# EXISTENCE OF THE SPECTRAL GAP FOR ELLIPTIC OPERATORS

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Abstract. Let $M$ be a connected, noncompact, complete Riemannian manifold, consider the operator $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $\exp[V]$ integrable w.r.t. the Riemannian volume element. This paper studies the existence of the spectral gap of $L$. As a consequence of the main result, let $\rho$ be the distance function from a point $o$, then the spectral gap exists provided $\lim_{\rho\to\infty} \sup L\rho < 0$ while the spectral gap does not exist if $o$ is a pole and $\lim_{\rho\to\infty} \inf L\rho \ge 0$. Moreover, the elliptic operators on $\mathbb{R}^d$ are also studied.

## 1. Introduction

Let $\widetilde{M}$ be a $d$-dimensional, connected, noncompact, complete Riemannian manifold, and let $M$ be either $\widetilde{M}$ or an unbounded regular closed domain in $\widetilde{M}$. Next, consider $L = \Delta + \nabla V$ for some $V \in C^\infty(M)$ with $Z := \int_M \exp[V]\mathrm{d}x < \infty$. Let $\mathrm{d}\mu = Z^{-1}\exp[V]\mathrm{d}x$ be defined on $M$. The spectral gap of the operator $L$ (with Neumann boundary condition if $\partial M \neq \emptyset$) is characterized as

$$\lambda_1 = \inf\left\{\frac{\mu(|\nabla f|^2)}{\mu(f^2)-\mu(f)^2} : f \in C^1(M)\cap L^2(\mu), f \neq \text{constant}\right\}. \tag{1.1}$$

We say the spectral gap of $L$ exists if $\lambda_1 > 0$. From now on, we assume that $L$ is regular in the sense that $C_0^\infty(M)$ is dense in $W^{1,2}(M, \mathrm{d}\mu)$ with the Sobolev norm $\|\cdot\|_{L^2(\mu)} + \|\nabla\cdot\|_{L^2(\mu)}$.

According to Wang[12] and Chen-Wang[3], we have $\lambda_1 > 0$ provided the Ricci curvature is bounded below and $\mathrm{Hess}_V$ is uniformly negatively definite out of a compact domain. Actually, the recent work by the author [14] shows that this condition implies the logarithmic Sobolev inequality which is stronger than the existence of spectral gap. Moreover, [13] proved that the logarithmic Sobolev inequality is equivalent to an exponential integrability

1991 *Mathematics Subject Classification.* 35P15, 60H30.

*Key words and phrases.* Spectral gap, elliptic operator, essential spectrum.

Research supported in part by NSFC(19631060), Fok Ying-Tung Educational Foundation and Scientific Research Foundation for Returned Overseas Chinese Scholars. Research at MSRI is supported in part by NSF grant DMS-9701755.

Typeset by $\mathcal{AMS}$-TEX

of the distance function square, which naturally refers to the negativity of $\mathrm{Hess}_V$ along the radial direction.

On the other hand, we know that the spectral gap may exist if the distance function itself is exponential integrable. For instance, let $M = [0, \infty)$ and $L = \frac{\mathrm{d}^2}{\mathrm{d}r^2} - c\frac{\mathrm{d}}{\mathrm{d}r}, c > 0$, then (see [4; Example 2.8]) $\lambda_1 = c^2/4 > 0$. From this we may guess that the existence of spectral gap, unlike the logarithmic Sobolev inequality, essentially depends on the first order radial-direction derivative of $V$ rather than the second order derivative. This observation is now supported by Corollary 1.4 in the paper.

Our study is based on the fact that $\lambda_1 > 0$ is equivalent to $\inf \sigma_{ess}(-L) > 0$, where $\sigma_{ess}(-L)$ denotes the essential spectrum of $-L$ (with Neumann boundary condition if $\partial M \neq \emptyset$). To see this one need only to show that 0 is an eigenvalue with multiplicity 1, equivalently, for any $f \in L^2(\mu)$ with $Lf = 0$ and $\nu f|_{\partial M} = 0$ ( where $\nu$ denotes the inward unit normal vector field of $\partial M$ when it is nonempty), one has $f$ is constant. This is a consequence of a result in Sturm [10].

Next, for fixed $o \in \widetilde{M}$, let $\rho(x)$ be the Riemannian distance function from $o$. For $r > 0$, let $B_r = \{x \in M : \rho(x) < r\}$ and $B_r^c = M \setminus B_r$. by Donnely-Li's decomposition theorem (see [5]), one has $\inf \sigma_{ess}(-L) = \lim_{r\to\infty} \lambda^c(r)$, where

$$\lambda^c(r) = \inf\{\mu(|\nabla f|^2) : f \in C^1(M), \mu(f^2) = 1, f = 0 \text{ on } B_r\}.$$

Thence, $\lambda_1 > 0$ is equivalent to $\lambda^c(r) > 0$ for large $r$. More precisely, we have the following.

**Theorem 1.1.** (1) *If $\mu(B_r) > 0$, then*

$$\lambda_1 \leq \lambda^c(r)/\mu(B_r). \tag{1.2}$$

(2) *Let $\lambda(R)$ be the smallest positive Neumann eigenvalue of $-L$ on $B_R$. If $\lambda^c(r) > 0$, then*

$$\lambda_1 \geq \sup_{R>r} \frac{\lambda^c(r)\lambda(R)\mu(B_R)(R-r)^2 - 2\lambda(R)(1-\mu(B_R))}{2\lambda(R)(R-r)^2 + \lambda^c(r)(R-r)^2\mu(B_R) + 2\mu(B_R)} > 0. \tag{1.3}$$

We now go to estimate the quantity $\lambda^c(r)$. For $D \geq 0$, define

$$\gamma(r) = \sup_{\rho(x)=r, x\notin cut(o)} L\rho(x), \quad C(r) = \int_{D+1}^r \gamma(s)\mathrm{d}s, \ r > D.$$

Here and in what follows, the point $x$ runs over $M$.

**Theorem 1.2.** *Suppose that there exists $D > 0$ such that either $\partial M \subset B_D$ or $\nu\rho \leq 0$ on $\partial M \cap (B_D^c \setminus cut(o))$. For any $r_0 > D$ and positive function $f \in C[r_0, \infty)$, we have*

$$\lambda^c(r_0) \geq \inf_{t\geq r_0} f(t)\bigg\{\int_{r_0}^t \exp[-C(r)]\,dr \int_r^\infty \exp[C(s)]f(s)\,ds\bigg\}^{-1}. \tag{1.4}$$

*Consequently, we have $\lambda_1 > 0$ provided there exists positive $f \in C[D+1,\infty)$ such that*

$$\sup_{t\geq D+1} \frac{1}{f(t)} \int_{D+1}^t \exp[-C(r)]\,dr \int_r^\infty \exp[C(s)]f(s)\,ds < \infty. \tag{1.5}$$

We remark that the assumption in Theorem 1.2 holds if either $\partial M$ is bounded or $o \in M$ and $M$ is convex. Especially, for the case $M = [0,\infty)$ and $o = 0$, we have $C(r) = V(r)$. By [4; Theorem 2.1], if $f' > 0$ then

$$\lambda_1 \geq \inf_{t\geq 0} f'(t)\exp[C(t)]\bigg\{\int_t^\infty \exp[C(s)]f(s)\mathrm{d}s\bigg\}^{-1}. \tag{1.6}$$

Hence, the second assertion in Theorem 1.2 can be regarded as an extension of [4; Theorem 2.1] to Riemannian manifolds.

**Corollary 1.3.** *Under the assumption of Theorem* 1.2. *We have $\lambda_1 > 0$ if*

$$\sup_{t\geq D+1} \exp[-C(t)]\int_t^\infty \exp[C(s)]\,ds < \infty.$$

*Consequently, $\lambda_1 > 0$ provided $\int_{D+1}^\infty (\gamma+\varepsilon)^+(r)\,dr < \infty$ for some $\varepsilon > 0$.*

Now, it is the time to state the result mentioned in the abstract.

**Corollary 1.4.** (1) *Under the assumption of Theorem* 1.2. *We have $\lambda_1 > 0$ provided $\lim_{r\to\infty}\sup_{\rho(x)=r, x\notin cut(o)} L\rho(x) < 0$.*

(2) *Suppose that $\partial M$ is bounded and $o$ is a pole. If $\lim_{r\to\infty}\inf_{\rho(x)=r} L\rho(x) \geq 0$, then $\lambda_1 = 0$.*

*Remark.* 1) The first part of Corollary 1.4 follows from Corollary 1.3 directly. It was pointed out to the author by the referee that this can also be proved by using Cheeger's inequality (c.f. [1]), the argument goes as follows. Define the isoperimetric constant by

$$h_r = \inf \frac{A(\partial\Omega \cap \mathrm{interior}(M))}{\mu(\Omega)},$$

where $\Omega$ runs over all bounded open subset of $B_r^c$ and $A$ denotes the measure on $\partial\Omega$ induced by $\mu$. Then, one has $\lambda^c(r) \geq h_r^2/4$.

Next, let $\nu$ be the inward normal vector field of $\partial\Omega$. Noting that $\nu\rho \leq 0$ on $\partial M \cap \partial\Omega$, we obtain

$$A(\partial\Omega\cap \text{interior}(M)) \geq \int_{\partial\Omega} \nu\rho \mathrm{d}A - \int_{\partial\Omega\cap\partial M} \nu\rho\mathrm{d}A \geq \int_{\partial\Omega}\nu\rho\mathrm{d}A = -\int_{\Omega} L\rho\mathrm{d}\mu,$$

where $L\rho$ is understood in distribution sense in the case that $\mathrm{cut}(o)\neq\emptyset$. Then, under the condition we have $\lim_{r\to\infty}\lambda^c(r) > 0$.

2) The proof of Corollary 1.4 (2) is based on the following upper bound estimate (c.f. [9; Proposition 2.13]):

$$\lambda_1 \leq \frac{1}{4}\sup\{\varepsilon^2 : \mu(\exp[\varepsilon\rho]) < \infty\}. \tag{1.7}$$

This estimate can be proved by taking the test function $f_n = \exp[\varepsilon(\rho\wedge n)/2]$ and then letting $n\to\infty$, refer to the proof of Theorem 3.2 below.

The proofs of the above results are given in the next section, and along the same line, the spectral gap of elliptic operators on $\mathbb{R}^d$ is studied in section 3.

## 2. Proofs

*Proof of Theorem 2.1.* We prove (1.2) and (1.3) respectively.

a) The proof of (1.2) is modified from Thomas [10] which studies the upper bound of the spectral gap for discrete systems.

For any $\varepsilon > 0$, choose $f_\varepsilon\in C^1(M)$ with $f_\varepsilon|_{B_r} = 0$ such that $\mu(f_\varepsilon^2) = 1$ and $\mu(|\nabla f_\varepsilon|^2)\leq \varepsilon+\lambda^c(r)$. Noting that $\mu(f_\varepsilon) = \mu(f_\varepsilon 1_{B_r^c})\leq\sqrt{\mu(B_r^c)}$, we have $\mu(f_\varepsilon^2)-\mu(f_\varepsilon)^2\geq\mu(B_r)$, then

$$\lambda_1 \leq \frac{\mu(|\nabla f_\varepsilon|^2)}{\mu(f_\varepsilon^2)-\mu(f_\varepsilon)^2}\leq\frac{\varepsilon+\lambda^c(r)}{\mu(B_r)}.$$

This proves (1.2) by letting $\varepsilon\downarrow 0$.

b) Next, we go to prove (1.3). It suffices to show that for any $f\in C^1(M)$ with $\mu(f^2) = 1, \mu(f) = 0$ and any $R > r$,

$$\mu(|\nabla f|^2)\geq\frac{\lambda^c(r)\lambda(R)\mu(B_R)(R-r)^2 - 2\lambda(R)(1-\mu(B_R))}{2\lambda(R)(R-r)^2+\lambda^c(r)(R-r)^2\mu(B_R)+2\mu(B_R)}. \tag{2.1}$$

Let $a = \mu(f^2 1_{B_R})$. Noting that $\mu(f) = 0$, we obtain

$$\begin{aligned}\frac{1}{\lambda(R)}\mu_{B_R}(|\nabla f|^2) &\geq \mu_{B_R}(f^2 1_{B_R}) - \mu_{B_R}(f1_{B_R})^2 = \frac{a}{\mu(B_R)} - \frac{\mu(f1_{B_R^c})^2}{\mu(B_R)^2}\\ &\geq \frac{a}{\mu(B_R)} - \frac{(1-a)(1-\mu(B_R))}{\mu(B_R)^2} = \frac{a+\mu(B_R)-1}{\mu(B_R)^2},\end{aligned}$$

where $\mu_{B_R} = \mu/\mu(B_R)$. This implies

$$\mu(|\nabla f|^2) \geq \frac{\lambda(R)}{\mu(B_R)}\big(a + \mu(B_R) - 1\big) =: g_1(a). \tag{2.2}$$

Next, define

$$h(x) = \begin{cases} 0, & \text{if } \rho(x) \leq r, \\ 1, & \text{if } \rho(x) \geq R, \\ \frac{\rho(x)-r}{R-r}, & \text{otherwise.} \end{cases}$$

Then $fh = 0$ on $\{x \in M : \rho(x) = r\}$. By the definition of $\lambda^c(r)$,

$$1 - a \leq \mu(f^2h^2) \leq \frac{\mu(|\nabla(fh)|^2)}{\lambda^c(r)} \leq \frac{2}{\lambda^c(r)}\Big(\mu(|\nabla f|^2) + \frac{a}{(R-r)^2}\Big).$$

Therefore,

$$\mu(|\nabla f|^2) \geq \frac{\lambda^c(r)}{2} - \Big(\frac{\lambda^c(r)}{2} + \frac{1}{(R-r)^2}\Big)a =: g_2(a). \tag{2.3}$$

By combining (2.2) with (2.3) we obtain

$$\mu(|\nabla f|^2) \geq \inf_{\varepsilon \in [0,1]} \max\{g_1(\varepsilon), g_2(\varepsilon)\}. \tag{2.4}$$

Since $g_1(\varepsilon)$ is increasing in $\varepsilon$ while $g_2(\varepsilon)$ is decreasing in $\varepsilon$, the above infimum is attained at

$$\varepsilon_0 = \frac{\lambda^c(r)/2 + \lambda(R)(1-\mu(B_R))/\mu(B_R)}{\lambda(R)/\mu(B_R) + \lambda^c(r)/2 + 1/(R-r)^2}$$

which solves $g_1(\varepsilon) = g_2(\varepsilon)$. Then $\mu(|\nabla f|) \geq g_1(\varepsilon_0) = g_2(\varepsilon_0)$ which is equal to the right-hand side of (2.1). $\square$

*Proof of Theorem 1.2.* For any $m > r_0$, let $\Omega_m = B_m \setminus \bar{B}_{r_0}$. Since $L$ is regular,

$$\lambda^c(r_0) = \lim_{m\to\infty} \lambda_0(\Omega_m), \tag{2.5}$$

where $\lambda_0(\Omega_m)$ denotes the smallest eigenvalue of $-L$ on $\Omega_m$ with Neumann condition on interior$(\partial M \cap \partial\Omega_m)$ and Dirichlet condition on the remainder of $\partial\Omega_m$. Let $u(>0)$ be the corresponding eigenfunction. Define

$$F(t) = \int_{r_0}^t \exp[-C(r)]\mathrm{d}r \int_r^m \exp[C(s)]f(s)\mathrm{d}s, \quad t \in [r_0, m].$$

We claim that there exists $c(m) > 0$ such that $u(x) \le c(m)F(\rho(x))$ on $\Omega_m$. Actually, since $|\nabla u|$ is bounded on $\Omega_m$, it suffices to show that $u = 0$ on $S := \{x \in \partial\Omega_m : \rho(x) = r_0\}$. If there exists $x_0 \in S$ such that $u(x_0) > 0$, then $x_0 \in \text{interior}(\partial M \cap S)$ by the boundary conditions. This means that $\nu(x_0) = \nabla\rho(x_0)$ which contradicts the assumption that $\nu\rho(x_0) \le 0$.

Next, let $c := \inf_{t \ge r_0} f(t)/F(t)$, and let $x_t$ be the $L$-diffusion process with reflecting boundary on $\partial B_m$. By the assumption and the Itô's formula for $\rho(x_t)$ (see [7]), we have, before the time $\tau := \{t \ge 0 : \rho(x_t) = r_0\}$,

$$\mathrm{d}\rho(x_t) = \sqrt{2}\mathrm{d}b_t + L\rho(x_t)\mathrm{d}t - \mathrm{d}L_t, \quad x_0 \in \Omega_m, \tag{2.6}$$

where $b_t$ is an one-dimensional Brownian motion, $L\rho$ is taken to be zero on $\text{cut}(o)$ and $L_t$ is an increasing process with support contained in $\{t \ge 0 : x_t \in \text{cut}(o) \cup \partial B_m\}$. Noting that $L\rho(x_t) \le \gamma(\rho(x_t))$ for $x_t \notin \text{cut}(o)$, by (2.6) and Itô's formula we obtain

$$\begin{aligned} \mathrm{d}F \circ \rho(x_t) &\le \sqrt{2}F' \circ \rho(x_t)\mathrm{d}b_t - f(x_t)\mathrm{d}t \\ &\le \sqrt{2}F' \circ \rho(x_t)\mathrm{d}b_t - cF \circ \rho(x_t)\mathrm{d}t. \end{aligned}$$

This then implies

$$E^x F \circ \rho(x_{t\wedge\tau}) \le F \circ \rho(x) \exp[-ct].$$

Let $\tau' = \inf\{t \ge 0 : x_t \in \overline{\partial\Omega_m \setminus \partial M}\}$, we have $\tau' \le \tau$ and $u(x_{t\wedge\tau'}) \le u(x_t \wedge \tau)$. Noting that $E^x u(x_{t\wedge\tau'}) = u(x)\exp[-\lambda_0(\Omega_m)t]$, we obtain

$$u(x)\exp[-\lambda_0(\Omega_m)t] \le c(m)E^x F \circ \rho(x_t \wedge \tau) \le c(m)F \circ \rho(x)\exp[-ct].$$

This implies $\lambda_0(m) \ge c$ for any $m > r_0$. Therefore, $\lambda^c(r_0) \ge c$. $\square$

It was pointed out by the referee that there is an equivalent analysis proof of Theorem 1.2 (refer to [6; Lemma 1.1]). Let $F$ and $u$ be as in above with $\int_{\Omega_m} u^2 = 1$, then $LF\circ\rho \le -cF\circ\rho$ on $\Omega_m$ in the distribution sense. Let $f = u/F$, then $f$ is bounded as was shown in the proof of Theorem 1.2. We have $uf\nu F \le 0$ on $\partial\Omega_m$ since $\nu\rho \le 0$ on $\partial M$ and $u = 0$ on $\partial\Omega_m \setminus \partial M$. Therefore, by Green's formula, we obtain

$$\begin{aligned}\lambda_0(\Omega_m) &= -\int_{\Omega_m} uLu\mathrm{d}\mu = -\int_{\Omega_m} uL(fF)\mathrm{d}\mu\\ &= -\int_{\Omega_m} [ufLF + uFLf + 2u\langle\nabla f, \nabla F\rangle]\mathrm{d}\mu\\ &\geq c\int_{\Omega_m} [ufF + \langle\nabla(fF^2), \nabla f\rangle - 2u\langle\nabla f, \nabla F\rangle]\mathrm{d}\mu + \int_{\partial\Omega_m} [uF\nu f]\mathrm{d}A\\ &= c + \int_{\Omega_m} |\nabla f|^2 F\mathrm{d}\mu + \int_{\partial\Omega_m} [u\nu u - uf\nu F]\mathrm{d}A \geq c.\end{aligned}$$

Here, we have used the fact that $u\nu u = 0$ by the mixed boundary condition.

*Proof of Corollary 1.3.* The prove of the first assertion is essentially due to [4]. Under the condition we have

$$\int_t^\infty \exp[C(s)]\mathrm{d}s \leq c\exp[C(t)], \quad t \geq D+1$$

for some constant $c > 0$. This implies (see [4; Lemma 6.1])

$$\int_t^\infty \exp[\varepsilon s + C(s)]\mathrm{d}s \leq \frac{c}{1-c\varepsilon}\exp[C(t)+\varepsilon t], \quad \varepsilon \in (0, c^{-1}).$$

By taking $f(r) = \exp[r/(2c)]$ in (1.4), we prove the first assertion.

Next, if there exists $\varepsilon > 0$ such that $c_1 := \int_{D+1}^\infty (\gamma+\varepsilon)^+\mathrm{d}s < \infty$. Let $C_\varepsilon(r) = C(r) - \int_{D+1}^r (\gamma+\varepsilon)^+(s)\mathrm{d}s$. Then $C_\varepsilon'(r) = \gamma(r) - (\gamma+\varepsilon)^+(r) \leq -\varepsilon$. Therefore,

$$\begin{aligned}\exp[-C(t)]\int_t^\infty \exp[C(s)]\mathrm{d}s &\leq \exp[-C_\varepsilon(t)]\int_t^\infty \exp[C_\varepsilon(r) + c_1]\mathrm{d}r\\ &\leq \exp[-C_\varepsilon(t)]\int_t^\infty \exp[C_\varepsilon(t) - \varepsilon(r-t) + c_1]\mathrm{d}r\\ &= \exp[c_1]/\varepsilon < \infty.\end{aligned}$$

Hence $\lambda_1 > 0$ by the first assertion. $\square$

*Remark.* ¿From (1.3) we may derive explicit lower bounds of $\lambda_1$. For instance, assume that $B_R$ is convex for any $R$, let $K \geq 0$ be such that Ric$-$Hess$_V \geq -K$. We have[3]

$$\lambda(R) \geq \frac{\pi^2}{8}K\Big\{\exp\big[KR^2/2\big] - 1\Big\}^{-1}. \tag{2.7}$$

Next, if $\lim_{r\to\infty}\sup\gamma(r) < 0$, let $\beta(r) = \inf_{s\geq r}(-\gamma(s))^+$, by taking $f(t) = \exp[\beta(r)t/2]$ in (1.4), we obtain

$$\begin{aligned}\lambda^c(r) &\geq \inf_{t\geq r}\exp\left[\frac{\beta(r)t}{2}\right]\left\{\int_r^t \mathrm{d}s\int_s^\infty \exp\left[\frac{\beta(r)u}{2}+\int_s^u\gamma(v)\mathrm{d}v\right]\mathrm{d}u\right\}^{-1}\\ &\geq \inf_{t\geq r}\exp\left[\frac{\beta(r)t}{2}\right]\left\{\int_r^t \mathrm{d}s\int_s^\infty \exp\left[\frac{\beta(r)u}{2}-\beta(r)(u-s)\right]\mathrm{d}u\right\}^{-1}\\ &= \frac{\beta(r)^2}{4}.\end{aligned}\tag{2.8}$$

Here, in the second step, we have assumed that $\beta(r)>0$ so that $\gamma(v)\leq-\beta(r)$ for $v\geq r$. Then the estimate $\lambda^c(r)\geq\beta(r)^2/4$ is true for any $r$ since $\beta(r)$ is nonnegative. The explicit lower bound of $\lambda_1$ then follows from (1.3), (2.7) and (2.8).

*Proof of Corollary 1.4 (2).* Suppose that $\partial M\subset B_D$. Under the polar coordinate at $o$, we have $x=(r,\xi)$ for $r=\rho(x)$ and $\xi\in\mathbb{S}^{d-1}$, the $(d-1)$-dimensional unit sphere which is considered as the bundle of unit tangent vectors at $o$. Under this coordinate, the Riemannian volume element can be written as $\mathrm{d}x=g(r,\xi)\mathrm{d}r\mathrm{d}\xi$ and $\Delta\rho=\frac{\partial}{\partial r}(\log g(r,\xi))|_{r=\rho}$. Suppose that $\lim_{r\to\infty}\inf_{\rho(x)=r}L\rho\geq 0$. Then, for any $\varepsilon>0$ there exists $r_1>D$ such that

$$\frac{\partial}{\partial r}\log g(r,\xi)\geq-\frac{\varepsilon}{2}-\frac{\partial}{\partial r}V(r,\xi),\quad r\geq r_1.$$

This implies

$$\begin{aligned}g(r,\xi)&\geq g(r_1,\xi)\exp\left[-\frac{\varepsilon}{2}(r-r_1)-V(r,\xi)+V(r_1,\xi)\right]\\ &\geq c\exp\left[-\frac{\varepsilon}{2}\rho-V\right],\quad r\geq r_1\end{aligned}$$

for some constant $c>0$. Therefore

$$\begin{aligned}\mu(\exp[\varepsilon\rho])&\geq\int_{[D,\infty)\times\mathbb{S}^{d-1}}\exp[\varepsilon r+V(r,\xi)]g(r,\xi)\mathrm{d}r\mathrm{d}\xi\\ &\geq c\int_{[r_1,\infty)\times\mathbb{S}^{d-1}}\exp\left[\frac{\varepsilon}{2}r\right]\mathrm{d}r\mathrm{d}\xi=\infty.\end{aligned}$$

By (1.7), we have $\lambda_1=0$. $\square$

*Remark.* 1) According to the above proof, the function $\rho$ in Corollary 1.4 (2) can be replaced by the distance from any bounded regular domain such that the outward-pointing normal exponential map on the boundary induces a diffeomorphism. See e.g. Kumura [8] for some discussions on such manifolds.

2) In general, for any $r > D > 0$, let

$$\Xi_r^D = \{\xi \in \mathbb{S}^{d-1} : \exp[s\xi]|_{[0,r]} \text{ is minimal and } \exp[s\xi] \in M, s \in [D, r]\}.$$

Then $\Xi_r^D$ is nonincreasing in $r$. Let $\nu = \mathrm{d}\xi$ be the standard measure on $\mathbb{S}^{d-1}$, the assumption of Corollary 1.4 (2) can be replaced by: there exists $D > 0$ such that

$$\lim_{r\to\infty} \nu(\Xi_r^D)\exp[\varepsilon r] = \infty \qquad \text{for any } \varepsilon > 0. \tag{2.9}$$

## 3. Spectral gap for elliptic operators on $\mathbb{R}^d$

This section is a continuation of [2] and [4] in which the lower bound estimates are studied for the spectral gap of elliptic operators on $\mathbb{R}^d$.

Consider the operator $L = \sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^d b_i(x)\partial_i$, where $\partial_i = \frac{\partial}{\partial x_i}$, $a(x) := (a_{ij}(x))$ is positively definite, $a_{ij} \in C^2(\mathbb{R}^d)$ and $b_i = \sum_{j=1}^d (a_{ij}\partial_j V + \partial_j a_{ij})$ for some $V \in C^2(\mathbb{R}^d)$ with $Z := \int \exp[V]\mathrm{d}x < \infty$. The specific form of $b$ implies that $L$ is symmetric with respect to $\mathrm{d}\mu = Z^{-1}\exp[V]\mathrm{d}x$. In the present setting, the spectral gap of $L$ is described as

$$\lambda_1(a, V) = \inf\{\mu(\langle a\nabla f, \nabla f\rangle) : f \in C^1(\mathbb{R}^d), \mu(f) = 0, \mu(f^2) = 1\}. \tag{4.1}$$

Moreover, we assume that $L$ is regular in the sense that $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d, \mathrm{d}\mu)$ with the Sobolev norm $\|\cdot\|_{L^2(\mu)} + \|\sqrt{\langle a\nabla\cdot, \nabla\cdot\rangle}\|_{L^2(\mu)}$.

Obviously, if $a \geq \alpha I$ for some constant $\alpha > 0$, then $\lambda_1(a, V) \geq \alpha\lambda_1(I, V)$. From this one may transform the present setting to the manifold case. But this comparison only works for the case $a$ is uniformly positively definite, and it will lead to some loss if $a$ is very different from $I$, see e.g. Examples 3.1 and 3.2 below. Hence, it should be worthy to study $L$ directly as in previous sections.

Define

$$\gamma(r) = \sup_{|x|=r} \frac{r(\mathrm{tr}(a(x)) + \langle b(x), x\rangle)}{\langle a(x)x, x\rangle} - \frac{1}{r},$$
$$C(r) = \int_1^r \gamma(s)\mathrm{d}s, \quad \alpha(r) = \inf_{|x|=r} \frac{1}{r^2}\langle a(x)x, x\rangle, \quad r > 0.$$

The main result in this section is the following.

**Theorem 3.1.** *If there exists positive $f \in C[1,\infty)$ such that*

$$\sup_{t\geq 1} \frac{1}{f(t)} \int_1^t \exp[-C(r)]dr \int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}ds < \infty, \tag{3.2}$$

*then* $\lambda_1 > 0$.

*Proof.* For $g \in C^2(\mathbb{R})$, we have

$$\begin{aligned} Lg(|x|) =& \frac{1}{|x|^3}\big(|x|^2\mathrm{tr}(a(x)) + |x|^2\langle b(x), x\rangle - \langle a(x)x, x\rangle\big)g'(|x|) \\ &+ \frac{1}{|x|^2}\langle a(x)x, x\rangle g''(|x|), \quad |x| > 0. \end{aligned}$$

For positive $f \in C[1,\infty)$ with $\int_1^\infty \exp[C(r)]\frac{f(r)}{\alpha(r)}\mathrm{d}r < \infty$, let

$$g(t) = \int_1^t \exp[-C(r)]\mathrm{d}r \int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\mathrm{d}s.$$

Then

$$Lg(|x|) \le -f(|x|), \quad |x| \ge 1.$$

Therefore, the proof of Theorem 1.2 implies that $\lambda^c(1) > 0$ provided (3.2) holds, then $\lambda_1 > 0$. $\square$

*Remark.* Theorem 3.1 remains true for unbounded regular domains with bounded boundary. As for the unbounded boundary case, for the estimation of $\lambda^c(r)$, one has to consider the normal vector field induced by the metric $\langle \partial_i, \partial_j\rangle = (a^{-1})_{ij}$, this will cause difficulty for general $a$.

For the case $M = [0,\infty)$, one has $\gamma = \frac{b}{a}, \alpha = a$. Then, by Theorem 3.1, we have $\lambda_1 > 0$ if there exists positive $f \in C[1,\infty)$ with $f' < 0$ such that

$$\sup_{t>1} \frac{1}{f'(t)} \exp[-C(t)] \int_t^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\mathrm{d}s < \infty. \tag{3.3}$$

This is just the condition in [4; Theorem 2.1]. Therefore, Theorem 3.1 is the exact extension of [4; Theorem 2.1] to high dimensions.

Next, the following examples shows that Theorem 3.1 can be better than comparing $a$ with a constant matrix.

*Example 3.1.* Take $a(x) = (1+|x|^2)^\alpha I, b(x) = 0, \alpha \ge (1+d)/2$. Then $L$ is regular. It is easy to see that $V = -\alpha\log(1+|x|^2)$. Noting that $a \ge I$, by the comparison procedure, we may consider the operator $\bar{L} = \Delta - \nabla V$. But by (1.7) the spectral gap of $\bar{L}$ does not exists since $\mu(\exp[\varepsilon|x|]) = \infty$ for any $\varepsilon > 0$. Hence the comparison procedure does not work for this example.

On the other hand, one has $\alpha(r) = (1+r^2)^\alpha, \gamma(r) = \frac{d-1}{r}, C(r) = r^{d-1}$. Take $f = \sqrt{t}$, we obtain

$$\int_1^t \exp[-C(r)]\mathrm{d}r \int_r^\infty \exp[C(s)]\frac{f(s)}{\alpha(s)}\mathrm{d}s \le 2\int_1^t r^{3/2-2\alpha}\mathrm{d}r \le 4f(t)$$

since $2\alpha \ge 1+d$. By Theorem 3.1 we have $\lambda_1 > 0$.

*Example 3.2.* Take $a(x) = \frac{1}{|x|+1}I, V(x) = -|x|^2$ for $|x| \ge 1$. Then the comparison procedure does not apply. Now we go to check the condition of Theorem 3.1. Obviously, $\alpha(r) = \frac{1}{1+r}, \langle b(x), x\rangle = -\frac{r}{(1+r)^2} - \frac{2r^2}{1+r}$. Then

$$\gamma(r) = \frac{d-1}{r} - \frac{1}{1+r} - 2r, \quad \exp[C(r)] = \frac{c_1 r^{d-1}}{r+1}\exp[-4r^2], \quad r \ge 1$$

for some $c_1 > 0$. Take $f(r) = r^{1-d}\exp[-r+4r^2]$, there exists $c_2 > 0$ such that

$$\frac{1}{f(t)}\int_1^t \exp[-C(r)]\mathrm{d}r \int_r^\infty \exp[C(s)]f(s)\mathrm{d}s \le c_2\frac{1}{f(t)}\int_1^t \exp[-r+4r^2]\frac{1+r}{r^{d-1}}\mathrm{d}r$$

which goes to $c_2/8$ as $t \to \infty$. Therefore, Theorem 3.1 implies that $\lambda_1 > 0$.

Finally, we present an upper bound estimate as (1.7).

**Theorem 3.2.** *Let $\beta(r) = \sup_{|x|=r}\frac{1}{r^2}\langle a(x)x, x\rangle$, we have*

$$\lambda_1 \le \frac{1}{4}\sup\left\{\varepsilon^2 : \mu\Big(\exp\Big[\varepsilon\int_0^{|x|}\frac{1}{\sqrt{\beta(r)}}dr\Big]\Big) < \infty\right\}.$$

*Proof.* Let $h(r) = \int_0^r \frac{1}{\sqrt{\beta(s)}}\mathrm{d}s$. If $\mu(\exp[\varepsilon h(|x|)]) = \infty$, we go to prove $\lambda_1 \le \varepsilon^2/4$. Let $f(x) = \exp[\frac{\varepsilon}{2}(h(|x|)\wedge n)],\ n \ge 1$. By (3.1), we have

$$\lambda_1 \le \frac{\varepsilon^2\mu(f^2)}{4(\mu(f^2)-\mu(f)^2)}. \tag{3.4}$$

Next, for any $m > 1$, choose $r_m > 0$ such that $\mu(\{h(|x|) \ge r_m\}) = 1/m$, we have

$$\mu(1_{\{|x|\ge r_m\}}f^2)^{1/2} \ge \sqrt{m}\mu(1_{\{|x|\ge r_m\}}f) \ge \sqrt{m}\mu(f) - \sqrt{m}\exp[\varepsilon h(r_m)/2].$$

Then

$$\mu(f)^2 \le \Big(\sqrt{\mu(f^2)}/\sqrt{m} + \exp\big[\varepsilon h(r_m)/2\big]\Big)^2. \tag{3.5}$$

Noting that $\mu(f^2) \to \infty$ as $n \to \infty$, by combining (3.4) with (3.5), we obtain

$$\lambda_1 \le \frac{\varepsilon^2}{4(1-1/m)}, \quad m > 1.$$

Therefore, $\lambda_1 \leq \varepsilon^2/4$ since $m$ is arbitrary. $\square$

*Acknowledgement.* The author is grateful to the referee whose suggestions improved the quality of the paper. The revised version was finished during the author's visit to MSRI in Berkeley.

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