4).

Finally, we show the uniqueness. Suppose $(\mathbf{E}, \mathbf{H})$ and $(\mathbf{E}^*, \mathbf{H}^*)$ are two solutions of (1.1)-(1.4). Let

$$\hat{\mathbf{E}} = \mathbf{E} - \mathbf{E}^*, \hat{\mathbf{H}} = \mathbf{H} - \mathbf{H}^*.$$

Similar to the calculation in deriving energy estimates, we find

$$\sup_{0 \leq t \leq T} \int_\Omega [|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{H}}|^2] dx + \int\int_{Q_T} [\sigma(x, |\mathbf{E}|)\mathbf{E} - \sigma(x, |\mathbf{E}^*|)\mathbf{E}^*] \cdot [\mathbf{E} - \mathbf{E}^*] \, dxdt$$
$$\leq 0.$$

The monotonicity of $\sigma(x,s)$ implies that the second term in the above inequality is nonnegative. It follows that

$$\sup_{0 \leq t \leq T} \int_\Omega [|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{H}}|^2] dx \leq 0.$$

Therefore, the uniqueness follows immediately.

Q.E.D.

3. Singular Limit Problem

In this section we shall show that the solution of (1.1)-(1.4) has a limit as $\varepsilon \to 0$, which solves Maxwell's equations in quasi-stationary fields, i.e. the system (1.1)-(1.4) with $\varepsilon = 0$. A weak solution of the quasi-stationary system is defined as in Definition 2.1 with $\varepsilon = 0$.

From now on we denote by $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$ the weak solution of the system (1.1)-(1.4).

Theorem 3.1: *The limit of* $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$ *as* $\varepsilon \to 0$ *solves the quasi-stationary system (1.1)-(1.4) with* $\varepsilon = 0$ *in the weak sense. Moreover, the weak solution is unique if* $\sigma(x,s) > 0$ *for all* $(x,s) \in \Omega \times R^+$.

Proof: The crucial step in proving the convergence is to show

$$\sigma(|\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon \to \sigma(|\mathbf{E}|)\mathbf{E},\ a.e. \text{ in } Q_T \text{ as } \varepsilon \to 0.$$

The monotonicity of $\sigma(x,s)$ in s plays a key rule.

First of all, from Lemma 2.1 and the weak compactness we see

$$\begin{aligned}
&\mathbf{E}_\varepsilon \to \mathbf{E},\ \mathbf{H}_{\varepsilon t} \to \mathbf{H}_t, \text{weakly in } L^2(Q_T),\\
&\nabla \times \mathbf{H}_\varepsilon \to \nabla \times \mathbf{H}, \text{ weakly in } L^{\frac{p+2}{p+1}}(Q_T),\\
&\sigma(x, |\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon \to \mathbf{J}(x,t), \qquad \text{weakly in } L^{\frac{p+2}{p+1}}(Q_T),
\end{aligned}$$

where $\mathbf{J}(x,t) \in L^{\frac{p+2}{p+1}}(Q_T)$. Moreover, as $div\mathbf{H}_\varepsilon(x,t) = 0$, by the decomposition property of $H^1(\Omega)$ property, after extracting a subsequence if necessary we see that

$$\mathbf{H}_\varepsilon \to \mathbf{H}, \text{ strongly in } L^2(Q_T)$$

and

$$\mathbf{H}_\varepsilon \to \mathbf{H},\ a.e. \text{ in } Q_T.$$

Next we show

$$\mathbf{J}(x,t) = \sigma(x, |\mathbf{E}|)\mathbf{E}, \qquad a.e. \text{in } Q_T.$$

We use a monotonicity argument. As a first step, we show

$$\lim_{\varepsilon\to 0} \int\int_{Q_T} \sigma(x, |\mathbf{E}_\varepsilon|)|\mathbf{E}_\varepsilon|^2 dxdt = \int\int_{Q_T} \mathbf{J}\cdot\mathbf{E} dxdt.$$

Here we adopt an idea from [10]. Let $\lambda(t)$ be a nonnegative smooth function and

$$\lambda'(t) \le 0, \lambda(0) = 1, \lambda(T) = 0.$$

Define an operator L in $L^{p+2}(Q_T)^3$ as follows:

$$L[\mathbf{E}] = \sigma(x, |\mathbf{E}|)\mathbf{E}.$$

Since $\sigma(x, s)$ is monotonic increasing in s, then the operator L is monotonic increasing, that is,

$$< L[\mathbf{E}_\varepsilon] - L[\mathbf{E}], \mathbf{E}_\varepsilon - \mathbf{E} > \ge 0.$$

It is clear that

$$\begin{aligned} &< L[\mathbf{E}_\varepsilon] - L[\mathbf{E}], \mathbf{E}_\varepsilon - \mathbf{E} > \\ &=< L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon > - < L[\mathbf{E}_\varepsilon], \mathbf{E} > - < L[\mathbf{E}], \mathbf{E}_\varepsilon > + < L[\mathbf{E}], \mathbf{E} > \end{aligned}$$

It follows that

$$\lim_{\varepsilon \to 0} inf < L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon > \ge < \mathbf{J}, \mathbf{E} > . \tag{3.1}$$

On the other hand, from the system (1.1)-(1.2) we have

$$\begin{aligned} &\int_0^T \int_\Omega \sigma(\mathbf{E}_\varepsilon)|\mathbf{E}_\varepsilon|^2 \lambda(t) dxdt \\ &= -\int_0^T \int_\Omega [\varepsilon \lambda(t) \mathbf{E}_{\varepsilon t} \cdot \mathbf{E}_\varepsilon + \lambda \mathbf{H}_{\varepsilon t} \cdot \mathbf{H}_\varepsilon] \, dxdt + \int \int_{Q_T} \mathbf{F} \cdot \mathbf{E}_\varepsilon dxdt \\ &= -\int_0^T \int_\Omega \left\{ \frac{\partial}{\partial t} \left[\frac{1}{2} (\varepsilon |\mathbf{E}_\varepsilon|^2 + |\mathbf{H}_\varepsilon|^2) \lambda(t) \right] - \lambda'(t) \left[\frac{1}{2} (\varepsilon |\mathbf{E}_\varepsilon|^2 + |\mathbf{H}_\varepsilon|^2) \right] \right\} dxdt + \\ &\int \int_{Q_T} \mathbf{F} \cdot \mathbf{E}_\varepsilon dxdt \\ &\le \int_0^T \int_\Omega \left\{ \lambda'(t) \left[\frac{1}{2} |\mathbf{H}_\varepsilon|^2 \right] \right\} dxdt + \frac{1}{2} \int_\Omega \left[\varepsilon |\mathbf{E}_\varepsilon(x,0)|^2 + |\mathbf{H}_\varepsilon(x,0)|^2 \right] dx + \\ &\int \int_{Q_T} \mathbf{F} \cdot \mathbf{E}_\varepsilon dxdt. \end{aligned} \tag{3.2}$$

Since $\lambda'(t) \leq 0$, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} sup \int_0^T \int_\Omega \sigma(|\mathbf{E}_\varepsilon|)|\mathbf{E}_\varepsilon|^2 \lambda'(t) dxdt \\
&\leq \frac{1}{2}\int_0^T \int_\Omega \lambda(t)|\mathbf{H}|^2 dxdt + \frac{1}{2}\int_\Omega |\mathbf{H}_0(x)|^2 dx + \int\int_{Q_T} \mathbf{E}\cdot\mathbf{F} dxdt. \qquad (3.3)
\end{aligned}$$

Recall from Definition 2.1 that $(\mathbf{E}_\varepsilon, \mathbf{H}_\varepsilon)$ satisfies the following integral equations:

$$\begin{aligned}
&\int_0^T \int_\Omega \left[-\varepsilon\mathbf{E}_\varepsilon\cdot\mathbf{\Phi}_t + \sigma(x,|\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon\cdot\mathbf{\Phi}\right] dxdt \\
&= \int_0^T \int_\Omega \mathbf{H}_\varepsilon\cdot(\nabla\times\mathbf{\Phi}) dxdt + \int\int_{Q_T} \mathbf{F}\cdot\mathbf{\Phi}_\varepsilon dxdt + \int_\Omega \varepsilon\mathbf{E}_0\cdot\mathbf{\Phi}(x,0)dx, \\
&\int_0^T \int_\Omega \left[-\mathbf{H}_\varepsilon\cdot\mathbf{\Psi}_t + \mathbf{E}_\varepsilon\cdot(\nabla\times\mathbf{\Psi})\right] dxdt = \int_\Omega \left[\mathbf{H}_0(x)\cdot\mathbf{\Psi}(x,0)\right] dx.
\end{aligned}$$

Note by Lemma 2.1 that

$$\begin{aligned}
&|\int_0^T \int_\Omega \varepsilon\mathbf{E}_{\varepsilon t}\cdot\mathbf{\Phi} dxdt| \\
&\leq \left(\int_0^T \int_\Omega \varepsilon|\mathbf{E}_{\varepsilon t}|^2 dxdt\right)^{1/2} \left(\int_0^T \int_\Omega \varepsilon|\mathbf{\Phi}|^2 dxdt\right)^{1/2} \\
&\to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

We take the limit as $\varepsilon \to 0$ to obtain

$$\int_0^T \int_\Omega \left[\mathbf{J}\cdot\mathbf{\Phi}\right] dxdt = \int_0^T \int_\Omega \mathbf{H}\cdot(\nabla\times\mathbf{\Phi}) dxdt + \int\int_{Q_T} \mathbf{F}\cdot\mathbf{\Phi} dxdt \qquad (3.4)$$

$$\int_0^T \int_\Omega \left[-\mathbf{H}\cdot\mathbf{\Psi}_t + \mathbf{E}\cdot(\nabla\times\mathbf{\Psi})\right] dxdt = \int_\Omega \left[\mathbf{H}_0(x)\cdot\mathbf{\Psi}(x,0)\right] dx. \qquad (3.5)$$

Now by choosing $\mathbf{\Phi} = \lambda(t)\mathbf{E}$ and $\mathbf{\Psi} = \lambda(t)\mathbf{H}$ (note that the condition at $t = T$ is satisfied since $\lambda(T) = 0$), we obtain

$$\int_0^T \int_\Omega [\lambda(t)\mathbf{J}\cdot\mathbf{E}] dxdt = \frac{1}{2}\int_0^T \int_\Omega [\lambda'(t)|\mathbf{H}|^2] dxdt + \frac{1}{2}\int_\Omega |\mathbf{H}_0(x)|^2] dx + \int\int_{Q_T} \mathbf{E}\cdot\mathbf{F} dxdt.$$

It follows by (3.3) that

$$\lim_{\varepsilon\to 0} sup < \lambda(t)L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon > \leq < \lambda(t)\mathbf{J}, \mathbf{E} > . \qquad (3.6)$$

Combination of (3.1) and (3.6) yields

$$\lim_{\varepsilon \to 0} < \lambda(t) L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon > = < \lambda(t)\mathbf{J}, \mathbf{E} > . \tag{3.7}$$

Consequently, by choosing $\lambda(t)$ properly we have

$$\lim_{\varepsilon \to 0} < L[\mathbf{E}_\varepsilon], \mathbf{E}_\varepsilon > = < \mathbf{J}, \mathbf{E} > . \tag{3.8}$$

For any vector field $\mathbf{W} \in L^2(Q_T) \bigcap L^{p+1}(Q_T)$, the monotonicity of $\sigma(x, s)$ in s implies

$$\int\int_{Q_T} [\sigma(x, |\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon - \sigma(x, |\mathbf{W}|)\mathbf{W}] \cdot [\mathbf{E}_n - \mathbf{W}] dxdt \geq 0. \tag{3.9}$$

We rewrite the above inequality to the following form:

$$\begin{aligned} &\int\int_{Q_T} \left\{\sigma(|\mathbf{E}_\varepsilon|)|\mathbf{E}_\varepsilon|^2 - \sigma(|\mathbf{E}_\varepsilon|)\mathbf{E}_\varepsilon \cdot \mathbf{W}\right\} dxdt \\ &\geq \int\int_{Q_T} [\sigma(x, |\mathbf{W}|)\mathbf{W}] \cdot [\mathbf{E}_\varepsilon - \mathbf{W}] dxdt. \end{aligned} \tag{3.10}$$

We take the limit as $\varepsilon \to 0$ and use (3.8) for the first term in (3.10) to obtain

$$\begin{aligned} &\int\int_{Q_T} \{\mathbf{J} \cdot \mathbf{E} - \mathbf{J} \cdot \mathbf{W}\} dxdt \\ &\geq \int\int_{Q_T} [\sigma(x, |\mathbf{W}|)\mathbf{W}] \cdot [\mathbf{E} - \mathbf{W}] dxdt. \end{aligned}$$

Equivalently,

$$\int\int_{Q_T} \{[\mathbf{J} - \sigma(x, |\mathbf{W}|)\mathbf{W}] \cdot [\mathbf{E} - \mathbf{W}]\} dxdt \geq 0. \tag{3.11}$$

Set $\mathbf{W} = \mathbf{E} + \delta\mathbf{Y}$, where $\delta > 0$ is small parameter and $\mathbf{Y} \in L^2(Q_T) \bigcap L^{p+1}(Q_T)$ is arbitrary.

With the above choice of $\mathbf{W}$ in the equality (3.10), we obtain

$$\int\int_{Q_T} \{\mathbf{Y} \cdot [\mathbf{J} - \sigma(|\mathbf{E} + \delta\mathbf{Y}|)[\mathbf{E} + \delta\mathbf{Y}]\} dxdt \geq 0.$$

When $\sigma(x,s)$ is continuous in s, then we let $\delta \to 0$ to obtain

$$\mathbf{J}(x,t) = \sigma(x, |\mathbf{E}|)\mathbf{E},$$

since $\mathbf{Y}(x,t)$ is arbitrary in $L^2(Q_T) \bigcap L^{p+1}(Q_T)$.

When $\sigma(x,s)$ has a jump at a point, say, $s = 1$. Then as in §2 we understand that the value of $\sigma(x,s)$ at $s = 1$ is

$$\sigma(x, 1-) \leq \sigma(x,1) \leq \sigma(x, 1+).$$

By using the same procedure as in §2, we can derive the above inequality.

Finally, by taking limit for (2.1)-(2.2) we see that $(\mathbf{E},\mathbf{H})$ is a weak solution of the quasi-stationary system.

To prove the uniqueness, we assume that $(\mathbf{E},\mathbf{H})$ and $(\mathbf{E}^*,\mathbf{H}^*)$ are two weak solutions to the quasi-stationary system. Let $\hat{\mathbf{E}} = \mathbf{E} - \mathbf{E}^*$ and $\hat{\mathbf{H}} = \mathbf{H} - \mathbf{H}^*$. Then the energy estimate implies

$$\sup_{0\leq t\leq T} \int_\Omega |\hat{\mathbf{H}}|^2 dx + \int\int_{Q_T} [\sigma(x, |\mathbf{E}|)\mathbf{E} - \sigma(x, |\mathbf{E}^*|)\mathbf{E}^*] \cdot [\mathbf{E} - \mathbf{E}^*] dx dt \leq 0.$$

The monotonicity of $\sigma(x,s)$ in s implies the second term in the above inequality is nonnegative. It follows that

$$\hat{\mathbf{H}} = 0, \qquad a.e. \text{in } Q_T.$$

From the definition of weak solution, we have $\hat{\mathbf{E}} = 0$ as long as $\sigma > 0$.

Q.E.D.

To conclude this section, we consider two special classes of electric conductivity $\sigma(x,s) = s^p$ and

$$\sigma(x,s) = \begin{cases} a, \text{ if } |s| \leq 1; \\ b, \text{ if } |s| > 1, \end{cases}$$

where $0 < a < b$.

For the first case with $\mathbf{F} = 0$, it is easy to see that

$$\mathbf{E} = |\nabla \times \mathbf{H}|^{-\frac{p}{p+1}} \nabla \times \mathbf{H}.$$

It follows from Theorem 3.1 that there exists a unique weak solution to the following evolution system

$$\mathbf{H}_t + \nabla \times [|\nabla \times \mathbf{H}|^{-\frac{p}{p+1}} \nabla \times \mathbf{H}] = \mathbf{G}, (x,t) \in Q_T,$$

subject to the initial-boundary conditions:

$$\begin{aligned} &\mathbf{N} \times (\nabla \times \mathbf{H}) = 0, && (x,t) \in \partial\Omega \times [0,T], \\ &\mathbf{H}(x,0) = \mathbf{H}_0(x), && x \in \Omega, \end{aligned}$$

where $\mathbf{G}$ is a known exterior magnetic field. This existence result is not covered in [16]. More regularity of the weak solution can be established as in [16]. We shall not repeat it here.

For the second case, we see that

$$\mathbf{E} = \frac{1}{a}\nabla \times \mathbf{H} + \frac{1}{a}\mathbf{F},$$

in the region $Q_T^- = \{(x,t) : |\mathbf{E}| < 1\}$ and

$$\mathbf{E} = \frac{1}{b}\nabla \times \mathbf{H} + \frac{1}{b}\mathbf{F},$$

in the region $Q_T^+ = \{(x,t) : |\mathbf{E}| > 1\}$. Note that

$$div\mathbf{H}(x,t) = div\mathbf{H}_0(x) = 0,$$

we see that

$$\nabla \times \nabla \times \mathbf{H} = -\Delta\mathbf{H}.$$

It follows that $\mathbf{H}$ satisfies the parabolic equation:

$$\mathbf{H}_t - \frac{1}{a}\Delta\mathbf{H} = -\frac{1}{a}\nabla \times \mathbf{F}, \qquad (x,t) \in Q_T^-$$

and

$$\mathbf{H}_t - \frac{1}{b}\Delta\mathbf{H} = -\frac{1}{b}\nabla \times \mathbf{F}, \qquad (x,t) \in Q_T^+.$$

The regularity theory of parabolic equations implies that $\mathbf{H}(x,t)$ is smooth in $Q_T^{\pm}$.

The interface between Q_T^- and Q_T^+ is defined by

$$\Gamma = \{(x,t) : |\mathbf{E}| = 1\},$$

which is a free boundary.

Remark 3.1: *We may allow that $\sigma(s) = 0$ if $|\mathbf{E}| < 1$ in the above example. In this case, one must consider the full system (1.1)-(1.2) in order to define a weak solution. However, in this case the uniqueness of the weak solution does not hold.*
Remark 3.2: *It is not clear whether or not Γ is indeed a hypersurface in $R^3 \times (0,\infty]$. It would be of great interesting to study the smoothness of the interface Γ and to find the free boundary conditions for $\mathbf{H}$.*

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