SUPERSIMPLE ω-CATEGORICAL GROUPS

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Abstract. An ω-categorical supersimple group is finite-by-abelian-by-finite, and has finite SU-rank. Every definable subgroup is commensurable with an $\mathrm{acl}(\emptyset)$-definable subgroup.

It is well-known that an ω-categorical superstable group is abelian-by-finite [1] and has finite rank (in fact by [3] every ω-categorical superstable theory is even one-based of finite Morley rank). In the case of a supersimple ω-categorical theory, much less is known except that things do not turn out that nicely: recently, Hrushovski [5] has constructed a simple ω-categorical theory of SU-rank 1 whose geometry is not locally modular. It is unknown whether a supersimple ω-categorical theory can have infinite rank. The question therefore arises whether at least in the group case the results from the superstable case generalize. This is indeed the case; we shall prove the following:

Theorem 1. *An ω-categorical supersimple group is finite-by-abelian-by-finite, and has finite SU-rank.*

The proof will proceed in two stages: we shall first prove that an ω-categorical group of finite SU-rank is finite-by-abelian-by-finite. (That the argument given in [7] for ω-categorical groups of finite Morley rank can be generalized to the present context has also been noted by David Evans [4].) For this we shall also need some information about definable subgroups of an ω-categorical group of finite SU-rank. It turns out that as in the superstable case, every definable subgroup is commensurable with one definable over $\mathrm{acl}(\emptyset)$. However, it is not true that an ω-categorical group of finite SU-rank is necessarily one-based: Pourmahdian [8] has fused Hrushovski's example mentioned above with an elementary abelian 2-group, and obtained an ω-categorical group of SU-rank 1 which is not one-based. Finally, we shall use the information obtained for ω-categorical groups of finite SU-rank to show that ω-categorical supersimple groups of infinite rank cannot exist.

One of our main tools is the following Theorem due to Schlichting [9], and rediscovered by Bergmann and Lenstra [2]:

Fact 2. *Let G be a group and $\mathfrak{F}$ a family of subgroups such that for any $F_1, F_2 \in \mathfrak{F}$ the index $|F_1 : F_1 \cap F_2|$ is bounded. Then there is a subgroup F^c, invariant under all automorphisms of G which stabilize $\mathfrak{F}$ setwise, and commensurable with all $F \in \mathfrak{F}$.*

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In fact, it is easy to see that if all the elements of $\mathfrak{F}$ are definable, then F^c is definable as a finite extension of a finite intersection of groups in $\mathfrak{F}$. (In our case definability also follows immediately from ω-categoricity.)

Corollary 3. *Let F be a definable subgroup of an ω-categorical group. Then there is a definable subgroup F^c, commensurable with F, such that F^c is stabilized under any automorphism of G stabilizing the commensurability class of F.*

Proof: By ω-categoricity, there is a bound on the indices $|F : F \cap F'|$, for F' a commensurable conjugate of F. Now apply Fact 2. $\square$

In fact, the same conclusion holds for any relatively definable subgroup of a group type-definable in a simple theory, see [10, Corollary 1.8]. Clearly, F^c is defined over the same parameters as the original F.

We shall first look at abelian groups of finite SU-rank. The following definition differs from the stable case.

Definition 1. A group is *minimal* if every definable subgroup of infinite index is finite.

Note that in a supersimple theory every definable group contains a definable minimal subgroup (any infinite definable subgroup of minimal possible SU-rank will do).

Theorem 4. *Let G be an ω-categorical finite-by-abelian-by-finite group of finite SU-rank. Then any definable subgroup of G is commensurable with one definable over $acl(\emptyset)$.*

Proof: Suppose N is a normal subgroup of finite index in G and with finite commutator group N'. If N_1 is a conjugate of N by some automorphism of G, then $(NN_1)'$ is finite as well; we may thus assume that N is characteristic, and hence $\emptyset$-definable, as is N'. Replacing a definable subgroup H of G by the definable subgroup $(H \cap N)/N'$ of N/N', we can reduce the problem to the case when G is abelian.

For abelian G, consider first a minimal subgroup A of G. By the finiteness of rank, there are finitely many conjugates of A, say $(A_i : i < n)$, such that every conjugate of A intersects $A^0 := \sum_{i<n} A_i$ in a subgroup of finite index. We may choose the A_i almost linearly independent, i.e. $A_i \cap \sum_{j \neq i} A_j$ is finite for all $i < n$.

Definition 2. A *virtual isogeny* f between an abelian minimal group A and an abelian minimal group B is a definable isomorphism $f : D/K \to I/C$, where the *domain* D has finite index in A, the *image* I has finite index in B, and K (the *kernel*) and C (the *cokernel*) are finite. Two virtual isogenies f_1 and f_2 are *equivalent* if the derived maps from $D_1 \cap D_2$ to $(I_1 + I_2)/(C_1 + C_2)$ agree on a subgroup of finite index in A. This is an equivalence relation, and in fact a congruence.

Lemma 5. *In an ω-categorical theory, the equivalence class of every virtual isogeny from A to B (both abelian minimal $\emptyset$-definable groups) is algebraic over $\emptyset$ (as an imaginary element).*

Proof: It is standard that the family of virtual isogenies from A to A modulo equivalence, together with the zero endogeny, forms a ring R with identity, called the ring of virtual endogenies; it is easy to see from minimality that R has no zero divisors. If A is ω-categorical, R must be locally finite, whence a locally finite field, and every element of R is algebraic. Furthermore, if there is a virtual isogeny f between A and B, then composing any virtual isogeny r between A and B with f^{-1} (which is a virtual isogeny between B and A) yields a virtual isogeny of A; clearly r is algebraic over rf^{-1} and f. Hence the equivalence class $[r]$ of r is algebraic over $[f]$. Choosing another virtual isogeny f' between A and B with $\mathrm{acl}(f') \cap \mathrm{acl}(f) = \mathrm{acl}(\emptyset)$, we see that $[r]$ must be algebraic over $\emptyset$. $\square$

Now consider any conjugate A' of A. Every $a \in A' \cap A^0$ can be written as $\sum_{i<n} a_i$ with $a_i \in A_i$; since $A' \cap A^0$ is infinite by maximality of n, there must be some $i = i(A') < n$ such that $A_i \cap (A' + \sum_{k \neq i} A_k)$ is infinite. For every $j \neq i$ we now define an endogeny $r(A', j)$ from A_i to A_j: if $A_j \cap (A' + \sum_{k \neq j} A_k)$ is finite, then $r(A', j) = 0$; otherwise $r(A', j)(x) := \{y \in A_j : x - y \in A' + \sum_{k \neq i,j} A_k\} = A_j \cap (x + A' + \sum_{k \neq i,j} A_k)$ (and it is easy to see that this is a virtual isogeny). But if A'' is such that $i(A'') = i(A')$, and $r(A', j)$ and $r(A'', j)$ are equivalent virtual endogenies for all $j \neq i(A')$, then one can check that A' and A'' must be commensurable. It follows that there are only finitely many commensurability classes among the conjugates of A.

Consider the group A^c given by Corollary 3. It has only finitely many conjugates, corresponding to the commensurability classes of conjugates of A, and must therefore be definable over $\mathrm{acl}(\emptyset)$. This proves the assertion for minimal groups.

If $H \leq G$ is not minimal, then by supersimplicity it contains a minimal subgroup A which is commensurable with some $\mathrm{acl}(\emptyset)$-definable A^c. But HA^c/A^c is a subgroup of G/A^c of smaller SU-rank; by induction it is commensurable with an $\mathrm{acl}(\emptyset)$-definable group H_c/A^c, whose preimage H^c in G is as required. $\square$

Theorem 6. *An ω-categorical group of finite SU-rank is finite-by-abelian-by-finite.*

Proof: Suppose not, and let G be a counterexample. By ω-categoricity, we may assume that G has no proper $\emptyset$-definable subgroups of finite index.

Claim: *We may assume that $G'/(G' \cap Z(G))$ is finite.*

Proof of Claim: By simplicity, G does not have the strict order property; by [6, Theorem 1.2] it is nilpotent. By ω-categoricity, the subgroups G^n in the lower central series are $\emptyset$-definable and there is some minimal $n > 1$ such that G'/G^n is infinite. (Note that G/G' must be infinite since G has no characteristic subgroup of finite index, and G' is infinite since G is not finite-by-abelian.)

We claim that G/G^n is not finite-by-abelian-by-finite. So suppose otherwise, and let N be a normal subgroup of G of finite index (not necessarily definable) such that N'/G^n is finite. If N_1 is a conjugate of N by some automorphism of G, then $(NN_1)'/G^n$ is finite as well; we may thus assume that N is characteristic. But since G has no proper characteristic subgroup of finite index, $N = G$, and G'/G^n is finite, a contradiction.

As G'/G^{n-1} is finite by minimality of n, G^{n-1}/G^n is infinite; since G^{n-1}/G^n is contained in $Z(G/G^n)$, we may replace G by G/G^n and are done. $\square$

Since $G'/(G' \cap Z(G))$ is finite, $G_g := C_{G/Z(G)}(g/Z(G))$ has finite index in $G/Z(G)$ for every $g \in G$. Put $H_g := \{(h, [h, g]) : h \in G_g\}$, a subgroup of $G/Z(G) \times Z(G)$

(note that the commutator $[h, g]$ is well-defined). Since $G/Z(G) \times Z(G)$ is finite-by-abelian, H_g is commensurable with an $\mathrm{acl}(\emptyset)$-definable group by Lemma 4; it follows by compactness that there are only finitely many commensurability classes in the family $\{H_g : g \in G\}$. Note that $[h, g] = [h, g']$ for any $h \in \pi_1(H_g \cap H_{g'})$, whence $[h, g'g^{-1}] = 1$. However, we may choose g and g' to be two independent generic elements such that H_g and $H_{g'}$ are commensurable. Then $\pi_1(H_g \cap H_{g'})$ is a subgroup of finite index in $G/Z(G)$, and $g^{-1}g'$ is a generic element of G with $|G : C_G(g^{-1}g')|$ finite.

The set of all $g \in G$ such that $C_G(g)$ has finite index in G is a subgroup of G, which is $\emptyset$-definable by ω-categoricity; by the above it has finite index in G. As G has no $\emptyset$-definable subgroups of finite index by assumption, $C_G(g)$ has finite index in G for all $g \in G$. But then $[g, G]$ is finite for all $g \in G$. If $g \mathop{\smile\!\!\!\!\!\!\;|}\, g'$, then $[g, g'] \in \mathrm{acl}(g) \cap \mathrm{acl}(g') = \mathrm{acl}(\emptyset)$. Since every element g' of G can be written as $g' = g_1 g_2$ with $g \mathop{\smile\!\!\!\!\!\!\;|}\, g_1$ and $g \mathop{\smile\!\!\!\!\!\!\;|}\, g_2$, we obtain $[g, g'] = [g, g_1 g_2] = [g, g_2][g, g_1][g, g_1, g_2]$. Then $[g, g_1] \in \mathrm{acl}(\emptyset)$ is independent of g_2, and $[g, g_1, g_2] \in \mathrm{acl}(\emptyset)$ as well. Thus there are only finitely many commutators and G' is finite, a contradiction. $\square$

Theorem 7. *An ω-categorical supersimple group has finite SU-rank.*

Proof: Let G be an ω-categorical supersimple group of infinite rank, and let $\varphi(x)$ be a formula isolating a type p of rank ω. Let n be a maximal finite rank of a 1-type over the parameters used so far.

Let $\bar{a}$ be a tuple of independent realizations of p such that there is a non-algebraic type of finite SU-rank greater than n over $\bar{a}$. (For example, take a forking extension q of p of rank greater than n; a sufficiently long initial segment of a Morley sequence in q will do.) Let $\bar{p}$ be the type of $\bar{a}$, and let m be the maximal rank of an extension of p of finite SU-rank over $2 \cdot |\bar{a}|$ elements. Let $\bar{q}$ be a non-algebraic extension of $\bar{p}$ of finite SU-rank, such that the SU-rank of every co-ordinate over the domain of $\bar{q}$ and the other co-ordinates is greater than m. Consider a Morley sequence $I := (\bar{a}_i : i < \omega)$ in $\bar{q}$. Then $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\;|}\, \bar{a}_1$ by assumption on the SU-ranks of the co-ordinates of $\bar{q}$. Choose a maximal independent subset A of I containing $\bar{a}_0$ (independent as singletons); this is a finite set. Let $H := \{g \in G : SU(g/\bar{a}_0) < \omega\}$ and $K := \{g \in G : SU(g/A) < \omega\}$. By ω-categoricity, H is an $\bar{a}_0$-definable subgroup and K is an A-definable subgroup of G, with $H \leq K$ and $SU(K) < \omega$.

As I is a Morley sequence, for every $j < \omega$ there is an automorphism σ_j of G which maps $\bar{a}_i$ to $\bar{a}_{i+j}$ for all $i < \omega$. Then $SU(\sigma_j(A)) = SU(A) = \omega \cdot |A|$, but $SU(\bar{a}_0, \ldots, \bar{a}_n) < SU(A) + \omega$ for all $n < \omega$. It follows that $SU(A/\sigma_j(A)) < \omega$, so σ_j stabilizes K (setwise). However, it maps $\bar{a}_0$ to $\bar{a}_j$. By Theorem 4 there are only finitely many commensurability classes among K-conjugates of H, so there is $i < j$ such that $\sigma_i(H)$ and $\sigma_j(H)$ are commensurable, whence $\sigma_i(H^c) = \sigma_j(H^c)$.

Now H^c is definable over $\bar{a}$, so $\sigma_i(H^c)$ is definable over $\mathrm{dcl}(\bar{a}_i) \cap \mathrm{dcl}(\bar{a}_j)$. But $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\;|}\, \bar{a}_1$ implies $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\;|}\, \bar{a}_j$ by indiscernibility. Thus $\sigma_i(H^c)$ is a non-trivial group of finite SU-rank definable over $\mathrm{acl}(\emptyset)$. If $r \in S_1(\mathrm{acl}(\emptyset))$ is a generic type for $\sigma_i(H^c)$, then $SU(r) = SU(\sigma_i(H^c)) = SU(H^c) = SU(H) > n$. This contradicts the maximal choice of n, as r does not fork over $\emptyset$. $\square$

References

[1] Walter Baur, Gregory Cherlin and Angus Macintyre, Totally categorical groups and rings, J. Algebra, 57 (1979), 407–440.

[2] George M. Bergman and Hendrik W. Lenstra, Jr., Subgroups close to normal subgroups, J. Algebra, 127 (1989), 80–97.

[3] Gregory Cherlin, Leo Harrington and Alistair Lachlan, $\aleph_0$-categorical $\aleph_0$-stable structures, Ann. Pure Appl. Logic, 28 (1985), 103–135.

[4] David M. Evans, $\aleph_0$-categorical groups of finite rank, notes (1998).

[5] Ehud Hrushovski, Simplicity and the Lascar group, notes (1997).

[6] H. D. Macpherson, Absolutely ubiquitous structures and $\aleph_0$-categorical groups, Quart. J. Math. Oxford (2), 39 (1988), 483–500.

[7] Bruno Poizat, Groupes Stables, Nur al mantiq wal marifah, 1988.

[8] Massoud Pourmahdian, notes 1998.

[9] G. Schlichting, Operationen mit periodischen Stabilisatoren, Arch. Math. (Basel), 34 (1980), 97–99.

[10] Frank O. Wagner, Groups in simple theories, submitted to the J. Symb. Logic (1997).

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