CHEEGER'S INEQUALITIES FOR GENERAL SYMMETRIC FORMS AND EXISTENCE CRITERIA FOR SPECTRAL GAP

Mu-Fa Chen and Feng-Yu Wang

(Beijing Normal University)

Abstract. In this paper, some new forms of the Cheeger's inequalities are established for general (maybe unbounded) symmetric forms (Theorem 1.1 and Theorem 1.2), the resulting estimates improve and extend the ones obtained by Lawler and Sokal (1988) for bounded jump processes. Furthermore, some existence criteria for spectral gap of general symmetric forms or general reversible Markov processes are presented (Theorem 1.4 and Theorem 3.1), based on the Cheeger's inequalities and a relationship between the spectral gap and the first Dirichlet and Neumann eigenvalues on local region.

1. Introduction

The Cheeger's inequalities[1] are well known and widely used in geometric analysis, they provide a practical way to estimate the first eigenvalue of Laplacian in terms of volumes. These inequalities were then established for bounded jump processes by Lawler and Sokal[7] (in which, a detail comment on the earlier study and references is included). The first aim of the paper is to establish the inequalities for general (maybe unbounded) symmetric forms.

Let $(E, \mathcal{E})$ be a measurable space with reference probability measure π. Consider the symmetric form D with domain $\mathcal{D}(D)$:

$$D(f,g) = \frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)(f(x)-f(y))(g(x)-g(y)) + \int K(\mathrm{d}x) f(x) g(x), \qquad f,g \in \mathcal{D}(D)$$
$$\mathcal{D}(D) = \{f \in L^2(\pi) :\, D(f,f) < \infty\}.$$

where J is a symmetric measure: $J(\mathrm{d}x, \mathrm{d}y) = J(\mathrm{d}y, \mathrm{d}x)$. Without loss of generality, we assume that $J(\{(x,x)\} : x \in E) = 0$.

We are interested in the following two quantities:

$$\lambda_0 = \inf\{D(f,f) :\, \pi(f^2) = 1\}, \tag{1.1}$$
$$\lambda_1 = \inf\{D(f,f) :\, \pi(f) = 0,\ \pi(f^2) = 1\}. \tag{1.2}$$

1991 *Mathematics Subject Classification.* 60J25, 60J75, 47A75.

Key words and phrases. Cheeger's inequality, spectral gap, Neumann and Dirichlet eigenvalue, jump process.

Research supported in part by NSFC (No. 19631060), Qiu Shi Sci. & Tech. Found., DPFIHE, MCSEC and MCMCAS. Research at MSRI is supported in part by NSF grant DMS-9701755.

We remark that in these definitions, the usual condition "$f \in \mathcal{D}(D)$" is not needed since $D(f,f)=\infty$ for all $f \in L^2(\pi)\setminus\mathcal{D}(D)$. We even do not assume in some cases the density of $\mathcal{D}(D)$ in $L^2(\pi)$. In what follows, whenever λ_1 is considered, the killing measure $K(\mathrm{d}x)$ is setting to be zero. In this case, we have $\lambda_0=0$ and λ_1 is known as the spectral gap of the symmetric form $(D,\mathcal{D}(D))$.

Define the Cheeger's constants as follows:

$$h=\inf_{\pi(A)>0}\frac{J(A\times A^c)+K(A)}{\pi(A)}, \tag{1.3}$$

$$k=\inf_{\pi(A)\in(0,1)}\frac{J(A\times A^c)}{\pi(A)\pi(A^c)}, \tag{1.4}$$

$$k'=\inf_{\pi(A)\in(0,1/2]}\frac{J(A\times A^c)}{\pi(A)}=\inf_{\pi(A)\in(0,1)}\frac{J(A\times A^c)}{\pi(A)\wedge\pi(A^c)}, \tag{1.5}$$

where $a\wedge b=\min\{a,b\}$. Clearly, $k/2\le k'\le k$ and it is easy to see that k' can be varied over whole $(k/2,k)$. For instance, take $E=\{0,1\}$, $K=0$, $J(\{i\}\times\{j\})=1$ for $i\ne j$ and $\pi(0)=p\le 1/2$, $\pi(1)=1-p$. Then $k'/k=1-p$.

Recall that for a given reversible jump process, we have a q-pair $(q(x),q(x,\mathrm{d}y))$: $q(x,E)\le q(x)\le\infty$ for all $x\in E$. Throughout the paper, we assume that $q(x)<\infty$ for all $x\in E$. The reversibility simply means that the measure $\pi(\mathrm{d}x)q(x,\mathrm{d}y)$ is symmetric, which gives us automatically a measure J. Then, the killing measure is given by $K(\mathrm{d}x)=\pi(\mathrm{d}x)d(x)$, where $d(x)=q(x)-q(x,E)$ is called the non-conservative quantity in the context of jump processes. A jump process is called bounded if $\sup_{x\in E}q(x)<\infty$. In this case (or more generally, if $\|J(\cdot,E)+K\|_{\mathrm{op}}<\infty$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm from $L^1_+(\pi):=\{f\in L^1(\pi):\ f\ge 0\}$ to $\mathbb{R}_+$, then), for the corresponding form, we have $\mathcal{D}(D)=L^2(\pi)$. For more details, refer to [2].

Theorem (Lawler & Sokal). *Take $J(\mathrm{d}x,\mathrm{d}y)=\pi(\mathrm{d}x)q(x,\mathrm{d}y)$ and suppose that $\|J(\cdot,E)+K/2\|_{\mathrm{op}}\le M<\infty$. Then, we have*

$$h\ge\lambda_0\ge\frac{h^2}{2M}. \tag{1.6}$$

Next, if additionally $K=0$, then

$$k\ge\lambda_1\ge\max\left\{\frac{\kappa k^2}{8M},\frac{k'^2}{2M}\right\}, \tag{1.7}$$

where

$$\kappa=\inf_{X,Y}\sup_{c\in\mathbb{R}}\frac{(\mathbb{E}|(X+c)^2-(Y+c)^2|)^2}{1+c^2}\ge 1,$$

the infimum is taken over all i.i.d. random variables X and Y with $\mathbb{E}X=0$ and $\mathbb{E}X^2=1$.

In what follows, we consider directly the general symmetric measure J whenever it is possible. In other words, we do not require the existence of a kernel of a modification of $J(\mathrm{d}x,\cdot)/\pi(\mathrm{d}x)$, for which some extra conditions on $(E,\mathcal{E})$ are needed.

We now turn to discuss our general setup. Note that the lower bounds given in (1.6) and (1.7) decrease to zero as $M\uparrow\infty$. So the results would lost their meaning if we go directly from bounded case to the unbounded forms. More seriously, when we adopt a general approximation procedure to reduce the unbounded case to the bounded one (cf. [2; Theorem 9.12]), the lower bounds given above usually vanish as we go to the limit. To overcome the difficulty, one needs some trick. Here we propose a comparison technique. That is, comparing the original form with some other forms introduced below.

Take and fix a non-negative, symmetric function $r \in \mathscr{E} \times \mathscr{E}$ and a non-negative function $s \in \mathscr{E}$ such that

$$\|J^{(1)}(\cdot, E) + K^{(1)}\|_{\text{op}} \le 1, \qquad L_+^1(\pi) \to \mathbb{R}_+, \tag{1.8}$$

where

$$J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y) = I_{\{r(x,y)>0\}} \frac{J(\mathrm{d}x, \mathrm{d}y)}{r(x,y)^\alpha}, \qquad K^{(\alpha)}(\mathrm{d}x) = I_{\{s(x)>0\}} \frac{K(\mathrm{d}x)}{s(x)^\alpha}, \qquad \alpha \ge 0.$$

For jump processes, one may simply choose

$$r(x,y) = q(x) \vee q(y) = \max\{q(x), q(y)\} \quad \text{and} \quad s(x) = d(x).$$

We remark that when $\alpha < 1$, the operator $J^{(\alpha)}(\cdot, E) + K^{(\alpha)}$ from $L_+^1(\pi)$ to $\mathbb{R}_+$ may no longer be bounded. Correspondingly, we have symmetric forms $D^{(\alpha)}$ defined by $(J^{(\alpha)}, K^{(\alpha)})$. Therefore, with respect to the form $D^{(\alpha)}$, according to (1.1)—(1.5), we can define $\lambda_0^{(\alpha)}$, $\lambda_1^{(\alpha)}$ and the Cheeger's constants $h^{(\alpha)}$, $k^{(\alpha)}$ and $k^{(\alpha)'}$ ($\alpha \ge 0$). However, in what follows, we need only three cases $\alpha = 0$, $1/2$ and 1. When $\alpha = 0$, we return to the original form and so the superscript "(α)" is omitted from our notations.

The next two results are our new forms of the Cheeger's inequalities.

Theorem 1.1. *Suppose that (1.8) holds. We have*

$$\lambda_0 \ge \frac{{h^{(1/2)}}^2}{2 - \lambda_0^{(1)}} \ge \frac{{h^{(1/2)}}^2}{1 + \sqrt{1 - {h^{(1)}}^2}}. \tag{1.9}$$

Theorem 1.2. *Let $K = 0$ and* (1.8) *hold. Then, we have*

$$\lambda_1 \ge \left(\frac{k^{(1/2)}}{\sqrt{2} + \sqrt{2 - \lambda_1^{(1)}}} \right)^2, \tag{1.10}$$

$$\lambda_1 \ge \frac{{k^{(1/2)}}'^2}{1 + \sqrt{1 - {k^{(1)}}'^2}}. \tag{1.11}$$

When $\|J(\cdot, E) + K\|_{\text{op}} \le M < \infty$, the simplest choice of r and s are: $r(x,y) \equiv M$ and $s(x) \equiv M$. Then, (1.8) holds and moreover $h^{(1/2)} = h/\sqrt{M}$, ${k^{(1/2)}}' = k'/\sqrt{M}$, $h^{(1)} = h/M$ and ${k^{(1)}}' = k'/M$. Hence, by (1.9) and (1.11), we get

$$\lambda_0 \ge M\big(1 - \sqrt{1 - h^2/M^2}\,\big) = \frac{h^2}{M\big(1 + \sqrt{1 - h^2/M^2}\,\big)} \in \left[\frac{h^2}{2M}, \frac{h^2}{M}\right].$$

and

$$\lambda_1 \geq M\Big(1-\sqrt{1-k'^2/M^2}\Big) = \frac{k'^2}{M\big(1+\sqrt{1-k'^2/M^2}\big)} \in \left[\frac{k'^2}{2M}, \frac{k'^2}{M}\right]. \tag{1.12}$$

Therefore, for the lower bounds, (1.9) improves (1.6) and (1.11) improves (1.7). More essentially, the lower bound (1.11) is often good enough so that the approximation procedure [2; Theorem 9.12] mentioned above becomes practical. However, we will not go to this direction. In the context of Markov chains on finite graphs, (1.12) was obtained before by Chung [5]. Applying (1.12) to $J^{(1)}$, we get $\lambda_1^{(1)} \geq 1-\sqrt{1-{k^{(1)}}'^2}$. From this and (1.10), we obtain $\lambda_1 \geq \left(\dfrac{k^{(1/2)}}{\sqrt{2}+\sqrt{1+\sqrt{1-{k_1^{(1)}}'^2}}}\right)^2$ which is indeed controlled by (1.11) since $k^{(\alpha)} \leq 2{k^{(\alpha)}}'$. This means that (1.11) is usually more practical than (1.10) except a good lower bound of $\lambda_1^{(1)}$ is known in advance. However, (1.10) and (1.11) are not comparable even in the case of $E=\{0,1\}$. See also the discussion in the second paragraph below Lemma 2.2.

In view of Theorem 1.2, we have $\lambda_1>0$ whenever $k^{(1/2)}>0$. We now study some more explicit conditions for the Cheeger's constants appeared in Theorem 1.2. To state the result, we should use the operators corresponding to the forms. For a jump process, the operator corresponding to $(D^{(\alpha)}, \mathcal{D}(D^{(\alpha)}))$ can be expressed by the following simple form

$$\Omega^{(\alpha)} f(x) = \int I_{[r(x,y)>0]}\frac{q(x,\mathrm{d}y)}{r(x,y)^\alpha}[f(y)-f(x)] + I_{[s(x)>0]}\frac{d(x)}{s(x)^\alpha}f(x).$$

Next, we need some local quantities of λ_0 and λ_1. First, for $B\in\mathcal{E}$ with $\pi(B)\in(0,1)$, let $\lambda_1^{(\alpha)}(B)$ and $k^{(\alpha)}(B)$ be defined by (1.2) and (1.4) with E, π and D replaced respectively by B, $\pi^B := \pi(\cdot\cap B)/\pi(B)$ and

$$D_B^{(\alpha)}(f,f) = \frac{1}{2}\int_{B\times B} J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)(f(y)-f(x))^2. \tag{1.13}$$

Second, define

$$\lambda_0^{(\alpha)}(B) = \inf\big\{D^{(\alpha)}(f,f):\ \pi(f^2)=1,\ f|_{B^c}=0\big\}.$$

As usual, we call $\lambda_0^{(\alpha)}(B)$ and $\lambda_1^{(\alpha)}(B)$ respectively the (generalized) first Dirichlet and Neumann eigenvalue on B. It is a simple matter to check that as in (1.7), $k^{(\alpha)}(B)\geq\lambda_1^{(\alpha)}(B)$.

For $A\in\mathcal{E}$, put $M_A^{(\alpha)} = \underset{A}{\text{ess sup}_\pi} J^{(\alpha)}(\mathrm{d}x, A^c)/\pi(\mathrm{d}x)$, where ess sup_π denotes the essential supremum with respect to π.

Theorem 1.3. *Let $K=0$. Given $\alpha\geq 0$ and $B\in\mathcal{E}$ with $\pi(B)>1/2$. Suppose that there exist a function $\varphi^{(\alpha)}$ with $\delta_1^{(\alpha)}\big(\varphi^{(\alpha)}\big) := \text{ess sup}_{J^{(\alpha)}}|\varphi^{(\alpha)}(x)-\varphi^{(\alpha)}(y)|<\infty$ and a symmetric operator $\big(\Omega^{(\alpha)},\mathcal{D}\big(\Omega^{(\alpha)}\big)\big)$ corresponding to the form $\big(D^{(\alpha)},\mathcal{D}\big(D^{(\alpha)}\big)\big)$ such that $\mathcal{D}\big(\Omega^{(\alpha)}\big)\supset\{I_A:\ A\in\mathcal{E},\ A\subset B\}$ and $\gamma_{B^c}^{(\alpha)} := -\sup_{B^c}\Omega^{(\alpha)}\varphi^{(\alpha)}>0$. Then, we have*

$$k^{(\alpha)} \geq {k^{(\alpha)}}' \geq \frac{k^{(\alpha)}(B)\,\gamma_{B^c}^{(\alpha)}\,[2\pi(B)-1]}{k^{(\alpha)}(B)\,\delta_1^{(\alpha)}\big(\varphi^{(\alpha)}\big)\,[2\pi(B)-1]+\pi(B)^2\big[\delta_1^{(\alpha)}\big(\varphi^{(\alpha)}\big)M_B^{(\alpha)}+\gamma_{B^c}^{(\alpha)}\big]}.$$

Usually, for locally compact E, we have $k^{(\alpha)}(B) > 0$ and $M_B^{(\alpha)} < \infty$ for all compact B. Then the result means that $k^{(\alpha)'} > 0$ provided $\delta_1^{(\alpha)}\big(\varphi^{(\alpha)}\big) < \infty$ and $\gamma_{B^c}^{(\alpha)} > 0$ for large enough B.

Up to now, we have discussed the lower bound of λ_1 by using the Cheeger's constants. However, Theorem 1.3 is indeed a modification of the second approach we are going to study. That is, estimating λ_1 in terms of local λ_0 and λ_1 on subsets of E. The last method has been used recently in the context of diffusions by Wang [9] and it indeed works for general reversible processes. The details of the next two results for general situation are delayed to Section 3. Here, we restrict ourselves to the symmetric forms introduced above.

It is the position to state our first criterion for $\lambda_1 > 0$.

Theorem 1.4. *Let $K = 0$. Then for any $A \subset B$ with $0 < \pi(A),\, \pi(B) < 1$, we have*

$$\frac{\lambda_0(A^c)}{\pi(A)} \geq \lambda_1 \geq \frac{\lambda_1(B)[\lambda_0(A^c)\pi(B) - 2M_A\pi(B^c)]}{2\lambda_1(B) + \pi(B)^2[\lambda_0(A^c) + 2M_A]}. \tag{1.14}$$

As we mentioned before, usually, $\lambda_1(B) > 0$ for all compact B. Hence the result means that $\lambda_1 > 0$ iff $\lambda_0(A^c) > 0$ for some compact A. Because, we can first fix such an A and then make B large enough so that the right-hand side of (1.14) becomes positive.

Finally, we present an upper bound of λ_1 which provides us a necessary condition for $\lambda_1 > 0$ and can qualitatively be sharp as illustrated by Example 4.5.

Theorem 1.5. *Let $K = 0$, $r > 0$, J-a.e. and (1.8) hold. If there exists $\varphi \geq 0$ such that*

$$0 < \delta_2(\varphi) := \text{ess sup}_J |\varphi(x) - \varphi(y)|^2 r(x,y) < \infty,$$

then

$$\lambda_1 \leq \frac{\delta_2(\varphi)}{4} \sup\Big\{\varepsilon^2 : \varepsilon \geq 0,\, \pi\big(e^{\varepsilon\varphi}\big) < \infty\Big\}.$$

Consequently, $\lambda_1 = 0$ if there exists $\varphi \geq 0$ with $0 < \delta_2(\varphi) < \infty$ such that $\pi\big(e^{\varepsilon\varphi}\big) = \infty$ for all $\varepsilon > 0$. In particular, when $J(\mathrm{d}x, \mathrm{d}y) = \pi(\mathrm{d}x)q(x, \mathrm{d}y)$, $\delta_2(\varphi)$ can be replaced by $\delta_2'(\varphi) := \text{ess sup}_\pi \int |\varphi(x) - \varphi(y)|^2 q(x, \mathrm{d}y) < \infty$, without using the function r and (1.8).

We mention that the study on the leading eigenvalue of a bounded integral operator is indeed included in our general setup. Consider the operator P on $L^2(\pi)$: $Pf(x) = \int p(x, \mathrm{d}y) f(y)$, generated by an arbitrary kernel $p(x, \mathrm{d}y)$ with $M := \sup_x p(x, E) < \infty$. Let $\pi(\mathrm{d}x)p(x, \mathrm{d}y)$ be symmetric for a moment. Clearly, the spectrum of P on $L^2(\pi)$ is determined by the one of $M - P$. Note that

$$\big\langle f,\, (M - P)f\big\rangle_\pi = \frac{1}{2}\int \pi(\mathrm{d}x)p(x, \mathrm{d}y)\big[f(x) - f(y)\big]^2 + \int \pi(\mathrm{d}x)\big[M - p(x, E)\big] f(x)^2.$$

Thus, the largest (non-trivial) eigenvalue of the integral operator P can be deduced from λ_0 or λ_1 treated in the paper. Finally, by using a symmetrizing procedure, all the results presented here can be extended to the non-symmetric forms. Refer to [2; Chapter 9] or [7] for instance.

The remainder of the paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.1—1.3. At the end of the section, a different approach to handle the unbounded symmetric forms is presented. A general existence criterion for spectral gap is presented in Section 3, which also contains the proofs of Theorems 1.4 and 1.5. All the results concerning with the spectral gap are illustrated by Markov chains in the last section.

2. Proofs of Theorems 1.1—1.3

We begin this section with the functional representation of the Cheeger's constants. The proof is essential the same as in [7] and [8; §3.3] for the bounded situation and hence omitted.

Lemma 2.1. *For every $\alpha \ge 0$, we have*

$$\begin{aligned}
h^{(\alpha)} &= \inf\left\{\frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f(x)-f(y)| + K^{(\alpha)}(f) : f\ge 0,\, \pi(f)=1\right\},\\
k^{(\alpha)} &= \inf\left\{\int J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f(x)-f(y)| : f\in L^1_+(\pi),\, \int\pi(\mathrm{d}x)\pi(\mathrm{d}y)|f(x)-f(y)|=1\right\}\\
&= \inf\left\{\int J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f(x)-f(y)| : f\in L^1_+(\pi),\, \pi(|f-\pi(f)|)=1\right\},\\
k^{(\alpha)\prime} &= \inf\left\{\frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f(x)-f(y)| : f\in L^1_+(\pi),\, \min_{c\in\mathbb{R}}\pi(|f-c|)=1\right\}.
\end{aligned}$$

Proof of Theorem 1.1. The idea of the proof is based on [7].

Let $E^* = E\cup\{\infty\}$. For any $f\in\mathscr{E}$, define f^* on E^* by setting $f^* = fI_E$. Next, define $J^{*(\alpha)}$ on $E^*\times E^*$ by

$$J^{*(\alpha)}(C) = \begin{cases} J^{(\alpha)}(C), & C\in\mathscr{E}\times\mathscr{E},\\ K^{(\alpha)}(A), & C = A\times\{\infty\} \text{ or } \{\infty\}\times A,\ A\in\mathscr{E},\\ 0, & C=\{\infty\}\times\{\infty\}.\end{cases}$$

We have $J^{*(\alpha)}(\mathrm{d}x,\mathrm{d}y) = J^{*(\alpha)}(\mathrm{d}y,\mathrm{d}x)$ and

$$\int J^{(\alpha)}(\mathrm{d}x,E)f(x)^2 + K^{(\alpha)}(f^2) = \int J^{*(\alpha)}(\mathrm{d}x,\mathrm{d}y)f^*(x)^2, \tag{2.1}$$

$$D^{(\alpha)}(f,f) = \frac{1}{2}\int J^{*(\alpha)}(\mathrm{d}x,\mathrm{d}y)(f^*(y)-f^*(x))^2, \tag{2.2}$$

$$\frac{1}{2}\int J^{(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f(y)-f(x)| + \int K^{(\alpha)}(\mathrm{d}x)|f(x)| = \frac{1}{2}\int J^{*(\alpha)}(\mathrm{d}x,\mathrm{d}y)|f^*(y)-f^*(x)|. \tag{2.3}$$

Therefore, for f with $\pi(f^2)=1$, by (2.1)–(2.3), (1.8), part (1) of Lemma 2.1 and Cauchy-Schwarz inequality,

$$\begin{aligned}
{h^{(1)}}^2 &\le \left\{\frac{1}{2}\int J^{*(1)}(\mathrm{d}x,\mathrm{d}y)\big|f^*(y)^2-f^*(x)^2\big|\right\}^2\\
&\le \frac{1}{2}D^{(1)}(f,f)\int J^{*(1)}(\mathrm{d}x,\mathrm{d}y)\big[f^*(y)+f^*(x)\big]^2\\
&= \frac{1}{2}D^{(1)}(f,f)\left\{2\int J^{*(1)}(\mathrm{d}x,\mathrm{d}y)\big[f^*(y)^2+f^*(x)^2\big] - \int J^{*(1)}(\mathrm{d}x,\mathrm{d}y)\big[f^*(y)-f^*(x)\big]^2\right\}\\
&\le D^{(1)}(f,f)\big[2-D^{(1)}(f,f)\big].
\end{aligned}$$

This implies that $D^{(1)}(f,f)\ge 1-\sqrt{1-{h^{(1)}}^2}$ and so

$$\lambda_0^{(1)} \ge 1-\sqrt{1-{h^{(1)}}^2}. \tag{2.4}$$

Next, by (1.8), part (1) of Lemma 2.1 and another use of the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} {h^{(1/2)}}^2 &\le \bigg\{\frac{1}{2}\int J^{*(1/2)}(\mathrm{d}x,\mathrm{d}y)\big|f^*(y)^2 - f^*(x)^2\big|\bigg\}^2 \\ &\le \frac{1}{2}D(f,f)\int J^{*(1)}(\mathrm{d}x,\mathrm{d}y)\big[f^*(y)+f^*(x)\big]^2 \\ &\le D(f,f)\big[2 - D^{(1)}(f,f)\big] \le D(f,f)\big[2-\lambda_0^{(1)}\big]. \end{aligned} \tag{2.5}$$

¿From this and (2.4), the required assertion follows. □

Proof of Theorem 1.2. a) First, we prove (1.10). Let $f \in \mathcal{D}(D)$ with $\pi(f) = 0$ and $\pi(f^2) = 1$. Set $g = f + c$, $c \in \mathbb{R}$. Similar to (2.5), we have

$$\begin{aligned} \bigg\{\int J^{(1/2)}(\mathrm{d}x,\mathrm{d}y)\big|g(y)^2 - g(x)^2\big|\bigg\}^2 &\le 4D(f,f)\big[2(1+c^2) - D^{(1)}(f,f)\big] \\ &\le 4D(f,f)[2(1+c^2)-\beta] \end{aligned}$$

for all $\beta:\ 0 \le \beta < \lambda_1^{(1)} \le 2$. Hence by Lemma 2.1, we have

$$D(f,f) \ge \frac{1}{4[2(1+c^2)-\beta]}\bigg\{\int J^{(1/2)}(\mathrm{d}x,\mathrm{d}y)\big|g(y)^2 - g(x)^2\big|\bigg\}^2 \ge \frac{\kappa_\beta}{4}{k^{(1/2)}}^2 \tag{2.6}$$

where κ_β is the same as κ defined below (1.7) but replacing the denominator $1+c^2$ with $2(1+c^2)-\beta$. To estimate κ_β, we adopt an optimizing procedure which will be used several times subsequently. Set $\gamma = \mathbb{E}|X| \in (0,1]$. It is known that

$$\lim_{c\to\pm\infty}\frac{\big(\mathbb{E}\big|(X+c)^2-(Y+c)^2\big|\big)^2}{2(1+c^2)-\beta} = 2(\mathbb{E}|X-Y|)^2 \ge 2(\mathbb{E}|X|)^2 = 2\gamma^2$$

and when $c = 0$, $\mathbb{E}\big|X^2 - Y^2\big| \ge 2(1-\mathbb{E}|X|) = 2(1-\gamma)$ (cf. [7] or [2; §9.2]). Thus,

$$\kappa_\beta \ge \inf_{\gamma\in(0,1]}\max\bigg\{2\gamma^2,\ \frac{4(1-\gamma)^2}{2-\beta}\bigg\}. \tag{2.7}$$

We now need an elementary fact.

Lemma 2.2. *Let f and g be continuous functions on $[0,1]$ and satisfy $f(0) < g(0)$ and $f(1) > g(1)$. Suppose that f is increasing and g is decreasing. Then*

$$\inf_{\gamma\in[0,1]}\max\{f(\gamma),\, g(\gamma)\} = f(\gamma_0),$$

where γ_0 is the unique solution to the equation $f = g$ on $[0,1]$.

Applying Lemma 2.2 to (2.7), we get

$$\kappa_\beta \ge \frac{4}{\big(\sqrt{2}+\sqrt{2-\beta}\big)^2}.$$

Combining this with (2.6) and then letting $\beta \uparrow \lambda_1^{(1)}$, we obtain (1.10).

It is worthy to mention that the estimate just proved can be sharp. To see this, simply consider $E = \{0, 1\}$, $J(\{i\}, \{j\}) = 1\,(i \neq j)$ and $\pi_0 = \pi_1 = 1/2$. Then $k^{(1/2)} = \lambda_1^{(1)} = \lambda_1 = 2$. Moreover, the same example shows that in contract to (1.9), the analog of (1.9) "$\lambda_1 \geq {k^{(1/2)}}^2 / \big[4\big(2 - \lambda_1^{(1)}\big)\big]$" or "$\lambda_1 \geq {k^{(1/2)'}}^2 / \big[2 - \lambda_1^{(1)}\big)\big]$" does not hold.

b) Define

$$\widetilde{D}_B^{(\alpha)}(f, g) = \frac{1}{2} \int_{B \times B} J^{(\alpha)}(\mathrm{d}x, \mathrm{d}y)[f(y) - f(x)]^2 + \int_B J^{(\alpha)}(\mathrm{d}x, B^c) f(x)^2.$$

It is easy to see that

$$\lambda_0(B) = \inf\big\{\widetilde{D}_B(f, f) : \pi(f^2 I_B) = 1\big\}.$$

Let

$$h_B^{(\alpha)} = \inf_{A \subset B,\, \pi(A) > 0} \frac{J^{(\alpha)}(A \times (B \setminus A)) + J^{(\alpha)}(A \times B^c)}{\pi(A)} = \inf_{A \subset B,\, \pi(A) > 0} \frac{J^{(\alpha)}(A \times A^c)}{\pi(A)}. \tag{2.8}$$

Then by Theorem 1.1, we have $\lambda_0(B) \geq {h_B^{(1/2)}}^2 \Big/ \Big[1 + \sqrt{1 - {h_B^{(1)}}^2}\,\Big]$.

Next, we prove that

$$\lambda_1 \geq \inf_{\pi(B) \leq 1/2} \lambda_0(B).$$

For each $\varepsilon > 0$, choose f_ε with $\pi(f_\varepsilon) = 0$ and $\pi(f_\varepsilon^2) = 1$ such that $\lambda_1 + \varepsilon \geq D(f_\varepsilon, f_\varepsilon)$. Next, choose c_ε such that $\pi(f_\varepsilon < c_\varepsilon)$, $\pi(f_\varepsilon > c_\varepsilon) \leq 1/2$. Set $f_\varepsilon^\pm = (f_\varepsilon - c_\varepsilon)^\pm$ and $B_\varepsilon^\pm = \{f_\varepsilon^\pm > 0\}$. Then

$$\begin{aligned}
\lambda_1 + \varepsilon &\geq D(f_\varepsilon - c_\varepsilon,\, f_\varepsilon - c_\varepsilon) \\
&= \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y)\big[\big|f_\varepsilon^+(y) - f_\varepsilon^+(x)\big| + \big|f_\varepsilon^-(y) - f_\varepsilon^-(x)\big|\big]^2 \\
&\geq \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y)\big(f_\varepsilon^+(y) - f_\varepsilon^+(x)\big)^2 + \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y)\big(f_\varepsilon^-(y) - f_\varepsilon^-(x)\big)^2 \\
&\geq \lambda_0\big(B_\varepsilon^+\big)\pi\big(\big(f_\varepsilon^+\big)^2\big) + \lambda_0\big(B_\varepsilon^-\big)\pi\big(\big(f_\varepsilon^-\big)^2\big) \\
&\geq \inf_{\pi(B) \leq 1/2} \lambda_0(B)\pi\big(\big(f_\varepsilon^+\big)^2 + \big(f_\varepsilon^-\big)^2\big) = (1 + c_\varepsilon^2) \inf_{\pi(B) \leq 1/2} \lambda_0(B) \geq \inf_{\pi(B) \leq 1/2} \lambda_0(B).
\end{aligned}$$

Because ε is arbitrary, we obtain the required conclusion.

Finally, combining the above two assertions together, we obtain

$$\begin{aligned}
\lambda_1 &\geq \inf_{\pi(B) \leq 1/2} \frac{{h_B^{(1/2)}}^2}{1 + \sqrt{1 - {h_B^{(1)}}^2}} \geq \inf_{\pi(B) \leq 1/2} \frac{\inf_{\pi(B) \leq 1/2} {h_B^{(1/2)}}^2}{1 + \sqrt{1 - {h_B^{(1)}}^2}} \\
&\geq \frac{\inf_{\pi(B) \leq 1/2} {h_B^{(1/2)}}^2}{1 + \sqrt{1 - \inf_{\pi(B) \leq 1/2} {h_B^{(1)}}^2}} = \frac{{k^{(1/2)'}}^2}{1 + \sqrt{1 - {k^{(1)'}}^2}} \qquad \square
\end{aligned}$$

Proof of Theorem 1.3. The proof is split into two lemmas given below. Noticing that α is fixed, we may and will omit the superscript "(α)" everywhere in the next two lemmas and their proofs for simplicity. $\square$

Lemma 2.3. *Let $B \in \mathcal{E}$ with $2\pi(B) > 1$. Then*

$$k' \geq \frac{h_{B^c} k(B)(2\pi(B) - 1)}{k(B)(2\pi(B) - 1) + 2\pi(B)^2(M_B + h_{B^c})},$$

where h_B is defined by (2.8).

Proof. We need only to consider the case that $h_{B^c} k(B) > 0$. For any $A \in \mathcal{E}$ with $\pi(A) \in (0, 1/2]$, let $\gamma = \pi(AB)/\pi(A)$. Then

$$\begin{aligned}\frac{J(A \times A^c)}{\pi(A)} &= \frac{1}{2\pi(A)} \int J(\mathrm{d}x, \mathrm{d}y) \big[I_A(y) - I_A(x)\big]^2 \geq \frac{1}{2\pi(A)} \int_{B \times B} J(\mathrm{d}x, \mathrm{d}y) \big[I_A(y) - I_A(x)\big]^2 \\ &\geq \frac{k(B)\pi^B(A)\pi^B(A^c)}{\pi(A)} \geq \frac{\pi(B) - 1/2}{\pi(B)^2} k(B)\gamma. \qquad (2.9)\end{aligned}$$

Here, in the last step, we have used $\pi(AB) \leq \pi(A) \leq 1/2$. On the other hand, we have

$$h_{B^c} \pi(AB^c) \leq \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y) \big[I_{AB^c}(x) - I_{AB^c}(y)\big]^2 = \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y) \big|I_{A^c \cup B}(x) - I_{A^c \cup B}(y)\big|.$$

Noticing that J is symmetric and

$$\big|I_{A^c \cup B}(x) - I_{A^c \cup B}(y)\big| \leq \big|I_{A^c}(x) - I_{A^c}(y)\big| + I_{B \times B^c + B^c \times B} \big|I_{AB}(x) - I_{AB}(y)\big|,$$

we obtain

$$h_{B^c}(1 - \gamma) = \frac{h_{B^c} \pi(AB^c)}{\pi(A)} \leq \frac{J(A \times A^c)}{\pi(A)} + M_B \gamma.$$

Combining this with (2.9) and applying Lemma 2.2, we get

$$\begin{aligned}\frac{J(A \times A^c)}{\pi(A)} &\geq \inf_{\gamma \in [0,1]} \max\Big\{(\pi(B) - 1/2)\pi(B)^{-2} k(B)\gamma,\ h_{B^c} - \big(M_B + h_{B^c}\big)\gamma\Big\} \\ &= \frac{h_{B^c} k(B)(2\pi(B) - 1)}{k(B)(2\pi(B) - 1) + 2\pi(B)^2(M_B + h_{B^c})}. \qquad \square\end{aligned}$$

Lemma 2.4. *Let φ satisfy $\delta_1(\varphi) < \infty$. If $\gamma_B = -\sup_B \Omega\varphi > 0$, then $h_B \geq \gamma_B/\delta_1(\varphi) > 0$.*

Proof. For any $A \subset B$, we have

$$\begin{aligned}\gamma_B \pi(A) &\leq \int_A [-\Omega\varphi] \mathrm{d}\pi = \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y)(I_A(x) - I_A(y))(\varphi(x) - \varphi(y)) \\ &\leq \frac{\delta_1(\varphi)}{2} \int J(\mathrm{d}x, \mathrm{d}y)|I_A(x) - I_A(y)| = \delta_1(\varphi) J(A \times A^c).\end{aligned}$$

Hence, $h_B \geq \gamma_B/\delta_1(\varphi)$. $\square$

To conclude this section, we discuss a different way to deal with the general symmetric forms. In contrast to the previous approach, we now keep (J, K) to be the same but change the L^2-space. To do so, let p be a measurable function and satisfy $\alpha_p := \text{ess inf}_\pi p > 0$, $\beta_p := \pi(p) < \infty$ and $\|J(\cdot, E) + K\|_{\text{op}} \leq \beta_p$ ($L^1_+(\pi_p) \to \mathbb{R}_+$), where $\pi_p = p\pi/\beta_p$. For jump processes, one may take $p(x) = q(x) \vee r$ for some $r \geq 0$. From this, one sees the main restriction of the present approach: $\int \pi(\mathrm{d}x) q(x) < \infty$, since we require that $\pi(p) < \infty$. Except this point, the approach is not comparable with the previous one (see Example 4.5 and Example 4.7 given below).

Next, define h_p, k_p and k'_p by (1.3)–(1.5) respectively with π replaced by π_p and then divided by β_p. For instance, $k'_p = \inf_{\pi_p(A) \leq 1/2} J(A \times A^c)/\pi(pI_A)$.

Theorem 2.5. *Let p, α_p, β_p and π_p be given above. Define $\lambda_{p,i}\,(i=0,1)$ by (1.1) and (1.2) with π replaced by π_p. Then, we have*

$$\lambda_i \ge \frac{\alpha_p}{\beta_p}\,\lambda_{p,i}, \qquad i=0,\ 1. \tag{2.10}$$

In particular,

$$\lambda_0 \ge \alpha_p\Big(1-\sqrt{1-h_p^2}\,\Big) \tag{2.11}$$

and when $K=0$,

$$\lambda_1 \ge \max\left\{\frac{\kappa}{8}\alpha_p k_p^2,\ \alpha_p\Big(1-\sqrt{1-k_p'^2}\,\Big)\right\}. \tag{2.12}$$

Proof. a) We prove that $L^\infty(\pi)$ is dense in $\mathcal{D}(D)$ in the D-norm: $\|f\|_D^2 = D(f,f)+\pi(f^2)$. The proof is similar to [2; Lemma 9.7]. First, we show that $L^\infty(\pi)\subset\mathcal{D}(d)$. Because $1\in L^1(\pi_p)$ and $\|J(\cdot,E)+K\|_{\rm op}\le\beta_p$, we have $J(E,E)+K(E)\le\beta_p<\infty$. Thus,

$$\begin{aligned} D(f,f) &\le \int J(\mathrm{d}x,\mathrm{d}y)\big[f(y)^2+f(x)^2\big]+\int K(\mathrm{d}x)f(x)^2\\ &\le 2\|f\|_\infty^2\big(J(E,E)+K(E)\big)<\infty, \end{aligned}$$

and hence $f\in\mathcal{D}(D)$. Next, let $f\in\mathcal{D}(D)$ and set $f_n=(-n)\vee(f\wedge n)$. Then $f_n\in\mathcal{D}(D)$,

$$|f_n(y)-f_n(x)|\le|f(y)-f(x)| \qquad\text{and}\qquad |f_n(x)|\le|f(x)| \tag{2.13}$$

for all x, y and n. Clearly, $\pi\big((f_n-f)^2\big)\to 0$. Moreover, since $D(f_n-f,f_n-f)\le 4D(f,f)<\infty$ by (2.13), we have $D(f_n-f,f_n-f)\to 0$ by (2.13) and the dominated convergence theorem. Therefore, $\|f_n-f\|_D\to 0$.

b) Here, we prove (2.10) for $i=1$ only since the proof for $i=0$ is similar and even simpler. Then, (2.11) and (2.12) follows from (1.7) and the comment right after Theorem 1.2 with $M=\beta_p$.

Because $L^\infty(\pi)\subset L^2(\pi_p)$ and $L^2(\pi_p)$ is just the domain of the form $D(f,f)$ on $L^2(\pi_p)$, by definition of λ_1 and $\lambda_{1,p}$, it suffices to show that $\pi_p(f^2)-\pi_p(f)^2\ge\big[\pi(f^2)-\pi(f)^2\big]\alpha_p/\beta_p$ for every $f\in L^\infty(\pi)$. The proof goes as follows.

$$\begin{aligned} \pi_p\big(f^2\big)-\pi_p(f)^2 &= \inf_{c\in\mathbb{R}}\int\big(f(x)-c\big)^2\pi_p(\mathrm{d}x)\\ &= \beta_p^{-1}\inf_{c\in\mathbb{R}}\int\big(f(x)-c\big)^2 p(x)\pi(\mathrm{d}x)\\ &\ge \frac{\alpha_p}{\beta_p}\inf_{c\in\mathbb{R}}\int\big(f(x)-c\big)^2\pi(\mathrm{d}x)\\ &= \frac{\alpha_p}{\beta_p}\big[\pi\big(f^2\big)-\pi(f)^2\big]. \quad\square \end{aligned}$$

3. A Criterion for the Existence of Spectral Gap. Proofs of Theorem 1.4 and Theorem 1.5

To state our main criterion, we need some preparation.

Let E be a locally compact separable metric space with Borel field $\mathcal{E}$ and $\mathrm{supp}(\pi) = E$. Denote by $C_b(E)$ (resp. $C_0(E)$) the set of all bounded continuous functions (resp. with compact support) on E.

Next, let $(D, \mathcal{D}(D))$ be a regular conservative Dirichlet form on $L^2(\pi)$. By Beurling-Deny's formulae, the form can be expressed as follows

$$D(f,f) = D^{(c)}(f,f) + \frac{1}{2}\int J(\mathrm{d}x,\mathrm{d}y)(f(x)-f(y))^2, \qquad f \in \mathcal{D}(D)\cap C_0(E) \tag{3.1}$$

where $\mathcal{D}(D^{(c)}) = \mathcal{D}(D)\cap C_0(E)$ and satisfies a strong local property; J is a symmetric Radon measure on the product space $E\times E$ off diagonal. Moreover, there exists a finite, non-negative Radon measure $\mu^c_{\langle f\rangle}$ such that

$$D^{(c)}(f,f) = \frac{1}{2}\int_E \mathrm{d}\mu^c_{\langle f\rangle}, \qquad f \in \mathcal{D}(D)\cap C_b(E).$$

Theorem 3.1. *Let $\mathcal{C}\subset\mathcal{D}(D)\cap C_0(E)$ be dense in $\mathcal{D}(D)$ in the D-norm: $\|f\|_D^2 = D(f,f)+\pi(f^2)$. Set $\mathcal{C}_L = \{f+c:\ f\in\mathcal{C},\ c\in\mathbb{R}\}$. Given $A,\ B\in\mathcal{E}$, $A\subset B$ with $0<\pi(A),\ \pi(B)<1$. Suppose that the following conditions hold.*

1. *There exists a conservative Dirichlet form $(D_B,\mathcal{D}(D_B))$ on the square-integrable functions on B with respect to π^B such that $\mathcal{C}|_B\subset\mathcal{D}(D_B)$ and*
$$D(f,f)\ge D_B(fI_B,fI_B), \qquad f\in\mathcal{C}_L.$$
2. *There exists a function $h\in\mathcal{C}_L$: $0\le h\le 1$, $h|_A=0$ and $h|_{B^c}=1$ such that*

$$c(h) := \sup_{f\in\mathcal{C}_L}\frac{1}{\pi\big(f^2I_B\big)}\bigg[\frac{1}{2}\int f^2\mathrm{d}\mu^c_{\langle h\rangle} + \int_{B\times A^c} J(\mathrm{d}x,\mathrm{d}y)\big[f(1-h)(y)-f(1-h)(x)\big]^2\bigg] < \infty.$$

Then, we have
$$\frac{\lambda_0(A^c)}{\pi(A)} \ge \lambda_1 \ge \frac{\lambda_1(B)[\lambda_0(A^c)\pi(B) - 2c(h)\pi(B^c)]}{2\lambda_1(B)+\pi(B)^2[\lambda_0(A^c)+2c(h)]}.$$

Proof. The upper bound is easy. Simply take $f\in\mathcal{C}_L$ with $f|_A=0$ and $\pi(f^2)=1$. Then

$$\pi(f^2)-\pi(f)^2 = 1-\pi\big(fI_{A^c}\big)^2 \ge 1-\pi\big(f^2\big)\pi\big(A^c\big) = 1-\pi\big(A^c\big) = \pi(A).$$

Hence $\lambda_1\le D(f,f)/\pi(A)$ which gives us $\lambda_1\le\lambda_0\big(A^c\big)/\pi(A)$.

For the lower bound, let $f\in\mathcal{C}_L$ with $\pi(f)=0$ and $\pi\big(f^2\big)=1$. Set $\gamma=\pi\big(f^2I_B\big)$.

a) By condition (1), we have

$$\begin{aligned}
D(f,f) &\ge D_B(fI_B,fI_B)\ge\lambda_1(B)\pi(B)^{-1}\big[\pi\big(f^2I_B\big)-\pi(B)^{-1}\pi\big(fI_B\big)^2\big]\\
&=\lambda_1(B)\pi(B)^{-1}\big[\pi\big(f^2I_B\big)-\pi(B)^{-1}\pi\big(fI_{B^c}\big)^2\big]\\
&\ge\lambda_1(B)\pi(B)^{-1}\big[\gamma-\pi(B)^{-1}\pi\big(f^2I_{B^c}\big)\pi\big(B^c\big)\big]\\
&=\lambda_1(B)\pi(B)^{-2}\big[\gamma-\pi\big(B^c\big)\big].
\end{aligned}\tag{3.2}$$

b) Let ρ be the metric in E. By the construction of $\mu^c_{\langle f\rangle}$ ([6; §3.2]), there exist a sequence of relatively compact open sets G_ℓ increasing to E, a sequence of symmetric, non-negative Radon measures σ_{β_n} and a sequence δ_ℓ such that

$$\int_E g\mathrm{d}\mu^c_{\langle f\rangle} = \lim_{\ell\to\infty}\lim_{\beta_n\to\infty}\beta_n\int_{G_\ell\times G_\ell,\,\rho(x,y)<\delta_\ell}[f(x)-f(y)]^2 g(x)\sigma_{\beta_n}(\mathrm{d}x,\mathrm{d}y)$$
$$f,g\in\mathcal{D}(D)\cap C_0(E).$$

¿From this and

$$[(fh)(x)-(fh)(y)]^2\le 2h(y)^2[f(x)-f(y)]^2+2f(x)^2[h(x)-h(y)]^2,$$

it follows that

$$\int\mathrm{d}\mu^c_{\langle fh\rangle}\le 2\int h^2\mathrm{d}\mu^c_{\langle f\rangle}+2\int f^2\mathrm{d}\mu^c_{\langle h\rangle},$$

first for $f,\,g\in\mathcal{D}(D)\cap C_0(E)$ and then for $f,\,g\in\mathcal{D}(D)\cap C_b(E)$ (cf. [6; §3.2]). Hence

$$D^{(c)}(fh,fh)=\frac{1}{2}\int\mathrm{d}\mu^c_{\langle fh\rangle}\le 2D^{(c)}(f,f)+\int f^2\mathrm{d}\mu^c_{\langle h\rangle}. \tag{3.3}$$

On the other hand, since

$$|(fh)(x)-(fh)(y)|\le|f(x)-f(y)|+I_{B\times A^c\cup A^c\times B}(x,y)|f(1-h)(x)-f(1-h)(y)|,$$

we have

$$\begin{aligned}\int J(\mathrm{d}x,\mathrm{d}y)[(fh)(x)-(fh)(y)]^2&\le 2\int J(\mathrm{d}x,\mathrm{d}y)[f(x)-f(y)]^2\\&\quad+4\int_{B\times A^c}J(\mathrm{d}x,\mathrm{d}y)[f(1-h)(x)-f(1-h)(y)]^2.\end{aligned}\tag{3.4}$$

Thus, combining (3.1), (3.3), (3.4) with condition (2) together, we get

$$\begin{aligned}D(fh,fh)&\le 2D(f,f)+\int f^2\mathrm{d}\mu^c_{\langle h\rangle}+2\int_{B\times A^c}J(\mathrm{d}x,\mathrm{d}y)\big[f(1-h)(x)-f(1-h)(y)\big]^2\\&\le 2D(f,f)+2c(h)\pi\big(f^2I_B\big)\\&\le 2D(f,f)+2\gamma c(h).\end{aligned}$$

That is,

$$\begin{aligned}D(f,f)&\ge\frac{1}{2}D(fh,fh)-\gamma c(h)\ge\frac{1}{2}\lambda_0\big(A^c\big)\pi\big(f^2h^2\big)-\gamma c(h)\\&\ge\frac{1}{2}\lambda_0\big(A^c\big)\pi\big(f^2I_{B^c}\big)-\gamma c(h)=\frac{1}{2}\lambda_0\big(A^c\big)(1-\gamma)-\gamma c(h).\end{aligned}\tag{3.5}$$

Combining (3.2) with (3.5) together, we obtain

$$\begin{aligned}D(f,f)&\ge\inf_{\gamma\in[0,1]}\max\left\{\frac{\lambda_1(B)}{\pi(B)^2}\big(\gamma-\pi\big(B^c\big)\big),\,\frac{1}{2}\lambda_0\big(A^c\big)(1-\gamma)-\gamma c(h)\right\}\\&=\lambda_1(B)\pi(B)^{-2}\big(\gamma_0-\pi\big(B^c\big)\big).\end{aligned}\tag{3.6}$$

The assertion of the theorem now follows from (3.6) and Lemma 2.2. □

Theorem 3.1 is effective for diffusions was shown in [9] with a more direct proof (in this case the Dirichlet form is explicit). We now apply the theorem to jump processes.

Proof of Theorem 1.4. First, the topological assumptions of Theorem 3.1 are unnecessary in the present context. To see that condition (1) is fulfilled, simply take D_B to be the one defined by (1.13). For condition (2), take $h = I_{A^c}$. Then

$$\int_{B\times A^c} J(\mathrm{d}x, \mathrm{d}y)[(fI_A)(x) - (fI_A)(y)]^2 = \int_{A\times A^c} J(\mathrm{d}x, \mathrm{d}y) f(x)^2 \le M_A \pi\big(f^2 I_A\big) \le M_A \pi\big(f^2 I_B\big).$$

This means that condition (2) holds with $c(h) = M_A$. We have thus proved Theorem 1.4. □

The application of Theorem 3.1 (or Theorem 1.4) requires some estimates of $\lambda_0(A^c)$ and $\lambda_1(B)$, which may be obtained from Theorems 1.1—1.2. These estimates are usually in the qualitative sense good enough for $\lambda_1(B)$, for which there are also quite a lot of publications including the authors' study in the past years. However, for $\lambda_0(A^c)$, the bound presented above may not be sharp enough, especially in the unbounded situation. For this reason, we now introduce a different result.

Theorem 3.2. *Let E be a metric space with Borel field $\mathscr{E}$ and let (x_t) be a reversible right-continuous Markov process valued in E with weak generator Ω. Suppose that the corresponding Dirichlet form is regular. Next, fix a closed set B. Suppose additionally that the following conditions hold.*

(1) *There exists a positive function φ, $\varphi|_B = 0$ and*

$$\sup_{B^c} \frac{1}{\varphi}\Omega\varphi =: \delta < 0.$$

(2) *There exists a sequence of open sets (E_n): $E_0 \supset B$, $E_n \uparrow E$ such that φ is bounded below on each $E_n \setminus B$ by a positive constant.*
(3) *The first Dirichlet eigenfunction of Ω on each $E_n \setminus B$ is bounded above.*

Then we have $\lambda_0\big(B^c\big) \ge \delta$. In particular, for jump processes, the condition "$\varphi|_B = 0$" given in (1) can be removed.

Clearly, conditions (2) and (3) with compact B are fulfilled for diffusions or Markov chains. Thus, the key condition here is the first one.

Proof of Theorem 3.2. The last assertion follows by replacing φ with φI_{B^c}. Indeed,

$$\begin{aligned}\Omega\big(\varphi I_{B^c}\big)(x) &= \int q(x, \mathrm{d}y)\big[\big(\varphi I_{B^c}\big)(y) - \big(\varphi I_{B^c}\big)(x)\big]\\ &\le \int q(x, \mathrm{d}y)\varphi(y) - \big(\varphi I_{B^c}\big)(x) = \Omega\varphi(x) \le -\delta\big(\varphi I_{B^c}\big)(x) \quad \text{on} \quad B^c.\end{aligned}$$

We are now going to prove the main assertion of the theorem. Set $\tau_B = \inf\{t \ge 0 : x_t \in B\}$. Then, by condition (1) plus a truncating argument if necessary, we get

$$\mathbb{E}^x \varphi(x_{t\wedge\tau_B}) \le \varphi(x), \qquad t \ge 0,\ x \notin B.$$

Next, let u_n (≥ 0) be the first Dirichlet eigenfunction of Ω on $E_n \setminus B$. Set $\tau = \inf\{t \geq 0 : x_t \notin E_n \setminus B\}$. Then, by conditions (2) and (3), there exists $c_1 > 0$ such that $u(x_{t\wedge\tau}) \leq c_1\varphi(x_{t\wedge\tau_B})$ and so

$$u_n(x)e^{-\lambda_0(E_n\setminus B)t} = \mathbb{E}^x u_n(x_{t\wedge\tau}) \leq c_1\mathbb{E}^x\varphi(x_{t\wedge\tau_B}) \leq c_1\varphi(x)e^{-\delta t}, \qquad x \in E_n \setminus B.$$

This implies that $\lambda_0(E_n \setminus B) \geq \delta$. Finally, because the Dirichlet form is regular, it is easy to show that $\lambda_0(B^c) = \lim_{n\to\infty}\lambda_0(E_n \setminus B)$ and so the required assertion follows. $\square$

For the remainder of this section, we turn to study the upper bound of λ_1.

Let $(D, \mathcal{D}(D))$ be a general conservative Dirichlet form and let $P(t, x, \mathrm{d}y)$ be the corresponding transition probability. Fix $\varphi \geq 0$. Suppose that $\varphi \wedge n \in \mathcal{D}(D)$ for every $n \geq 1$. Set $f_n = \exp[\varepsilon(\varphi \wedge n)/2]$. Since the function $e^{\alpha x}$ is locally Lipschitz continuous and $\varphi \wedge n$ is bounded, by the elementary spectral representation theory, we have

$$\begin{aligned}
D(f_n, f_n) &= \lim_{t\to 0}\frac{1}{2t}\int \pi(\mathrm{d}x)P(t,x,\mathrm{d}y)[f_n(x) - f_n(y)]^2\\
&\leq \frac{\varepsilon^2}{4}C(\varphi,n)\lim_{t\to 0}\frac{1}{2t}\int \pi(\mathrm{d}x)P(t,x,\mathrm{d}y)[(\varphi\wedge n)(x) - (\varphi\wedge n)(y)]^2\\
&\leq \frac{\varepsilon^2}{4}C(\varphi,n)D(\varphi\wedge n, \varphi\wedge n) < \infty,
\end{aligned}$$

where $C(\varphi, n)$ is the Lipschitz norm of $e^{\varepsilon x/2}$ on the range of $\varphi\wedge n$. This leads us to introduce the following constant

$$\delta(\varepsilon,\varphi) = \varepsilon^{-2}\sup_{n\geq 1} D(f_n, f_n)/\pi\big(f_n^2\big).$$

Theorem 3.3. *Let $(D, \mathcal{D}(D))$, φ, f_n and $\delta(\varepsilon,\varphi)$ be as above. Then, we have*

$$\lambda_1 \leq \sup\big\{\varepsilon^2\delta(\varepsilon,\varphi) : \pi\big(e^{\varepsilon\varphi}\big) < \infty\big\}.$$

Proof. We need to show that if $\pi\big(e^{\varepsilon\varphi}\big) = \infty$, then $\lambda_1 \leq \varepsilon^2\delta(\varepsilon,\varphi)$. For $n \geq 1$, we have

$$\lambda_1 \leq \frac{1}{2}\frac{\int J(\mathrm{d}x,\mathrm{d}y)[f_n(x) - f_n(y)]^2}{\pi\big(f_n^2\big) - \pi(f_n)^2}. \tag{3.7}$$

For every $m \geq 1$, choose $r_m > 0$ such that $\pi(\varphi \geq r_m) \leq 1/m$. Then

$$\pi\big(I_{[\varphi\geq r_m]}f_n^2\big)^{1/2} \geq \sqrt{m}\,\pi\big(I_{[\varphi\geq r_m]}f_n\big) \geq \sqrt{m}\,\pi(f_n) - \sqrt{m}\,e^{\varepsilon r_m/2}.$$

Hence

$$\pi\big(f_n\big)^2 \leq \Big[\sqrt{\pi\big(f_n^2\big)}/\sqrt{m} + e^{\varepsilon r_m/2}\Big]^2. \tag{3.8}$$

On the other hand, by assumption, we have

$$D(f_n, f_n) \leq \varepsilon^2\delta(\varepsilon,\varphi)\pi\big(f_n^2\big). \tag{3.9}$$

Noticing that $\pi\big(f_n^2\big) \uparrow \infty$, combining (3.9) with (3.7) and (3.8) and then letting $n \uparrow \infty$, we obtain

$$\lambda_1 \leq \varepsilon^2\delta(\varepsilon,\varphi)/\big[1 - m^{-1}\big].$$

The proof is completed by setting $m \uparrow \infty$. $\square$

Proof of Theorem 1.5. It suffices to prove the first assertion because the remainder of the proof is similar. Let f_n be given as in Theorem 3.3. Note that by the Mean Value Theorem, $|e^A - e^B| \le |A-B| e^{A\vee B} = |A-B|(e^A \vee e^B)$ for all $A,\, B \ge 0$. Hence

$$\begin{aligned} D(f_n, f_n) &= \frac{1}{2}\int J(\mathrm{d}x,\mathrm{d}y)[f_n(x) - f_n(y)]^2 \\ &\le \frac{\varepsilon^2}{8}\int J^{(1)}(\mathrm{d}x,\mathrm{d}y)[\varphi(x)-\varphi(y)]^2 r(x,y)\big[f_n(x)\vee f_n(y)\big]^2 \\ &\le \frac{\varepsilon^2}{4}\delta_2(\varphi)\pi\big(f_n^2\big). \end{aligned}$$

The conclusion now follows from Theorem 3.3 with $\delta(\varepsilon,\varphi) = \frac{1}{4}\delta_2(\varphi)$. $\square$

4. Existence of Spectral Gap for Markov Chains.

Usually, the power of a result for general jump processes should be justified by Markov chains.

Let E be countable and (q_{ij}) be a regular and irreducible Q-matrix, reversible with respect to $\pi = (\pi_i)$. As usual, let $q_i = \sum_{j\ne i} q_{ij}$. Assume that $q_i > 0$ for all i to rule out the reducible case. Then $K = 0$ and $\Omega f(i) = \sum_{j\ne i} q_{ij}[f_j - f_i]$. The density of the symmetric measure with respect to the counting measure becomes $J(i,j) = \pi_i q_{ij}$ $(i \ne j)$. For simplicity, we consider only two typical situations: $E = \mathbb{Z}_+$ or $E = \mathbb{Z}^d$ and take $r(i,j) = 1/(q_i \vee q_j)$. Denote by $|i|$ the L^1-norm, i.e., $|i| = \sum_{k=1}^d |i_k|$ for $i = (i_1, \cdots, i_d) \in \mathbb{Z}^d$.

A combination of Theorem 1.2 and the next result provides us a simple condition for the existence of spectral gap for birth-death processes and the result seems to be new from our knowledge even in such a simple situation (cf. [3]).

Theorem 4.1. *Consider the birth-death process on $\mathbb{Z}_+$ with birth rates (b_i) and death rates (a_i).*

(1) *Take $r_{ij} = (a_i + b_i)\vee(a_j + b_j)$ $(i \ne j)$. Then $k^{(\alpha)'} > 0$ (equivalently, $k^{(\alpha)} > 0$) iff there exists a constant $c > 0$ such that*

$$\frac{\pi_i a_i}{[(a_i+b_i)\vee(a_{i-1}+b_{i-1})]^\alpha} \ge c\sum_{j\ge i}\pi_j, \qquad i \ge 1. \tag{4.1}$$

Then, we indeed have $k^{(\alpha)'} \ge c$. Furthermore

$$k^{(\alpha)} \ge \inf_{i\ge 1}\frac{\pi_i a_i}{[(a_i+b_i)\vee(a_{i-1}+b_{i-1})]^\alpha (1-\pi_i)\sum_{j\ge i}\pi_j}.$$

(2) *Let $\sum_i \pi_i(a_i+b_i) < \infty$. Take $p_i = a_i + b_i$. Then we have $k_p' > 0$ (equivalently, $k_p > 0$) iff $\inf_{i\ge 1}\dfrac{\pi_i a_i}{\sum_{j\ge i}\pi_j p_j} > 0$ and moreover*

$$k_p' \ge \inf_{i\ge 1}\frac{\pi_i a_i}{\sum_{j\ge i}\pi_j p_j}, \qquad k_p \ge \inf_{i\ge 1}\frac{\pi_i a_i}{(1-\pi_i p_i/\beta_p)\sum_{j\ge i}\pi_j p_j}.$$

Roughly speaking, (4.1) holds if π_j has exponential decay. For polynomial decay, (4.1) can still be true when $\alpha = 1/2$. See Example 4.5.

Proof of Theorem 4.1. Here we prove part (1) only since the proof of part (2) is similar.

a) Let $k^{(\alpha)} > 0$. Take $A = I_i = \{i, i+1, \cdots\}$ for a fixed $i > 0$ and

$$J^{(\alpha)}(i,j) = \frac{\pi_i q_{ij}}{[q_i \vee q_j]^\alpha} = \begin{cases} \dfrac{\pi_i a_i}{[(a_i+b_i)\vee(a_{i-1}+b_{i-1})]^\alpha} =: \pi_i \tilde{a}_i, & \text{if } j = i-1 \\ \dfrac{\pi_i a_i}{[(a_i+b_i)\vee(a_{i+1}+b_{i+1})]^\alpha} =: \pi_i \tilde{b}_i, & \text{if } j = i+1. \end{cases}$$

Then

$$k^{(\alpha)\prime} \le k^{(\alpha)} \le \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\pi(A^c)} = \frac{\pi_i \tilde{a}_i}{\left(\sum_{j\ge i}\pi_j\right)\left(\sum_{j<i}\pi_j\right)} \le \frac{\pi_i \tilde{a}_i}{\pi_0 \sum_{j\ge i}\pi_j}.$$

This proves the necessity of the condition.

b) Next, assume that the condition holds. Then for each A with $\pi(A) \in (0,1)$, since the symmetry of $J^{(\alpha)}$, we may assume that $0 \notin A$. Set $i_0 = \min A \ge 1$. Then, $A \subset I_{i_0}$, $A^c \subset E \setminus \{i_0\}$ and so

$$\frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)} \ge \frac{\pi_{i_0}\tilde{a}_{i_0}}{\sum_{j\ge i_0}\pi_j} \ge c, \qquad \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\pi(A^c)} \ge \frac{\pi_{i_0}\tilde{a}_{i_0}}{(1-\pi_{i_0})\sum_{j\ge i_0}\pi_j}.$$

Because A is arbitrary, we obtain the required assertions. □

Theorem 4.2. *Let $E = \mathbb{Z}_+$. Suppose that (q_{ij}) has finite range R, i.e., $q_{ij} = 0$ whenever $|i-j| > R$. Then, we have $\lambda_1 > 0$ provided*

$$\varlimsup_{i\to\infty} \sum_j \frac{q_{ij}}{\sqrt{q_i \vee q_j}}(j-i) < 0.$$

Proof. Simply take $\varphi_i = i+1$ and $B = \{0, 1, \cdots, n\}$ for large n in Theorem 1.3 and then apply Theorem 1.2. □

Similarly, we have the following result.

Theorem 4.3. *Let $E = \mathbb{Z}^d$. Suppose that (q_{ij}) has finite range R. Then, we have $\lambda_1 > 0$ provided*

$$\varlimsup_{|i|\to\infty} \sum_j \frac{q_{ij}}{\sqrt{q_i \vee q_j}} \sum_{k=1}^d \big[|j_k|\vee R - |i_k|\vee R\big] < 0.$$

Proof. Take $\varphi_i = \sum_{k=1}^d |i_k| \vee R + 1$ in Theorem 1.3 and then apply Theorem 1.2. □

Theorem 4.4. *Let $E = \mathbb{Z}^d$. If there exists a positive function φ such that*

$$\varlimsup_{|i|\to\infty} \frac{1}{\varphi}\Omega\varphi < 0,$$

then $\lambda_1 > 0$.

Proof. Apply Theorem 1.2, Theorem 3.2 and then Theorem 1.4 to the finite sets $\{i : |i| \le n\}$. □

The following example, taken from [3], is especially rare and interesting since it exhibits the critical phenomena for the existence of spectral gap. It is now used to justify the power of our results and we should see soon what will happen. Similar example for diffusion was given in [4] and [9].

Example 4.5. *Let $E=\mathbb{Z}_+$ and $a_i=b_i=i^\gamma$ $(i\ge 1)$ for some $\gamma>0$, $a_0=0$ and $b_0=1$. Then $\lambda_1>0$ iff $\gamma\ge 2$.*

Proof. a) By part (1) of Theorem 4.1, we have $k^{(1/2)}>0$ iff $\gamma\ge 2$. Thus, by Theorem 1.2, we have $\lambda_1>0$ for all $\gamma\ge 2$.

b) Applying Theorem 1.5 to $\varphi_i=1+i^{1-\gamma/2}$, it follows that $\lambda_1=0$ for all $\gamma\in(1,2)$.

c) The conditions of Theorem 4.2 hold whenever $\gamma\ge 2$. Hence $\lambda_1>0$ for all $\gamma\ge 2$.

d) Next, taking $\varphi_i=\sqrt{i}$ $(i\ge 1)$, we see that $\Omega\varphi(i)/\varphi_i=-\frac{1}{2\gamma(\gamma-1)}i^{\gamma-2}+O\big(i^{\gamma-3}\big)$. Then

$$\lim_{i\to\infty}\frac{1}{\varphi_i}\Omega\varphi(i)=\begin{cases}-\infty, & \gamma>2\\ -\frac{1}{4}, & \gamma=2.\end{cases}$$

By Theorem 4.4, we have $\lambda_1>0$ for all $\gamma\ge 2$.

On the other hand, take $f_n(i)=i^{\frac{\gamma-1}{2}}\wedge n^{\frac{\gamma-1}{2}}$ and $A=\{0\}$. Then

$$\begin{aligned}\lambda_0(A^c)&\le \varlimsup_{n\to\infty}\frac{\sum_{i,j\ge 0}\pi_i q_{ij}[f_n(j)-f_n(i)]^2}{2\sum_{i\ge 0}\pi_i f_n(i)^2}=\varlimsup_{n\to\infty}\frac{\sum_{i\ge 0}\pi_i q_{i,i+1}[f_n(i+1)-f_n(i)]^2}{2\sum_{i\ge 0}\pi_i f_n(i)^2}\\ &\le \varlimsup_{n\to\infty}\frac{1+(\gamma-1)^2\sum_{i=1}^n i^{\gamma-3}}{\sum_{i=1}^n i^{-1}}=0,\qquad 1<\gamma<2.\end{aligned}$$

By Theorem 1.4, we get $\lambda_1\le\lambda_0(A^c)/\pi(A)=0$. The case that $\gamma\le 1$ can be ignored since then the chain is not positive recurrent. $\square$

Thus, we have seen that all the results presented in the paper, except Theorem 2.5 which does not work for this example, are qualitatively sharp for this example since every one covers the required region and there is no gap left. Finally, taking $\alpha=0$ in part (1) of Theorem 4.1, we obtain $k\ge\big(\sum_{i=1}^\infty i^{-\gamma}\big)^{-1}>0$ for all $\gamma>1$. In other words, we have $k>0$ but $\lambda_1=0$ for all $\gamma\in(1,2)$. Therefore, the condition "$k>0$" is not but "$k^{(1/2)}>0$" is sufficient for $\lambda_1>0$.

The next two examples show that the two approaches used in the paper for the Cheeger's inequalities may all attain sharp estimates but they are not comparable (remember that Theorem 2.5 is not suitable for Example 4.5). We mention that as far as we know, no optimal estimate provided by the Cheeger's technique ever appeared before.

Example 4.6. *Let $E=\mathbb{Z}_+$ and take $a_i\equiv a>0$ and $b_i\equiv b>0$. Then, both Theorem 1.2 and Theorem 2.5 are sharp.*

Proof. This is a standard example which is often used to justify the power of a method. It is well known that $\lambda_1=\big(\sqrt{a}-\sqrt{b}\big)^2$ (cf. [2; Example 9.22] and [3]).

a) By part (1) of Theorem 4.1, we have

$$k^{(\alpha)'}\ge\inf_{i\ge 1}\frac{\pi_i a_i}{(a+b)^\alpha\sum_{j\ge i}\pi_j}=\frac{a-b}{(a+b)^\alpha}.$$

Then, by Theorem 1.2, we get $\lambda_1\ge\big(\sqrt{a}-\sqrt{b}\big)^2$.

b) Take $p_i\equiv a+b$. Then by part (2) of Theorem 4.1,

$$k_p'\ge\inf_{i\ge 1}\frac{\pi_i a_i}{\sum_{j\ge i}\pi_j p_j}=\frac{a-b}{a+b}.$$

The same estimate as in a) now follows from Theorem 2.5. $\square$

Example 4.7. *Let $E=\mathbb{Z}_+$ and take $q_{0k}=\beta_k>0$ (be careful to distinguish the sequence (β_k) and the constant β_p), $q_{k0}=1/2$ for $k\ge 1$ and $q_{ij}=0$ for all other $i\ne j$. Assume that $q_0=\sum_{k\ge 1}\beta_k<\infty$. Then, Theorem 2.5 is sharp for all q_0 but Theorem 1.2 is sharp only for $q_0\le 1/2$.*

Proof. From $\pi_0 q_{0k}=\pi_k q_{k0}$, it follows that $\pi_k=2\pi_0\beta_k$, $k\ge 1$ and $\pi_0=(1+2q_0)^{-1}$. An interesting point of the example is that the decay of $\sum_{j\ge i}\pi_j$ as $i\to\infty$ can be arbitrary slow, not necessarily exponential. The last condition is necessary for $\lambda_1>0$ for the birth-death processes with rates bounded below (by a positive constant) and above (cf. [2; Corollary 9.19 (4)]).

a) Take $p_i=q_i\vee(1/2)$, then $\alpha_p=1/2$. Without loss of generality, assume that $0\notin A$. Then

$$\begin{aligned}\frac{1}{\beta_p}\cdot\frac{J(A\times A^c)}{\pi_p(A)\wedge\pi_p(A^c)}&=\frac{\sum_{i\in A}\pi_i q_{i0}}{\Big(\sum_{i\in A}2\pi_0\beta_i p_i\Big)\wedge\Big(\pi_0 p_0+\sum_{i\notin A,\,i\ne 0}2\pi_0\beta_i p_i\Big)}\\&=\frac{\sum_{i\in A}\beta_i}{\Big(\sum_{i\in A}2\beta_i p_i\Big)\wedge\Big(p_0+\sum_{i\notin A,\,i\ne 0}2\beta_i p_i\Big)}\\&=\frac{\sum_{i\in A}\beta_i}{\Big(\sum_{i\in A}\beta_i\Big)\wedge\Big(p_0+\sum_{i\notin A,\,i\ne 0}\beta_i\Big)}\ge 1.\end{aligned}$$

This gives us $k_p'\ge 1$ and hence by Theorem 2.5,

$$\lambda_1\ge\alpha_p\Big(1-\sqrt{1-{k_p'}^2}\Big)\ge 1/2.$$

Actually, every equality in the last line must hold.

b) Again, assume that $0\notin A$. Then

$$\begin{aligned}\frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)}&=\frac{\sum_{i\in A}\pi_i q_{i0}(q_i\vee q_0)^{-\alpha}}{\Big(\sum_{i\in A}2\pi_0\beta_i\Big)\wedge\Big(\pi_0+\sum_{i\notin A,\,i\ne 0}2\pi_0\beta_i\Big)}\\&=\frac{1}{2}\cdot\frac{\sum_{i\in A}2\beta_i}{\big(\frac12\vee q_0\big)^{\alpha}\Big[\Big(\sum_{i\in A}2\beta_i\Big)\wedge\Big(1+\sum_{i\notin A,\,i\ne 0}2\beta_i\Big)\Big]}\\&=\frac{1}{2}\cdot\frac{1}{\big(\frac12\vee q_0\big)^{\alpha}}\cdot\frac{\sum_{i\in A}\beta_i}{\Big(\sum_{i\in A}\beta_i\Big)\wedge\Big(1/2+\sum_{i\notin A,\,i\ne 0}\beta_i\Big)}\\&=\frac{1}{2}\cdot\frac{1}{\big(\frac12\vee q_0\big)^{\alpha}}\cdot\frac{1}{1\wedge\Big[\Big(1/2+\sum_{i\notin A,\,i\ne 0}\beta_i\Big)\Big/\sum_{i\in A}\beta_i\Big]}.\end{aligned}$$

Because $\Big(1/2+\sum_{i\notin A,\,i\ne 0}\beta_i\Big)\Big/\sum_{i\in A}\beta_i$ decreases when A increases, by setting $A=\{i\}$ for a large enough $i\ne 0$, it follows that

$$k^{(\alpha)\prime}=\inf_{A:\,0\notin A}\frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)}=\frac{1}{2}\Big(\frac{1}{2}\vee q_0\Big)^{-\alpha}.$$

By Theorem 1.2, we get

$$\lambda_1 \geq \frac{1}{2}\Big\{1 \vee (2q_0) + \sqrt{\big(1 \vee (2q_0)\big)^2 - 1}\,\Big\}^{-1}.$$

Thus, the lower bound is equal to $1/2 = \lambda_1$ iff $q_0 \leq 1/2$. □

The following counterexample shows the limitation of the Cheeger's inequalities. Of course, the example can be easily handled with the help of some comparison technique. However, this suggests us that sometimes it is necessary to examine a model carefully before applying the inequalities.

Example 4.8. *Consider the birth-death process with $a_{2i-1} = (2i-1)^2$, $a_{2i} = (2i)^4$ and $b_i = a_i$ for all $i \geq 1$. Then, we have $k^{(1/2)'} = 0$ and so Theorem 1.2 is not applicable.*

Proof. First, applying Theorem 4.4 to $\varphi_i = \sqrt{i}$ or comparing the chain with the one with rates $a_i = b_i = (2i)^2$, one sees that $\lambda_1 > 0$. Next, because $\mu_i = 1/a_i$ (and hence $\pi_i = \mu_i/Z$, where Z is the normalizing constant), we have $\sum_{j\geq i}\mu_j = O(i^{-1})$. However, $\sqrt{a_i \vee a_{i-1}} = O(i^2)$. Hence $\sup_{i\geq 1}\sqrt{a_i \vee a_{i-1}}\sum_{j\geq i}\mu_j = \infty$. This gives us $k^{(1/2)'} = 0$ by part (1) of Theorem 4.1.

Note that the choice $r_{ij} = q_i \vee q_j$ $(i \neq j)$ is usually not optimal in the sense for which (1.8) often becomes inequality rather than equality. However, the improvement provided by an optimal r_{ij} is still not enough to cover this example and so the problem is really due to the limitation of the technique. □

References

1. Cheeger, J. (1970), *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis, a symposium in honor of S. Bochner 195–199, Princeton U. Press, Princeton.
2. Chen, M. F. (1992), *From Markov Chains to Non-Equilibrium Particle Systems*, Singapore, World Scientific.
3. Chen, M. F. (1996), *Estimation of spectral gap for Markov chains*, Acta Math. Sin. New Ser. 12:4, 337–360.
4. Chen, M. F. and Wang, F. Y. (1997), *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. 349, 1209–1237.
5. Chung, F. R. K. (1997), *Spectral Graph Theory*, CBMS, **92**, AMS, Providence, Rhode Island.
6. Fukushima, M., Oshima, Y. and Takeda, M. (1994), *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter & Co.
7. Lawler, G. F. and Sokal, A. D. (1988), *Bounds on the L^2 spectrum for Markov chain and Markov processes: a generalization of Cheeger's inequality*, Trans. Amer. Math. Soc. **309**, 557–580.
8. Saloff-Coste, L. (1997), *Lectures on Finite Markov Chains*, LNM **1665**, Springer-Verlag.
9. Wang, F. Y. (1997), *Existence of spectral gap for diffusion processes*, preprint.

5. Appendix. Proof of Lemma 2.1, for referee's reference

Because $\alpha \geq 0$ is fixed, we can omit the superscript "(α)" everywhere in the proof. Denote by $\tilde{h}$, $\tilde{k}$ and $\tilde{k}'$ the right-hand sides of the above quantities. By taking $f = I_A$, we obtain $h \geq \tilde{h}$, $k \geq \tilde{k}$ and $k' \geq \tilde{k}'$. We now prove the reverse inequalities.

a) For any $f \geq 0$ with $\pi(f) = 1$, let $A_\gamma = \{f > \gamma\}$, $\gamma \geq 0$. By the symmetry of J, we have

$$\begin{aligned}
&\frac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)|f(y) - f(x)| + K(f) = \int_{\{f(x)>f(y)\}} J(\mathrm{d}x, \mathrm{d}y)[f(x) - f(y)] + K(f)\\
&= \int_0^\infty \mathrm{d}\gamma\Big\{J\big(\{f(x) > \gamma \geq f(y)\}\big) + K\big(\{f > \gamma\}\big)\Big\}\\
&= \int_0^\infty \big[J\big(A_\gamma \times A_\gamma^c\big) + K(A_\gamma)\big]\mathrm{d}\gamma \geq h\int_0^\infty \pi\big(A_\gamma\big)\mathrm{d}\gamma = h\pi(f) = h.
\end{aligned}$$

Hence $\tilde{h} \geq h$.

b) For any $f \in L^1_+(\pi)$ with $\int \pi(\mathrm{d}x)\pi(\mathrm{d}y)|f(x) - f(y)| = 1$, by a), we have

$$\begin{aligned}
&\int J(\mathrm{d}x, \mathrm{d}y)|f(x) - f(y)| = 2\int \mathrm{d}\gamma J(A_\gamma \times A_\gamma^c)\\
&\geq k\int_0^\infty \mathrm{d}\gamma\ (\pi \times \pi)(A_\gamma \times A_\gamma^c) = k\int_0^\infty \pi(\mathrm{d}x)\pi(\mathrm{d}y)|f(x) - f(y)| = k.
\end{aligned}$$

This proves the first equality of $k^{(\alpha)}$.

Next, we show that

$$\int |f - \pi(f)|\mathrm{d}\pi = \sup_{g:\, \pi(g)=0,\, \inf_{c\in\mathbb{R}}\|g-c\|_\infty\leq 1} \int fg\mathrm{d}\pi, \tag{5.1}$$

where $\|\cdot\|_p$ denotes the L^p-norm. First, let $\pi(g) = 0$ with $\inf_{c\in\mathbb{R}}\|g - c\|_\infty \leq 1$. Then, because $\pi(g) = 0$ and $\pi(f - \pi(f)) = 0$, we have

$$\int fg\mathrm{d}\pi = \int (f - \pi(f))g\mathrm{d}\pi = \int (f - \pi(f))(g - c)\mathrm{d}\pi$$

for all $c \in \mathbb{R}$. Hence, by Hölder inequality, we have

$$\left|\int fg\mathrm{d}\pi\right| \leq \|f - \pi(f)\|_1\|g - c\|_\infty$$

for all c. This gives us

$$\left|\int fg\mathrm{d}\pi\right| \leq \|f - \pi(f)\|_1 \inf_c \|g - c\|_\infty \leq \|f - \pi(f)\|_1.$$

On the other hand, for a given $f \in L^1(\pi)$, set $A_f^+ = \{f \geq \pi(f)\}$ and $A_f^- = \{f < \pi(f)\}$. Take $g_0 = I_{A_f^+} - I_{A_f^-} - \pi(A_f^+) + \pi(A_f^-)$. Then, $g_0 \in L^\infty(\pi)$ and $\pi(g_0) = 0$. Finally, take $c_0 = 1 - 2\pi(A_f^+)$. Then, it is easy to check that $\inf_c \|g_0 - c\|_\infty = \|g_0 - c_0\|_\infty = 1$. Therefore, we have $\int fg_0\mathrm{d}\pi = \int |f - \pi(f)|\mathrm{d}\pi$ as required.

We now prove the second equality of $k^{(\alpha)}$. Let $f \ge 0$ and set $A_\gamma = \{f \ge \gamma\}$. Again, by using a) and (5.1), we have

$$\begin{aligned}
&\int J(\mathrm{d}x, \mathrm{d}y)|f(y) - f(x)| \ge 2k \int_0^\infty \mathrm{d}\gamma \pi(A_\gamma)\pi(A_\gamma^c) = k \int_0^\infty \mathrm{d}\gamma \int \big|I_{A_\gamma} - \pi(A_\gamma)\big| \mathrm{d}\pi \\
&= k \int_0^\infty \mathrm{d}\gamma \sup_{g:\, \pi(g)=0,\, \inf_{c\in\mathbb{R}} \|g-c\|_\infty \le 1} \int I_{A_\gamma} g \mathrm{d}\pi \\
&\ge k \sup_{g:\, \pi(g)=0,\, \inf_{c\in\mathbb{R}} \|g-c\|_\infty \le 1} \int_0^\infty \mathrm{d}\gamma \int I_{A_\gamma} g \mathrm{d}\pi \\
&= k \sup_{g:\, \pi(g)=0,\, \inf_{c\in\mathbb{R}} \|g-c\|_\infty \le 1} \int f g \mathrm{d}\pi = k \int |f - \pi(f)| \mathrm{d}\pi.
\end{aligned}$$

Therefore, we obtain $\tilde{k} \ge k$.

c) Choose $c_0 \in \mathbb{R}$ such that $\pi(f < c_0),\ \pi(f > c_0) \le 1/2$. Let $f_\pm = (f - c_0)^\pm$. Then we have $f_+ + f_- = |f - c_0|$ and $\pi(|f - c_0|) = \min_c \pi(|f - c|)$. For any $\gamma \ge 0$, define $A_\gamma^\pm = \{f_\pm > \gamma\}$. We have

$$\begin{aligned}
&\frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y)|f(y) - f(x)| = \frac{1}{2} \int J(\mathrm{d}x, \mathrm{d}y) \big[|f_+(y) - f_+(x)| + |f_-(y) - f_-(x)|\big] \\
&= \int_0^\infty \big[J\big(A_\gamma^+ \times A_\gamma^{+c}\big) + J\big(A_\gamma^- \times A_\gamma^{-c}\big)\big] \mathrm{d}\gamma \\
&\ge k' \int_0^\infty \big[\pi\big(A_\gamma^+\big) + \pi\big(A_\gamma^-\big)\big] \mathrm{d}\gamma = k'\pi(f_+ + f_-) = k'\pi(|f - c_0|) = k' \min_c \pi(|f - c|).
\end{aligned}$$

This implies that $\tilde{k}' \ge k'$. $\square$

Acknowledgement. The second named author would like to thank MSRI for a happy stay.

DEPARTMENT OF MATHEMATICS, BEIJING NORMAL UNIVERSITY, BEIJING 100875, THE PEOPLE'S REPUBLIC OF CHINA.