SOME PROPERTIES OF THE RANGE OF SUPER-BROWNIAN MOTION

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Abstract. We consider a super-Brownian motion X. Its canonical measures can be studied through the path-valued process called the Brownian snake. We obtain the limiting behavior of the volume of the ε-neighborhood for the range of the Brownian snake, and as a consequence we derive the analogous result for the range of super-Brownian motion and for the support of the integrated super-Brownian excursion. Then we prove the support of X_t is capacity-equivalent to $[0,1]^2$ in $\mathbb{R}^d$, $d \geq 3$, and the range of X, as well as the support of the integrated super-Brownian excursion are capacity-equivalent to $[0,1]^4$ in $\mathbb{R}^d$, $d \geq 5$.

Introduction

Super-Brownian motion, denoted here by $X = (X_t, t \geq 0)$, is a measure-valued process in $\mathbb{R}^d$. It can be obtained as a limit of branching Brownian particle systems. We refer to Dynkin [8] for such an approximation in a more general setting. Another way to study super-Brownian motion, is to use the path-valued process, called the Brownian snake, which was introduced by Le Gall [9, 12]. Furthermore this approach allows us to study also the integrated super-Brownian excursion (ISE). This process appears naturally when one consider the limit of rescaled lattice trees in high dimension (see Derbez and Slade [4, 3]). For every bounded Borel set $A \subset \mathbb{R}^d$, we denote by $A^\varepsilon = \left\{x \in \mathbb{R}^d; d(x,A) \leq \varepsilon\right\}$ and by $|A|$ the Lebesgue measure of the set A. Recently Tribe [19] (see also Perkins [16]) proved a convergence result for the volume of the ε-neighborhood of the support at time $t > 0$, supp X_t, of super-Brownian motion in dimension $d \geq 3$. More precisely, Tribe showed that the quantity $\varepsilon^{2-d} \left|(\text{supp } X_t)^\varepsilon \cap A\right|$ converges a.s. to a deterministic constant times $\int \mathbf{1}_A(x) X_t(dx)$. Using results of Le Gall [11] on hitting probabilities for the Brownian snake, we give a similar result for the range of the Brownian snake. We then derive an analogous result (theorem 2.1) for the range of super-Brownian motion after time $t > 0$, $\mathcal{R}_t(X)$ defined as the closure of $\cup_{s \geq t} \text{supp } X_s$. More precisely, we show that there exists a positive constant C_0 depending only on d such that for every Borel set $A \subset \mathbb{R}^d$, $d \geq 4$, for every $t > 0$, we have a.s.

$$\lim_{\varepsilon \to 0} \varphi_d(\varepsilon) \left|\mathcal{R}_t(X)^\varepsilon \cap A\right| = C_0 \int_t^\infty ds \int \mathbf{1}_A(z) X_s(dz),$$

where $\varphi_4(\varepsilon) = \log(1/\varepsilon)$ and $\varphi_d(\varepsilon) = \varepsilon^{4-d}$ if $d \geq 5$. We also give a similar result for the support of ISE (corollary 2.4).

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Pemantle and Peres [14] defined the notion of capacity-equivalence for two random Borel sets, and later Pemantle and al. [15] showed that the range of Brownian motion in $\mathbb{R}^d$, $d \geq 3$, is capacity-equivalent to $[0,1]^2$. As an application of the previous results , we show (proposition 4.3) that a.s. on $\{X_t \neq 0\}$, the set supp $X_t \subset \mathbb{R}^d$, $d \geq 3$, is capacity-equivalent to $[0,1]^2$, and that a.s. the range $\mathcal{R}_t(X) \subset \mathbb{R}^d$ and the support of ISE for $d \geq 5$ are capacity-equivalent to $[0,1]^4$.

Let us now describe more precisely the contents of the following sections. In section 1, we recall the definition of the path-valued process $W = (W_s, s \geq 0)$ called the Brownian snake. We denote by ζ_s the lifetime of the path W_s. We recall the links between the Brownian snake, super-Brownian motion and ISE.

In section 1.3, we introduce the main tools concerning the Brownian snake. In particular, we consider $T_{(x,\varepsilon)}$ the hitting time for the Brownian snake of $\bar{B}(x,\varepsilon)$, the closed ball with center x and radius ε:

$$T_{(x,\varepsilon)} = \inf\left\{s \geq 0; \exists t \in [0,\zeta_s], W_s(t) \in \bar{B}(x,\varepsilon)\right\}.$$

The function $u_\varepsilon(x) = \mathbb{N}_0\left[T_{(x,\varepsilon)} < \infty\right]$, where $\mathbb{N}_0$ is the excursion measure of the Brownian snake away from the trivial path 0, is the maximal nonnegative solution of $\Delta u = 4u^2$ on $\mathbb{R}^d \backslash B(0,\varepsilon)$ (see also Dynkin [7]). The study of $|\mathcal{R}(W)^\varepsilon \cap A| = \int_A dx\ \mathbf{1}_{\left\{T_{(x,\varepsilon)}<\infty\right\}}$, where $\mathcal{R}(W)$ is the range of the Brownian snake, relies on the explicit law of the first hitting path $\left(W_{T_{(x,\varepsilon)}}, \zeta_{T_{(x,\varepsilon)}}\right)$ under the excursion measure. This law has been computed by Le Gall [11, 13]. It is closely related to the law of the process $(x_t^\varepsilon, 0 \leq t \leq \tau^\varepsilon)$, defined as the unique strong solution of

$$dx_t^\varepsilon = d\beta_t + \frac{\nabla u_\varepsilon(x_t^\varepsilon - x)}{u_\varepsilon(x_t^\varepsilon - x)}dt, \quad \text{for} \quad 0 \leq t \leq \tau^\varepsilon,$$

where β is a Brownian motion in $\mathbb{R}^d$ started at $\beta_0 = 0$ and $\tau^\varepsilon = \inf\{t \geq 0; |x_t^\varepsilon - x| = \varepsilon\}$.

In section 2, we state the main result on the convergence of the volume of the ε-neighborhood of $\mathcal{R}_t(X)$. The method of the proof is completely different from the one used by Tribe in [19]. It is derived from the convergence of the volume of the ε-neighborhood of the range of the Brownian snake in $L^2(\mathbb{N}_0)$ (proposition 2.3).

Section 3 is devoted to the proof of the latter convergence. The proof of the $L^2(\mathbb{N}_0)$ convergence is somewhat technical because we need a precise rate of convergence. The derivation of this estimate relies heavily on the explicit law of $\left(W_{T_{(x,\varepsilon)}}, \zeta_{T_{(x,\varepsilon)}}\right)$ under $\mathbb{N}_0$. It also depends on precise information on the behavior of the function u_1 at infinity. In particular we give the asymptotic expansion of u_1 at infinity in the appendix.

In section 4 we prove the results on capacity-equivalence for the support and the range of super-Brownian motion and for the support of ISE. Let $f : [0,\infty) \to [0,\infty]$ be a decreasing function. We define the energy of a Radon measure ν on $\mathbb{R}^d$ with respect to the kernel f by: $\mathcal{I}_f(\nu) = \iint f(|x-y|)\nu(dx)\nu(dy)$, and the capacity of a set $\Lambda \subset \mathbb{R}^d$ by $\operatorname{cap}_f(\Lambda) = \left[\inf_{\nu(\Lambda)=1} \mathcal{I}_f(\nu)\right]^{-1}$. Following the terminology introduced in [14], we say that two sets Λ_1 and Λ_2 are capacity-equivalent if there exist two positive constants c and C such that for every kernel f, we have

$$c \operatorname{cap}_f(\Lambda_1) \leq \operatorname{cap}_f(\Lambda_2) \leq C \operatorname{cap}_f(\Lambda_1).$$

Proposition 4.3 states that a.s. the set supp $X_t \subset \mathbb{R}^d$, $d \geq 3$, is capacity-equivalent to $[0,1]^2$, and that a.s. the range $\mathcal{R}_t(X) \subset \mathbb{R}^d$, as well as the support of ISE for $d \geq 5$ are capacity-equivalent to $[0,1]^4$. The proof follows the method of [15].

1. Preliminaries on the Brownian snake and super-Brownian motion

We first introduce some notation. We denote by $(M_f, \mathcal{M}_f)$ the space of all finite measures on $\mathbb{R}^d$, endowed with the topology of weak convergence. We denote by $\mathcal{B}_{b+}(\mathbb{R}^p)$, respectively $\mathcal{B}_{b+}(\mathbb{R}^+ \times \mathbb{R}^p)$, the set of all real bounded nonnegative measurable functions defined on $\mathbb{R}^p$, respectively on $\mathbb{R}^+ \times \mathbb{R}^p$. We also denote by $\mathcal{B}(\mathbb{R}^p)$ the Borel σ-field on $\mathbb{R}^p$. For $A \in \mathcal{B}(\mathbb{R}^p)$, let $\mathcal{C}l(A) = \bar{A}$ be the closure of A. For every measure $\nu \in M_f$, and $f \in \mathcal{B}_{b+}(\mathbb{R}^d)$, we shall write $\int f(y)\nu(dy) = (\nu, f)$. We also denote by supp ν the closed support of the measure ν. If S is a Polish space, we denote by $C(I,S)$ the set of all continuous functions from $I \subset \mathbb{R}$ into S.

1.1. **The Brownian snake.**

We recall some facts about the Brownian snake, a path-valued Markov process introduced by Le Gall [9, 12]. A stopped path is a continuous function $\mathrm{w} : [0,\zeta] \to \mathbb{R}^d$, where $\zeta = \zeta_{(\mathrm{w})}$ is called the lifetime of the path. We shall denote by $\hat{\mathrm{w}}$ the end point $\mathrm{w}(\zeta)$. Let $\mathcal{W}$ be the space of all stopped paths in $\mathbb{R}^d$. When equipped with the metric

$$d(\mathrm{w},\mathrm{w}') = \left|\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}\right| + \sup_{s\geq 0} \left|\mathrm{w}(s\wedge\zeta_{(\mathrm{w})}) - \mathrm{w}'(s\wedge\zeta_{(\mathrm{w}')})\right|,$$

the space $\mathcal{W}$ is a Polish space.

Let $\mathrm{w} \in \mathcal{W}$ and $a, b \geq 0$, such that $a \leq b \wedge \zeta_{(\mathrm{w})}$. There exists a unique probability measure on $\mathcal{W}$ denoted by $Q^{\mathrm{w}}_{a,b}(d\mathrm{w}')$ such that:

(i) $\zeta_{(\mathrm{w}')} = b$, $Q^{\mathrm{w}}_{a,b}(d\mathrm{w}')$-a.s.
(ii) $\mathrm{w}'(t) = \mathrm{w}(t)$ for every $0 \leq t \leq a$, $Q^{\mathrm{w}}_{a,b}(d\mathrm{w}')$-a.s.
(iii) The law of $(\mathrm{w}'(t+a), 0 \leq t \leq b-a)$ under $Q^{\mathrm{w}}_{a,b}(d\mathrm{w}')$ is the law of Brownian motion in $\mathbb{R}^d$ started at $\mathrm{w}(a)$ and stopped at time $b-a$.

We shall also consider $Q^{\mathrm{w}}_{a,b}(d\mathrm{w}')$ as a probability on the space $C([0,b],\mathbb{R}^d)$. We set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W}; \mathrm{w}(0) = x\}$ for $x \in \mathbb{R}^d$. Let $\mathrm{w} \in \mathcal{W}_x$. We restate theorem 1.1 from [9]:

Theorem 1.1 (Le Gall). *There exists a continuous strong Markov process with values in $\mathcal{W}_x$, $W = (W_s, s \geq 0)$, whose law is characterized by the following two properties.*

(i) *The lifetime process $\zeta = \big(\zeta_s = \zeta_{(W_s)}, s \geq 0\big)$ is a reflecting Brownian motion in $\mathbb{R}^+$.*
(ii) *Conditionally given $(\zeta_s, s \geq 0)$, the process $(W_s, s \geq 0)$ is a time-inhomogeneous continuous Markov process, whose transition kernel between times s and $s' \geq s$ is*

$$P_{s,s'}(\mathrm{w}, d\mathrm{w}') = Q^{\mathrm{w}}_{m(s,s'),\zeta_{s'}}(d\mathrm{w}'),$$

where $m(s,s') := \inf_{r\in[s,s']} \zeta_r$.

From now on we shall consider the canonical realization of the process W defined on the space $C(\mathbb{R}^+, \mathcal{W}_x)$. The law of W started at w is denoted by $\mathcal{E}_{\mathrm{w}}$. We will use the following consequence of (ii): outside a $\mathcal{E}_{\mathrm{w}}$-negligible set, for every $s' > s$, one has $W_s(t) = W_{s'}(t)$ for every $t \in [0, m(s,s')]$. We shall write $\mathcal{E}^*_{\mathrm{w}}$ for the law of the process W killed when its lifetime reaches zero. The distribution of W under $\mathcal{E}^*_{\mathrm{w}}$ can be characterized as in theorem

1.1, except that its lifetime process is distributed as a linear Brownian motion killed at its first hitting time of $\{0\}$. The state space for $(W, \mathcal{E}^*_{\rm w})$ is the space $\mathcal{W}^*_x = \mathcal{W}_x \cup \partial$, where ∂ is a cemetery point. The trivial path $\mathbf{x}$ such that $\zeta_{(\mathbf{x})} = 0$, $\mathbf{x}(0) = x$ is clearly a regular point for the process $(W, \mathcal{E}_{\rm w})$. Following [2] chapter 3, we can consider the excursion measure, $\mathbb{N}_x$, outside $\{\mathbf{x}\}$. The distribution of W under $\mathbb{N}_x$ can be characterized as in theorem 1.1, except that now the lifetime process ζ is distributed according to Itô measure of positive excursions of linear Brownian motion. We normalize $\mathbb{N}_x$ so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x \left[\sup_{s \geq 0} \zeta_s > \varepsilon \right] = \frac{1}{2\varepsilon}.$$

The Brownian snake enjoys a scaling property: if $\lambda > 0$, the law of the process $W^{(\lambda)}_s(t) = \lambda^{-1} W_{\lambda^4 s}(\lambda^2 t)$ under $\mathbb{N}_x$ is $\lambda^{-2} \mathbb{N}_{\lambda^{-1} x}$.

We recall the strong Markov property for the snake under $\mathbb{N}_x$ (see [12]). Let T be a stopping time of the natural filtration $\mathcal{F}^W$ of the process W. Assume $T > 0$ $\mathbb{N}_x$-a.e., and let F, H nonnegative measurable functionals on $C(\mathbb{R}^+, \mathcal{W}^*_x)$ such that F is $\mathcal{F}^W_T$ measurable. Then if θ denotes the usual shift operator, we have

$$\mathbb{N}_x \left[T < \infty; F \cdot H \circ \theta_T \right] = \mathbb{N}_x \left[T < \infty; F \cdot \mathcal{E}^*_{W_T}[H] \right].$$

Let $\sigma = \inf \{ s > 0; \zeta_s = 0 \}$ denote the duration of the excursion of ζ under $\mathbb{N}_x$. The range $\mathcal{R} = \mathcal{R}(W)$ of W is defined under $\mathbb{N}_x$ by

$$\mathcal{R} = \{ W_s(t); 0 \leq t \leq \zeta_s, 0 \leq s \leq \sigma \} = \left\{ \hat{W}_s; 0 \leq s \leq \sigma \right\}.$$

For every nonnegative measurable function F on $\mathcal{W}^*_x$, we have

$$\mathbb{N}_x \left[\int_0^\sigma F(W_s, \zeta_s) ds \right] = \int_0^\infty \mathbb{E}_x \left[F(\beta_{[0,t]}, t) \right] dt,$$

where $\beta_{[0,t]}$ is under $\mathbb{P}_x$ the restriction to $[0,t]$ of a Brownian motion in $\mathbb{R}^d$ started at $\beta_0 = x$. Now consider under $\mathbb{N}_x$ the continuous version $\left(l^t_s, t > 0, s \geq 0 \right)$ of the local time of ζ at level t and time s. We define a measure valued process Y on $\mathbb{R}^d$ by setting for every $t > 0$, for every $\varphi \in \mathcal{B}_{b+}(\mathbb{R}^d)$,

$$(Y_t, \varphi) = \int_0^\sigma dl^t_s \; \varphi(\hat{W}_s).$$

We shall sometimes write $Y_t(W)$ to recall that Y_t is a function of the Brownian snake. From the joint continuity of the local time and the continuity of the map $s \mapsto \hat{W}_s$, we get that $\mathbb{N}_x$-a.e., the process Y is continuous on $(0, \infty)$ for the Prohorov distance on M_f. Let $\varphi \in \mathcal{B}_{b+}(\mathbb{R}^d)$. We define on $\mathbb{R}^+ \times \mathbb{R}^d$ the function $v(t,x) = \mathbb{N}_x \left[1 - \exp -(Y_t, \varphi) \right]$, if $t > 0$, and $v(0,x) = \varphi(x)$. We will write $v(t)$ for the function $v(t, \cdot)$. We recall that the function v is the unique nonnegative measurable solution of the integral functional equation

$$v(t) + 2 \int_0^t ds \; P_s \left[v(t-s)^2 \right] = J(t) \qquad t \geq 0, \tag{1}$$

where $J(t,x) = P_t[\varphi](x)$, and $(P_t, t \geq 0)$ is the Brownian semi-group in $\mathbb{R}^d$. A few other remarks on the solution of (1) are presented in section 5.1 below.

1.2. **Super-Brownian motion and ISE.** Let us now recall the definition of super-Brownian motion and its connection with the Brownian snake. The second part of the next theorem is lemma 4.1 from [6]. Let $\nu \in M_f$.

Theorem 1.2. *There exists a continuous strong Markov process $X = (X_s, s \geq 0)$ defined on the canonical space $C(\mathbb{R}^+, M_f)$, whose law is characterized by the two following properties under $\mathbb{P}_\nu^X$.*

(i) $X_0 = \nu$, $\mathbb{P}_\nu^X$*-a.s.*
(ii) *For every $\varphi \in \mathcal{B}_{b+}(\mathbb{R}^d)$, $t \geq s > 0$, we have*

$$\mathbb{E}_\nu^X \left[\exp\left[-(X_t, \varphi)\right] \mid \sigma(X_u, 0 \leq u \leq s)\right] = \exp\left[-(X_s, v(t-s))\right],$$

where the function v is the unique nonnegative solution of (1) with $J(t) = P_t[\varphi]$.

Furthermore, for every integer $m \geq 1$, $t_m > \cdots > t_1 \geq 0$, $\varphi_1, \ldots, \varphi_m \in \mathcal{B}_{b+}(\mathbb{R}^d)$, we have

$$\mathbb{E}_\nu^X \left[\exp\left[- \sum_{\{i; t_i \leq t\}} (X_{t-t_i}, \varphi_i)\right]\right] = \exp\left[-(\nu, v(t))\right], \tag{2}$$

where v is the unique nonnegative solution to the integral equation (1) with right-hand side $J(t) = \sum_{\{i;t_i \leq t\}} P_{t-t_i}[\varphi_i]$.

Theorem 1.3 (Le Gall [9, 12]). *Let $\sum_{i\in I} \delta_{W^i}$ be a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int \nu(dx)\mathbb{N}_x[\cdot]$, then the process Z defined by $Z_0 = \nu$ and $Z_t = \sum_{i\in I} Y_t(W^i)$ if $t > 0$, is distributed according to $\mathbb{P}_\nu^X$.*

We deduce from the normalization of $\mathbb{N}_x$ that $\mathbb{N}_x[Y_t \neq 0] = 1/2t < \infty$. This implies that for every $t > 0$, there is only a finite number of indices $i \in I$ such that the process $(Y_s(W^i), s \geq t)$ is nonzero.

We now recall the connection between ISE and Brownian snake. There exists a unique collection $\left(\mathbb{N}_0^{(r)}, r > 0\right)$ of probability measure on $C(\mathbb{R}^+, \mathcal{W}_0^*)$ such that:

1. For every $r > 0$, $\mathbb{N}_0^{(r)}[\sigma = r] = 1$.
2. For every $\lambda > 0$, $r > 0$, F, nonnegative measurable functional on $C(\mathbb{R}^+, \mathcal{W}_0^*)$,
$$\mathbb{N}_0^{(r)}\left[F(W^{(\lambda)})\right] = \mathbb{N}_0^{(\lambda^{-4}r)}\left[F(W)\right].$$
3. For every nonnegative measurable functional F on $C(\mathbb{R}^+, \mathcal{W}_0^*)$,

$$\mathbb{N}_0[F] = \frac{1}{\sqrt{2\pi}} \int_0^\infty dr\; r^{-3/2} \mathbb{N}_0^{(r)}[F]. \tag{3}$$

The measurability of the mapping $r \mapsto \mathbb{N}_0^{(r)}[F]$ follows from the scaling property 2. Under $\mathbb{N}_0^{(1)}$, the distribution of W is characterized as in theorem 1.1, except that the lifetime process is distributed according to the normalized Itô measure. The law of the ISE is the law of the continuous tree associated to $\sqrt{2}W$, under $\mathbb{N}_0^{(1)}$ (see corollary 4 in [10] and [1]). In particular the law of the support of ISE is the law of $\sqrt{2}\mathcal{R}$ under $\mathbb{N}_0^{(1)}$, where we set $\lambda A = \{x; \lambda^{-1}x \in A\}$.

1.3. **Hitting probabilities for the Brownian snake.** We now recall a few results from [11]. Let $\mathrm{w} \in \mathcal{W} \cup C(\mathbb{R}^+, \mathbb{R}^d)$, we introduce the first hitting time of $A \in \mathcal{B}(\mathbb{R}^d)$:

$$\tau_A(\mathrm{w}) = \inf \{t \geq 0; \mathrm{w}(t) \in A\},$$

with the usual convention $\inf \emptyset = \infty$. We omit w when there is no risk of confusion. Consider the Brownian snake W, and set

$$T_{(y,\varepsilon)} = \inf \left\{s \geq 0; \exists t \in [0, \zeta_s], W_s(t) \in \bar{B}(y, \varepsilon)\right\},$$

where $B(y,\varepsilon)$ is the open ball in $\mathbb{R}^d$ centered at y with radius $\varepsilon > 0$, and $\bar{B}(y,\varepsilon)$ its closure. We know from [12] that the function defined on $\mathbb{R}^d \backslash \bar{B}(0,\varepsilon)$,

$$u_\varepsilon(y) := \mathbb{N}_0 \left[T_{(y,\varepsilon)} < \infty\right] = \mathbb{N}_0 \left[\mathcal{R} \cap \bar{B}(y,\varepsilon) \neq \emptyset\right] = \mathbb{N}_{-y} \left[\mathcal{R} \cap \bar{B}(0,\varepsilon) \neq \emptyset\right],$$

is the maximal nonnegative solution on $\mathbb{R}^d \backslash \bar{B}(0,\varepsilon)$ of

$$\Delta u = 4u^2.$$

This result was first proved in a more general setting by Dynkin [7] in terms of superprocesses. The function u_ε is strictly positive on $\mathbb{R}^d \backslash \bar{B}(0,\varepsilon)$. For every $y_0 \in \partial B(0,\varepsilon)$, we have

$$\lim_{y \in \bar{B}(0,\varepsilon)^c; y \to y_0} u_\varepsilon(y) = \infty.$$

Scaling and symmetry arguments show that for every $y \in \mathbb{R}^d \backslash \bar{B}(0,\varepsilon)$,

$$u_\varepsilon(y) = \varepsilon^{-2} u_1\left(\frac{|y|}{\varepsilon}\right), \tag{4}$$

where the function $u_1(r)$, $r \in (1,\infty)$ is the maximal nonnegative solution on $(1,\infty)$ of

$$u_1''(r) + \frac{d-1}{r} u_1'(r) = 4u_1^2(r).$$

It is easy to see that the function u_1 is decreasing. In section 5.2 we give the asymptotic expansion of u_1 at infinity.

We give the following result on the probability of the event $\left\{T_{(y,\varepsilon)} < \infty\right\}$ (see lemma 2.1 of [11]). Assume $x_0 \notin \bar{B}(y,\varepsilon)$. Then $\mathbb{N}_{x_0}$-a.e. for every $T \geq 0$, we have

$$\begin{aligned} \mathcal{E}^*_{W_T}\left[T_{(y,\varepsilon)} < \infty\right] &= 2\int_0^{\zeta_T \wedge \tau_{B(y,\varepsilon)}(W_T)} dt\; u_\varepsilon(W_T(t) - y))\, \mathrm{e}^{\left[-2\int_0^t u_\varepsilon(W_T(s)-y)ds\right]} \\ &= 1 - \exp\left[-2\int_0^{\zeta_T \wedge \tau_{B(y,\varepsilon)}(W_T)} u_\varepsilon(W_T(s) - y) ds\right]. \end{aligned} \tag{5}$$

Let $x_0, x \in \mathbb{R}^d$. We will now describe the law of $W_{T_{(x,\varepsilon)}}$ under $\mathbb{N}_{x_0}[\cdot \mid T_{(x,\varepsilon)} < \infty]$. First of all we denote by β a Brownian motion in $\mathbb{R}^d$ started at x_0 under $\mathbb{P}_{x_0}$. Assume $x_0 \notin \bar{B}(x,\varepsilon)$. Corollary 2.3 from [11] ensures that there exists $\mathbb{P}_{x_0}$-a.s. a unique continuous process $x^\varepsilon = (x^\varepsilon_t, 0 \leq t \leq \tau^\varepsilon)$ taking values in $\mathbb{R}^d$ such that for every $\eta \in (0, |x - x_0| - \varepsilon)$, for every $t \leq \tau^\varepsilon_\eta = \inf\{s \geq 0; |x^\varepsilon_s - x| \leq \varepsilon + \eta\}$,

$$x^\varepsilon_t = \beta_t + \int_0^t \frac{\nabla u_\varepsilon(x^\varepsilon_s - x)}{u_\varepsilon(x^\varepsilon_s - x)} ds,$$

furthermore, $\mathbb{P}_{x_0}$-a.s. $\tau^\varepsilon = \lim_{\eta\to 0}\tau^\varepsilon_\eta < \infty$ and $|x^\varepsilon_{\tau^\varepsilon} - x| = \varepsilon$. We also recall that thanks to Girsanov's theorem, we have for every nonnegative measurable function F on $C([0,t],\mathbb{R}^d)$

$$\mathbb{E}_{x_0}\left[\tau^\varepsilon > t; F\left(x^\varepsilon_{[0,t]}\right)\right] = \mathbb{E}_{x_0}\left[\tau_{B(x,\varepsilon)}(\beta) > t; F\left(\beta_{[0,t]}\right)\frac{u_\varepsilon(\beta_t - x)}{u_\varepsilon(x_0 - x)}\exp\left[-2\int_0^t u_\varepsilon(\beta_s - x)ds\right]\right],$$

where $x^\varepsilon_{[0,t]}$ and $\beta_{[0,t]}$ are the restriction of x^ε and β to $[0,t]$. The law of x^ε under $\mathbb{P}_{x_0}$ can be interpreted as a probability measure on $\mathcal{W}^*_{x_0}$. Consider the closed set

$$A = \left\{\mathrm{w} \in \mathcal{W}^*_{x_0}; \tau_{\bar{B}(x,\varepsilon)}(\mathrm{w}) < \infty\right\}.$$

It has been proved in [11] that its capacitary measure with respect to the Brownian snake with initial point x_0 is exactly $u_\varepsilon(x_0 - x)$ times the law of x^ε under $\mathbb{P}_{x_0}$. It is not hard to check however that the normalized capacitary measure can be interpreted as the hitting distribution under $\mathbb{N}_{x_0}$ (cf [13], this is proved in a way similar to the classical interpretation of the capacitary measure as a last exit distribution, see e.g. Port and Stone [17]). Thus we deduce that for every nonnegative measurable function F on $\mathcal{W}^*_{x_0}$, we have

$$\mathbb{N}_{x_0}\left[T_{(x,\varepsilon)} < \infty; F(W_{T_{(x,\varepsilon)}}, \zeta_{T_{(x,\varepsilon)}})\right] = u_\varepsilon(x_0 - x)\mathbb{E}_{x_0}\left[F(x^\varepsilon, \tau^\varepsilon)\right].$$

Hence for every $t \geq 0$, and every nonnegative measurable function F on $C([0,t],\mathbb{R}^d)$, we have

$$\mathbb{N}_{x_0}\left[T_{(x,\varepsilon)} < \infty; \zeta_{T_{(x,\varepsilon)}} > t; F\left((W_{T_{(x,\varepsilon)}}(s), s \in [0,t])\right)\right] = \mathbb{E}_{x_0}\left[\tau_{B(x,\varepsilon)} > t; F\left(\beta_{[0,t]}\right)u_\varepsilon(\beta_t - x)\exp\left[-2\int_0^t u_\varepsilon(\beta_s - x)ds\right]\right]. \tag{6}$$

Finally we shall use the following inequality, that can be derived from the Feynman-Kac formula (use the fact that u_ε solves $\Delta u = 4u_\varepsilon u$)

$$u_\varepsilon(x) \geq 2\mathbb{E}_0\left[\int_0^{\tau_{B(x,\varepsilon)}} dt\;\; u_\varepsilon(\beta_t - x)^2\exp\left[-4\int_0^t u_\varepsilon(\beta_s - x)ds\right]\right]. \tag{7}$$

There is in fact equality in (7) (see the remark on page 293 of [11]).

2. A property of the range of super-Brownian motion

For $A \in \mathcal{B}(\mathbb{R}^d)$, $\varepsilon > 0$, we set $A^\varepsilon := \left\{x \in \mathbb{R}^d; d(x,A) \leq \varepsilon\right\}$, with $d(x,A) = \inf\{|x-y|; y \in A\}$. We will write $|A|$ for the Lebesgue measure of A. We also set

$$C_0 = \mathrm{a}_0 2\pi^{d/2}\Gamma([d-2]/2)^{-1},$$

where the constant a_0 is defined in lemma 5.1 (see also the remark below the lemma). We set $\mathcal{R}_t(X) = \mathcal{C}l\left(\bigcup_{s\geq t}\operatorname{supp} X_s\right)$. Let $\varphi_d(\varepsilon) = \varepsilon^{4-d}$ if $d \geq 5$ and $\varphi_4(\varepsilon) = \log(1/\varepsilon)$ for $\varepsilon > 0$.

Theorem 2.1. *Let $\nu \in M_f$. For every Borel set $A \subset \mathbb{R}^d$, $d \geq 4$, for every $t > 0$, $\mathbb{P}^X_\nu$-a.s.*

$$\lim_{\varepsilon\to 0}\varphi_d(\varepsilon)\,|\mathcal{R}_t(X)^\varepsilon \cap A| = C_0\int_t^\infty ds\;(X_s, \mathbf{1}_A). \tag{8}$$

If there exists $\rho < 4$ such that $\lim_{\varepsilon\to 0}\varepsilon^{\rho-d}\,|(\operatorname{supp}\nu)^\varepsilon| = 0$ then (8) holds with $t = 0$.

Let K a compact subset of $\mathbb{R}^d$. We consider the measure $\phi(K)$ defined by $\phi(K)(A) = |K \cap A|$. Since the set $\mathcal{R}_t(X)$ is compact for $t > 0$, the theorem implies that a.s. the sequence of measures $(\varphi_d(\varepsilon)\phi(\mathcal{R}_t(X)^\varepsilon), \varepsilon > 0)$ converges weakly to $C_0 \int_t^\infty ds\ (X_s, \mathbf{1}_A)$.

Let us recall the main theorem of [19] (see also [16]).

Theorem 2.2 (Tribe). *Let A a bounded Borel set in $\mathbb{R}^d$, $d \geq 3$. Fix $t > 0$ and $\nu \in M_f$. Then there exists a positive constant α_0 depending only on d such that*

$$\lim_{\varepsilon \to 0} \varepsilon^{2-d} \left|(\text{supp } X_t)^\varepsilon \cap A\right| = \alpha_0 (X_t, \mathbf{1}_A),$$

where the convergence holds $\mathbb{P}_\nu^X$-a.s. and in $L^2(\mathbb{P}_\nu^X)$.

We shall deduce theorem 2.1 from the next proposition on the range of the Brownian snake, whose proof will be given in the next section. For $\theta \in (0,1)$, we set $h_{d,\theta}(\varepsilon) = \varepsilon^{1-\theta}$ if $d \geq 5$ and $h_{4,\theta}(\varepsilon) = \log(1/\varepsilon)^{-1/\theta}$ for $\varepsilon \in (0,1)$. For short we will write h_d for $h_{d,\theta}$.

Proposition 2.3. *Let $d \geq 4$. For every $\theta \in (0, 1/d)$ and every $R_0 > 0$, there exists a constant $\kappa = \kappa(\theta) > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, for every x_0 with $|x_0| \leq R_0$, and every Borel set $A \subset \bar{B}(0, R_0)$, we have*

$$\left| \mathbb{N}_{x_0} \left[\varphi_d(\varepsilon) \left| \mathcal{R}(W)^\varepsilon \cap A \cap \bar{B}(x_0, h_d(\varepsilon))^c \right| - C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_A) \right] \right| \leq h_d(\varepsilon)^{\kappa/2},$$

and

$$\mathbb{N}_{x_0} \left[\left[\varphi_d(\varepsilon) \left| \mathcal{R}(W)^\varepsilon \cap A \cap \bar{B}(x_0, h_d(\varepsilon))^c \right| - C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_A) \right]^2 \right] \leq h_d(\varepsilon)^\kappa.$$

Remark. We have trivially $B(x_0, \varepsilon) \subset \mathcal{R}(W)^\varepsilon$, $\mathbb{N}_{x_0}$-a.e. Since $\mathbb{N}_{x_0}$ is an infinite measure, $\mathbb{N}_{x_0}\left[|\mathcal{R}(W)^\varepsilon \cap B(x_0, \delta)|\right] = \infty$ for every $\varepsilon, \delta > 0$. This is the reason why we consider $A \cap \bar{B}(x_0, h_d(\varepsilon))^c$ rather than A in the previous proposition.

We first give a consequence of this proposition.

Corollary 2.4. *Let $d \geq 4$. For every Borel set $A \subset \mathbb{R}^d$, $\mathbb{N}_{x_0}$-a.e., we have*

$$\lim_{\varepsilon \to 0} \varphi_d(\varepsilon) \left| \mathcal{R}(W)^\varepsilon \cap A \right| = C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_A).$$

The results holds $\mathbb{N}_0^{(1)}$-a.s. if $|\partial A| = 0$.

Proof of corollary 2.4. Since $\mathbb{N}_{x_0}$-a.e. the range $\mathcal{R}(W)$ is bounded, we only need to consider a bounded Borel set A. Let $\kappa > 0$ be fixed as in proposition 2.3. Let ε_n such that $h_d(\varepsilon_n) = n^{-2/\kappa}$ for $n \geq 1$. Using the Borel-Cantelli lemma and the second upper bound of proposition 2.3, we get that the sequence $(\varphi_d(\varepsilon_n) \left|\mathcal{R}(W)^{\varepsilon_n} \cap A\right|, n \geq 1)$ converges $\mathbb{N}_{x_0}$-a.e. to $C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_A)$. But for $\varepsilon' \leq \varepsilon$, since $\mathcal{R}(W)^{\varepsilon'} \subset \mathcal{R}(W)^\varepsilon$, we have

$$\varphi_d(\varepsilon') \left| \mathcal{R}(W)^{\varepsilon'} \cap A \right| \leq \varphi_d(\varepsilon) \left| \mathcal{R}(W)^\varepsilon \cap A \right| \varphi_d(\varepsilon')/\varphi_d(\varepsilon).$$

A monotonicity argument using the fact that $\varphi_d(\varepsilon_{n+1})/\varphi_d(\varepsilon_n)$ converges to 1, completes the proof of the first part.

The above result implies that $\mathbb{N}_0$-a.e. the sequence of measures $(\varphi_d(\varepsilon)\phi(\mathcal{R}(W)^\varepsilon), \varepsilon > 0)$ converges weakly to $C_0 \int_0^\infty ds\, Y_s$. Using (3) we see this convergence also holds dr-a.e. $\mathbb{N}_0^{(r)}$-a.s. By the scaling property the Brownian snake and the family $(\mathbb{N}_0^{(r)}, r > 0)$, we get this convergence holds $\mathbb{N}_0^{(1)}$-a.s. Thus we have for every Borel set $A \subset \mathbb{R}^d$, $\mathbb{N}_0^{(1)}$-a.s.

$$C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_{\mathrm{Int}(A)}) \leq \liminf_{\varepsilon\to 0} \varphi_d(\varepsilon)\, |\mathcal{R}(W)^\varepsilon \cap A| \\ \leq \limsup_{\varepsilon\to 0} \varphi_d(\varepsilon)\, |\mathcal{R}(W)^\varepsilon \cap A| \leq C_0 \int_0^\infty ds\ (Y_s, \mathbf{1}_{\bar{A}}),$$

where $\mathrm{Int}(A)$ denotes the interior of A. To prove the second part of the corollary we just need to check that if $|\partial A| = 0$ then $\int_0^\infty ds\ (Y_s, \mathbf{1}_{\mathrm{Int}(A)}) = \int_0^\infty ds\ (Y_s, \mathbf{1}_{\bar{A}})$. It is enough to prove that $|A| = 0$ implies $\int_0^\infty ds\ (Y_s, \mathbf{1}_A) = 0$ $\mathbb{N}_0^{(1)}$-a.s. Conditioning by the lifetime process, we get

$$\mathbb{N}_0^{(1)} \left[\int_0^\infty ds\ (Y_s, \mathbf{1}_A)\right] = \mathbb{N}_0^{(1)} \left[\int_0^1 dt\ \mathbf{1}_A(\hat{W}_t)\right] = \int_0^1 dt\ \mathbb{N}_0^{(1)} \left[P_{\zeta_t}[\mathbf{1}_A](0)\right].$$

This is equal to zero if $|A| = 0$. This ends the proof of the second part of the corollary. □

As a byproduct of the proof we get that $\mathbb{N}_{x_0}$-a.e. and $\mathbb{N}_0^{(1)}$-a.s. the sequence of measures $(\varphi_d(\varepsilon)\phi(\mathcal{R}(W)^\varepsilon), \varepsilon > 0)$ converges weakly to $C_0 \int_0^\infty ds\, Y_s$.

We first state some straightforward consequences of (4) and lemma 5.1. We say that $\varepsilon_0 > 0$ satisfies the condition (C) if $\varepsilon_0^{-\theta} \geq 4/3$ if $d \geq 5$ or $\log(1/\varepsilon_0) \geq 4\log(2/\theta)/\theta$ if $d = 4$. For $\theta \in (0, 1/4)$ this implies that for $\varepsilon \in (0, \varepsilon_0)$, $h_4(\varepsilon)/\varepsilon \geq 4/3$ and that

$$\log(\log(1/\varepsilon))/[\theta \log(1/\varepsilon)] \leq 1/2. \tag{9}$$

For $d \geq 4$, $\theta \in (0, 1/d)$, there exists a constant b_1 such that for every ε satisfying (C), $x \notin B(0, h_d(\varepsilon))$ we have

$$u_\varepsilon(x) \leq \mathrm{b}_0 \varphi_d(\varepsilon)^{-1} |x|^{2-d}, \tag{10}$$

$$u_\varepsilon(x) \leq \varphi_d(\varepsilon)^{-1} |x|^{2-d} \left[\mathrm{a}_0 + \mathrm{b}_1 h_d(\varepsilon)^{\theta/2}\right]. \tag{11}$$

For $|x| > \varepsilon$, we have

$$u_\varepsilon(x) \geq \mathrm{a}_0 \varphi_d(\varepsilon)^{-1} |x|^{2-d} \quad \text{if } d \geq 5, \tag{12}$$

$$u_\varepsilon(x) \geq \mathrm{a}_0 \varphi_4(\varepsilon)^{-1} |x|^{-2} \left[1 + \log(2|x|)/\log(1/\varepsilon)\right]^{-1} \quad \text{if } d = 4. \tag{13}$$

We will also often use the following inequality for ε satisfying (C): $\varphi_d(\varepsilon) h_d(\varepsilon)^d \leq h_d(\varepsilon)^3$.

Proof of theorem 2.1. Recall that for every $t > 0$, $\mathbb{P}_\nu^X$ a.s. the set $\mathcal{R}_t(X)$ is bounded. Thus we only need to consider a bounded Borel set A. Thanks to the Markov property of X at time t and theorem 2.2 it is clearly enough to prove the second part of theorem 2.1. Let $\nu \in M_f$ and $\rho < 4$ such that $\lim_{\varepsilon\to 0} \varepsilon^{\rho-d} |(\mathrm{supp}\ \nu)^\varepsilon| = 0$. For short we write a.s. for $\mathbb{P}_\nu^X$-a.s.

First step. Recall we can write for every $t > 0$, $X_t = \sum_{i\in I} Y_t(W^i)$, where $\sum_{i\in I} \delta_{W^i}$ is a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity measure $\int \nu(dx)\mathbb{N}_x[\cdot]$. We let x_0^i denote the starting point of the Brownian snake W^i (i.e. $x_0^i = W_0^i(0)$). Notice that a.s. for every $i \in I$, $x_0^i \in \mathrm{supp}\ \nu$, which is bounded thanks to the hypothesis on $\mathrm{supp}\ \nu$. Fix $\theta \in (0, 1/d)$ such that $d-\rho \geq (d-4)/(1-\theta)$ (and $\theta < 4-\rho$ if $d=4$). Fix R_0 such that supp $\nu \subset B(0,R_0)$. Let κ and $\varepsilon_0 < 1$ fixed as in proposition 2.3. We notice that for every bounded Borel set $A \subset B(0,R_0)$,

$$\varphi_d(\varepsilon)\,|\mathcal{R}_0(X)^\varepsilon \cap A| \leq \sum_{i\in I} V_\varepsilon(W^i) + \varphi_d(\varepsilon)\left|A \cap (\text{supp } \nu)^{h_d(\varepsilon)}\right|,$$

where

$$V_\varepsilon(W^i) = \varphi_d(\varepsilon)\left|\mathcal{R}(W^i)^\varepsilon \cap A \cap \bar{B}(x_0^i, h_d(\varepsilon))^c\right|.$$

We set $V_0(W^i) = C_0 \int_0^\infty ds\; (Y_s(W^i), \mathbf{1}_A)$. We use the second moment formula for a Poisson measure to get:

$$\mathbb{E}^X_\nu\left[\left[\sum_{i\in I} V_\varepsilon(W^i) - \sum_{i\in I} V_0(W^i)\right]^2\right] = \int \nu(dx)\mathbb{N}_x\left[[V_\varepsilon(W) - V_0(W)]^2\right]$$
$$+ \left[\int \nu(dx)\mathbb{N}_x\left[V_\varepsilon(W) - V_0(W)\right]\right]^2.$$

We deduce from proposition 2.3 that for every $\varepsilon \in (0,\varepsilon_0]$,

$$\mathbb{E}^X_\nu\left[\left[\sum_{i\in I} V_\varepsilon(W^i) - \sum_{i\in I} V_0(W^i)\right]^2\right] \leq [(\nu,\mathbf{1}) + (\nu,\mathbf{1})^2] h_d(\varepsilon)^\kappa.$$

Notice the hypothesis on supp ν and θ imply that $\lim_{\varepsilon\to 0} \varphi_d(\varepsilon)\left|(\text{supp } \nu)^{h_d(\varepsilon)}\right| = 0$. Arguments similar to those used in the first part of the proof of corollary 2.4 show then a.s.

$$\lim_{\varepsilon\to 0} \sum_{i\in I} V_\varepsilon(W^i) = \sum_{i\in I} V_0(W^i).$$

Notice we have $\sum_{i\in I} V_0(W^i) = C_0 \int_0^\infty ds\; (X_s, \mathbf{1}_A)$. Using the above remark on supp ν, we deduce that a.s.

$$\limsup_{\varepsilon\to 0} \varphi_d(\varepsilon)\,|\mathcal{R}_0(X)^\varepsilon \cap A| \leq C_0 \int_0^\infty ds\; (X_s, \mathbf{1}_A).$$

Second step. To get a lower bound, consider an increasing sequence $(E_p, p\geq 1)$ of measurable subsets of $E = C(\mathbb{R}^+, \mathcal{W})$ such that $\bigcup_{p\geq 1} E_p = E$ and $\int \nu(dx)\mathbb{N}_x[E_p] = \alpha_p < \infty$. (For instance we can take $E_p = \left\{W; \sup_{s\geq 0} \zeta_s \geq 1/p\right\}$.) Then a.s. the set $I_p = \left\{i \in I; W^i \in E_p\right\}$ is finite. We have

$$\varphi_d(\varepsilon)\,|\mathcal{R}_0(X)^\varepsilon \cap A| \geq \sum_{i\in I_p} V_\varepsilon(W^i) - \sum_{(i,j)\in I_p^2;\, i\neq j} U_\varepsilon(W^i, W^j),$$

where

$$U_\varepsilon(W^i,W^j) = \varphi_d(\varepsilon)\left|\mathcal{R}(W^i)^\varepsilon \cap \mathcal{R}(W^j)^\varepsilon \cap A \cap \bar{B}(x_0^i, h_d(\varepsilon))^c \cap \bar{B}(x_0^j, h_d(\varepsilon))^c\right|$$
$$= \varphi_d(\varepsilon) \int_{A\cap \bar{B}(x_0^i,h_d(\varepsilon))^c \cap \bar{B}(x_0^j,h_d(\varepsilon))^c} dy\; \mathbf{1}_{\left\{T_{(y,\varepsilon)}(W^i)<\infty\right\}} \mathbf{1}_{\left\{T_{(y,\varepsilon)}(W^j)<\infty\right\}}.$$

Arguments similar to those of the first step show that a.s.

$$\lim_{\varepsilon\to 0}\sum_{i\in I_p} V_\varepsilon(W^i) = \sum_{i\in I_p} V_0(W^i) = \sum_{i\in I_p} C_0 \int_0^\infty ds\ (Y_s(W^i), \mathbf{1}_A).$$

Now conditionally on the cardinal of I_p, the Brownian snakes $(W^i, i \in I_p)$ are independent and have the same law: $\mu_p = \alpha_p^{-1}\int \nu(dx)\mathbb{N}_x[\cdot \cap E_p]$. For two independent Brownian snakes (W, W') under $\mu_p \otimes \mu_p$, we get using (10), that for ε satisfying (C),

$$\begin{aligned}
\mu_p \otimes \mu_p[U_\varepsilon(W,W')] &\leq \alpha_p^{-2} \iint \nu(dx_0)\nu(dx_0')\mathbb{N}_{x_0}\otimes \mathbb{N}_{x_0'}[U_\varepsilon(W,W')] \\
&\leq \varphi_d(\varepsilon)\alpha_p^{-2}\iint \nu(dx_0)\nu(dx_0')\int_{A\cap \bar B(x_0,h_d(\varepsilon))^c\cap \bar B(x_0',h_d(\varepsilon))^c} dy \\
&\qquad\qquad \left[\mathrm{b}_0\varphi_d(\varepsilon)^{-1}\left|y-x_0\right|^{2-d}\right]\left[\mathrm{b}_0\varphi_d(\varepsilon)^{-1}\left|y-x_0'\right|^{2-d}\right] \\
&\leq \varphi_d(\varepsilon)^{-1}\alpha_p^{-2}(\nu,\mathbf{1})^2\mathrm{b}_0^2 \sup_{x_0\in\mathbb{R}^d}\int_{\bar B(0,R_0)\setminus \bar B(x_0,h_d(\varepsilon))} dy\left|y-x_0\right|^{4-2d} \\
&\leq c\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)^{4-d} \quad \text{if } d\geq 5 \\
&\leq c\varphi_4(\varepsilon)^{-1}\log(\log(1/\varepsilon)) \quad \text{if } d=4 \\
&\leq ch_d(\varepsilon)^{\theta/2} \quad \text{if } d\geq 4,
\end{aligned}$$

where the constant c is independent of ε and A. Using the Borel-Cantelli lemma for the sequence $(h_d(\varepsilon_n) = n^{-4/\theta}, n \geq 1)$, and a monotonicity argument, we get that $\mu_p \otimes \mu_p$-a.s. $\lim_{\varepsilon\to 0} U_\varepsilon(W,W') = 0$. Then since the cardinal of I_p is a.s. finite, we get that for every integer $p \geq 1$, a.s.,

$$\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_p^2;\ i\neq j} U_\varepsilon(W^i, W^j) = 0.$$

We deduce that for every integer $p \geq 1$, a.s.

$$\liminf_{\varepsilon\to 0}\varphi_d(\varepsilon)\left|\mathcal{R}_0(X)^\varepsilon\cap A\right| \geq \sum_{i\in I_p} C_0\int_0^\infty ds\ (Y_s(W^i),\mathbf{1}_A).$$

We get the lower bound by letting $p \to \infty$. This and the upper bound of the first step ends the proof of the theorem. □

3. Proof of proposition 2.3

We shall use many times in the sequel the fact that $\int_0^\infty ds\ (Y_s, \mathbf{1}_A) = \int_0^\sigma ds\ \mathbf{1}_A(\hat W_s)$ $\mathbb{N}_{x_0}$-a.e. We assume $d \geq 4$. We recall easy equalities, which can readily be deduced from the results of section 5.1. For every $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathbb{N}_x\left[\int_0^\sigma ds\ \mathbf{1}_A(\hat W_s)\right] = \int_A dy\ G(x,y), \tag{14}$$

where G is the Green kernel in $\mathbb{R}^d$: $G(x,y)=2^{-1}\pi^{-d/2}\Gamma([d-2]/2)\,|x-y|^{2-d}$, and

$$\mathbb{N}_x\left[\left[\int_0^\sigma ds\ \mathbf{1}_A(\hat{W}_s)\right]^2\right]=4\int dy\ G(x,y)\left[\int_A dz\ G(y,z)\right]^2. \tag{15}$$

We can also compute the first moment under $\mathcal{E}^*_{\mathrm{w}}$. For every $A\in\mathcal{B}(\mathbb{R}^d)$, $\mathrm{w}\in\mathcal{W}$, we have with $\zeta=\zeta_{(\mathrm{w})}$,

$$\mathcal{E}^*_{\mathrm{w}}\left[\int_0^\sigma ds\ \mathbf{1}_A(\hat{W}_s)\right]=2\int_0^\zeta dt\ \mathbb{N}_{\mathrm{w}(t)}\left[\int_0^\sigma ds\ \mathbf{1}_A(\hat{W}_s)\right]=2\int_0^\zeta dt\int_A dy\ G(\mathrm{w}(t),y). \tag{16}$$

Thanks to the space invariance of the law of the Brownian snake, we shall only consider the case $x_0=0$ and $A\subset\bar{B}(0,R_0)$, for R_0 fixed. We fix $\theta\in(0,1/d)$ and $R_0>1$. Let $\varepsilon'_0>0$ satisfying (C). We consider $\varepsilon\in(0,\varepsilon'_0)$. In this section, we denote by c, $c_1,c_2,\dots$ positive constants whose values depend only on d,θ and R_0. The value of c may vary from line to line. For short we shall write $A_\varepsilon=A\cap\bar{B}(0,h_d(\varepsilon))^c$ (not to be confused with A^ε) and $\mathcal{R}$ for $\mathcal{R}(W)$.

We first consider the case $d\geq 5$. Notice that

$$\mathbb{N}_0\left[|\mathcal{R}^\varepsilon\cap A_\varepsilon|\right]=\int_{A_\varepsilon}dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty\right]=\int_{A_\varepsilon}dx\ u_\varepsilon(x).$$

Thus we deduce from (12) and (11), that for $\varepsilon\in(0,\varepsilon'_0)$,

$$\begin{aligned}\mathrm{a}_0\varepsilon^{d-4}\int_A dx\ |x|^{2-d}&-\mathrm{a}_0\varepsilon^{d-4}\int_{B(0,\varepsilon^{1-\theta})}dx\ |x|^{2-d}\\&\leq\mathbb{N}_0\left[|\mathcal{R}^\varepsilon\cap A_\varepsilon|\right]\leq\varepsilon^{d-4}[\mathrm{a}_0+\mathrm{b}_1h_d(\varepsilon)^{\theta/2}]\int_A dx\ |x|^{2-d}.\end{aligned}$$

Therefore using also (14), we have

$$\left|\mathbb{N}_{x_0}\left[\varepsilon^{4-d}\,|\mathcal{R}(W)^\varepsilon\cap A_\varepsilon|-C_0\int_0^\infty ds\ (Y_s,\mathbf{1}_A)\right]\right|\leq ch_d(\varepsilon)^{\theta/2}.$$

Thus we get the first bound of proposition 2.3 (take $\kappa<\theta$ and ε_0 small enough). The proof is similar for $d=4$ (use (29) instead of (28) and the fact that $|x|$ is bounded by R_0).

Now we will prove the second bound. To this end we have to find an upper bound on $I=\mathbb{N}_0\left[|\mathcal{R}^\varepsilon\cap A_\varepsilon|^2\right]$ and a lower bound on $J=\mathbb{N}_0\left[|\mathcal{R}^\varepsilon\cap A_\varepsilon|\int_0^\sigma ds\ \mathbf{1}_A(\hat{W}_s)\right]$.

3.1. **An upper bound on I.** The term I can also be written

$$I=\iint_{A_\varepsilon\times A_\varepsilon}dx\ dy\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty;T_{(y,\varepsilon)}<\infty\right].$$

Consider the above integral as the sum of the integral over $|x-y|\leq 2h_d(\varepsilon)$ (denoted by I_1) and the one over $|x-y|>2h_d(\varepsilon)$ (denoted by I_2). Using (10) we easily obtain an upper bound on I_1:

$$I_1 \le |B(0,2h_d(\varepsilon))| \int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty\right]$$
$$\le c h_d(\varepsilon)^d \int_{A_\varepsilon} dx\ \varphi_d(\varepsilon)^{-1}\mathrm{b}_0\,|x|^{2-d} \le c_1\varphi_d(\varepsilon)^{-2}h_d(\varepsilon)^3.$$

Notice the event $\left\{T_{(x,\varepsilon)}<\infty; T_{(y,\varepsilon)}<\infty\right\}$ is a subset of

$$\left\{T_{(x,\varepsilon)}<\infty; T_{(y,\varepsilon)}\circ\theta_{T_{(x,\varepsilon)}}<\infty\right\}\cup\left\{T_{(y,\varepsilon)}<\infty; T_{(x,\varepsilon)}\circ\theta_{T_{(y,\varepsilon)}}<\infty\right\},$$

where θ_t is the usual shift operator. By symmetry, we get

$$(17) I_2 \le 2\iint_{A_\varepsilon\times A_\varepsilon} dx\ dy\ \mathbf{1}_{\{|x-y|>2h_d(\varepsilon)\}}\mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; T_{(y,\varepsilon)}\circ\theta_{T_{(x,\varepsilon)}}<\infty\right].$$

Using the strong Markov property of the Brownian snake under $\mathbb{N}_0$ at the stopping time $T_{(x,\varepsilon)}$ and (5), we see that the quantity $\mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; T_{(y,\varepsilon)}\circ\theta_{T_{(x,\varepsilon)}}<\infty\right]$ is equal to

$$\mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; 2\int_0^{\zeta_{T_{(x,\varepsilon)}}\wedge\tau_{B(y,\varepsilon)}(W_{T(x,\varepsilon)})} dt\ u_\varepsilon\left(W_{T_{(x,\varepsilon)}}(t)-y\right)\mathrm{e}^{\left[-2\int_0^t u_\varepsilon\left(W_{T_{(x,\varepsilon)}}(s)-y\right)ds\right]}\right].$$

Finally the law of the stopped path $W_{T_{(x,\varepsilon)}}$ under $\mathbb{N}_0$ is given by (6). Thus the previous expression is equal to

$$2\int_0^\infty dt\ \mathbb{E}_0\left[\tau_{B(x,\varepsilon)}>t;\tau_{B(y,\varepsilon)}>t; u_\varepsilon(\beta_t-x)u_\varepsilon(\beta_t-y)\,\mathrm{e}^{\left[-2\int_0^t ds\ [u_\varepsilon(\beta_s-x)+u_\varepsilon(\beta_s-y)]\right]}\right].$$

We substitute this last expression for $\mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; T_{(y,\varepsilon)}\circ\theta_{T_{(x,\varepsilon)}}<\infty\right]$ in (17), and then decompose the right-hand side of (17) in three terms by considering the integral in $dxdy$ on the sets $|\beta_t-x|\wedge|\beta_t-y|>h_d(\varepsilon)$ (integral I_{21}), $|\beta_t-x|\le h_d(\varepsilon)$ (integral I_{22}), and $|\beta_t-y|\le h_d(\varepsilon)$ (integral I_{23}) (recall $|x-y|>2h_d(\varepsilon)$).

An upper bound on I_{21}. We shall need the following notation:

$$I_0 = 4\mathrm{a}_0^2\int dz\ G(0,z)\left[\int_A dx\ \ |z-x|^{2-d}\right]^2.$$

We use (11) to bound I_{21} above by: for $\varepsilon\in(0,\varepsilon_0')$,

$$4\iint_{A_\varepsilon\times A_\varepsilon} dx\ dy\ \mathbf{1}_{\{|x-y|>2h_d(\varepsilon)\}}\int_0^\infty dt\ \mathbb{E}_0\Big[\,|\beta_t-x|>h_d(\varepsilon);|\beta_t-y|>h_d(\varepsilon);$$
$$\varphi_d(\varepsilon)^{-2}\,|\beta_t-x|^{2-d}\,|\beta_t-y|^{2-d}\left(\mathrm{a}_0+\mathrm{b}_1h_d(\varepsilon)^{\theta/2}\right)^2\Big]$$
$$\le 4\varphi_d(\varepsilon)^{-2}\left[\mathrm{a}_0^2+ch_d(\varepsilon)^{\theta/2}\right]\iint_{A\times A} dx\ dy\ \int dz\ G(0,z)\,|z-x|^{2-d}\,|z-y|^{2-d}$$
$$\le \varphi_d(\varepsilon)^{-2}I_0 + c_2\varphi_d(\varepsilon)^{-2}h_d(\varepsilon)^{\theta/2}.$$

An upper bound on I_{22} and I_{23}. By symmetry we have $I_{22} = I_{23}$. Before getting an upper bound on I_{22}, notice that $|\beta_t - x| \leq h_d(\varepsilon)$ and $|x-y| > 2h_d(\varepsilon)$ imply $|\beta_t - y| > h_d(\varepsilon)$. Furthermore thanks to (10), we get

$$\begin{aligned}\int_{A_\varepsilon} dy\ \mathbf{1}_{\{|\beta_t - y| > h_d(\varepsilon)\}} u_\varepsilon(\beta_t - y)\, \mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_s - y)ds} &\leq \int_A dy\ \left[\mathrm{b}_0 \varphi_d(\varepsilon)^{-1} |\beta_t - y|^{2-d}\right] \\ &\leq \mathrm{b}_0 \varphi_d(\varepsilon)^{-1} \int_{B(0,R_0)} dy\ |y|^{2-d} = c_3 \varphi_d(\varepsilon)^{-1}.\end{aligned}$$

Thus the sum $I_{22} + I_{23}$ is bounded above by

$$8c_3 \varphi_d(\varepsilon)^{-1} \int_{A_\varepsilon} dx\ \int_0^\infty dt\ \mathbb{E}_0 \left[\tau_{B(x,\varepsilon)} > t; \mathbf{1}_{\{|\beta_t - x| \leq h_d(\varepsilon)\}} u_\varepsilon(\beta_t - x)\, \mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_s - x)ds}\right].$$

Using the Cauchy-Schwarz inequality and formula (7), we get

$$\begin{aligned} I_{22} + I_{23} &\leq 8c_3 \varphi_d(\varepsilon)^{-1} \left[\int_{A_\varepsilon} dx\ \int_0^\infty dt\ \mathbb{P}_0 \left[|\beta_t - x| \leq h_d(\varepsilon)\right]\right]^{1/2} \\ &\qquad \times \left[\int_{A_\varepsilon} dx\ \int_0^\infty dt\ \mathbb{E}_0 \left[\tau_{B(x,\varepsilon)} > t; u_\varepsilon(\beta_t - x)^2\, \mathrm{e}^{-4\int_0^t u_\varepsilon(\beta_s - x)ds}\right]\right]^{1/2} \\ &\leq 8c_3 \varphi_d(\varepsilon)^{-1} \left[\int_A dx\ \int dz\ G(0,z) \mathbf{1}_{\{|z-x| \leq h_d(\varepsilon)\}}\right]^{1/2} \left[\int_{A_\varepsilon} dx\ 2^{-1} u_\varepsilon(x)\right]^{1/2}.\end{aligned}$$

Then thanks to (10), we get $I_{22} + I_{23} \leq c_4 \varphi_d(\varepsilon)^{-3/2} h_d(\varepsilon)^{d/2} \leq c_4 \varphi_d(\varepsilon)^{-2} h_d(\varepsilon)^{3/2}$.

Conclusion on the upper bound on I. By combining the previous results, we get for $d \geq 4$

$$I \leq c_1 \varphi_d(\varepsilon)^{-2} h_d(\varepsilon)^3 + \varphi_d(\varepsilon)^{-2} I_0 + c_2 \varphi_d(\varepsilon)^{-2} h_d(\varepsilon)^{\theta/2} + c_4 \varphi_d(\varepsilon)^{-2} h_d(\varepsilon)^{3/2}.$$

Thus we get $\varphi_d(\varepsilon)^2 I \leq I_0 + c_5 h_d(\varepsilon)^{\theta/2}$.

3.2. **A lower bound on J.** We shall need the last hitting time of $\bar{B}(x,\varepsilon)$ under $\mathbb{N}_0$ for the Brownian snake:

$$L_{(x,\varepsilon)} = \sup\left\{s \geq 0; \exists t \in [0, \zeta_s], W_s(t) \in \bar{B}(x,\varepsilon)\right\}.$$

We then get

$$\begin{aligned} J = \int_{A_\varepsilon} dx\ \mathbb{N}_0 &\left[T_{(x,\varepsilon)} < \infty; \int_0^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right] \\ &+ \int_{A_\varepsilon} dx\ \mathbb{N}_0 \left[T_{(x,\varepsilon)} < \infty; \int_{T_{(x,\varepsilon)}}^{\sigma} ds\ \mathbf{1}_A(\hat{W}_s)\right] \\ &\qquad - \int_{A_\varepsilon} dx\ \mathbb{N}_0 \left[T_{(x,\varepsilon)} < \infty; \int_{T_{(x,\varepsilon)}}^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right].\end{aligned}$$

The time-reversal invariance property of the Itô measure and the characterization of the excursion measure $\mathbb{N}_x$ readily imply that the latter itself enjoys the same invariance property. Thus the first two terms of the right-hand side are equal. We shall denote their sum by J_1. Let J_2 denote the third term.

A lower bound on J_1. Let us use the strong Markov property of the Brownian snake at time $T_{(x,\varepsilon)}$, then (16) and (6), to get

$$\begin{aligned} J_1 &= 2\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; 2\int_0^{\zeta_{T_{(x,\varepsilon)}}} dt\ \int_A dy\ G\left(W_{T_{(x,\varepsilon)}}(t),y\right)\right] \\ &= 4\int_{A_\varepsilon} dx\int_A dy\int_0^\infty dt\ \mathbb{E}_0\left[\tau_{B(x,\varepsilon)}>t; G(\beta_t,y)u_\varepsilon(\beta_t-x)\,\mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_s-x)ds}\right]. \end{aligned}$$

Fatou's lemma gives that $\liminf_{\varepsilon\to 0}\varphi_d(\varepsilon)J_1\geq J_0$, where

$$J_0 = 4\mathrm{a}_0\iint_{A\times A} dxdy\int dz\ G(0,z)G(z,y)\left|z-x\right|^{2-d}.$$

Unfortunately, we need an estimate on the rate of convergence. This requires some technical calculations. Notice that on $\{\tau_{B(x,h_d(\varepsilon))}(\beta)>t\}$, inequalities (12), (13) and (10) imply

$$\mathrm{a}_0\varphi_d(\varepsilon)^{-1}F_d(\beta_t-x)\left|\beta_t-x\right|^{2-d}\leq u_\varepsilon(\beta_t-x)\leq \mathrm{b}_0\varphi_d(\varepsilon)^{-1}\left|\beta_t-x\right|^{2-d},$$

where $F_d(z) = 1$ if $d\geq 5$ and $F_4(z) = \left[1+\log(2\left|z\right|)/\log(1/\varepsilon)\right]^{-1}$. For short we write $\Gamma_t = 2\mathrm{b}_0\varphi_d(\varepsilon)^{-1}\int_0^t\left|\beta_s-x\right|^{2-d}ds$. Then $\varphi_d(\varepsilon)J_1$ is bounded below by

$$J_1' = 4\mathrm{a}_0\int_{A_\varepsilon} dx\int_A dy\int_0^\infty dt\ \mathbb{E}_0\left[\tau_{B(x,h_d(\varepsilon))}>t; G(\beta_t,y)\left|\beta_t-x\right|^{2-d}F_d(\beta_t-x)\,\mathrm{e}^{-\Gamma_t}\right].$$

In order to obtain an upper bound on $|J_1'-J_0|$, we have to find an upper bound on

$$\iint_{A\times A} dxdy\int_0^\infty dt\ \mathbb{E}_0\left[G(\beta_t,y)\left|\beta_t-x\right|^{2-d}\left[1-\mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{\left\{\tau_{B(x,h_d(\varepsilon))}>t\right\}}F_d(\beta_t-x)\,\mathrm{e}^{-\Gamma_t}\right]\right].$$

Thus we shall decompose $1-\mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{\left\{\tau_{B(x,h_d(\varepsilon))}\,>\,t\right\}}F_d(\beta_t-x)\,\mathrm{e}^{-\Gamma_t}$ into a sum of four terms:

$$\begin{aligned} &[1-\mathbf{1}_{A_\varepsilon}(x)]+\mathbf{1}_{A_\varepsilon}(x)\left[1-\mathbf{1}_{\left\{\tau_{B(x,h_d(\varepsilon))}>t\right\}}\right] \\ &\quad+\mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{\left\{\tau_{B(x,h_d(\varepsilon))}>t\right\}}\left[1-F_d(\beta_t-x)\right]+\mathbf{1}_{A_\varepsilon}(x)\mathbf{1}_{\left\{\tau_{B(x,h_d(\varepsilon))}>t\right\}}F_d(\beta_t-x)\left[1-\mathrm{e}^{-\Gamma_t}\right]. \end{aligned}$$

We denote by J_{11}, J_{12}, J_{13} and J_{14} the corresponding integrals. The integral

$$J_{11} = \int_{A\setminus A_\varepsilon} dx\int_A dy\int_0^\infty dt\ \mathbb{E}_0\left[G(\beta_t,y)\left|\beta_t-x\right|^{2-d}\right]$$

is easily bounded above by

$$\int_{B(0,h_d(\varepsilon))} dx\ \int_{B(0,R_0)} dy\int dz\ G(0,z)G(z,y)\left|z-x\right|^{2-d}\leq c_6h_d(\varepsilon)^2.$$

We bound J_{12} by applying the strong Markov property of Brownian motion at time $\tau_{B(x,h_d(\varepsilon))}$,

$$\begin{aligned} J_{12} &= \int_{A_\varepsilon} dx\int_A dy\int_0^\infty dt\ \mathbb{E}_0\left[\tau_{B(x,h_d(\varepsilon))}\leq t; G(\beta_t,y)\left|\beta_t-x\right|^{2-d}\right] \\ &\leq \int_{A_\varepsilon} dx\int_A dy\ \mathbb{E}_0\left[\tau_{B(x,h_d(\varepsilon))}<\infty;\int dz\ G(\beta_{\tau_{B(x,h_d(\varepsilon))}},z)G(z,y)\left|z-x\right|^{2-d}\right]. \end{aligned}$$

An easy calculation shows that there exists a constant c_7 such that for every $(x,x') \in B(0,2R_0)\times B(0,2R_0)$, $|x-x'|\leq 1/2$,

$$\int_{B(0,R_0)} dy \int dz\; G(x',z)G(z,y)\,|z-x|^{2-d} \leq c_7\varphi_d(\left|x'-x\right|)$$

Furthermore we have for every $r\in(0,1)$,

$$\int_{B(0,R_0)} dx\;\mathbb{P}_0\left[\tau_{B(x,r)}<\infty\right] = \int_{B(0,R_0)} dx\;\left[\left(\frac{r}{|x|}\right)^{d-2}\wedge 1\right] \leq cr^{d-2}. \tag{18}$$

We deduce from the previous remarks that if $d\geq 5$,

$$J_{12}\leq ch_d(\varepsilon)^{4-d}\int_{A_\varepsilon} dx\;\mathbb{P}_0\left[\tau_{B(x,h_d(\varepsilon))}<\infty\right] \leq ch_d(\varepsilon)^{4-d+d-2} = ch_d(\varepsilon)^2,$$

and if $d=4$, $J_{12}\leq c\log(1/h_d(\varepsilon))h_d(\varepsilon)^2$. Thus we get that for $d\geq 4$, $J_{12}\leq c_8h_d(\varepsilon)^{3/2}$.
If $d\geq 5$ then $J_{13}=0$. For $d=4$ thanks to (9) we have for $|z|\geq h_4(\varepsilon)$, $|1-F_4(z)|\leq 2\left|\log(2\left|z\right|)\right|/\log(1/\varepsilon)$. We deduce that

$$\begin{aligned}
J_{13} &\leq \log(1/\varepsilon)^{-1}\iint_{A\times A} dxdy\int_0^\infty dt\;\mathbb{E}_0\left[\tau_{B(x,h_d(\varepsilon))}>t; G(\beta_t,y)2\left|\log(2\left|\beta_t-x\right|)\right|\left|\beta_t-x\right|^{-2}\right]\\
&\leq c\log(1/\varepsilon)^{-1}\iint_{A\times A} dxdy\int dz\;G(0,z)\left|\log(2\left|z-x\right|)\right|\left|z-x\right|^{-2}G(z,y)\\
&\leq c\log(1/\varepsilon)^{-1}\leq c_9h_d(\varepsilon)^\theta.
\end{aligned}$$

Notice first that thanks to (9), $F_d(z)\leq 2$ for $|z|\geq h_d(\varepsilon)$. We have, using the Markov property for Brownian motion at time s,

$$\begin{aligned}
J_{14} &\leq 2\iint_{A\times A} dxdy\int_0^\infty dt\\
&\qquad\mathbb{E}_0\left[\tau_{B(x,h_d(\varepsilon))}>t; G(\beta_t,y)\left|\beta_t-x\right|^{2-d}2\mathrm{b}_0\varphi_d(\varepsilon)^{-1}\int_0^t\left|\beta_s-x\right|^{2-d}ds\right]\\
&\leq c\varphi_d(\varepsilon)^{-1}\iint_{A\times A} dxdy\int_0^\infty ds\int_0^\infty dt\\
&\qquad\mathbb{E}_0\left[\left|\beta_s-x\right|^{2-d}\mathbb{E}_{\beta_s}\left[\left|\beta_t-x\right|>h_d(\varepsilon); G(\beta_t,y)\left|\beta_t-x\right|^{2-d}\right]\right]\\
&\leq c\varphi_d(\varepsilon)^{-1}M(d,h_d(\varepsilon)),
\end{aligned}$$

where

$$M(d,\varepsilon)=\iint_{B(0,R_0)^2} dxdy\iint dzdz'\;G(0,z)\left|z-x\right|^{2-d}G(z,z')G(z',y)\left|z'-x\right|^{2-d}\mathbf{1}_{|z'-x|>\varepsilon}.$$

An easy computation shows there exists a constant c such that for $\varepsilon\in(0,1]$,

$$M(d,\varepsilon)\leq\begin{cases} c & \text{if } d\in\{4,5\},\\ c+c\log(1/\varepsilon) & \text{if } d=6,\\ c\varepsilon^{6-d} & \text{if } d\geq 7.\end{cases} \tag{19}$$

Thus we easily deduce that $J_{14}\leq c_{10}h_d(\varepsilon)^\theta$.

We have $\varepsilon^{4-d}J_1 \geq J_0 - 4\mathrm{a}_0(J_{11}+J_{12}+J_{13}+J_{14})$. Putting together the previous results, we get for $d \geq 4$,

$$\varepsilon^{4-d}J_1 \geq J_0 - 4\mathrm{a}_0[c_6 h_d(\varepsilon)^2 + c_8 h_d(\varepsilon)^{3/2} + c_9 h_d(\varepsilon)^\theta + c_{10} h_d(\varepsilon)^\theta] \geq J_0 - c_{11} h_d(\varepsilon)^\theta .$$

An upper bound on J_2. We will first recall the decomposition of the Brownian snake under $\mathcal{E}^*_{\mathrm{w}}$ (see theorem 2.5 in [12]). We denote by (α_i,β_i), $i\in I$, the excursion intervals of ζ above its minimum process (i.e. of the process $(\zeta_t - \inf_{s\in[0,t]}\zeta_s)$ above 0) before σ under $\mathcal{E}^*_{\mathrm{w}}$. For $i\in I$ the paths $W_s, s\in[\alpha_i,\beta_i]$ coincide over $[0,\zeta_{\alpha_i}]$. For every $i\in I$, and $s\geq 0$ we set $W^i_s(t) = W_{(\alpha_i+s)\wedge\beta_i}(t+\zeta_{\alpha_i})$, $t\in[0,\zeta^i_s]$ with $\zeta^i_s = \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}$. Then W^i_s is a stopped path ($W^i_s\in\mathcal{W}$) with initial point $W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i}) = \hat{W}_{\alpha_i} = \mathrm{w}(\zeta_{\alpha_i})$.

Proposition 1 (Le Gall). *The random measure $\sum_{i\in I}\delta_{(\zeta_{\alpha_i},W^i)}$ is under $\mathcal{E}^*_{\mathrm{w}}$ a Poisson point measure on $[0,\zeta_{(\mathrm{w})}]\times C(\mathbb{R}^+,\mathcal{W})$ with intensity $2dt\ \mathbb{N}_{\mathrm{w}(t)}[\cdot]$.*

The process $\sum_{i\in I,\zeta_{\alpha_i}\leq t}\delta_{W^i}$ for $t\in[0,\zeta_{\mathrm{w}}]$ is a Poisson point process with inhomogeneous intensity. We will now describe the law under $\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}$ of the first excursion $(\zeta_{\alpha_{i_0}},W^{i_0})$ which hits the ball $\bar{B}(x,\varepsilon)$, that is, with evident notation, the first excursion for which $T_{(x,\varepsilon)}(W^i)$ is finite. We first notice that under $\mathbb{N}_0[\cdot \mid T_{(x,\varepsilon)}<\infty]$, $\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}$-a.s. there are such excursions. Indeed we have thanks to lemma 2.1 of [11] that $\mathbb{N}_0[. \mid T_{(x,\varepsilon)}<\infty]$-a.s.

$$\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}[\exists i\in I, T_{(x,\varepsilon)}(W^i)<\infty] = 1-\exp -2\int_0^{\zeta_{T_{(x,\varepsilon)}}} dt\ u_\varepsilon(W_{T_{(x,\varepsilon)}}(t)-x) = 1.$$

Since the integral $\int_0^r dt\ u_\varepsilon(W_{T_{(x,\varepsilon)}}(t)-x)$ is finite for $r<\zeta_{T_{(x,\varepsilon)}}$, we deduce there exists a unique first excursion i_0 which hits $\bar{B}(x,\varepsilon)$. Classical arguments on Poisson point process implies that the law of $(\zeta_{\alpha_{i_0}},W^{i_0})$ is $2\mathbf{1}_{[0,\zeta_{T_{(x,\varepsilon)}})}(t)dt\ \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}[T_{(x,\varepsilon)}<\infty,\cdot]$. We introduce the random time $M_{(x,\varepsilon)} = \inf\{s>T_{(x,\varepsilon)};\zeta_s = m(T_{(x,\varepsilon)},L_{(x,\varepsilon)})\}$. It is clear from the definition of the excursion i_0 that $\alpha_{i_0} = M_{(x,\varepsilon)}$ under $\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}$. We will now express J_2 using the excursion i_0. We have

$$\begin{aligned}
J_2 &= 2\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty;\int_{M_{(x,\varepsilon)}}^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right]\\
&= 2\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty;\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}\left[\int_{M_{(x,\varepsilon)}}^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right]\right]\\
&= 2\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty;\mathcal{E}^*_{W_{T_{(x,\varepsilon)}}}\left[\int_{\alpha_{i_0}}^{L_{(x,\varepsilon)}(W^{i_0})} ds\ \mathbf{1}_A(\hat{W}^{i_0}_s)\right]\right]\\
&= 4\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty;\int_0^{\zeta_{T_{(x,\varepsilon)}}} dt\ \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\left[T_{(x,\varepsilon)}<\infty;\int_0^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right]\right].
\end{aligned}$$

We used the time reversal property of the Brownian snake for the first equality, then the strong Markov property and at last the definition of the excursion i_0 and its law. We will distinguish according to $\{t\geq\tau_{B(x,h_d(\varepsilon))}\}$ (integral J_{21}) and $\{t<\tau_{B(x,h_d(\varepsilon))}\}$ (integral J_{22}). Notice that since $x\in A_\varepsilon$ we have $\tau_{B(x,h_d(\varepsilon))}(W_{T_{(x,\varepsilon)}})<\zeta_{T_{(x,\varepsilon)}}$ $\mathbb{N}_0$-a.e.

We now bound J_{21} using (14).

$$\begin{aligned}
J_{21} &= 4\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; \int_{\tau_{B(x,h_d(\varepsilon))}}^{\zeta_{T_{(x,\varepsilon)}}} dt\ \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\left[T_{(x,\varepsilon)}<\infty; \int_0^{L_{(x,\varepsilon)}} ds\ \mathbf{1}_A(\hat{W}_s)\right]\right]\\
&\leq 4\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; \int_{\tau_{B(x,h_d(\varepsilon))}}^{\zeta_{T_{(x,\varepsilon)}}} dt\ \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\left[\int_0^{\sigma} ds\ \mathbf{1}_A(\hat{W}_s)\right]\right]\\
&= 4\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; \int_{\tau_{B(x,h_d(\varepsilon))}}^{\zeta_{T_{(x,\varepsilon)}}} dt\ \int_A dy\ G(W_{T_{(x,\varepsilon)}}(t),y)\right].
\end{aligned}$$

Now we use (6), the Cauchy-Schwarz inequality and (7) to get

$$\begin{aligned}
J_{21} &\leq 4\int_{A_\varepsilon} dx \int_0^\infty dt\ \mathbb{E}_0\left[\tau_{B(x,\varepsilon)}>t\geq \tau_{B(x,h_d(\varepsilon))}; \int_A dy\ G(\beta_t,y)u_\varepsilon(\beta_t-x)\,\mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_r-x)dr}\right]\\
&\leq 4\left[2^{-1}\int_{A_\varepsilon} dx\ u_\varepsilon(x)\right]^{1/2}\left[\int_{A_\varepsilon} dx\int_0^\infty dt\ \mathbb{E}_0\left[t\geq \tau_{B(x,h_d(\varepsilon))}; \left(\int_A dy\ G(\beta_t,y)\right)^2\right]\right]^{1/2}\\
&\leq c\varphi_d(\varepsilon)^{-1/2}\left[\int_{A_\varepsilon} dx\ \mathbb{P}_0\left[\tau_{B(x,h_d(\varepsilon))}<\infty\right]\sup_{x'\in B(0,2R_0)}\int dz\ G(z,x')\left(\int_A dy\ G(z,y)\right)^2\right]^{1/2}\\
&\leq c\varphi_d(\varepsilon)^{-1/2}h_d(\varepsilon)^{(d-2)/2}.
\end{aligned}$$

We used the strong Markov property at time $\tau_{B(x,h_d(\varepsilon))}$ and (18) for the last two inequalities. This implies that $J_{21}\leq c_{12}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)^{1/2}$.

Using the time reversal property of the Brownian snake, the strong Markov property at time $T_{(x,\varepsilon)}$ and (16) we get

$$\begin{aligned}
J_{22} &= 4\int_{A_\varepsilon} dx\ \mathbb{N}_0\left[T_{(x,\varepsilon)}<\infty; \int_0^{\tau_{B(x,h_d(\varepsilon))}} dt\ \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\left[T_{(x,\varepsilon)}<\infty; \int_{T_{(x,\varepsilon)}}^{\sigma} ds\ \mathbf{1}_A(\hat{W}_s)\right]\right]\\
&= 8\int_{A_\varepsilon} dx\ \mathbb{N}_0\Bigg[T_{(x,\varepsilon)}<\infty; \int_0^{\tau_{B(x,h_d(\varepsilon))}} dt\\
&\qquad\qquad \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\left[T_{(x,\varepsilon)}<\infty; \int_0^{\zeta_{T_{(x,\varepsilon)}}} ds\int_A dy\ G(W_{T_{(x,\varepsilon)}}(s),y)\right]\Bigg].
\end{aligned}$$

We will distinguish according to $\{s \geq \tau_{B(x,h_d(\varepsilon))}\}$ (integral J_{23}) and $\{s < \tau_{B(x,h_d(\varepsilon))}\}$ (integral J_{24}). We now bound J_{23}. We have

$$
\begin{aligned}
J_{23} &= 8\int_{A_\varepsilon} dx\, \mathbb{N}_0\Bigg[T_{(x,\varepsilon)} < \infty; \int_0^{\tau_{B(x,h_d(\varepsilon))}} dt \\
&\qquad \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\Bigg[T_{(x,\varepsilon)} < \infty; \int_{\tau_{B(x,h_d(\varepsilon))}}^{\zeta_{T_{(x,\varepsilon)}}} ds \int_A dy\; G(W_{T_{(x,\varepsilon)}}(s), y)\Bigg]\Bigg] \\
&= 8\int_{A_\varepsilon} dx \int_0^\infty dt\; \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))} > t; u_\varepsilon(\beta_t - x)\,\mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_r - x)dr} \int_0^\infty ds \\
&\qquad \mathbb{E}_{\beta_t}\Bigg[\tau_{B(x,\varepsilon)} > s \geq \tau_{B(x,h_d(\varepsilon))}; \int_A dy\; G(\beta_s, y) u_\varepsilon(\beta_s - x)\,\mathrm{e}^{-2\int_0^s u_\varepsilon(\beta_v - x)dv}\Bigg]\Bigg] \\
&\leq c\varphi_d(\varepsilon)^{-1}\int_{A_\varepsilon} dx \int_0^\infty dt\; \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))} > t; |\beta_t - x|^{2-d} \\
&\qquad \left[\int_0^\infty ds\; \mathbb{E}_{\beta_t}\left[\tau_{B(x,\varepsilon)} > s; u_\varepsilon(\beta_s - x)^2\,\mathrm{e}^{-4\int_0^s u_\varepsilon(\beta_v - x)dv}\right]\right]^{1/2} \\
&\qquad \left[\int_0^\infty ds\; \mathbb{E}_{\beta_t}\left[s \geq \tau_{B(x,h_d(\varepsilon))}; \left(\int_A dy\; G(\beta_s, y)\right)^2\right]\right]^{1/2}\Bigg] \\
&\leq c\varphi_d(\varepsilon)^{-1}\int_{A_\varepsilon} dx \int_0^\infty dt\; \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))} > t; |\beta_t - x|^{2-d}\left[2^{-1}u_\varepsilon(\beta_t - x)\right]^{1/2} \\
&\qquad \left[\mathbb{E}_{\beta_t}\left[\tau_{B(x,h_d(\varepsilon))} < \infty; \mathbb{E}_{\beta_{\tau_{B(x,h_d(\varepsilon))}}}\left[\int_0^\infty ds\; \left(\int_A dy\; G(\beta_s, y)\right)^2\right]\right]\right]^{1/2}\Bigg] \\
&\leq c\varphi_d(\varepsilon)^{-3/2}\int_{A_\varepsilon} dx \int_0^\infty dt\; \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))} > t; |\beta_t - x|^{(6-3d)/2}\,\mathbb{P}_{\beta_t}\left[\tau_{B(x,h_d(\varepsilon))} < \infty\right]^{1/2}\Bigg] \\
&\qquad \left[\sup_{x' \in B(0,2R_0)} \int dz' G(x', z')\left(\int_A dy\; G(z', y)\right)^2\right]^{1/2} \\
&\leq c\varphi_d(\varepsilon)^{-3/2}\int_{A_\varepsilon} dx \int_{|z-x| \geq h_d(\varepsilon)} dz\; G(0,z)\,|z - x|^{(6-3d)/2}\,h_d(\varepsilon)^{(d-2)/2}\,|z - x|^{(2-d)/2}.
\end{aligned}
$$

We used (6) twice for the second equality, (10) and Cauchy-Schwarz inequality for the first inequality, (7) and the strong Markov property at time $\tau_{B(x,h_d(\varepsilon))}$ for the second and (18) for the last. We easily deduce that $J_{23} \leq c_{13}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)$.

For J_{24} we have using (6) twice and (10) twice,

$$\begin{aligned}
J_{24} &= 8\int_{A_\varepsilon} dx\ \mathbb{N}_0\Bigg[T_{(x,\varepsilon)}<\infty; \int_0^{\tau_{B(x,h_d(\varepsilon))}} dt \\
&\qquad\qquad \mathbb{N}_{W_{T_{(x,\varepsilon)}}(t)}\Bigg[T_{(x,\varepsilon)}<\infty; \int_0^{\tau_{B(x,h_d(\varepsilon))}} ds \int_A dy\ G(W_{T_{(x,\varepsilon)}}(s),y)\Bigg]\Bigg] \\
&= 8\int_{A_\varepsilon} dx\ \int_0^\infty dt \int_0^\infty ds\ \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))}>t; u_\varepsilon(\beta_t-x)\,\mathrm{e}^{-2\int_0^t u_\varepsilon(\beta_r-x)dr} \\
&\qquad\qquad \mathbb{E}_{\beta_t}\Bigg[\tau_{B(x,h_d(\varepsilon))}>s; \int_A dy\ G(\beta_s,y)u_\varepsilon(\beta_s-x)\,\mathrm{e}^{-2\int_0^s u_\varepsilon(\beta_v-x)dv}\Bigg]\Bigg] \\
&\leq c\varphi_d(\varepsilon)^{-2}\int_{A_\varepsilon} dx\ \int_0^\infty dt \int_0^\infty ds\ \mathbb{E}_0\Bigg[\tau_{B(x,h_d(\varepsilon))}>t; |\beta_t-x|^{2-d} \\
&\qquad\qquad \mathbb{E}_{\beta_t}\Bigg[\tau_{B(x,h_d(\varepsilon))}>s; \int_A dy\ G(\beta_s,y)\,|\beta_s-x|^{2-d}\Bigg]\Bigg] \\
&\leq c\varphi_d(\varepsilon)^{-2}M(d,h_d(\varepsilon)).
\end{aligned}$$

Using (19) we get $J_{24}\leq c_{14}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)^\theta$. As a conclusion we get

$$J_2\leq c_{12}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)^{1/2}+c_{13}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)+c_{14}\varphi_d(\varepsilon)^{-1}h_d(\varepsilon)^\theta.$$

Conclusion on the lower bound on J.
By combining the previous results, we get for $d\geq 4$,

$$\varphi_d(\varepsilon)J\geq J_0-c_{11}h_d(\varepsilon)^\theta-\varphi_d(\varepsilon)J_2\geq J_0-c_{15}h_d(\varepsilon)^\theta.$$

3.3. **End of the proof of proposition 2.3.** We deduce from formula (15), that

$$J_0=C_0\mathbb{N}_0\left[\left[\int_0^\sigma \mathbf{1}_A(\hat{W}_s)ds\right]^2\right],\quad\text{and}\quad I_0={C_0}^2\mathbb{N}_0\left[\left[\int_0^\sigma \mathbf{1}_A(\hat{W}_s)ds\right]^2\right].$$

Thus we get from section 3.1 and 3.2 that for ε small enough

$$\mathbb{N}_0\left[\left[\varphi_d(\varepsilon)\,|\mathcal{R}^\varepsilon\cap A_\varepsilon|-C_0\int_0^\sigma ds\ \mathbf{1}_A(\hat{W}_s)\right]^2\right]\leq c_5h_d(\varepsilon)^{\theta/2}+2c_{15}h_d(\varepsilon)^\theta.$$

Take $\kappa<\theta/2$ and ε_0 small to get the second upper bound of proposition 2.3. □

4. Capacity equivalence for the support and the range of X

Let $f:(0,\infty)\to[0,\infty)$ be a decreasing function. We put $f(0)=\lim_{r\downarrow 0}f(r)\in[0,\infty]$. We define the energy of a Radon measure ν on $\mathbb{R}^d$ with respect to the kernel f by: $\mathcal{I}_f(\nu)=\iint f(|x-y|)\nu(dx)\nu(dy)$, and the capacity of a set $\Lambda\in\mathcal{B}(\mathbb{R}^d)$ by

$$\mathrm{cap}_f(\Lambda)=\left[\inf_{\nu(\Lambda)=1}\mathcal{I}_f(\nu)\right]^{-1}.$$

Following [14], we say that two sets Λ_1 and Λ_2 are capacity-equivalent if there exist two positive constants c and C such that for every kernel f, we have

$$c\,\mathrm{cap}_f(\Lambda_1) \le \mathrm{cap}_f(\Lambda_2) \le C\,\mathrm{cap}_f(\Lambda_1)\,.$$

The next lemma is an immediate consequence of the remarks in [15] p.385.

Lemma 4.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set. Suppose there exist two positive constants c' and γ such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma-d}\,|\Lambda^\varepsilon| = c'.$$

Then there exists a constant C such that for every kernel f, we have

$$\mathrm{cap}_f(\Lambda) \le C\left[\int_0^1 f(r)r^{\gamma-1}dr\right]^{-1}.$$

For every measure $\mu \in M_f$, we set

$$S_\varepsilon(\mu) = \iint \mu(dx)\mu(dy)\; p(\varepsilon^2, x-y),$$

where p is the Brownian transition density in $\mathbb{R}^d$: $p(t,x) = (2\pi t)^{-d/2}\,\mathrm{e}^{-|x|^2/2t}$, $(t,x) \in (0,\infty)\times\mathbb{R}^d$. The next lemma is also an immediate consequence of [15] (p.387).

Lemma 4.2. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set. Suppose there exist two positive constants c' and γ and a measure $\mu \in M_f$ such that $\mu(\Lambda^c) = 0$ and*

$$\lim_{\varepsilon\to 0} \varepsilon^{d-\gamma} S_\varepsilon(\mu) = c'.$$

Then there exists a constant c such that for every kernel f, we have

$$c\left[\int_0^1 f(r)r^{\gamma-1}dr\right]^{-1} \le \mathrm{cap}_f(\Lambda)\,.$$

For example, for every integer $p \le d$, we can consider the cube $[0,1]^p$ as a subset of $\mathbb{R}^d$, and then we obviously have

$$\lim_{\varepsilon\to 0} \varepsilon^{p-d}\,|([0,1]^p)^\varepsilon| = 2\pi^{(d-p)/2}/\Gamma((d-p)/2),$$

and if μ is Lebesgue measure on $[0,1]^p$,

$$\lim_{\varepsilon\to 0} \varepsilon^{d-p} S_\varepsilon(\mu) = (2\pi)^{(p-d)/2}.$$

Thus we deduce from lemma 4.1 and 4.2 that there exist two positive constants c'_p, C'_p, such that for every kernel f,

$$c'_p\left[\int_0^1 f(r)r^{p-1}dr\right]^{-1} \le \mathrm{cap}_f([0,1]^p) \le C'_p\left[\int_0^1 f(r)r^{p-1}dr\right]^{-1}. \tag{20}$$

We shall prove the following result on super-Brownian motion and ISE.

Proposition 4.3. (i) *Assume $d \ge 3$. Let $t > 0$, $\nu \in M_f$. $\mathbb{P}^X_\nu$-a.s. on $\{X_t \neq 0\}$, the set* $\mathrm{supp}\, X_t$ *is capacity-equivalent to $[0,1]^2$.*

(ii) *Assume* $d \geq 5$. *Let* $t > 0$, $\nu \in M_f$. $\mathbb{P}_\nu^X$*-a.s. on* $\{X_t \neq 0\}$, *the set* $\mathcal{R}_t(X)$ *is capacity-equivalent to* $[0,1]^4$. *Furthermore, if there exists a positive number* $\rho < 4$ *such that* $\lim_{\varepsilon\to 0} \varepsilon^{\rho-d} |(\text{supp } \nu)^\varepsilon| = 0$, *then* $\mathbb{P}_\nu^X$*-a.s. the set* $\mathcal{R}_0(X)$ *is capacity-equivalent to* $[0,1]^4$.

(iii) *Assume* $d \geq 5$. *The set* $\mathcal{R}_t(W)$ *is capacity-equivalent to* $[0,1]^4$ $\mathbb{N}_0^{(1)}$*-a.s.*

Proof of proposition 4.3 (i). Let $d \geq 3$. It is well-known that for $t > 0$, $\mathbb{P}_\nu^X$-a.s. the set supp X_t is bounded. Thus, thanks to theorem 2.2, $\mathbb{P}_\nu^X$-a.s., we have

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d} |(\text{supp } X_t)^\varepsilon| = \alpha_0(X_t, \mathbf{1}).$$

Now apply lemma 4.1 to $\Lambda = \text{supp } X_t$, with $\gamma = 2$ and take $p = 2$ in (20). We get that $\mathbb{P}_\nu^X$-a.s., on $\{X_t \neq 0\}$, there exists a (random) constant $C_1 > 0$, such that for every kernel f,

$$\text{cap}_f(\text{supp } X_t) \leq C_1 \,\text{cap}_f([0,1]^2)\,.$$

For the second part of (i), we use lemma 4.4 below. Recall notation Y_t from section 1.1.

Lemma 4.4. *Fix* $t > 0$ *and* $x \in \mathbb{R}^d$, $d \geq 3$. *Then we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{d-2} (2\pi)^{d/2} S_\varepsilon(Y_t) = \frac{4}{d-2} (Y_t, \mathbf{1}),$$

where the convergence holds $\mathbb{N}_x$*-a.e. and in* $L^2(\mathbb{N}_x)$.

Let us explain how the proof is completed using lemma 4.4. Thanks to lemma 4.2, the above lemma and (20) imply that $\mathbb{N}_x$-a.e. on $\{Y_t \neq 0\}$, there exists a positive constant c_1 such that for every kernel f,

$$\text{cap}_f(\text{supp } Y_t) \geq c_1 \,\text{cap}_f([0,1]^2)\,.$$

Now remember that for $t > 0$, under $\mathbb{P}_\nu^X$, we can write $X_t = \sum_{i\in I} Y_t(W^i)$, where $\sum_{i\in I} \delta_{W^i}$ is a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int \nu(dx)\, \mathbb{N}_x[\cdot]$. On $\{X_t \neq 0\}$, there exists i_0 such that $Y_t(W^{i_0}) \neq 0$. Then we have supp $Y_t(W^{i_0}) \subset$ supp X_t. Thus the previous lemma entails that there exists a.s. a positive constant $c_1(W^{i_0})$ such that for every kernel f,

$$\text{cap}_f(\text{supp } X_t) \geq \text{cap}_f(\text{supp } Y_t(W^{i_0})) \geq c_1(W^{i_0})\, \text{cap}_f([0,1]^2)$$

This completes the proof of (i). □

Proof of proposition 4.3 (ii). Let $d \geq 5$. We argue as in the proof of (i) using theorem 2.1 instead of theorem 2.2 and the following lemma instead of lemma 4.4.

Lemma 4.5. *Fix* $t \geq 0$ *and* $x \in \mathbb{R}^d$, $d \geq 5$. *Then we have for every* $T > t \geq 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{d-4} (2\pi)^{d/2} S_\varepsilon\left(\int_t^T ds\; Y_s\right) = \frac{16}{(d-2)(d-4)} \int_t^T ds\; (Y_s, \mathbf{1}),$$

where the convergence holds $\mathbb{N}_x$*-a.e. and in* $L^2(\mathbb{N}_x)$.

□

Proof of proposition 4.3 (iii). Let $d \geq 5$. For the first part we argue as in the proof of (i) using the second part of corollary 2.4 instead of theorem 2.2. Notice that thanks to (3) and the scaling property of the family $(\mathbb{N}_0^{(r)}, r > 0)$, the convergence in lemma 4.5 also holds

$\mathbb{N}_0^{(1)}$-a.s. The second part of (iii) is then a direct consequence of lemma 4.2 (with $\mu = \int_0^T ds\, Y_s$ and $\gamma = 4$) and (20) (with $p = 4$). $\square$

The proofs of lemma 4.4 and lemma 4.5 are very similar. We shall only prove the latter. The former uses the same techniques in a simpler way.

Proof of lemma 4.5. We first want to show the convergence in $L^2(\mathbb{N}_x)$. Fix $T > t \geq 0$. By standard monotone class arguments, we deduce from the results of section 5.1 an explicit expression for

$$\mathbb{N}_x \left[\int_0^T \cdots \int_0^T ds_1 \dots ds_4 \int \cdots \int Y_{s_1}(dx_1) \dots Y_{s_4}(dx_4) g(s_1, \dots, s_4, x_1, \dots, x_4) \right],$$

where g is any measurable positive function on $(\mathbb{R}^+)^4 \times (\mathbb{R}^d)^4$. Specializing to the case $g(s_1, \dots, s_4, x_1, \dots, x_4) = \prod_{i=1}^4 \mathbf{1}_{[t,T]}(s_i) p(\varepsilon^2, x_1 - x_2) p(\varepsilon^2, x_3 - x_4)$, we get

$$\begin{aligned}
&\mathbb{N}_x \left[S_\varepsilon \left(\int_t^T ds\, Y_s \right)^2 \right] \\
&= \frac{1}{3} 4! 2^3 \int_0^T ds \int dy\, p(s, x - y) \Bigg\{ 4 \int_{(t-s)_+}^{T-s} ds_1 \int dy_1\, p(s_1, y - y_1) \int_0^{T-s} ds_2 \\
&\qquad \int dy_2\, p(s_2, y - y_2) \int_{(t-s-s_2)_+}^{T-s-s_2} ds_3 \int dy_3\, p(s_3, y_2 - y_3) \int_0^{T-s-s_2} ds_4 \\
&\qquad \int dy_4\, p(s_4, y_2 - y_4) \int_{(t-s-s_2-s_4)_+}^{T-s-s_2-s_4} ds_5 \int dy_5\, p(s_5, y_4 - y_5) \\
&\qquad \int_{(t-s-s_2-s_4)_+}^{T-s-s_2-s_4} ds_6 \int dy_6\, p(s_6, y_4 - y_6) \\
&\qquad\qquad [p(\varepsilon^2, y_1 - y_3) p(\varepsilon^2, y_5 - y_6) + p(\varepsilon^2, y_1 - y_5) p(\varepsilon^2, y_3 - y_6) \\
&\qquad\qquad\qquad + p(\varepsilon^2, y_1 - y_6) p(\varepsilon^2, y_3 - y_5)] \\
&\quad + \int_0^{T-s} ds_7 \int dy_7\, p(s_7, y - y_7) \int_{(t-s-s_7)_+}^{T-s-s_7} ds_8 \int dy_8\, p(s_8, y_7 - y_8) \\
&\qquad\qquad \int_{(t-s-s_7)_+}^{T-s-s_7} ds_9 \int dy_9\, p(s_9, y_7 - y_9) \\
&\qquad \int_0^{T-s} ds_{10} \int dy_{10}\, p(s_{10}, y - y_{10}) \int_{(t-s-s_{10})_+}^{T-s-s_{10}} ds_{11} \int dy_{11}\, p(s_{11}, y_{10} - y_{11}) \\
&\qquad\qquad \int_{(t-s-s_{10})_+}^{T-s-s_{10}} ds_{12} \int dy_{12}\, p(s_{12}, y_{10} - y_{12}) \\
&\qquad [p(\varepsilon^2, y_8 - y_9) p(\varepsilon^2, y_{11} - y_{12}) + p(\varepsilon^2, y_8 - y_{11}) p(\varepsilon^2, y_9 - y_{12}) \\
&\qquad\qquad + p(\varepsilon^2, y_8 - y_{12}) p(\varepsilon^2, y_9 - y_{11})] \Bigg\}
\end{aligned}$$

We write J_1, J_2, J_3, J_4, J_5, and J_6, respectively for the integrals corresponding to the integrands $p(\varepsilon^2, y_1-y_3)p(\varepsilon^2, y_5-y_6)$, $p(\varepsilon^2, y_1-y_5)p(\varepsilon^2, y_3-y_6)$, $p(\varepsilon^2, y_1-y_6)p(\varepsilon^2, y_3-y_5)$, $p(\varepsilon^2, y_8-y_9)p(\varepsilon^2, y_{11}-y_{12})$, $p(\varepsilon^2, y_8-y_{11})p(\varepsilon^2, y_9-y_{12})$, and $p(\varepsilon^2, y_8-y_{12})p(\varepsilon^2, y_9-y_{11})$ respectively. As we shall see the integral J_4 gives the main contribution. Before proceeding to the calculations, we give three useful bounds: for every positive real numbers s, $\varepsilon^2 < 2^{-1}(T^{-1}\wedge T)$, we have for $d \geq 5$

$$\int_0^T \left(\varepsilon^2+s+r\right)^{-d/2} dr \leq \frac{2}{d-2}\left(\varepsilon^2+s\right)^{1-d/2}, \tag{21}$$

$$\int_0^T \left(\varepsilon^2+s+r\right)^{1-d/2} dr \leq \frac{2}{d-4}\left(\varepsilon^2+s\right)^{2-d/2}, \tag{22}$$

$$\int_0^T \left(\varepsilon^2+r\right)^{2-d/2} dr \leq H_T(\varepsilon) := \begin{cases} 2(d-6)^{-1}\varepsilon^{6-d} & \text{if } d \geq 7, \\ 4\ln \varepsilon^{-1} & \text{if } d=6, \\ \sqrt{6T} & \text{if } d=5. \end{cases} \tag{23}$$

From now on, we assume that $\varepsilon^2 < 2^{-1}(T^{-1}\wedge T)$ and also $\varepsilon^2 \ln \varepsilon^{-1} < T$ if $d=6$. Let us derive an upper bound on J_1. By repeated applications of the Chapman-Kolmogorov identities, we get

$$\begin{aligned} J_1 \leq 2^8 \int_0^T \cdots \int_0^T & ds \ldots ds_6 \int dy\, p(s, x-y) \int dy_1\, p(s_1, y-y_1) \\ & \int dy_2\, p(s_2, y-y_2) \int dy_3\, p(s_3, y_2-y_3) \int dy_4\, p(s_4, y_2-y_4) \\ & \quad \int dy_5\, p(s_5, y_4-y_5) \int dy_6\, p(s_6, y_4-y_6) p(\varepsilon^2, y_1-y_3) p(\varepsilon^2, y_5-y_6) \\ \leq 2^8 \int_0^T \cdots \int_0^T & ds \ldots ds_6\, p(\varepsilon^2+s_1+s_2+s_3, 0) p(\varepsilon^2+s_5+s_6, 0). \end{aligned}$$

We can apply (21), (22) and (23) to get:

$$\begin{aligned} J_1 &\leq \frac{2^8}{(2\pi)^d} T \int_0^T ds_1 \frac{4}{(d-2)(d-4)} (\varepsilon^2+s_1)^{2-d/2}\ \frac{4}{(d-2)(d-4)} T\varepsilon^{4-d} \\ &\leq c_1 T^2 \varepsilon^{4-d} H_T(\varepsilon), \end{aligned}$$

where the constant c_1 depends only on d. We can use the same method for J_2:

$$\begin{aligned} J_2 \leq 2^8 \int_0^T \cdots \int_0^T & ds \ldots ds_6 \int dy\, p(s, x-y) \int dy_1\, p(s_1, y-y_1) \\ & \int dy_2\, p(s_2, y-y_2) \int dy_3\, p(s_3, y_2-y_3) \int dy_4\, p(s_4, y_2-y_4) \\ & \quad \int dy_5\, p(s_5, y_4-y_5) \int dy_6\, p(s_6, y_4-y_6) p(\varepsilon^2, y_1-y_5) p(\varepsilon^2, y_3-y_6) \\ \leq 2^8 \int_0^T \cdots \int_0^T & ds \ldots ds_6 \int dz\, p(s_4, z) p(\varepsilon^2+s_1+s_2+s_5, z) p(\varepsilon^2+s_3+s_6, z), \end{aligned}$$

where we made the change of variables $z = y_2 - y_4$. Since $p(\varepsilon^2 + s_3 + s_6, z) \leq p(\varepsilon^2 + s_3 + s_6, 0)$ and $p(\varepsilon^2 + s_1 + s_2 + s_5, z) \leq p(\varepsilon^2 + s_1 + s_2 + s_5, 0)$, we can argue as for J_1 to get:

$$\begin{aligned} J_2 &\leq 2^8 \int_0^T \cdots \int_0^T ds \ldots ds_6 \; p(\varepsilon^2 + s_1 + s_2 + s_5, 0) p(\varepsilon^2 + s_3 + s_6, 0). \\ &\leq c_1 T^2 \varepsilon^{4-d} H_T(\varepsilon). \end{aligned}$$

By symmetry, we get $J_2 = J_3$. We want now to find an upper bound on J_4. Using (21), (22) and (23) we get:

$$\begin{aligned} J_4 &= 2^6 \int_0^T ds \int dy \; p(s, x-y) \Bigg[\int_0^{T-s} ds_7 \int_{(t-s-s_7)_+}^{T-s-s_7} ds_8 \int_{(t-s-s_7)_+}^{T-s-s_7} ds_9 \\ &\qquad \int dy_7 \; p(s_7, y - y_7) \int dy_8 \; p(s_8, y_7 - y_8) \int dy_9 \; p(s_9, y_7 - y_9) p(\varepsilon^2, y_8 - y_9) \Bigg]^2 \\ &= 2^6 \int_0^T ds \left[\int_0^{T-s} ds_7 \int_{(t-s-s_7)_+}^{T-s-s_7} ds_8 \int_{(t-s-s_7)_+}^{T-s-s_7} ds_9 \; p(\varepsilon^2 + s_8 + s_9, 0) \right]^2 \\ &\leq 2^6 (2\pi)^{-d} \int_0^T ds \left[\int_0^{T-s} ds_7 \frac{4}{(d-2)(d-4)} \left[\varepsilon^2 + 2(t-s-s_7)_+ \right]^{2-d/2} \right]^2 \\ &= \frac{2^{10}}{(2\pi)^d \left[(d-2)(d-4) \right]^2} \\ &\qquad \int_0^T ds \left[\varepsilon^{4-d} [(T-s) - (t-s)_+] + \int_0^{(t-s)_+} ds_7 \left[\varepsilon^2 + 2(t-s-s_7)_+ \right]^{2-d/2} \right]^2 \\ &\leq \frac{2^{10}}{(2\pi)^d \left[(d-2)(d-4) \right]^2} \int_0^T ds \left[\varepsilon^{4-d} [(T-s) \wedge (T-t)] + 2^{-1} H_{2T}(\varepsilon) \right]^2 \\ &\leq \frac{2^{10}}{(2\pi)^d} \left[\frac{\varepsilon^{4-d}}{(d-2)(d-4)} \right]^2 \left[\frac{(T-t)^3}{3} + (T-t)^2 t \right] + c_2 T^2 \varepsilon^{4-d} H_T(\varepsilon), \end{aligned}$$

where the constant c_2 depends only on d. We now compute an upper bound on J_5:

$$\begin{aligned} J_5 &\leq 2^6 \int_0^T \cdots \int_0^T ds \ldots ds_{12} \int dy \; p(s, x-y) \int dy_7 \; p(s_7, y - y_7) \\ &\qquad \int dy_8 \; p(s_8, y_7 - y_8) \int dy_9 \; p(s_9, y_7 - y_9) \int dy_{10} \; p(s_{10}, y - y_{10}) \\ &\qquad\qquad \int dy_{11} \; p(s_{11}, y_{10} - y_{11}) \int dy_{12} \; p(s_{12}, y_{10} - y_{12}) p(\varepsilon^2, y_8 - y_{11}) p(\varepsilon^2, y_9 - y_{12}) \\ &\leq 2^6 \int_0^T \cdots \int_0^T ds \ldots ds_{12} \int dz \; p(s_7 + s_{10}, z) p(\varepsilon^2 + s_8 + s_{11}, z) p(\varepsilon^2 + s_9 + s_{12}, z), \end{aligned}$$

where we made the change of variables $z = y_{10} - y_7$. Since $p(\varepsilon^2+s_9+s_{12}, z) \leq p(\varepsilon^2+s_9+s_{12}, 0)$, and $p(\varepsilon^2 + s_7 + s_8 + s_{10} + s_{11}, 0) \leq p(\varepsilon^2 + s_7 + s_8 + s_{10}, 0)$, we can argue as for J_1, and get:

$$J_5 \leq c_1 T^2 \varepsilon^{4-d} H_T(\varepsilon).$$

By symmetry we get $J_6 = J_5$. Combining the previous bounds leads to

$$\mathbb{N}_x \left[S_\varepsilon \left(\int_0^t ds\ Y_s \right)^2 \right] \leq \frac{2^{10}}{(2\pi)^d} \left[\frac{\varepsilon^{4-d}}{(d-2)(d-4)} \right]^2 \left[\frac{(T-t)^3}{3} + (T-t)^2 t \right] + c_3 T^2 \varepsilon^{4-d} H_T(\varepsilon),$$

where the constant c_3 depends only on d.

We shall now find a lower bound for $\mathbb{N}_x \left[S_\varepsilon (\int_t^T ds\ Y_s) \int_t^T ds\ (Y_s, \mathbf{1}) \right]$. Using similar arguments as in the beginning of the proof, we get

$$\begin{aligned}
I :=& \mathbb{N}_x \left[S_\varepsilon \left(\int_t^T ds\ Y_s \right) \int_t^T ds\ (Y_s, \mathbf{1}) \right] \\
=& \frac{1}{3} 3! 2^3 \int_0^T ds \int dy\ p(s, x-y) \int_{(t-s)_+}^{T-s} ds_1 \int dy_1\ p(s_1, y-y_1) \\
& \qquad \int_0^{T-s} ds_2 \int dy_2\ p(s_2, y-y_2) \int_{(t-s-s_2)_+}^{T-s-s_2} ds_3 \int dy_3\ p(s_3, y_2-y_3) \int_{(t-s-s_2)_+}^{T-s-s_2} ds_4 \\
& \qquad\qquad \int dy_4\ p(s_4, y_2-y_4) \left[p(\varepsilon^2, y_1-y_3) + p(\varepsilon^2, y_1-y_4) + p(\varepsilon^2, y_3-y_4) \right].
\end{aligned}$$

Since we are looking for a lower bound, we restrict our attention to the term $p(\varepsilon^2, y_3 - y_4)$. We get

$$\begin{aligned}
I \geq& 2^4 \int_0^T ds \int_{(t-s)_+}^{T-s} ds_1 \int_0^{T-s} ds_2 \int_{(t-s-s_2)_+}^{T-s-s_2} ds_3 \int_{(t-s-s_2)_+}^{T-s-s_2} ds_4\ p(\varepsilon^2 + s_3 + s_4, 0) \\
\geq& \frac{2^4}{(2\pi)^{d/2}} \frac{4}{(d-2)(d-4)} \int_0^T ds\ \left[(T-s) \wedge (T-t) \right] \\
& \qquad \int_0^{T-s} ds_2 \left[\left(\varepsilon^2 + 2(t-s-s_2)_+ \right)^{2-d/2} - 2 \left(\varepsilon^2 + (T-s-s_2) \right)^{2-d/2} \right] \\
\geq& \frac{2^6}{(2\pi)^{d/2}} \frac{1}{(d-2)(d-4)} \int_0^T ds\ \left[(T-s) \wedge (T-t) \right] \left[\varepsilon^{4-d} (T-s-(t-s)_+) - 2 H_T(\varepsilon) \right] \\
\geq& \frac{2^6}{(2\pi)^{d/2}} \frac{\varepsilon^{4-d}}{(d-2)(d-4)} \left[\frac{(T-t)^3}{3} + (T-t)^2 t \right] - c_4 T^2 H_T(\varepsilon),
\end{aligned}$$

where c_4 depends only on d. Finally we deduce from section 5.1, with $\varphi(s) = \mathbf{1}_{[0,T-t]}(s)$, that

$$\mathbb{N}_x \left[\left[\int_t^T ds\ (Y_s, \mathbf{1}) \right]^2 \right] = 4 \left[\frac{(T-t)^3}{3} + (T-t)^2 t \right].$$

Combining the previous results, we get for ε small enough

$$\mathbb{N}_x\left[\left[\varepsilon^{d-4}(2\pi)^{d/2}S_\varepsilon\left(\int_t^T ds\; Y_s\right) - \frac{2^4}{(d-2)(d-4)}\int_t^T ds\;(Y_s,\mathbf{1})\right]^2\right] \le c_5T^2\varepsilon^{d-4}H_T(\varepsilon)$$
$$\le c_6T^2\varepsilon,$$

where c_6 depends only on d. This gives the convergence in $L^2(\mathbb{N}_x)$. Now $S_\varepsilon\left(\int_t^T ds\; Y_s\right)$ is monotone decreasing in ε (cf lemma 5.3 in [15]). The $\mathbb{N}_x$-a.e. convergence then follows from the previous estimate by an application of the Borel-Cantelli lemma and monotonicity arguments. □

5. Appendix

5.1. **Formula for moments of the Brownian snake.** For the reader's convenience, we recall some explicit formulas for moments of the Brownian snake. These formulas are well-known, at least in the context of superprocesses (see e.g. Dynkin [5]). We can compute the Laplace functional of $\int_0^t ds(Y_s,\varphi(s))$ for $\varphi\in\mathcal{B}_{b+}(\mathbb{R}^+\times\mathbb{R}^d)$. To this end start from the finite dimensional Laplace functional (2) with $t_i=i/m$, $\varphi_i=\frac{1}{m}\varphi(i/m)$ for a nonnegative continuous function φ with compact support on $\mathbb{R}^+\times\mathbb{R}^d$. Thanks to the continuity of the process X, by a suitable passage to the limit, we get for $\nu\in M_f$

$$\mathbb{E}_\nu^X\left[\exp\left[-\int_0^t (X_{t-s},\varphi(s))ds\right]\right] = \exp\left[-(\nu,v(t))\right],$$

where v is a nonnegative solution of (1) with right-hand side $J(t,x)=\int_0^t ds\; P_{t-s}[\varphi(s)](x)$. This can be extended by monotone class arguments to any $\varphi\in\mathcal{B}_{b+}(\mathbb{R}^+\times\mathbb{R}^d)$. The uniqueness of the solution is easily established using arguments similar to the classical Gronwall lemma. Then we get $v(t,x)=\mathbb{N}_x\left[1-\exp\left[-\int_0^t ds\;(Y_{t-s},\varphi(s))\right]\right]$, thanks to theorem 1.3.

Now we introduce an auxiliary power series. Let us consider the analytic function $f(\lambda)=1-\sqrt{1-\lambda}$ for $|\lambda|<1$. It is easy to check that for $|\lambda|<1$, we have

$$f(\lambda)=\sum_{n=1}^\infty \gamma_n\lambda^n,$$

where the sequence $(\gamma_n, n\ge 1)$ is defined by $\gamma_1=1/2$ and the recurrence

$$\gamma_n=\frac{1}{2}\sum_{k=1}^{n-1}\gamma_k\gamma_{n-k}\quad\text{for}\quad n\ge 2$$

(use the fact that f solves $2f(\lambda)=f(\lambda)^2+\lambda$). Now let $T>0$ and J a nonnegative measurable function on $\mathbb{R}^+\times\mathbb{R}^d$, such that $M_T=\sup_{[0,T]\times\mathbb{R}^d}J(t,x)<\infty$. We define the family of measurable functions $(h_n, n\ge 1)$ on $\mathbb{R}^+\times\mathbb{R}^d$, by the initial condition

$$h_1(t)=J(t),$$

and the recurrence

$$h_n(t) = 2\sum_{k=1}^{n-1}\int_0^t ds\ P_s\left[h_k(t-s)h_{n-k}(t-s)\right] \quad \text{for} \quad n \geq 2. \tag{24}$$

We clearly have for every $n \geq 1$,

$$\sup_{[0,T]\times\mathbb{R}^d} |h_n| \leq [4T]^{n-1}[2M_T]^n\gamma_n.$$

Thus the power series $w(\lambda,t) = \sum(-1)^{n+1}\lambda^n h_n(t)$ is normally convergent on $[0,T]\times\mathbb{R}^d$ for $|\lambda| < [8TM_T]^{-1}$. And it clearly solves the integral equation on $[0,T]\times\mathbb{R}^d$

$$w(t) + 2\int_0^t ds\ P_s\left[w(t-s)^2\right] = \lambda J(t). \tag{25}$$

To get the uniqueness of the solution to the previous integral equation, use arguments similar to Gronwall's lemma. Finally we can compute the moments for the process Y under $\mathbb{N}_x$. Indeed, let $\varphi \in \mathcal{B}_{b+}(\mathbb{R}^+\times\mathbb{R}^d)$. We have shown that for $\lambda > 0$, the function $v_\lambda(t,x) = \mathbb{N}_x\left[1-\exp -\lambda\int_0^t (Y_{t-s},\varphi(s))ds\right]$ is the unique solution to (25) on $\mathbb{R}^+\times\mathbb{R}^d$ with $J(t,x) = \int_0^t ds\ P_s[\varphi(t-s)](x)$. Thus for $\lambda \geq 0$ small enough, we have $v_\lambda(t) = w(\lambda,t)$. Then from the series expansion for $w(\lambda,t)$, we get for every integer $n \geq 1$

$$\mathbb{N}_x\left[\left(\int_0^t ds\ (Y_{t-s},\varphi(s))\right)^n\right] = n!h_n(t,x),$$

where the functions h_n are defined by $h_1(t) = \int_0^t ds\ P_s[\varphi(t-s)]$, and the recurrence (24). In the same way it can be shown that for every $\varphi \in \mathcal{B}_{b+}(\mathbb{R}^d)$, for every $t \geq 0$, $n \geq 1$,

$$\mathbb{N}_x\left[(Y_t,\varphi)^n\right] = n!h_n(t,x),$$

where the functions are defined by $h_1(t) = P_t[\varphi]$, and the recurrence (24).

5.2. **Some properties of the function u_1.** We consider the function u_1, which is the maximal solution on $(1,\infty)$ of the non linear differential equation

$$u''(r) + \frac{d-1}{r}u'(r) = 4u(r)^2.$$

Lemma 5.1. *There exist positive constants* a_0, b_0 *and* $\mathrm{b'}_1$, *depending only on* d, *such that*

$$\lim_{r\to\infty} r^{d-2}u_1(r) = \mathrm{a}_0 \quad \text{if } d \geq 5, \quad \lim_{r\to\infty} r^2\log(r)\ u_1(r) = \mathrm{a}_0 = 1/2 \quad \text{if } d = 4;$$

furthermore for every $r > 1$,

$$u_1(r) \geq \mathrm{a}_0 r^{2-d} \quad \text{if } d \geq 5, \quad u_1(r) \geq \mathrm{a}_0 r^{-2}\log(2r)^{-1} \quad \text{if } d = 4; \tag{26}$$

and for every $r \geq 4/3$,

$$u_1(r) \leq \mathrm{b}_0 r^{2-d} \quad \text{if } d \geq 5, \quad u_1(r) \leq \mathrm{b}_0[2r^2\log(r)]^{-1} \quad \text{if } d = 4; \tag{27}$$

$$u_1(r) \leq \mathrm{a}_0 r^{2-d} + \mathrm{b'}_1 r^{6-2d} \quad \text{if } d \geq 5, \tag{28}$$

$$u_1(r) \leq \mathrm{a}_0 r^{-2}\log(r)^{-1} + \mathrm{b'}_1 r^{-2}\log(r)^{-2}\log(\log(r)) \quad \text{if } d = 4. \tag{29}$$

We will prove this lemma by giving the asymptotic expansion of u_1 at ∞. For $d \geq 5$, we will see the constant a_0 can be expressed as the radius of convergence of a series. We introduce the auxiliary function

$$z(t) = 4(d-2)^{-2(d-1)/(d-2)} t \; u_1 \left[\left(\frac{t}{d-2} \right)^{1/(d-2)} \right], \quad \text{for } t > d-2.$$

This function is solution on $(d-2, \infty)$ of

$$y''(t) = t^{-\delta-2} y(t)^2, \tag{30}$$

where $\delta = \dfrac{d-4}{d-2}$. We deduce from [18] p.132 that the function $z(t)$ (i.e. $r^{d-2}u_1(r)$) is decreasing.

Proof in the case $d \geq 5$. We deduce from theorem 1.1 of [18] that the limit $q = \lim_{t\to\infty} z(t)$ exists and is positive. Hence by integrating (30) twice from t to ∞, we get for $t > d-2$,

$$z(t) - q = \int_t^\infty (r-t) r^{-\delta-2} z(r)^2 dr. \tag{31}$$

Now consider the sequence $(q_n, n \geq 0)$ defined by $q_0 = 1$ and the recurrence

$$q_n = \frac{1}{n\delta(n\delta+1)} \sum_{k=0}^{n-1} q_k q_{n-k-1}, \quad \text{for} \quad n \geq 1.$$

Clearly we have for every $n \geq 0$, $q_n \leq 2\,[4/\delta]^n\, \gamma_{n+1}$, where the sequence $(\gamma_n, n \geq 1)$ has been introduced in section 5.1. The power series $\sum q_n q^{n+1} t^{-\delta n}$ is convergent and even C^∞ as a function of t at least for $t > t_1 = [4q/\delta]^{1/\delta}$. This power series also solves (31) for $t > t_1$. The same arguments as in the proof of the Gronwall lemma show that equation (31) possesses a unique solution bounded in a neighborhood of infinity. Thus the function z and the power series agree for $t > t_1$. Since the function z is analytic on $(d-2, \infty)$ and since q and the coefficients q_n are positive, we deduce that the radius of convergence of the series $\sum q_n s^n$ is $q(d-2)^{-\delta}$ and that for $t > d-2$

$$z(t) = \sum_{n=0}^\infty q_n q^{n+1} t^{-n\delta}.$$

Thus we get with obvious notation for $r > 1$,

$$\begin{aligned} u_1(r) &= 4^{-1}(d-2)^{d/(d-2)} r^{2-d} \sum_{n=0}^\infty q_n q^{n+1} (d-2)^{-n(d-4)/(d-2)} r^{-n(d-4)} \\ &= r^{2-d} \sum_{n=0}^\infty \mathrm{a}_n r^{-n(d-4)}. \end{aligned}$$

Since the function $r^{d-2}u_1(r)$ is decreasing we get (26). Since the real numbers $(\mathrm{a}_n, n \geq 0)$ are positive, (27) and (28) follow easily. Notice that $4(d-2)^{-2}\mathrm{a}_0$ is the radius of convergence of the series $\sum q_n s^n$. □

Proof in the case $d = 4$. We write $f(t) \sim g(t)$ at $0+$ when the real function f and g are positive or negative on $I = (0, 0+\varepsilon)$ for some $\varepsilon > 0$ and $\lim_{t\in I, t\to 0} f(t)/g(t) = 1$. We also write

$f(t) \sim g(t)$ at ∞ when $f(1/t) \sim g(1/t)$ at $0+$. We know from [18] p.133 that $z(t) \sim \log(t)^{-1}$ at ∞. We deduce from (30) that $z'(t)$ is negative on $(2, \infty)$ and $z''(t) \sim [t\log(t)]^{-2}$ at ∞. By integration, we get $z'(t) \sim t^{-1}\log(t)^{-2}$ at ∞. We now consider the function $w(s) = z(\mathrm{e}^s)$ which solves $w'' - w' = w^2$ on $(\log 2, \infty)$. Notice that the function w is positive decreasing and w' is negative. We also have $w(s) \sim s^{-1}$, $w'(s) \sim -s^{-2}$ and $w''(s) = o(s^{-2})$ at ∞. Thus the function defined on $(0, \infty)$ by

$$p(w(s)) = w'(s), \quad \text{for} \quad s \in (\log 2, \infty),$$

is well defined and even of class C^1, and $p'(w(s)) = w''(s)/w'(s)$. Thus the function p can be extended as a C^1 function on $[0, \infty)$ by setting $p(0) = 0$ and $p'(0) = 0$. Furthermore it solves

$$p(w)p'(w) - p(w) = w^2 \quad \text{on} \quad [0, \infty).$$

We also have $p(w) \sim -w^2$ at $0+$. We consider the sequence $(\rho_n, n \geq 2)$ defined by $\rho_2 = 1$ and the recurrence

$$\rho_n = \sum_{k=2}^{n-1} k\rho_k\rho_{n-k+1}, \quad \text{for} \quad n \geq 3.$$

The radius of convergence of the series $\sum(-1)^{n+1}\rho_n w^n$ is 0, nevertheless we will prove it is the asymptotic expansion of p at $0+$. We set $H_n(w) = \sum_{k=2}^n (-1)^{k+1}\rho_k w^k$ for $n \geq 2$. We now prove by recurrence that $p(w) = H_n(w) + h_n(w)$, where $h_n(w) = o(w^n)$ at $0+$. This is true for $n = 2$. Let us assume it is true at stage n. Let $g_{n,\alpha}(w) = (1-\alpha)(-1)^n\rho_{n+1}w^{n+1} - h_n(w)$. We easily have

$$\begin{aligned} g'_{n,\alpha}(w)p(w) + g_{n,\alpha}(w)[H'_n(w) - 1] &= \alpha(-1)^n\rho_{n+1}w^{n+1} + o(w^{n+1}), \\ &= (-1)^{n+1}\rho_{n+2}w^{n+2} + o(w^{n+2}), \quad \text{if} \quad \alpha = 0. \end{aligned}$$

Let us assume n is even. For $\alpha = 0$, the above right hand side is negative on $(0, \varepsilon]$, for ε small enough. Since p is negative and $[H'_n(w) - 1] < 0$ on $[0, \varepsilon]$, for ε small, we see that $g_{n,0}(w) < 0$ implies $g'_{n,0}(w) \geq 0$. As $g_{n,0}(0) = 0$, we get by contradiction that $g_{n,0} \geq 0$ on $[0, \varepsilon]$. This implies $h_n(w) \leq \rho_{n+1}w^{n+1}$. Similar arguments for $\alpha > 0$ implies that $g_{n,\alpha} \leq 0$ on $[0, \varepsilon_\alpha]$ for $\varepsilon_\alpha > 0$ small enough. Since this holds for any $\alpha > 0$ and since $h_n(w) \leq \rho_{n+1}w^{n+1}$ for w small enough, we deduce that $h_{n+1}(w) = h_n(w) - \rho_{n+1}w^{n+1} = o(w^{n+1})$. If n is odd the proof is similar.

From the definition of p, we then have $w'(s) = H_n(w(s)) + O(w(s)^{n+1})$ at ∞. For $n = 3$ this gives $w'(s) = -w(s)^2 + 2w(s)^3 + O(w(s)^4)$ at ∞. Since $w(s) \sim s^{-1}$ at $+\infty$, we deduce by integration that

$$\frac{1}{w(s)} - 2\log w(s) + O(1) = s \text{ at infinity.}$$

Standard arguments yields $w(s) = s^{-1} + 2s^{-2}\log(s) + O(s^{-2})$ at infinity. Thus we have

$$u_1(r) = \frac{1}{r^2}\left[\frac{1}{2\log(r)} + \frac{\log(\log(r))}{4\log(r)^2} + O\left(\log(r)^{-2}\right)\right] \text{ at } +\infty.$$

Notice the previous calculation can be continued to give an asymptotic expansion of u_1 at infinity. The inequalities (27) and (29) follow easily. We will now prove that for every $r > 1$, $u_1(r) \geq [2r^2\log(2r)]^{-1}$. We consider the function $w(r) = u_1(r) - [2r^2\log(2r)]^{-1}$. The function w is positive at least over $(1, 1+\eta) \cap (\eta^{-1}, \infty)$ for η small. Let us assume that w achieves its minimum at r_0 and that $w(r_0) \leq 0$. Then we have $r_0 \in [1+\eta, \eta^{-1}]$, $w'(r_0) = 0$ and $w''(r_0) \geq 0$. An easy computation gives

$$w''(r) = 4w(r)\left[u_1(r) + \frac{1}{2r^2\log(2r)}\right] - \frac{3}{r}w'(r) - \frac{1}{2r^4(\log(2r))^3}.$$

Evaluation at $r = r_0$ implies that $w''(r_0) < 0$. This contradicts the assumption. Hence w is positive, that is we get (26) for $d = 4$. □

References

[1] D. ALDOUS. Tree based models for random distribution of mass. *J. Statist. Phys.*, 73(3-4):625–641, 1993.

[2] R. BLUMENTHAL. *Excursions of Markov processes.* Birkhäuser, Boston, 1992.

[3] E. DERBEZ and G. SLADE. The scaling limit of lattice trees in high dimension. Preprint, 1996.

[4] E. DERBEZ and G. SLADE. Lattice trees and super-Brownian motion. *Canad. Math. Bull.*, 40(1):19–38, 1997. To appear.

[5] E. DYNKIN. Representation for functionals of superprocesses by multiple stochastic integrals, with application to self-intersection local times. *Astérisque*, 157-158:1147–1171, 1988.

[6] E. DYNKIN. Branching particle systems and superprocesses. *Ann. Probab.*, 19:1157–1194, 1991.

[7] E. DYNKIN. A probabilistic approach to one class of nonlinear differential equations. *Probab. Th. Rel. Fields*, 89:89–115, 1991.

[8] E. DYNKIN. *An introduction to branching measure-valued processes*, volume 6 of *CRM Monograph series.* Amer. Math. Soc., Providence, 1994.

[9] J.-F. LE GALL. A class of path-valued Markov processes and its applications to superprocesses. *Probab. Th. Rel. Fields*, 95:25–46, 1993.

[10] J.-F. LE GALL. The uniform random tree in a Brownian excursion. *Probab. Th. Rel. Fields*, 96:369–383, 1993.

[11] J.-F. LE GALL. Hitting probabilities and potential theory for the Brownian path-valued process. *Ann. Inst. Four.*, 44:277–306, 1994.

[12] J.-F. LE GALL. A path-valued Markov process and its connections with partial differential equations. In *Proceedings in First European Congress of Mathematics*, volume II, pages 185–212. Birkhäuser, Boston, 1994.

[13] J.-F. LE GALL. Communication personnelle.

[14] R. PEMANTLE and Y. PERES. Galton-Watson trees with the same mean have the same polar sets. *Ann. Probab.*, 23(3):1102–1124, 1995.

[15] R. PEMANTLE, Y. PERES, and J. W. SHAPIRO. The trace of spatial Brownian motion is capacity-equivalent to the unit square. *Probab. Th. Rel. Fields*, 106(3):379–400, 1996.

[16] E. A. PERKINS. The strong Markov property of the support of super-Brownian motion. In *The Dynkin Festschrift*, volume 34 of *Progr. Probab.*, pages 307–326, Boston, 1994. Birkhäuser.

[17] S. C. PORT and C. J. STONE. *Brownian motion and classical potential theory.* Academic Press, 1978.

[18] S. D. TALIAFERRO. Asymptotic behavior of solutions of $y'' = \phi(t)\ y^\lambda$. *J. Math. Analys. and Appl.*, 66:95–134, 1978.

[19] R. TRIBE. A representation for super-Brownian motion. *Stoch. Process. and Appl.*, 51:207–219, 1994.

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