HODGE THEORY AND COHOMOLOGY WITH COMPACT SUPPORTS

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ABSTRACT. This paper constructs a Hodge theory of noncompact topologically tame manifolds M. The main result is an isomorphism between the de Rham cohomology with compact supports of M and the kernel of the Hodge–Witten–Bismut Laplacian $\triangle_\mu$ associated to a measure $d\mu$ which has sufficiently rapid growth at infinity on M. This follows from the construction of a space of forms associated to $\triangle_\mu$ which satisfy an "extension by zero" property. The "extension by zero" property is proved for manifolds with cylindrical ends possessing gaussian growth measures.

1. INTRODUCTION

Let M be a real C^∞ manifold without boundary. We say M is *topologically tame* if there exists a compact manifold N with boundary such that M is diffeomorphic to the interior of N. For instance, vector bundles over compact base manifolds are topologically tame. Compact manifolds are trivially topologically tame. A surface of infinite genus is not topologically tame.

We will construct a Hodge theory of topologically tame noncompact manifolds M in this paper. First, we choose a metric g such that (M, g) is complete and the ends of M are metric cylinders. Then we define a particular (Hodge–Witten–Bismut) Laplacian operator $\triangle_\mu$ on certain differential forms on M. The subscript μ denotes an additional choice of measure $d\mu$. The operator $\triangle_\mu$ acts on forms which are square–integrable with respect to the measure $d\mu$.

The essential construct of this paper is that the measure $d\mu$ has certain rapid growth at infinity on M relative to Riemann–Lebesgue measure dx: $d\mu = e^{cr^2}dx$, $c > 0$. In this paper we prove (theorem 15):

Main Theorem. *With these choices, (M^n, g), $d\mu$, the $\triangle_\mu$–harmonic forms represent the de Rham cohomology with compact supports:*

$$\ker \triangle^p_\mu \cong H^p_{c,\text{de R}}(M), \qquad p = 0, \dots, n.$$

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The skeptical reader may observe that any nonzero $\omega \in \ker \triangle_\mu$, as an eigenform of an elliptic operator, *does not* have compact support. Nevertheless, we will exhibit a (not canonical!) linear surjection $j : \ker \triangle_\mu \to H_{c,\text{de R}}(M)$ and a (also not canonical) linear surjection $\Pi : H_{c,\text{de R}}(M) \to \ker \triangle_\mu$, in the situation covered by the main theorem.

In fact, we will consider a condition on more general measures $d\mu$ and on the domain of their associated operators $\triangle_\mu$. This condition is the extension by zero condition "property EZ" defined in the third section, and it is the main technique used in proving theorem 15:

Main Technique. *Choose a "nice" relatively compact open submanifold $M_0 \subset M$ such that M_0 is diffeomorphic to M under $\Psi : M \to M_0$. Show that if $\omega \in C^\infty(\triangle_\mu) = \bigcap_{k\le 1} \mathcal{D}_{(\triangle_\mu)^k}$ then $\left(\Psi^{-1}\right)^* \omega \in \Omega(M_0)$ has smooth extension by zero to M. (*"Property EZ"*) Show that this condition implies the existence of a surjection $j : \ker \triangle_\mu \to H_{\text{c,de R}}(M)$.*

Note that any eigenforms of $\triangle_\mu$ are in $C^\infty(\triangle_\mu)$, and that they decay very rapidly if $d\mu = e^{cr^2} dx$, $c > 0$ as prescribed.

Property EZ can hold only if M is topologically tame.

Theorem 13 is that property EZ implies the existence and surjectivity of j.

The remainder of the introduction defines $\triangle_\mu$ and motivates our choice of $d\mu$. We also recall the existing Hodge theory of $\triangle$, constructed in the usual way from dx, in the case of (M, g) with cylindrical ends. Comparison then shows the role of the densities $e^{\pm r^2}$ from the point of view of the Hodge theory of such (M, g).

Now, the metric g determines an inner product $(\cdot,\cdot)_x$ on the tangent space T_xM by definition, which induces an inner product $(\cdot,\cdot)_x$ on $\wedge T_x^*M$, the space of pointwise values of differential forms. The metric also determines a measure dx, by the coordinate formula $dx = \sqrt{\det g_{ij}}\, dx^1 \dots dx^n$.

Assume M is oriented. It is traditional to define an inner product on the space of smooth (C^∞) compactly–supported differential forms $\Omega_c(M)$ by the formula $\langle \omega, \nu \rangle = \int_M (\omega, \nu)_x \, dx = \int_M \omega \wedge \star\nu$. (See [**Wa**], for instance, for details.) Here $\star$ is the Hodge duality operator defined in terms of $(\cdot,\cdot)_x$ and the orientation on M.

Next we weight the inner product. Restricting ourselves to smooth, positive densities, if $h \in C^\infty(M)$ then define

$$d\mu = e^{2h} dx \qquad \text{and} \qquad \langle \omega, \nu \rangle_\mu = \int_M (\omega, \nu)_x \, d\mu.$$

Let $L^2_\mu\Omega$ be the completion of $\Omega_c(M)$ with respect to $\langle \cdot,\cdot \rangle_\mu$. The exterior derivative d has formal adjoint δ_μ: $\langle d\omega, \nu \rangle_\mu = \langle \omega, \delta_\mu \nu \rangle_\mu$ for $\omega, \nu \in \Omega_c$. In fact, the formula $\delta_\mu = e^{-2h} \delta e^{2h}$ holds, where $\delta = (-1)^{n(p+1)+1} \star d \star$ acting on Ω^p is the formal adjoint of d with respect to the usual inner product.

Let $\triangle_\mu = d\delta_\mu + \delta_\mu d$. It is well–known that

1. Theorem. *The operator $\triangle_\mu$ defined on Ω_c is densely–defined and symmetric with respect to $\langle \cdot,\cdot \rangle_\mu$, and is a second–order elliptic operator which maps Ω^p_c to* *itself. If (M, g) is complete, we can extend $\triangle_\mu$ to a unique nonnegative (self–adjoint) operator in $L^2_\mu\Omega$ ([**G**], [**C**]—see [**Bu**] for the weighted case).*

If M^n is in fact compact, *then $\triangle_\mu$ has compact resolvent (discrete spectrum). The Hodge decomposition of the smooth forms $\Omega^p(M)$, $p = 0, \dots, n$, follows:*

$$\Omega^p(M) = \ker \triangle^p_\mu \oplus \operatorname{im} d^{p-1} \oplus \operatorname{im} \delta^{p+1}_\mu, \qquad (M \ \textit{compact})$$

from which it follows that $H^p_{\mathrm{de\ R}}(M) \cong \ker \triangle^p_\mu$.

The weighting $d\mu = e^{2h}dx$ on L^2 spaces of forms has been considered in connection with the Witten treatment of the Morse inequalities [**Wi**]. Actually, our $\triangle_\mu$ on $L^2_\mu\Omega$ is unitarily–equivalent to the Laplacian usually considered in that context, as explicitly noted in [**Bi**]. There is the following unitary equivalence:

$$\begin{array}{cccc} & L^2_\mu\Omega & \xrightarrow{U} & L^2\Omega \\ (1) & \omega & \mapsto & e^h\omega \\ & d & & d_h = UdU^{-1} = e^h d e^{-h} \\ & \triangle_\mu = d\delta_\mu + \delta_\mu d & & \triangle_h = U\triangle_\mu U^{-1} = \triangle + |dh|^2 + A_h, \end{array}$$

where A_h is a certain zero order symmetric operator on $L^2\Omega(M)$ involving the second derivatives of h.

We want to consider noncompact topologically tame manifolds M. On such M we can explicitly construct metric and measure which yield nice spectral properties.

2. Theorem. *([**Bu**], Theorem 10.2) Let M be a C^∞, orientable, connected, and topologically tame manifold. There exists a metric g and for each $c \neq 0$ a measure $d\mu$ with smooth density with respect to dx such that*

(i) (M, g) is complete and

(ii) $\triangle_\mu$ in $L^2_\mu\Omega$ has compact resolvent (= discrete spectrum).

Under the chosen metric g, the ends of M are cylinders near infinity. The density of the chosen measure $d\mu$ behaves precisely as $\exp(cr^2)$ *near infinity on the ends, for any chosen $c \neq 0$.*

A certain Hodge decomposition is one consequence of (ii). It is sometimes misnamed a *strong* Hodge decomposition. It says:

3. Proposition. *If (M^n, g) is complete and $\triangle_\mu$ has compact resolvent, or merely a spectral gap (i.e. $\sigma(\triangle_\mu) \setminus \{0\} \subseteq [\epsilon, \infty)$ for some $\epsilon > 0$), then*

$$L^2_\mu\Omega^p = \ker \triangle^p_\mu \oplus \operatorname{im} d^{p-1} \oplus \operatorname{im} \delta^{p+1}_\mu, \qquad p = 0, \dots, n.$$

Proof. Classical. Use the spectral theorem. See [**Bu**] for details. □

The conclusion of proposition 3 is *not* strong in the sense that there is *a priori* no relation between $\ker \triangle_\mu$ and either $H_{\mathrm{de\ R}}(M)$ or $H_{c,\mathrm{de\ R}}(M)$ in the noncompact case. On must produce a separate proof of any correspondence. We will do so in the case covered by theorem 2—we will show that if $c > 0$ then $\ker \triangle_\mu \cong H_{c,\mathrm{de\ R}}(M)$ and if $c < 0$ then $\ker \triangle_\mu \cong H_{\mathrm{de\ R}}(M)$.

The widely studied case $d\mu = dx$ provides plenty of examples showing that $\ker \triangle$ and $H_{\mathrm{de\ R}}(M)$ at best stand in a very complicated relation to each other. The surveys [**An**] and [**L**] should serve as adequate illustration.

A positive example in the case $d\mu = dx$ is that if M has the additional structure of a Riemannian cover of a compact quotient manifold M/Γ, then the space $\ker \triangle$ is determined by the topology of M/Γ and by the group Γ. This situation was addressed by Atiyah [**At**]. (It should be clear that in general these noncompact M, with compact quotient M/Γ, need not be topologically tame. Conversely, many topologically tame noncompact M admit no metric under which there exists a group of isometries Γ with compact quotient M/Γ.)

The class of Riemannian manifolds (M, g) we consider in theorems 2 and 15—topologically tame and with cylindrical ends—has been a subject of Hodge–theoretic consideration. Again take $d\mu = dx$. The following theorem of Atiyah, Patodi, and Singer applies:

4. Theorem. *([**APS**], Proposition 4.9) Assume (M, g) is complete, is topologically tame and has cylindrical ends. Let $i : \Omega_c(M) \to \Omega(M)$ be the inclusion map. The space* $\ker \triangle$ *of L^2 harmonic forms is isomorphic to the* image of the induced map *i from $H_{c,\mathrm{de\ R}}(M)$ into $H_{\mathrm{de\ R}}(M)$.*

Although this result establishes a connection between the space of L^2 harmonic forms and the (topological) cohomologies, it shows that $\ker \triangle$ must vanish in circumstances where there is much cohomology to be represented. For instance, $\ker \triangle^p$ vanishes for all $0 \leq p \leq n$ if M^n is not compact and either $H^p_{\mathrm{de\ R}}(M) = 0$ for $1 \leq p \leq n/2$ or $H^p_{\mathrm{de\ R}}(M) = 0$ for $n/2 \leq p \leq n-1$. (Recall that $\ker \triangle^p \cong \ker \triangle^{n-p}$ for any (M, g) and that $H^n_{\mathrm{de\ R}}(M) = 0$ if M is not compact.)

Theorem 4 and the paper [**APS**] have motivated a family of results on manifolds with cylindrical ends. R. Melrose and coworkers (specifically, [**M**]) address the situation of a compact manifold with boundary, and a certain prescribed metric near the boundary which they call an "exact b–metric". Section 6.4 of [**M**] includes a Hodge theorem which may be equivalent to our theorem 15 after mapping the manifold with boundary and exact b–metric to a noncompact manifold with cylindrical ends. However, the technique given in section 3 of this paper applies to manifolds with noncylindrical ends, and furthermore directly addresses the complete noncompact manifold itself. For instance, the "boundary condition" here, that makes $\triangle_\mu$ self–adjoint, is exactly the condition that (M, g) is complete.

Now, we are motivated to consider measures with particular decay rates by the following example, based on well–known spectral properties of the quantum harmonic oscillator.

5. Example. Let $d\mu^+ = e^{+|x|^2}dx$, $d\mu^- = e^{-|x|^2}dx$ on Euclidean space $\mathbf{R}^n$. Then

$\triangle_{\mu^+}$ and $\triangle_\mu^-$ have compact resolvent and

$$\ker \triangle^p_{\mu^-} \cong \begin{cases} \mathbf{R}, & p = 0 \\ 0, & p > 0, \end{cases} \qquad \ker \triangle^p_{\mu^+} \cong \begin{cases} 0, & p < n \\ \mathbf{R}, & p = n. \end{cases}$$

Note that

$$H^p_{\text{de R}}(\mathbf{R}^n) \cong \begin{cases} \mathbf{R}, & p = 0 \\ 0, & p > 0, \end{cases} \qquad H^p_{c,\text{de R}}(\mathbf{R}^n) \cong \begin{cases} 0, & p < n \\ \mathbf{R}, & p = n. \end{cases}$$

In the example above, it is clear that the cases $d\mu^+ = e^{c|x|^2}dx$ for $c > 0$ and $d\mu^- = e^{c|x|^2}dx$ for $c < 0$ on $\mathbf{R}^n$ differ fundamentally, but we can easily show their close relation as well. In general, if $d\mu^+ = e^{+2h}dx$ and $d\mu^- = e^{-2h}dx$ on M, then from (1)

$$\triangle_{+h} = U_+ \triangle_{\mu^+} U_+^{-1} = \triangle + |dh|^2 + A_h,$$
$$\triangle_{-h} = U_- \triangle_{\mu^-} U_-^{-1} = \triangle + |dh|^2 - A_h,$$

are the unitarily equivalent versions of $\triangle_{\mu^+}$, $\triangle_{\mu^-}$ respectively, both acting on (unweighted) $L^2\Omega$. Thus the difference between $\triangle_{\mu^+}$ and $\triangle_{\mu^-}$ can be thought of as determined only by the sign of the potential term A_h (an endomorphism on $\wedge T^*M$), once we push $\triangle_{\mu^+}$, $\triangle_{\mu^-}$ back into the same space $L^2\Omega$. On the other hand, note that $L^2_{\mu^+}\Omega$ and $L^2_{\mu^-}\Omega$ contain forms of (in general) very different growth/decay at infinity.

In the rest of this paper, we will drop "de R" from our notation, but it is appropriate to note that our results relating analytic quantities to topological quantities are "modulo de Rham's theorem". Recall that de Rham's theorem identifies the de Rham space $\ker d/\operatorname{im} d$ defined in the C^∞ category with the real cohomology space $H(M;\mathbf{R})$ defined in the continuous category.

Now, if $\ker \triangle^p_\mu$ represents $H^p(M)$ for all $0 \le p \le n$ for a given noncompact M, then it will not also represent $H^p_c(M)$. However, as suggested by example 4, measures with inverse densities give a desirable relationship.

6. Proposition. Poincaré duality. *If M^n is orientable, then*

$$H^p(M) \cong \left(H_c^{n-p}(M)\right)^*.$$

Weighted Poincaré duality. *If (M^n, g) is orientable, and $d\mu^+ = e^{+2h}dx$, $d\mu^- = e^{-2h}dx$, then*

$$\ker \triangle^p_{\mu^-} \cong \ker \triangle^{n-p}_{\mu^+}.$$

Proof. For Poincaré duality, see [**BT**]. For this weighted version, see [**Bu**]. □

It follows that if we find a measure $d\mu = e^{2h}dx$ for which $\ker \triangle^p_\mu \cong H^p_c(M)$, then defining $d\mu^- = e^{-2h}dx$, we get (for free!) $\ker \triangle^p_{\mu^-} \cong H^p(M)$. Theorem 15 shows that this *is* the situation that holds for a topologically tame manifold, having chosen g and $d\mu$ for $c > 0$ and $c < 0$, respectively, as in theorem 2.

The methods of this paper are closely related to those in [**P**], where, in particular, the isomorphism between the L^2 cohomology of a vector bundle E, corresponding to a density $e^{c|x|^2}$ for $c > 0$, and the compact cohomology $H_c(E)$ was proven, and then used in a proof of the Morse–Bott inequalities.

However, the particular form of definitions 7, 9 and 11 and of theorem 13 are motivated by the following situation. If (M, g) is complete and has Ricci curvature bounded below, then the usual Laplace–Beltrami operator $-\frac{1}{2}\triangle$ on M has a unique heat kernel ρ. On Euclidean space $\mathbf{R}^n$, $\rho_t(x, y) = (2\pi t)^{-n/2}\exp(-|x-y|^2/2t)$. On M with Ricci bounded below, if $t > 0$ and $x_0 \in M$ then $d\mu^- \equiv \rho_t(x_0, x)dx$ is a probability measure on M. It is conjectured in [**Bu**] that $\ker\triangle_{\mu^-} \cong H^p(M)$ for this heat kernel measure $d\mu^-$, and certain evidence is given there. It is hoped that the methods of this paper will apply to $d\mu^+ \equiv (\rho_t(x_0, x))^{-1}\,dx$, in which case the conjecture would hold true. The conjecture implies an existence conjecture: if $b_p > 0$ then $\triangle^p_{\mu^-}$ has nonvanishing kernel.

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2. Forms of Very Rapid Decay C^∞_μ

7. Definition. *Assume (M, g) is complete, so $\triangle_\mu$ is self–adjoint. Define*

$$C^\infty_\mu \equiv C^\infty(\triangle_\mu) = \bigcap_{k\geq 1} \mathcal{D}_{(\triangle_\mu)^k}.$$

Clearly $\Omega_c(M) \subseteq C^\infty_\mu$, so C^∞_μ is never empty. If $\triangle_\mu$ has eigenforms, they too are in C^∞_μ: $\triangle_\mu\omega = \lambda\omega$ clearly implies $\omega \in \bigcap_k \mathcal{D}_{(\triangle_\mu)^k}$.

The ellipticity of $\triangle_\mu$ implies that $C^\infty_\mu \subseteq \Omega(M)$, i.e. that C^∞_μ forms are smooth. If M is compact, then $C^\infty_\mu = \Omega(M)$. If M is not compact, then C^∞_μ includes constraints on growth at infinity, and in fact $\Omega_c(M) \subset C^\infty_\mu \subset \Omega(M)$ are both proper inclusions.

It is appropriate to think of C^∞_μ as some kind of generalization to forms of the Schwartz space of functions, with the growth bounds determined by $d\mu$. More accurately, $e^h C^\infty_\mu = C^\infty(\triangle_h)$ generalizes Schwartz space—see section V.

Growth constraints are included in definition 7. Consider the example:

8. Example. Let $M = \mathbf{R}^n$ with the Euclidean metric, and let $d\mu = e^{+x^2}\,dx$. The unitarily equivalent operator $\triangle_h$ on $L^2\Omega$, corresponding to $\triangle_\mu$ on $L^2_\mu\Omega$, in this case is the harmonic oscillator Hamiltonian $\triangle^p_h = \triangle^p + |x|^2 + n - 2p$ (on $L^2\Omega^p$).

Considering only functions, it is well known that $C^\infty(\triangle^0_h)$ corresponds to the Schwarz space $\mathcal{S}(\mathbf{R}^n)$. In fact, the family of seminorms $\{\|f\|_k = \|(\triangle^0_h)^k f\|_{L^2}\}$ is equivalent to the usual family of seminorms $\{\|f\|_{\alpha,\beta} = \|x^\alpha D^\beta f\|_{L^2}\}$ ([**RS**], Appendix to V.3). If follows that $f \in C^\infty(\triangle^0_h)$ iff $\|f\|_k < \infty$ for all k iff $f \in \mathcal{S}(\mathbf{R}^n)$.

Therefore $g \in C_\mu^\infty$ iff $g = e^{-|x|^2/2} f$ for $f \in \mathcal{S}(\mathbf{R}^n)$. This shows that functions in C_μ^∞ have "very rapid" decay.

9. Definition. *H_μ is the cohomology of (C_μ^∞, d):*

$$H_\mu^p \equiv \frac{\ker d|_{C_\mu^{\infty,p}}}{\operatorname{im} d|_{C_\mu^{\infty,p-1}}}, \qquad p = 0, \ldots, n.$$

The closure of $\operatorname{im} d|_{C_\mu^\infty}$ is *not* taken in our definition of H_μ. Therefore $H_\mu \cong \ker \triangle_\mu$ if $\triangle_\mu$ has a spectral gap:

10. Corollary. *The Hodge decomposition of proposition 3 can be smoothed: If $\triangle_\mu$ has a spectral gap, then*

$$C_\mu^\infty = \ker \triangle_\mu \oplus \operatorname{im} d|_{C_\mu^\infty} \oplus \operatorname{im} \delta_\mu|_{C_\mu^\infty}$$

so $\ker \triangle_\mu \cong H_\mu$.

Proof. First, clearly $\ker \triangle_\mu \subset C_\mu^\infty$. Thus from proposition 3,

$$C_\mu^\infty = \ker \triangle_\mu \oplus C_\mu^\infty \cap (\operatorname{im} d \oplus \operatorname{im} \delta_\mu) .$$

Suppose $\omega \in C_\mu^\infty$, $\omega = d\phi + \delta_\mu \psi$, with ϕ,ψ not necessarily smooth. Recall that if $\nu \in \ker \triangle_\mu$ then $d\nu = \delta_\mu \nu = 0$. We have

$$\langle \nu, \omega \rangle_\mu = \langle \nu, d\phi \rangle_\mu + \langle \nu, \delta_\mu \psi \rangle_\mu = \langle \delta_\mu \nu, \phi \rangle_\mu + \langle d\nu, \psi \rangle_\mu = 0$$

for all $\nu \in \ker \triangle_\mu$. So $\omega \in (\ker \triangle_\mu)^\perp$. Since $\triangle_\mu$ has a spectral gap, $(\triangle_\mu)^{-1}$ is a bounded operator on $(\ker \triangle_\mu)^\perp$. Thus $\omega = \triangle_\mu \triangle_\mu^{-1} \omega = d\left(\delta_\mu \triangle_\mu^{-1} \omega\right) + \delta_\mu \left(d \triangle_\mu^{-1} \omega\right)$. That is, we can take $\phi = \delta_\mu \triangle_\mu^{-1} \omega$, $\psi = d \triangle_\mu^{-1} \omega$ in C_μ^∞. $\square$

3. Extension–by–Zero and Compactly–Supported Cohomology

We start with an obvious construction on a topologically tame manifold.

Suppose M is topologically tame and is equipped with a metric g making (M, g) a complete metric space. Then we can think of M as including a finite diameter part carrying all the interesting topology, and a finite list of infinite–length ends which are topological products. In fact, let $\Phi : \operatorname{int} N \to M$ be a diffeomorphism. Let $\hat{U} \subset N$ be any open neighborhood of ∂N which is small enough so that $\hat{U} \cong \partial N \times (0, \infty]$. The closure of $\hat{U}$ in N is compact. Decompose $\hat{U}$ into finitely–many connected pieces, so $\hat{U} = \bigsqcup_i Q_i \times (0, \infty]$, with each Q_i compact without boundary. The boundary ∂N is identified with points $(q \in Q_i, r = \infty)$ in this representation of $\hat{U}$. Define $U = \bigsqcup_i Q_i \times (0, \infty)$. Let $M_0 = M \setminus \overline{\Phi(U)}$, so M_0 has compact closure in M and finite diameter, and let $E_i = \Phi(Q_i \times (0, \infty))$. We can decompose $M = \overline{M_0} \sqcup (\bigsqcup_i E_i)$. Note that M_0 is diffeomorphic to M itself.

Now we consider the induced mapping $(\Psi^{-1})^*$, for $\Psi : M \to M_0$ a diffeomorphism, taking "very–rapid–decay" forms C_μ^∞ back to M_0.

11. Definition. Property EZ. *Let M be topologically tame, assume (M,g) is complete, and suppose a measure $d\mu = e^{2h}dx$ is given. Consider $M_0 \subset M$ an open submanifold of M with compact closure, such that $\partial M_0 \subset M$ is a smooth closed submanifold, and such that there exists a diffeomorphism $\Psi : M \to M_0$. Such M_0 exist since M is topologically tame. We say $d\mu$ has* Property EZ *if there exists M_0 and Ψ so that every $\omega \in (\Psi^{-1})^* C^\infty_\mu$ has a smooth extension by zero to all of M.*

From now on, we will write $H_c(M)$ for the cohomology of the complex $(\Omega_c(M), d)$, i.e. for the de Rham cohomology with compact supports. Note also that the cohomology spaces $H_c(M_0)$, $H_c(M)$ in the following lemma are finite–dimensional because M is topologically tame.

12. Lemma. *Let M be a topologically tame manifold. Suppose $M_0 \subset M$ is an open submanifold with compact closure, $\Psi : M \to M_0$ a diffeomorphism. Assume ∂M_0 is a smooth (closed) submanifold of M. Let $i : \Omega_c(M_0) \to \Omega_c(M)$ be the map which extends forms by zero. Then i induces an isomorphism $i : H_c(M_0) \to H_c(M)$.*

Proof. We construct a family Ψ_ϵ, $0 < \epsilon < 1$, of $M \to M_0$ diffeomorphisms which fix "large portions" of M_0. First, note that $\Psi(M_0)$ is diffeomorphic to M, since $\Psi|_{M_0} \circ \Psi$ is a diffeomorphism. Thus it is clear that $M \setminus \Psi(M_0)$ is diffeomorphic to ∂M_0 times an interval. Let $\alpha : \partial M_0 \times [-1, \infty) \to M \setminus \Psi(M_0)$ be such a diffeomorphism, chosen so that $\alpha(\partial M_0 \times [-1, 0)) = M_0 \setminus \Psi(M_0)$ and $\alpha(\partial M_0 \times [0, \infty)) = M \setminus M_0$, so $\alpha(\partial M_0 \times \{0\}) = \partial M_0$. Let $M_\epsilon = M_0 \setminus \alpha(\partial M_0 \times (-\epsilon, \infty))$ for $0 < \epsilon < 1$, and also define $\beta_\epsilon : [-1, \infty) \to [-1, 0)$ so that $\beta_\epsilon(\xi) = \xi$ for $-1 \le \xi \le -\epsilon$, and β_ϵ is smooth and monotone increasing with $\lim_{\xi \to \infty} \beta_\epsilon(\xi) = 0$.

Define $\Psi_\epsilon(x) = x$ for $x \in \Psi(M_0)$. For $x \in M \setminus \Psi(M_0)$, let $(y, \xi) = \alpha^{-1}(x)$, $y \in \partial M_0$, $\xi \in [-1, \infty)$. Define $\Psi_\epsilon(x) = \alpha(y, \beta_\epsilon(\xi))$. Then Ψ_ϵ fixes M_ϵ and is a diffeomorphism $M \to M_0$ for each $0 < \epsilon < 1$.

Now, Ψ_ϵ induces $\Psi_\epsilon^* : \Omega_c(M_0) \to \Omega_c(M)$ on forms and an isomorphism $\Psi_\epsilon^* : H_c(M_0) \to H_c(M)$.

If $\alpha \in H_c(M_0)$, $\alpha \ne 0$, we will show $i\alpha \ne 0$ in $H_c(M)$. This suffices, since $H_c(M_0)$, $H_c(M)$ are finite–dimensional. Choose any representative ω, so $[\omega] = \alpha$. Since ω has compact support in M_0, there exists $\epsilon > 0$ such that $\operatorname{supp} \omega \subset M_\epsilon$. Then $\Psi_\epsilon^* \omega = i\omega$. But $i\alpha = [i\omega] = [\Psi_\epsilon^* \omega] = \Psi_\epsilon^* \alpha \ne 0$ in $H_c(M)$ since Ψ_ϵ^* is an isomorphism. □

13. Theorem. *Suppose M^n is topologically tame, assume (M,g) is complete, and suppose the measure $d\mu = e^{2h}dx$ is given. If $d\mu$ has property EZ, then there exists a map $j : C^\infty_\mu \to \Omega_c(M)$, which restricts to a surjection*

$$j : H_\mu \to H_c(M).$$

Proof. Let M_0, Ψ be as in the definition of "property EZ". Define the map j to take $\omega \in C^\infty_\mu$ to the extension by zero of $(\Psi^{-1})^*\omega$. Let $k : \Omega_c(M_0) \to C^\infty_\mu$ be extension by zero. Then i factors through j: $i = j \circ k$.

By the lemma, i is an isomorphism in cohomology. At the level of cohomology,

$$H_c(M_0) \xrightarrow{k} H_\mu \xrightarrow{j} H_c(M), \qquad i = j \circ k \text{ isomorphism,}$$

it follows that k is an injection and j is a surjection. $\square$

Recalling Poincaré duality, we see that if $d\mu$ on (M^n, g) has property EZ then

$$\dim H^p_\mu \geq \dim H^p_c(M) = \dim H^{n-p}(M) \equiv b_{n-p}.$$

We do not exclude the possibility that $\dim H^p_\mu = \infty$, nor do we assume that H^p_μ has a Hilbert space structure. Also, it is not assumed that $\triangle_\mu$ has a spectral gap.

There is an available map going the other direction from j. Consider the orthogonal projection $\Pi : L^2_\mu\Omega \to \ker \triangle_\mu$. This map clearly restricts to $\Pi : \Omega_c(M) \to \ker \triangle_\mu$. We also use Π to denote the map on compactly–supported cohomology:

14. Definition. *Assume (M, g) is complete. Define*

$$\Pi : H_c(M) \to \ker \triangle_\mu \qquad \textit{by} \qquad [\omega] \mapsto \Pi(\omega).$$

Recall that if M is complete, then $\langle d\nu, \varphi\rangle_\mu = \langle \nu, \delta_\mu \varphi\rangle_\mu = 0$ for all $\nu \in \Omega_c(M)$ and $\varphi \in \ker \triangle_\mu$. This shows that Π is well–defined.

If $d\mu$ has property EZ, then the following method may also apply. First, the natural map $\eta : \ker \triangle_\mu \to H_\mu$, given by $\omega \mapsto [\omega]$ is always injective if M is complete, since if $\omega = d\nu$ for $\nu \in C^\infty_\mu$ then $\langle \omega, \varphi\rangle_\mu = \langle \nu, \delta_\mu \varphi\rangle_\mu = 0$ for all $\varphi \in \ker \triangle_\mu$. If $d\mu$ has property EZ, and if the composition

$$\ker \triangle_\mu \xrightarrow{\eta} H_\mu \xrightarrow{j} H_c(M) \xrightarrow{\Pi} \ker \triangle_\mu$$

is surjective, then Π is surjective. This reduces to checking whether an explicit endomorphism $\Pi \circ j \circ \eta$ on $\ker \triangle_\mu$, depending only on Ψ and $d\mu$, is surjective.

Whether Π is surjective or not is *a priori* independent of the existence of a spectral gap for $\triangle_\mu$.

In section 4 we will see that both j and Π are surjective in the case described by theorem 2 if $c > 0$, and thus that $H_\mu \cong H_c(M)$. That is, we prove that if: (i) M is topologically tame, (ii) the ends of (M, g) are cylinders, and (iii) $d\mu$ is a measure with density $\exp(cr^2)$, $c > 0$ on the ends, then $d\mu$ has property EZ. It follows that j is surjective, and we prove Π is surjective by the method described above.

4. Cylindrical Ends, Gaussian Growth Measure

Let us return to the setting of theorem 2. Let M be topologically tame, and define g as a metric under which (M, g) is complete and the ends of M are cylinders. Define $d\mu$ as in theorem 2 as well, with $c > 0$: $d\mu = e^{cr^2} dx$. (We will more explicitly define g and $d\mu$ in a moment.) In this section we prove that $d\mu$ on M has property EZ, and in fact we prove that both j and Π of the previous section are surjections. As we will see, the proof of property EZ ultimately depends on a fundamental estimate for C^∞_μ (actually C^∞_h) forms on (M, g). The estimate is found in theorem 16 and corollary 17 of section 5.

15. Theorem. *Let $c > 0$. Let M, g, $d\mu = e^{cr^2}dx$ be as in theorem 2. Then in addition to (i) and (ii), there exist surjections Π and j such that*

$$H_c(M) \xrightarrow{\Pi} \ker \triangle_\mu \cong H_\mu \xrightarrow{j} H_c(M).$$

It follows that

$$H_\mu \cong H_c(M).$$

Proof that j is surjective. We show that $d\mu$ has property EZ, so that by theorem 13, $j : H_\mu \to H_c(M)$ is a surjection.

Since M is topologically tame, $M = M_0 \cup (\bigsqcup_i E_i)$, with M_0 relatively compact, $E_i = (0,\infty) \times Q_i$, and Q_i compact without boundary. Let g be a product metric on each E_i and otherwise choose g arbitrarily.

Fix an end $E = (0,\infty) \times Q$, and let $Q = \bigcup_j U_j$ be a finite cover of Q by coordinate neighborhoods. Let $(x^1, x^2, \dots, x^n) = (x^1, \bar{x}) = (r, \bar{x})$ be coordinates on $(0,\infty) \times U_j$.

Define $d\mu = e^{cr^2}dx = e^{c(x^1)^2}dx$ on $(0,\infty) \times U_j$, therefore on E. Extend $d\mu$ to all of M in the obvious way, so $d\mu = e^{+2h}dx$ on M, where $h \in C^\infty(M)$ and $h = cr^2$ on each end, for the given $c > 0$.

Define Ψ on E as follows. Let $M_1 = M_0 \cup (\bigsqcup_i (0,1) \times Q_i)$, with coordinates $(s, \bar{y})$ on $(0,1) \times U_j$. We define $\Psi^{-1} : M_1 \to M$ by defining $\Psi^{-1} = \mathrm{id}$ on M_0 and by defining $\Psi^{-1} : (0,1) \times U_j \to (0,1) \times U_j$ as follows: $\Psi^{-1}(s, \bar{y}) = ((1-s)^{-1}, \bar{y})$ if $1/2 \le s < 1$, and smoothly and monotonically interpolate Ψ^{-1} for $0 \le s \le 1/2$. Clearly Ψ^{-1} is a diffeomorphism and $M_1 = \Psi(M)$ has compact closure.

To prove that $d\mu$ has property EZ, it suffices to prove that if $\omega \in C_\mu^\infty$ and $\operatorname{supp}\omega \subset E$, then $\left(\Psi^{-1}\right)^* \omega$ has smooth extension by zero to E. That is, if $\varphi_0, \{\varphi_i\}$ is a partition of unity subordinate to $M = M_1 \cup (\bigsqcup_i E_i)$ then $\omega \in C_\mu^\infty$ iff $\varphi_i\omega \in C_\mu^\infty$ for each i.

Now we arrange to use the properties of Schwartz forms on an end E—see section V—to prove the extension by zero on E for $\left(\Psi^{-1}\right)^* \omega$.

In fact, recall that $\triangle_h = e^h \triangle_\mu e^{-h} = \triangle + |dh|^2 + A_h$ on $L^2\Omega$ is unitarily–equivalent to $\triangle_\mu$ on $L^2_\mu\Omega$. Here $h = \frac{c}{2}r^2$. Let $\nu \in C_h^\infty = \bigcap_{k\ge 1} \mathcal{D}_{(\triangle_h)^k}$. Then $e^{-h}\nu \in C_\mu^\infty$. Suppose $\nu = \sum_{1\notin J} u_J dr \wedge d\bar{x}^J + \sum_{1 \notin I} u_I d\bar{x}^I$ on $(0,\infty) \times U_j$. Then since $\frac{\partial r}{\partial s} = (1-s)^{-2}$,

$$\begin{aligned}\omega' \equiv \left(\Psi^{-1}\right)^* \left(e^{-h}\nu\right) =& e^{-\frac{c}{2}(1-s)^{-2}}(1-s)^{-2}u_J((1-s)^{-1}, \bar{y})ds \wedge d\bar{y}^J \\ &+ e^{-\frac{c}{2}(1-s)^{-2}} u_I((1-s)^{-1}, \bar{y})d\bar{y}^I\end{aligned}$$

is in $\Omega(M_1)$.

Denote as $\frac{\partial}{\partial x^j} : \Omega^p((0,\infty) \times U_j) \to \Omega^p((0,\infty) \times U_j)$ the operator defined by $\frac{\partial}{\partial x^j}\omega = \sum \frac{\partial \omega_I}{\partial x^j} dx^I$ if $\omega = \sum \omega_I dx^I$. That is, $\frac{\partial}{\partial x^j}$ differentiates the coefficients of ω with respect to x^j. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ is a multiindex, define $\frac{\partial}{\partial x}^\alpha = \frac{\partial}{\partial x^1}^{\alpha_1} \cdots \frac{\partial}{\partial x^n}^{\alpha_n}$.

Clearly, if

$$|r^k\,(\partial/\partial x)^\alpha\,\nu(r,\bar{x})| \to 0 \quad \text{as } r\to\infty \tag{2}$$

for every $k \geq 0$ and every multiindex α, then it follows that

$$|\,(\partial/\partial y)^\alpha\,\omega'(s,\bar{y})| \to 0 \quad \text{as } s\to 1^- \tag{3}$$

for every α, where $(y^1, y^2, \ldots, y^n) = (y^1, \bar{y}) = (s, \bar{y})$ are coordinates on $(0,1)\times U_j$.

But (2) is exactly the conclusion of corollary 17, and (3) is exactly what is needed to prove property EZ on the end E. It follows from theorem 13 that $j : H_\mu \to H_c(M)$ is surjective.

(Remark: We have proven somewhat more rapid decay in (3) than the minimum needed to prove property EZ. However, the degree to which the measure $d\mu$ can be changed to a measure $d\mu'$ with slower growth rate $e^{2h'}$ depends on both the growth rate $e^{2h'}$ itself and the corresponding potential $V = |dh'|^2 + A_{h'}$ which plays the essential role in section 5.)

Proof that Π is surjective. Let $\{\omega^1, \ldots, \omega^N\}$ be an orthonormal basis for $\ker \triangle_\mu$. Note $N < \infty$ since $\triangle_\mu$ has compact resolvent. For each ω^i, we will find $\nu \in \Omega_c(M)$ such that $d\nu = 0$ and $\Pi(\nu) = \omega^i$.

Choose $\epsilon = \epsilon(N) > 0$ so that any $N\times N$ real matrix (a_{ij}) with $|a_{ij} - 1| < \epsilon$ for all i and $|a_{ij}| < \epsilon$ for $i \neq j$ is invertible. For instance, $\epsilon = N^{-1}$ suffices.

In what follows we will make a finite number of choices for R, using the finite–dimensionality of $\ker\triangle_\mu$ and the finite list of ends E_i, and we assume that the maximum of these R is used as needed.

For any $R \geq 0$, define Ψ_R analogously to Ψ on a particular end E: Ψ_R^{-1} is the identity on $\{s \leq R\} = M_R$, $\Psi_R^{-1}(s,\bar{x}) = (R + (1-(s-R))^{-1}, \bar{x})$ for $R + 1/2 \leq s < R+1$, and smoothly and monotonically interpolate Ψ^{-1} for $R \leq s \leq R+1/2$. Thus Ψ_R is a diffeomorphism and $\Psi_R(M) = M_{R+1}$. We may assume $|d\Psi_R^{-1}| \leq 4$ on $R \leq s \leq R+1/2$.

Next, we use property EZ: define j_R on forms $\varphi \in C_\mu^\infty$ as the extension by zero of $(\Psi_R^{-1})^*\varphi$. Let $\overline{\omega}^i = j_R\omega^i$. Each $\overline{\omega}^i \in \Omega_c(M)$ and $d\overline{\omega}^i = (\Psi_R^{-1})^* d\omega^i = 0$, so $[\overline{\omega}^i] \in H_c(M)$.

Clearly $|\left\langle \overline{\omega}^i, \omega^j\right\rangle_\mu| \leq \|\overline{\omega}^i - \omega^i\|_\mu$ if $i \neq j$ and $|\left\langle \overline{\omega}^i, \omega^i\right\rangle_\mu - 1| \leq \|\overline{\omega}^i - \omega^i\|_\mu$. We will show that we can choose $R \geq 0$ so that $\|\overline{\omega}^i - \omega^i\|_\mu < \epsilon$. It will follow that $A = \left(\left\langle \overline{\omega}^i, \omega^j\right\rangle_\mu\right)$ is invertible.

Note $\overline{\omega}^i|_{M_R} = \omega^i|_{M_R}$. Thus

$$\|\overline{\omega}^i - \omega^i\|_\mu^2 = \int_{R<s} |\overline{\omega}^i - \omega^i|^2 d\mu \leq 2\int_{R<s<R+1} |\overline{\omega}^i|^2 d\mu + 2\int_{R<s} |\omega^i|^2 d\mu.$$

Now,

$$\begin{aligned}\int_{R<s<R+1/2} |\overline{\omega}^i|^2 d\mu &\leq \int_{R<s<R+1/2} \left|\frac{\partial r}{\partial s}\right|^2 |\omega^i(r(s),\bar{y})|^2 e^{cs^2} ds\,d\bar{y} \\ &\leq 4^2 \int_{R<r} |\omega^i(r,\bar{x})|^2 e^{cr^2} dr\,d\bar{x}\end{aligned}$$

so the essential estimate is:

$$\begin{aligned}\int_{R+1/2<s<R+1} |\overline{\omega}^i|^2 d\mu &= \int_{R+1/2<s<R+1} (1-(s-R))^{-2} |\omega^i(r(s),\bar{x})|^2 e^{cs^2} ds d\bar{x} \\ &= \int_{R+2<r<\infty} |\omega^i(r,\bar{x})|^2 \exp\left(c(R+1-(r-R)^{-1})^2\right) dr d\bar{x} \\ &\le \int_{R+2<r<\infty} |\omega^i(r,\bar{x})|^2 \exp\left(c(R+1)^2\right) dr d\bar{x} \\ &\le \int_{R+1<r} |\omega^i(r,\bar{x})|^2 e^{cr^2} dr d\bar{x}.\end{aligned}$$

It follows that

$$\|\overline{\omega}^i - \omega^i\|_\mu^2 \le 36 \int_{R<r} |\omega^i|^2 d\mu.$$

Choose $R \ge 0$ so that $\|\overline{\omega}^i - \omega^i\|_\mu < \epsilon = \epsilon(N)$ for every $i = 1, \dots, N$. Then $A = \left(\left\langle \overline{\omega}^i, \omega^j \right\rangle_\mu\right)$ is invertible.

But

$$\Pi(\overline{\omega}^i) = \sum_j \left\langle \overline{\omega}^i, \omega^j \right\rangle_\mu \omega^j = A_{ij}\omega^j,$$

so if $\nu = A_{ij}^{-1}\overline{\omega}^j$ then

$$\Pi(\nu) = A_{ij}^{-1}\Pi(\overline{\omega}^j) = A_{ij}^{-1}A_{jk}\omega^k = \omega^i.$$

We have shown that Π is surjective, as in the method described in section 3. $\square$

5. Schwartz Forms on a Cylindrical End

Let $E = (0,\infty) \times Q$, where Q is compact without boundary. Give E a product metric, with $(0,\infty)$ given the Euclidean metric. (Then E is not complete, but we will compute here as though it is, since for our purposes E is an end of a complete manifold.) Let $Q = \bigcup_j U_j$ be a finite cover of Q by coordinate neighborhoods. Let $(x^1, x^2, \dots, x^n) = (x^1, \bar{x})$ be coordinates on $(0,\infty) \times U_j$.

Let $h = \frac{c}{2}(x^1)^2$ for $c > 0$.

We define $d_h = e^h d e^{-h}$, $d_h^* = e^{-h} d e^h$, and

$$\triangle_h = d_h d_h^* + d_h^* d_h = \triangle + |dh|^2 + A_h,$$

all acting in $L^2\Omega$ (with measure $dx = \sqrt{\det g_{ij}}dx^i dx^j$ as usual). Then $\triangle_h$ on $L^2\Omega$ and $\triangle_\mu$ on $L^2_\mu\Omega$ are unitarily–equivalent ([**Bi**]).

Denote $C_h^\infty = \bigcap_{k\ge 1} \mathcal{D}_{(\triangle_h)^k}$. Clearly, $\omega \in C_h^\infty$ iff $e^{-h}\omega \in C_\mu^\infty$. We will show that forms in C_h^∞ decay as fast as Schwartz functions on a cylindrical end E as $x^1 \to \infty$.

Define $\partial_j : \Omega^p((0,\infty)\times U_j) \to \Omega^p((0,\infty)\times U_j)$ by $\partial_j\omega = \sum \frac{\partial\omega_I}{\partial x^j} dx^I$ if $\omega = \sum \omega_I dx^I$. That is, ∂_j differentiates the coefficients of ω with respect to x^j. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ is a multiindex, let $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.

We prove the following result on forms in C_h^∞:

16. Theorem. *If $\omega \in C_h^\infty$ is supported in E, then for any $k \in \mathbf{N}$ and any multiindex α,*

$$\|(x^1)^k \partial^\alpha \omega\|_{L^2\Omega} < \infty. \tag{4}$$

Before proving (4), we note that it implies a pointwise result as well. That is, Sobolev inequalites (or even explicit Schwartz space calculations as in [**RS**], Appendix to V.3) imply:

17. Corollary. *If $\omega \in C_h^\infty$ is supported in E, then for any $k \in \mathbf{N}$, any multiindex α, and any $\bar{x}_0 \in U_j$,*

$$|(x^1)^k \partial^\alpha \omega|(x^1, \bar{x}_0) \to 0 \qquad \textit{as } x^1 \to \infty. \tag{5}$$

18. Remark. Properties (4) and (5) do not depend on the choice of local coordinates $\bar{x}$ on Q, since all transition functions and their derivatives are bounded on compact Q.

Also, since Q is compact, by using a partition of unity on Q it is enough to prove theorem 16 only for forms which have support completely inside one of the coordinate neighborhoods $(0,\infty)\times U_j$. Moreover, since inequality (4) is trivially true for smooth forms with compact support, we can assume (without loss of generality) that $\omega \equiv 0$ for $x^1 \leq 1$.

The proof of theorem 16 will proceed in several steps, starting with following lemmata.

19. Lemma. Weitzenbock's formula for $\triangle_h$. *Suppose ω is a smooth form on E, then*

$$\triangle_h \omega = \left(\sum_{i=1}^n \nabla_i^* \nabla_i + R(\bar{x}) + c^2 (x^1)^2 \pm c \right) \omega, \tag{6}$$

where $R(\bar{x}) : \wedge^p T_{\bar{x}}^* E \to \wedge^p T_{\bar{x}}^* E$ is a pointwise curvature endomorphism, and $\nabla_i = \nabla_{\partial/\partial x^i}$.

Proof. See [**CFKS**], Chapter 12.4. $\square$

20. Lemma. Gärding's inequality. *Let ω be a smooth form, suppose* $\operatorname{supp} \omega \subset (1,\infty)\times U_j$ *and suppose ω, $\triangle_h \omega \in L^2\Omega((1,\infty)\times Q)$. Then*

$$\sum_{i=1}^n \|\nabla_i \omega\|^2 + \|x^1 \omega\|^2 \leq C \left(\langle \triangle_h \omega, \omega \rangle + |\omega\|^2 \right). \tag{7}$$

Proof. Let $J(\tau) : (0,\infty) \to \mathbf{R}_+$ be a smooth non–increasing function such that $J(\tau) \equiv 1$ for $\tau \leq 2$ and $J(\tau) \equiv 0$ for $\tau \geq 3$. We define a family of cut–off functions

$$J_t : (0,\infty) \to \mathbf{R}_+$$

by

$$J_t(x^1, \bar{x}) = J\left(\frac{x^1}{t}\right).$$

We multiply both sides of (6) by $J_t^2\omega$ and integrate by parts:

$$\langle \triangle_h \omega, J_t^2\omega \rangle = \sum_{i=1}^{n} \langle \nabla_i \omega, \nabla_i (J_t^2 \omega) \rangle + \langle R(\bar{x})\omega, J_t^2\omega \rangle + c^2 \|x^1 J_t \omega\|^2 \pm c\|J_t\omega\|^2. \tag{8}$$

For each i we have

$$\langle \nabla_i \omega, \nabla_i (J_t^2\omega) \rangle = \|J_t \nabla_i \omega\|^2 + 2\langle J_t \frac{\partial J_t}{\partial x^i} \nabla_i \omega, \omega \rangle.$$

Therefore, substituting back into (8) we get

$$\begin{aligned} &\sum_{i=1}^{n} \|J_t \nabla_i \omega\|^2 + c^2 \|x^1 J_t \omega\|^2 \\ &\qquad = \langle \triangle_h \omega, J_t^2 \omega \rangle - 2 \sum_{i=1}^{n} \langle J_t \frac{\partial J_t}{\partial x^i} \nabla_i \omega, \omega \rangle - \langle R(\bar{x})\omega, J_t^2 \omega \rangle \pm c \|J_t \omega\|^2. \end{aligned} \tag{9}$$

Now we estimate:

$$\begin{aligned} 2\left| \langle J_t \frac{\partial J_t}{\partial x^i} \nabla_i \omega, \omega \rangle \right| &\le \sup_{\bar{x}} \left| \frac{\partial J_t}{\partial x^i} \right|^2 \|J_t \nabla_i \omega\|^2 + \|\omega\|^2 \\ &\le \frac{C}{t^2} \|J_t \nabla_i \omega\|^2 + C\|\omega\|^2 \end{aligned}$$

and

$$\left| \langle R(\bar{x})\omega, J_t^2 \omega \rangle \right| \le \sup_{\bar{x}} |R(\bar{x})| \|J_t \omega\|^2 \le C\|\omega\|^2.$$

For large t, the two inequalities above, together with (9), give:

$$\begin{aligned} \sum_{i=1}^{n} \|J_t \nabla_i \omega\|^2 + c^2 \|x^1 J_t \omega\|^2 &\le C\left(\langle \triangle_h \omega, J_t^2 \omega \rangle + \|J_t \omega\|^2 \right) \\ &\le C\left(\langle \triangle_h \omega, \omega \rangle + \|\omega\|^2 \right). \end{aligned} \tag{10}$$

Gärding's inequality now follows if we take $t \to \infty$ in the inequality above. $\square$

21. Lemma. *Let ω be a smooth form. Suppose* $\operatorname{supp} \omega \subset (1, \infty) \times U_j$ *and suppose ω, $\triangle_h \omega \in L^2\Omega((1,\infty) \times Q)$. Then*

$$\sum_{j=1}^{n} \|J_t \partial_j \omega\|^2 + \|J_t x^1 \omega\|^2 \le C\left(| \langle J_t^2 \triangle_h \omega, \omega \rangle | + \|J_t \omega\|^2 \right), \tag{11}$$

and

$$\sum_{j=1}^{n} \|\partial_j \omega\|^2 + \|x^1\omega\|^2 \leq C\left(\langle \triangle_h \omega, \omega\rangle + \|\omega\|^2\right). \tag{12}$$

Proof. Recall that

$$\nabla_i(\omega_I dx^I) = (\nabla_i \omega_I)dx^I + \omega_I \nabla_i(dx^I) = (\partial_i \omega_I)dx^I + \omega_I \Gamma_i(\bar{x})(dx^I), \tag{13}$$

where $\Gamma_i(\bar{x}) : \wedge T^*_{\bar{x}} \to \wedge T^*_{\bar{x}}$ is an endomorphism. The matrix of $\Gamma_i(\bar{x})$ can be expressed in terms of the Christoffel symbols $\Gamma^k_{ij}(\bar{x})$.

It follows from (13) that

$$\|J_t \nabla_j \omega\|^2 = \|J_t \partial_j \omega\|^2 + 2\left\langle J_t \Gamma_j(\bar{x})\omega, J_t \partial_j \omega\right\rangle + \|J_t \Gamma_j(\bar{x})\omega\|^2. \tag{14}$$

For some choice of small $\epsilon > 0$, we estimate:

$$2|\left\langle J_t \Gamma_j(\bar{x})\omega, J_t \partial_j \omega\right\rangle| \leq \frac{1}{\epsilon}\|J_t \Gamma_j \omega\|^2 + \epsilon\|J_t \partial_j \omega\|^2 \leq \frac{C}{\epsilon}\|J_t \omega\|^2 + \epsilon\|J_t \partial_j \omega\|^2.$$

Therefore, from (14)

$$\|J_t \partial_j \omega\|^2 \leq C\left(\|J_t \nabla_j \omega\|^2 + \|J_t \omega\|^2\right). \tag{15}$$

Inequality (11) follows from (10) and (15). Inequality (12) follows from (7) and (15) as $t \to \infty$. $\square$

Proof of theorem 16. Induction argument. For $\omega \in C^\infty_h$, $\operatorname{supp}\omega \subset (0,\infty) \times U_j$, inequality (4) follows from

$$\|\omega\|_l^2 := \sum_{\substack{k,\,\alpha \\ k+|\alpha| \leq l}} \|(x^1)^k \partial^\alpha \omega\|^2 \quad \leq \quad C(l)\left(\left\langle \triangle_h^l \omega, \omega\right\rangle + \|\omega\|^2\right) \tag{16}$$

where $0 \leq l < \infty$.

Remark. *Operator $\triangle_h^l$ is a PDO of order $2l$ with smooth coefficients which grow at most as $(x_1)^{2l}$. Therefore, for all $\omega \in C^\infty_h$ which satisfy $\|\omega\| \leq \infty$, we have*

$$\left\langle \triangle_h^l \omega, \omega\right\rangle \leq C_l \|\omega\|_l^2. \tag{17}$$

Moreover, for all $l_0 \leq l$

$$\left\langle \triangle_h^{l_0} \omega, \omega\right\rangle \leq C\left(\left\langle \triangle_h^l \omega, \omega\right\rangle + |\ \omega\|^2\right).$$

We will prove (16) by induction in l. Lemma 21 proves (16) for $l = 1$ and all $\omega \in C_h^\infty$ with $\operatorname{supp} \omega \subset (0,\infty) \times U_j$.

Suppose (16) is true for all $l \le l_0$ and all ω as above. Let $k + |\alpha| = l_0$. We substitute $(x^1)^k \partial^\alpha \omega$ instead of ω into (11) to get

$$\sum_{j=1}^{n} \|J_t \partial_j (x^1)^k \partial^\alpha \omega\|^2 + \|J_t (x^1)^{k+1} \partial^\alpha \omega\|^2 \le C \left(\left| \left\langle J_t^2 \triangle_h \left((x^1)^k \partial^\alpha \omega \right), (x^1)^k \partial^\alpha \omega \right\rangle \right| + \|J_t (x^1)^k \partial^\alpha \omega\|^2 \right). \tag{18}$$

Define $A = \left\langle J_t^2 \triangle_h \left((x^1)^k \partial^\alpha \omega \right), (x^1)^k \partial^\alpha \omega \right\rangle$. We write

$$\triangle_h \left((x^1)^k \partial^\alpha \omega \right) = (x^1)^k \partial^\alpha \triangle_h \omega + \left[\triangle_h, (x^1)^k \partial^\alpha \right] \omega, \tag{19}$$

where $[\cdot,\cdot]$ denotes the commutator of two differential operators. Then from (6),

$$\begin{aligned} \left[\triangle_h, (x^1)^k \partial^\alpha \right] \omega &= \left[\sum_{i=1}^{n} \nabla_i^* \nabla_i + R(\bar{x}), (x^1)^k \partial^\alpha \right] \omega + c^2 \left[(x^1)^2, (x^1)^k \partial^\alpha \right] \omega \\ &= P_1 \omega + P_2 \omega. \end{aligned} \tag{20}$$

Proposition.

a) *P_1 is a PDO of order at most $|\alpha| + 1$ with coefficients which grow at most as $(x^1)^k$.*

b) *P_2 is a PDO of order at most $|\alpha|$ with coefficients which grow at most as $(x^1)^{k+1}$.*

Proof of Proposition. This follows from the formula (13) for ∇_i and from two simple observations.

First,

$$\left[(x^1)^k, \partial_j \right] = \begin{cases} 0, & j \ne 1 \\ k (x^1)^{k-1}, & j = 1. \end{cases}$$

Second, the commutator of a smooth function $f(\bar{x})$ with ∂^α is a PDO of order $|\alpha| - 1$ with smooth (bounded) coefficients which depend only on $\bar{x}$. *Q.E.D.*

With the help of (19) and (20) we rewrite the quantity A as

$$A = \left\langle J_t^2 (x^1)^k \partial^\alpha \triangle_h \omega, (x^1)^k \partial^\alpha \omega \right\rangle + \left\langle J_t^2 P_1 \omega, (x^1)^k \partial^\alpha \omega \right\rangle + \left\langle J_t^2 P_2 \omega, (x^1)^k \partial^\alpha \omega \right\rangle, \tag{21}$$

and we estimate:

$$\begin{aligned} |A| \le{}& \|J_t (x^1)^k \partial^\alpha \triangle_h \omega\| \cdot \|J_t (x^1)^k \partial^\alpha \omega\| \\ &+ \epsilon \|J_t P_1 \omega\|^2 + \frac{1}{\epsilon} \|J_t (x^1)^k \partial^\alpha \omega\|^2 + \epsilon \|J_t P_2 \omega\|^2 + \frac{1}{\epsilon} \|J_t (x^1)^k \partial^\alpha \omega\|^2, \end{aligned} \tag{22}$$

where $\epsilon > 0$ is to be chosen later. Now we estimate the right–hand side of (22). For $\triangle_h\omega \in C_h^\infty$, by induction and by the remark above we have

$$\|J_t(x^1)^k\partial^\alpha\triangle_h\omega\| \cdot \|J_t(x^1)^k\partial^\alpha\omega\| \leq C\left(\left\langle(\triangle_h)^{l_0+1}\omega,\omega\right\rangle + \|\omega\|^2\right).$$

Moreover, from the proposition,

$$\|J_tP_1\omega\|^2 + \|J_tP_2\omega\|^2 \leq C\sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2.$$

Therefore,

$$|A| \leq C\Big(\left\langle(\triangle_h)^{l_0+1}\omega,\omega\right\rangle + \|\omega\|^2 + \epsilon\cdot\sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2 + \|\omega\|_{l_0}^2\Big).$$

The inequality (18) can be rewritten as

(23)
$$\begin{aligned}&\sum_{j=1}^n\|J_t(x^1)^k\partial_j\partial^\alpha\omega\|^2 + \|J_t(x^1)^{k+1}\partial^\alpha\omega\|^2\\ &\qquad\leq C\Big(\left\langle(\triangle_h)^{l_0+1}\omega,\omega\right\rangle + \|\omega\|^2 + \|\omega\|_{l_0}^2\Big) + C\epsilon\cdot\sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2.\end{aligned}$$

We can sum up inequalities (23) over all $k + |\alpha| = l_0$ and use the equality

$$\sum_{k+\alpha\leq l_0}\sum_{j=1}^n\|J_t(x^1)^k\partial_j\partial^\alpha\omega\|^2 + \|J_t(x^1)^{k+1}\partial^\alpha\omega\|^2 = \sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2$$

to get (24) from (23):

(24)
$$\begin{aligned}\sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2 &\leq C(l_0+1)\Big(\left\langle(\triangle_h)^{l_0+1}\omega,\omega\right\rangle + \|\omega\|^2 + \|\omega\|_{l_0}^2\Big)\\ &\quad + \epsilon\, C(l_0+1)\Big(\sum_{m+|\beta|\leq l_0+1}\|J_t(x^1)^m\partial^\beta\omega\|^2\Big).\end{aligned}$$

Finally, we choose ϵ small enough to move the right–hand $\sum_{m+|\beta|\leq l_0+1}$ over to the left.

Then (16) follows from (24) for $l = l_0 + 1$ if we take $t \to \infty$. $\square$

6. Application: Künneth Formula

22. Proposition. *Let M, N be orientable topologically tame manifolds. Then*

$$H_c(M \times N) \cong H_c(M) \otimes H_c(N).$$

Proof. Choose metrics g_M, g_N so that M, N have cylindrical ends as in theorem 2, and define $d\mu_M$, $d\mu_N$ as described there, choosing $c > 0$. Apply theorem 15 to M, N, and note that the same methods used to prove theorem 15 also show that $H_c(M \times N) \cong H_{\mu_M \times \mu_N}$, where $M \times N$ has been given the product metric and the product measure.

Now, $H_{\mu_M \times \mu_N}$ has the expected combinatorics, coming purely from analysis on Hilbert tensor products—more precisely Fredholm complexes. See [**BL**]. Specifically,

$$H_{\mu_M \times \mu_N} \cong H_{\mu_M} \otimes H_{\mu_N},$$

which proves our theorem. □

The above theorem applies only to what [**BT**] call manifolds with "finite good cover", but it is nonetheless a Hodge–theoretic proof of a Künneth formula for certain noncompact manifolds. See [**GH**], chapter 0, for proof of the Künneth formula on compact manifolds, using the same Hodge–theoretic technique of proof. The Künneth formula in fact holds for any pair of manifolds M, F for which the cohomology of F is finite–dimensional.

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