On the geometry of diffusion operators and stochastic flows

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Contents

Introduction

The concepts of second order semi-elliptic operator, Markov semi-group, diffusion process, diffusion measures on path spaces essentially give different pictures of the same fundamental objects, with related Riemannian or sub-Riemannian geometry. Here we consider a different layer of structure centred around the concepts of sums of squares of vector fields, stochastic differential equations, stochastic flows and Gaussian vector fields; again essentially equivalent, and this time with associated metric linear connections on tangent bundles and subbundles of tangent bundles. The difference between these two levels of structure can be seen from the fact that if a semi-elliptic differential operator on functions on a manifold M is given a representation as a sum of square of vector fields ("Hörmander form") it automatically gets an extension to an operator on differential forms. In exactly the same way representing a diffusion process as the one point motion of a stochastic flow determines a semi-group acting on differential forms (by pulling the form back by the flow and taking expectation.) Given a regularity condition there is an associated linear connection and adjoint 'semi-connection' in terms of which these operators can be simply described (e.g. by a Weitzenbock formula) as can many other important quantities (e.g. existence of moment exponents for stochastic flows). Moreover in the stochastic picture the connections remain relevant in the collapse from this level to the simpler one giving new results and new proofs of results e.g. on path space measures.

In more detail: Chapter 1 is connected with the construction of linear connections of vector bundles as push forwards of connections on trivial bundles. This is a direct analogue of the classical and elementary construction of the covariant derivative of a vector field on a submanifold of Euclidean space, leading to the Levi-Civita connections (Example 1B). Narasimhan & Ramadan's theorem of universal connections is evoked to assure us that all metric connections can be obtained this way (Theorem 1.1.2). We then go on to consider the various forms in which this construction will appear in situations described above. (E.g. how certain Gaussian fields of sections determine a connection.) Homogeneous spaces give a good class of examples described in some detail in §1.1 B. The notion of adjoint connection or semi-connection on a subbundle E of the tangent bundle TM to our underlying manifold M is described in §1.3. A semi-connection allows us to differentiate vector fields on M in E-directions. They play an important role in the theory. One difficulty is that the adjoint of a metric connection may not be metric for any metric (Corollary 1.3.7). In general Hörmander type hypoellipticity conditions on our generator $\mathcal{A}$, or equivalently on E, play little role in this article. However in Theorem 1.3.9 we show how they are related to the behaviour of parallel translations with respect to associated semi-connections.

In chapter 2 we concentrate on a generator $\mathcal{A}$ given in Hörmander form, and its associated stochastic differential equation (s.d.e.). A first result is Theorem 2.1.1 which shows in particular that (for $\dim M > 1$) any elliptic diffusion operator can be written as a sum of squares with no first order term, or equivalently that any elliptic diffusion is given by a Stratonovich equation with no drift term. The extension $\mathcal{A}^q$ of $\mathcal{A}$ to q-forms is shown to have the form $\mathcal{A}^q = -(d\hat{\delta} + \hat{\delta}d)$ for a certain operator $\hat{\delta}$ from q-forms to $q-1$ forms (Proposition 2.3.1) and also a Weitzenbock form $\mathcal{A}^q = \frac{1}{2}trace\hat{\nabla}^2 - \frac{1}{2}R^q$ (if there is no drift term A) (Theorem 2.4.2). Driver's notion of torsion skew symmetry is investigated in §2.2 in order to discuss the operators $\hat{\delta}$, and when they are L^2 adjoints of the exterior derivative d, in §2.3. Later, §3.3.3, the semigroups associated to these operators are used to obtain Böchner type vanishing theorems under positivity conditions on R^q.

The question of the symmetricity of $\mathcal{A}^q$ with respect to some measure on M is discussed in §2.5.2. Theorem 2.5.1 gives a fairly definitive result for $\mathcal{A}^q$ with the zero order terms removed. However conditions under which R^q is symmetric seem not so easy to find if $q > 1$. For $q = 1$ this reduces to symmetricity of the Ricci curvature $\breve{Ric}$ which is shown in Proposition C.6 of the Appendix to hold in the torsion skew symmetric case if and only if the torsion tensor $\breve{T}$ determines a coclosed differential 3-form, c.f. [Dri92]. These sections are not used later in this article.

The main applications in stochastic analysis start with Chapter 3. The basic idea is that the diffusion coefficient of an s.d.e often has a kernel: so that there is "redundant noise" from the point of view of the one point motion. We extend the results from the gradient case in [EY93] to our more general, possibly degenerate, situation giving a canonical decomposition of the noise into its redundant and non-redundant parts. We then show how this can be used to filter out the redundant noise in general situations. (This filtering out corresponds to the collapse in levels of structure mentioned above.) On the way we have to discuss conditional expectations of vector fields along the sample paths of our process, Definition 3.3.2. All this is done in some generality, e.g. allowing for the possibility of explosion. The main application is to the derivative process $T\xi_t$ of a stochastic flow: Theorem 3.3.7 and Theorem 3.3.8. When the redundant noise is filtered out the process becomes a "damped' or Dohrn-Guerra type parallel translation using the associated semi-connection. This procedure works equally for the derivative of the Itô map $\omega \mapsto \xi_t(x_0)(\omega)$ in the sense of Malliavin Calculus from which follow integration by parts theorems for possibly degenerate diffusion measures, Theorem 4.1.1. For gradient systems, using [EY93], this method was used by [EL96] and was suggested by [AE95]. The Levi-Civita connection appears in that case (which is why gradient systems behave so nicely), but in the degenerate case which is allowed here the connections are on arbitrary subbundles of TM and there is no unique particularly well behaved connection to use. Hypoellipticity is not assumed. The "admissible" vector fields are those which satisfy a natural "horizontality" condition, §4.1 B and §4.1 C. Closely related is a Clark-Ocone formula (Theorem 4.1.2) expressing suitably smooth functions on path space as stochastic integrals with respect to the predictable projection of their gradient. From this we use the method given in [CHL] to obtain a Logarithmic Sobolev inequality for our diffusion measures, Theorem 4.2.1. Our "damping" of the parallel translation means that no curvature constants appear: indeed since in general we have no Riemannian metric given on M it would be unnatural to have such constants. Logarithmic Sobolev inequalities automatically imply spectral gap inequalities and the constancy of functionals with vanishing gradient (or equivalently whose derivatives vanish on admissible vector fields), Corollary 4.1.3: a non-trivial result even for Frechet smooth functions on path space for the case of degenerate diffusions. In Theorem 4.1.1 the corresponding results are proved for the measures on paths on the diffeomorphism group Diff$M$ of M coming from stochastic flows, or equivalently from Wiener processes on Diff$M$ [Bax84].

Chapter 5 is concerned with applications to stability properties of stochastic flows. In particular upper and lower bounds for moment exponents are obtained in terms of the Weitzenbock curvatures of the associated connection and a generalization of the second fundamental form to our situations: Theorem 5.0.5. This gives a criterion for moment stability in terms of 'stochastic positivity' of a certain expression in the quantities with consequent topological implications: Corollary 5.0.6.

A weakness of these results is that we usually require the adjoint semi-connection to be metric for some metric. Theorem 5.0.7 shows that the lack of this condition really is reflected in the behaviour of the flow.

Chapter 6 consists of technical appendices. The first gives a detailed description of how the push forward construction of connections we use relates to Narasimhan & Ramanan's pull back of the universal connections. This is needed in the proof of Theorem 1.1.2. The other appendices give the notation of annihilation and creation operators used in the discussion of the Weitzenbock curvatures in section 2.4 and some basic formulae and curvature calculations for connections given in the L-W form.

The connection determined by a non-degenerate stochastic flow first appeared in [LJW84]: for this reason we have called it the LeJan-Watanabe or L-W connection. It was also discovered in the context of quantum flows in [AA96] and for sums of squares of vector fields in [PVB96]. It is used for analysis on loop spaces in [Aid96]. For the non-degenerate case many of the results given here were described in [ELJL97a] with announcements for degenerate situations in [ELJL97b]. They were stimulated by [EY93]. The Chentsov-Amari α-connections in statistics are rather different. They are in general non-metric if $\alpha \neq 0$ and torsion free, see [Ama85], pp42, 46.

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Chapter 1

Construction of connections

We consider connections on a C^∞ vector bundle E over a smooth manifold M determined by a split surjection of vector bundles $\underline{H} \overset{\overset{Y}{\longleftarrow}}{\longrightarrow} E \longrightarrow 0$ where $\underline{H} = M \times H$ is the trivial bundle with fibre a Hilbertable space H. The characterization of such a connection $\breve{\nabla}$ is that for each $x \in M$

$$\breve{\nabla}_v \left(X(\cdot)e\right) \equiv 0 \quad \text{all } v \in T_xM \text{ and } e \in \text{Image}Y(x) \ .$$

When M and E are finite dimensional and E has a Riemannian metric all metric connection on E can be obtained this way for some finite dimensional H. These connections can also be considered to be induced by Gaussian measures on the space of C^∞ sections of E. In §1.2 and §1.4 some basic examples are given. They describe the connections arising from certain Gaussian fields, operators in Hörmander form, stochastic differential equations, and homogeneous space structures.

For E a subbundle of TM there is also an adjoint 'semi-connection' $\hat{\nabla}$, investigated in §1.3. In particular we show $\hat{\nabla}$ is metric with respect to some Riemannian metric on M if and only if for one set of $x_0, y_0 \in M$ and $T > 0$ the parallel translation $\hat{//}_t$ using $\hat{\nabla}$ along $\{\xi_t^{T,y_0}(x_0)\}$, the solution $\xi_t(x_0)$ to the stochastic differential equation $dx_t = X(x_t) \circ dB_t$ conditioned to satisfy $\xi_T(x_0) = y_0$, is a bounded $L(T_{x_0}M, T_{y_0}M)$-valued process.

1.1 Construction of connections

A. Consider a $\mathcal{C}^\infty$ manifold M, a C^∞ vector bundle $\pi : E \to M$ over M and a C^∞ vector bundle homorphism $X : \underline{H} \to E$ of a trivial bundle $\underline{H} = M \times H$, where H is a Hilbertable space. We will consider only real bundles (and manifolds) here. At this stage M, E, H could be infinite dimensional (but separable, with M metrizable); however our main focus will be on cases with M and E finite dimensional. In this situation we shall write $n = \dim M$, $p =$ fibre dimension of E, with $m = \dim H$ if $\dim H < \infty$.

Suppose X is surjective and Y a chosen right inverse to X

$$\underline{H} \overset{Y}{\underset{\longrightarrow}{\longleftarrow}} E \longrightarrow 0.$$

Our situation is very similar to a special case of that of Harvey and Lawson in [HL93]. Let $\Gamma(E)$ denote the space of smooth sections of E, and set $E_x = \pi^{-1}(x), x \in M$. Write $X(x) = X(x, \cdot) : H \to E_x$. For u in E let Z^u be the section given by

$$Z^u(x) = X(x)Y(\pi(u))u. \tag{1.1.1}$$

Proposition 1.1.1 *There is a unique linear connection $\breve{\nabla}$ on E such that for all $u_0 \in E_{x_0}$, $x_0 \in M$ the covariant derivative of Z^{u_0} vanishes at x_0. It is the 'push forward' connection defined by*

$$\breve{\nabla}_{v_0} Z = X(x_0) d\left(Y(\cdot)Z(\cdot)\right)(v_0), \qquad v_0 \in T_{x_0}M,\ Z \in \Gamma(E) \tag{1.1.2}$$

where d refers to the usual derivative $d\left(Y(\cdot)Z(\cdot)\right) : TM \to H$ of the map $Y(\cdot)Z(\cdot) : M \to H$.

Proof. Certainly (1.1.2) defines a covariant differentiation. Let $\tilde{\nabla}$ be any linear connection on E. We have

$$Z(\cdot) = X(\cdot)Y(\cdot)Z(\cdot)$$

whence, for $v \in T_{x_0}M$,

$$\begin{aligned} \tilde{\nabla}_v Z &= X(x_0) d\left(Y(\cdot)Z(\cdot)\right)(v) + \tilde{\nabla}_v \left[X(\cdot)\ \left(Y(x_0)Z(x_0)\right)\right] \\ &= \breve{\nabla}_v Z + \tilde{\nabla}_v Z^{Z(x_0)}. \end{aligned} \tag{1.1.3}$$

Since $\tilde{\nabla}$ is assumed to be a genuine connection (not just a covariant differentiation: a point only relevant if E is infinite dimensional) and since also the map

$$\begin{array}{rcl} TM \times E & \to & E \\ (v, u) & \mapsto & \tilde{\nabla}_v Z^u \end{array}$$

gives a smooth section of the bundle of bilinear maps $L(TM, E; E)$ we see that $\breve{\nabla}$ is a smooth connection on E, (e.g. [Eli67]). Taking $\tilde{\nabla} = \breve{\nabla}$ in (1.1.3) we see that $\breve{\nabla}$ has the property required. Uniqueness also follows from (1.1.3). //

B. We shall be mainly interested in metric connections. These will arise in two, essentially equivalent, forms which we will call the *metric* form and the *Gaussian* form respectively. However the examples coming from homogeneous spaces are more easily understood in the more general non-metric framework and these will also be described below, in §1.2.

In the "metric" form H is now a Hilbert space, inner product $\langle , \rangle \equiv \langle , \rangle_H$ and so the surjective homomorphism X induces a Riemannian metric $\{\langle , \rangle_x : x \in M\}$ on E. The right inverse Y is chosen to be the adjoint of X, $Y = X^*$.

Theorem 1.1.2 *Let H be a Hilbert space, and Y the adjoint of X with respect to the induced metric on E by X. Then the connection $\breve{\nabla}$ is adapted to the Riemannian metric. Moreover if M and E are finite dimensional any metric connection for any Riemannian metric on E can be obtained this way from some such X with H some finite dimensional Hilbert space.*

Proof. Take a vector field U and a vector $v \in T_{x_0}M$. Then

$$\begin{aligned} d\langle U, U\rangle(v) &= 2\langle d(Y(\cdot)U(\cdot))(v), Y(x_0)U(x_0)\rangle \\ &= 2\langle X(x_0)(d(Y(\cdot)U(\cdot))(v), U(x_0)\rangle = 2\langle \breve{\nabla}_v U, U\rangle. \end{aligned}$$

This shows that $\breve{\nabla}$ is metric. The fact that any metric connection arises this way in the finite dimensional situation comes from Narasimhan and Ramanan's theorem [NR61] on universal connections. In the finite dimensional case the connection $\breve{\nabla}$ is precisely the pull back of the universal connection over the Grassmanian $G(m,p)$ of p-planes in H by the map $x \mapsto$ [image of $Y(x) : E_x \to H$]; for details see §A in the Appendix. Narasimhan and Ramanan show that any metric connection can be obtained as such a pull back. ■

In this situation we shall call $\breve{\nabla}$ the *LeJan-Watanabe* or *L-W*, connection determined by X, or by $(X, \langle,\rangle)$, for reasons described at the end of §1.3B below.

Example 1B (*Gradient systems*). Let $j : M \to \mathbb{R}^m$ be an immersion. Define $X(x) : \mathbb{R}^m \to T_xM$ to be the orthogonal projection of $\mathbb{R}^m$ on T_xM, identified with its image under the differential dj of j, so that $X(x)e = \operatorname{grad}\langle X(\cdot), e\rangle_{\mathbb{R}^m}$ using the induced metric on M. Then $Y(x) : T_xM \to \mathbb{R}^m$ is the inclusion, Tj, and we have the classical construction of the Levi-Civita connection for this metric.(That it has no torsion can also be seen from the formula (2.2.3) below.)

C. For the 'Gaussian form' suppose we have a mean zero Gaussian field W of sections of E. In its most general form W would be a section of the pull back p^*E of $\Gamma(E)$ over the projection $p : \Omega \times M \to M$ where $(\Omega, \mathcal{F}, P)$ is a probability space. Thus $W_x(\omega) := W(\omega, x) \in E_x$ for each $x \in M$, $\omega \in \Omega$. We will assume that $W(\omega, \cdot)$ is C^∞ for each $\omega \in \Omega$. The more concrete manifestation comes from a Gaussian measure γ on some subspace of the C^∞ sections of E. Then $(\Omega, \mathcal{F}, P) = (\Gamma(E), \mathcal{F}, \gamma)$, the canonical space, for $\mathcal{F}$ the σ-algebra of cylindrical subsets of $\Gamma(E)$, and we identify W_x with the evaluation map $\rho_x : \Gamma(E) \to E_x$, given by $\rho_x(U) = U(x)$. See [Bax76]. For any suitable function f on $\Gamma(E)$ we write $\mathbb{E}f$ or $\mathbb{E}f(W)$ for $\int_\Omega f(W(\omega,\cdot))P(d\omega)$ (equivalently $\int_{\Gamma(E)} f(U)d\gamma(U)$ in the canonical picture). Let γ_x be the law of W_x, a Gaussian measure on E_x. We make the nondegeneracy assumption that each γ_x is non-degenerate and so in the finite dimensional case has the form

$$\gamma_x(B) = (2\pi)^{-p/2} \int_B \mathrm{e}^{-\langle y,y\rangle_x^2} dy$$

for some $\langle,\rangle_x$ on E_x.

Proposition 1.1.3 *There is a unique connection ∇^γ on E such that the random variable $\nabla_v^\gamma W$ is independent of $W(x_0)$ for any $v \in T_{x_0}M$, $x_0 \in M$. It is given in terms of the conditional expectation by*

$$\nabla_v^\gamma Z = \frac{d}{dt}\, \mathbb{E}\, \{W(x_0)\, |W(\sigma(t)) = Z(\sigma(t))\}\, |_{t=0}; \tag{1.1.4}$$

or equivalently

$$\nabla_v^\gamma Z = \frac{d}{dt} \mathbb{E} W(x_0)\, \langle W(\sigma(t)), Z(\sigma(t))\rangle_{\sigma(t)}\Big|_{t=0} \tag{1.1.5}$$

for any C^1 curve σ with $\dot\sigma(0) = v$, $v \in T_{x_0}M$, and is adapted to the metric $\{\langle,\rangle_x, x \in M\}$. Moreover
(i) Let H_γ be the reproducing kernel Hilbert space of γ, then ∇^γ is the L-W connection for $(X, \langle,\rangle_{H_\gamma})$ where $X(x,h) = \rho_x(h) = h(x)$.
(ii) If E is a finite dimensional vector bundle over a finite dimensional M every metric connection can be considered as ∇^γ, given by (1.1.4), for some Gaussian measure γ on $\Gamma(E)$ with finite dimensional support.

Proof. Recall that the reproducing kernel Hilbert space H_γ of γ is the same as the Cameron-Martin space H of γ and is a Hilbert space, here necessarily consisting of C^∞ functions. Among its standard properties are:

(i) The restriction of ρ_x to H maps onto E_x, each $x \in M$ and induces the inner product $\langle,\rangle_x$. It will also be denoted by ρ_x.

(ii) The reproducing kernel k, a section of the vector bundle

$$\cup_{x \in M, y \in M} L(E_x, E_y) \to M \times M,$$

defined by the reproducing property that $k(x,\cdot)(v)$ belongs to H each $v \in E_x$ and for all $h \in H$,

$$\langle k(x,\cdot)v, h\rangle_{H_\gamma} = \langle h(x), v\rangle_x, \tag{1.1.6}$$

is also the covariance of γ:

$$k(x,y)v = \mathbb{E}\langle W(x), v\rangle_x W(y), \qquad v \in E_x,\ x, y \in M. \tag{1.1.7}$$

See [Bax76]. From (1.1.7) we see

$$k(x,y)W(x) = \mathbb{E}\{W(y)|W(x)\} \in T_y M \tag{1.1.8}$$

and so

$$k(x,\cdot)v = \mathbb{E}\{W(\cdot)\ |\, W(x) = v\} \quad \in H \subset \Gamma(E) \tag{1.1.9}$$

for all $v \in E_x$, and $x, y \in M$.

From this we see that the defining equation (1.1.4) or (1.1.5) for $\nabla^\gamma_v Z$ can be written

$$\begin{aligned} \nabla^\gamma_v Z &= \frac{d}{dt} k(\sigma(t), x_0) Z(\sigma(t)) \,|_{t=0} &(1.1.10)\\ &= \frac{d}{dt} \mathbb{E} W(x_0) \left\langle W(\sigma(t)), Z(\sigma(t)) \right\rangle_{\sigma(t)} \Big|_{t=0}. &(1.1.11)\end{aligned}$$

If we set $X(x) = \rho_x$ so $X : \underline{H} \to E$ we see from (1.1.6) that the adjoint map Y to X, using the induced Riemannian metric is just k:

$$\begin{aligned} Y(x) &= \rho_x^* = k(x, \cdot) : E_x \to H \\ k(x,y)(\cdot) &= X(y)Y(x) : E_x \to E_y. \end{aligned}$$

We are therefore in the 'metric' form discussed in Theorem 1.1.2 and $\tilde{\nabla}$ is just the L-W connection $\breve{\nabla}$ of ρ, $\langle , \rangle_{H_\gamma}$.

For the characterization in terms of the independence of $\tilde{\nabla}_Z W$ and W observe that for $u \in E_{x_0}$ the *reproducing vector field* $Z^{x_0,u}$, or Z^u as in (1.1.1) in the metric form, is given by

$$\begin{aligned} Z^u(y) &= X(y)Y(x_0)u = k(x_0, y)u &(1.1.12)\\ &= \mathbb{E}\{W(y) \mid W(x_0) = u\} &(1.1.13)\end{aligned}$$

so that for any linear connection $\tilde{\nabla}$ on E we have

$$\tilde{\nabla}_v Z^u = \mathbb{E}\left\{ \tilde{\nabla}_v W \mid W(x_0) = u \right\}.$$

Thus if $v \in T_{x_0}M$ we have $\tilde{\nabla}_v Z^u = 0$ for all u if and only if

$$\mathbb{E}\langle \tilde{\nabla}_v W, W(x_0) \rangle_{x_0} = 0$$

since $\tilde{\nabla}_v W$ and $W(x_0)$ are E_{x_0}-valued Gaussian random variables (this is exactly the condition for the independence of $W(x_0)$ and $\tilde{\nabla}_v W$). By Proposition 1.1.1 this proves uniqueness, i.e. that $\tilde{\nabla}_v W$ and $W(x_0)$ are independent for all x_0 and $v \in T_{x_0}M$ implies $\tilde{\nabla} = \nabla^\gamma$. It also shows that

$$\mathbb{E}\langle \nabla^\gamma_v W, W(x_0) \rangle_{x_0} = 0, \qquad \text{for all } v \in T_{x_0}M, \text{ all } x_0 \in M, \qquad (1.1.14)$$

which, again because they are Gaussian vectors, implies that $\nabla^\gamma_v W$ is independent of $W(x_0)$ for all $v \in T_{x_0}M$, $x_0 \in M$. (The fact that the processes $W(x_0)$ and $\nabla^\gamma_v W(x_0)$ both take values in fibres E_{x_0} of a bundle causes no difficulty in using the standard results we used: to reduce to the standard situation where the process takes values in a fixed Hilbert space H_0, say, either observe that we can find a measurable trivialization $\theta : E \to M \times H_0$ some $H_0, \langle \rangle_0$ with each $\theta_x : E_x \to W_0$ an isometry or simply note that we have given to us:

$$Y : E \to M \times H$$

which isometrically maps each E_x onto a subspace of H: so we can take $H_0 = H$. In this second way we can simply treat $\Gamma(E)$ as a subspace of the space of maps of M into H.)

Finally, to show that all such metric connections arise this way, let $\{\langle,\rangle_x : x \in M\}$ be a smooth metric on E, with a metric connection $\tilde{\nabla}$. By Theorem 1.1.2 there is a Euclidean space $\mathbb{R}^m, \langle\rangle$ and an $X : M \times \mathbb{R}^m \to E$ whose L-W connection is $\tilde{\nabla}$. Let γ_m be the standard Gaussian measure of $\mathbb{R}^m, \langle,\rangle$, and let γ be the image measure on $\Gamma(E)$ of γ under the map $\mathbb{R}^m \to \Gamma(E)$, $e \mapsto X(\cdot)e$. We claim $\nabla^\gamma = \tilde{\nabla}$. Indeed if k is the reproducing kernel for γ then if $u \in E_x$

$$\begin{aligned} k(x,y)u &= \int_{\mathbb{R}^m} \langle X(x)e, u\rangle_x X(y)(e)\, d\gamma_m(e) \\ &= X(y)Y(x)u \end{aligned}$$

for $Y(x) = X(x)^*$. Thus by (1.1.12) the definitions of the vector fields Z^u defined via X and via γ agree and so $\nabla^\gamma = \tilde{\nabla}$ by their defining property. ∎

Remark 1C. The proof above shows the essential equivalence between the "metric" and "Gaussian" forms. It also shows that the connection depends only on the law, γ, of the process (or equivalently on the subspace H of $\Gamma(E)$ together with its inner product) not on the process itself. The case of H a Hilbert space of sections is more intrinsic than that of a mapping of a Hilbert space into the space of sections, and often the Gaussian formulation is simpler to use, especially when H is infinite dimensional. However it is often the "metric" form which arises in practice, for example in the gradient systems of example 1B.

Example 1C: Gaussian vector fields on $\mathbb{R}^n$ are said to be *isotropic* if they are invariant in law under Euclidean transformations. The covariance of an isotropic Gaussian vector field on $\mathbb{R}^n$ is determined by two spectral measures F_L and F_N on $\mathbb{R}^+$. It is given by the formula

$$E\left(W^i(x)W^j(y)\right) = C^{ij}(x-y) \tag{1.1.15}$$

with

$$C^{ij}(z) = \int_{\mathbb{R}^+}\int_{S^{n-1}} \mathrm{e}^{\,i\rho\langle z,u\rangle} \left[u_i u_j \sigma(du)F_L(d\rho) + (\delta_{ij} - u_i u_j)\sigma(du)F_N(d\rho)\right],$$

σ being the uniform distribution on S^{n-1}. The vector field $W(x,\omega)$ has an almost sure C^∞ version when the measures F_L and F_N have moments of all orders. From formula (10) in [ELJL97a] or remarks in §1.3 B below (or rather from its generalization to the Gaussian field case) we see that

$$\breve{\Gamma}^i_{j,k}(x_0) = 0.$$

The connection so constructed is therefore trivial. The isotropic stochastic flows, associated with W was studied in [Jan85] and [BH86]. A special case had first been introduced in [Har81], using an approximation by discrete vortices.

However, even in that isotropic case, nontrivial connections arise if you consider the motion of several points:

In general, the Gaussian field W can be extended into a Gaussian field $W^{(d)}$ on $M^{(d)} = \{(x_1, x_2, \dots, x_d) \in M^d | x_i \neq x_j, \text{ for } i \neq j\}$ as follows. There is a canonical isomorphism between $T_{(x_1,x_2,\dots x_d)}M^{(d)}$ and the direct sum $\oplus_{\alpha=1}^d T_{x_\alpha}M$. With this identification, we can set $W^{(d)}(x_1, x_2, \dots, x_d) = \oplus_{\alpha=1}^d W(x_\alpha)$.

For the isotropic fields on $\mathbb{R}^n$,

$$W^{(d)}(x_1, x_2, \dots, x_d) = \oplus_{\alpha=1}^d \sum_{i=1}^n W^i(x_\alpha) \frac{\partial}{\partial x_i^\alpha}.$$

It is always a nondegenerate Gaussian vector except in dimension one, when the spectral measure is atomic. See e.g. Darling [Dar92]. We can define an associated metric on $(\mathbb{R}^n)^{(d)}$

$$\langle u, v \rangle_{(x_1,\dots,x_d)} = \langle K^{-1}u, v \rangle,$$

where $\langle , \rangle$ denotes the Euclidean metric and K^{-1} is the inverse of the matrix of covariances:

$$K_{(\alpha,i),(\beta,j)} = C^{ij}(x_\alpha - x_\beta), \qquad 1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq d.$$

The L-W connection $\breve{\nabla}^{(d)}$ is a metric connection on $(\mathbb{R}^n)^{(d)}$ for the metric constructed above. The computations below show it is not the Levi-Civita connection. From the result of §3.3 we can see it is related to a solution to a filtering problem: Given the paths of d points, the restriction to the path of one of the points of the damped adjoint parallel transport along the path of the d-point motion is the restriction of the derivative flow of d-point motion conditioned on the d-point motion.

The connection can be computed explicitly. For example when $d = 1$, by (1.1.4), the Christoffel symbols of $\breve{\nabla}^{(d)}$ are

$$\Gamma_{\alpha,\beta}^{\gamma} = \frac{\partial}{\partial t}|_{t=0}\, \mathbb{E}\{W(x_\gamma)\, |W(x_\lambda + t\delta_{\lambda\alpha}) = \delta_{\lambda\beta} \text{ for all } \lambda\}.$$

For $\alpha \neq \gamma$, $\Gamma_{\alpha,\beta}^{\gamma}$ clearly vanishes. Moreover

$$\Gamma_{\alpha,\beta}^{\alpha} = \frac{\partial}{\partial t}|_{t=0} \left([K(t,\alpha)]^{-1}\right)_{\beta,\lambda} C(x_\alpha - x_\lambda + t\,\delta_\lambda^\alpha)$$

with $([K(t,\alpha)])_{\lambda,\lambda'} = C\,(x_\lambda - x_{\lambda'} + t\delta_\lambda^\alpha - t\delta_{\lambda'}^\alpha)$. Hence, setting $K(0)_{\lambda\lambda'} = C(x_\lambda - x_{\lambda'})$ and $K'(0,\alpha) = \frac{\partial}{\partial t} K(t,\alpha)|_{t=0}$

$$\Gamma_{\alpha,\beta}^{\alpha} = \left(K(0)^{-1}K'(0,\alpha)\right)_{\beta,\alpha}.$$

For $d = 2$ one checks that in particular,

$$\Gamma_{12}^1 = \frac{-C'(x_1 - x_2)}{1 - C^2(x_1 - x_2)}$$

and

$$\Gamma_{11}^1 = \frac{-C(x_1 - x_2) \cdot C'(x_1 - x_2)}{1 - C^2(x_1 - x_2)}.$$

1.2 Basic Classes of Examples

Example A: Sums of squares of vector fields: operators in Hörmander form. For a particular manifestation of our basic class of examples consider a second order differential semi-elliptic operator $\mathcal{A}$ on M given in the Hörmander form

$$\mathcal{A} = \frac{1}{2}\sum_{1}^{m} L_{X^j} L_{X^j} + L_A. \tag{1.2.1}$$

where $X^1, \ldots, X^m, A$ are smooth vector fields on M, with L_V denoting Lie differentiation in the direction of a vector field V. We obtain

$$X : \underline{\mathbb{R}}^m \to TM$$

by

$$X(x)e = \sum_j \langle e, e_j\rangle X^j(x)$$

for $e_1, \ldots, e_m$ the standard orthonormal base for $\mathbb{R}^m$. If we assume $E_x := X(x)[\mathbb{R}^m]$ has constant rank we are in the metric form situation, obtaining a connection $\breve{\nabla}$ on the resulting subbundle $E =$ Image X of TM.

The geometry of such operators have been examined from a different viewpoint (see e.g. [Str86]), usually with a hypoellipticity assumption. They are usually considered as operators on functions. However the presentation of $\mathcal{A}$ in the form (1.2.1) shows how to extend it to more general tensors, in particular to differential forms on M. We show below in section 2.4 the relevance of $\breve{\nabla}$ to the analysis of these operators.

Example B: Connections arising in the theory of stochastic flows. Let $\mathcal{D}$ be the space of C^∞ diffeomorphisms of M with C^∞ topology making it a Polish topological space, (see [Bax84]). Following Baxendale [Bax84] consider a *Brownian motion*, $\{\xi_t : t \geq 0\}$, on $\mathcal{D}$, i.e. a stochastic process on D satisfying

1. almost sure continuity in t,
2. independent increments on the left, i.e. $\xi_t\xi_s^{-1}$ and $\xi_v\xi_u^{-1}$ are independent if $0 \leq s < t \leq u < v$,
3. time homogeneity;
4. ξ_0=identity.

For each $x \in M$ there is the process $\{\xi_t(x) : t \geq 0\}$ on M which was shown to be a diffusion with generator $\mathcal{A}$, say. Similarly on $M \times M$ there is a process $\{(\xi_t(x), \xi_t(y)) : t \geq 0\}$ for each $(x, y) \in M \times M$ giving a diffusion with generator

$\mathcal{A}^2$ say. It turns out that for $f, g : M \to \mathbb{R}$ both C^∞ with compact support then on $f \otimes g : M \times M \to \mathbb{R}$, defined by $(x, y) \mapsto f(x)g(y)$,

$$\mathcal{A}^2 (f \otimes g)(x, y) = \mathcal{A}(f)(x)g(y) + f(x)\mathcal{A}(g)(y) + \frac{1}{2}\Gamma^\xi ((df)_x, (dg)_y)$$

where $\Gamma^\xi : T^*M \times T^*M \to \mathbb{R}$ is the symmetric bilinear map given by

$$\Gamma^\xi ((df)_x, (dg)_y) = \lim_{h \downarrow 0} \mathbb{E} \frac{(f(\xi_h(x)) - f(x))(g(\xi_h(y)) - g(y))}{h}. \tag{1.2.2}$$

If we assume that $\mathcal{A}$ is elliptic, its symbol, which is quadratic and given by $\Gamma^\xi ((df)_x, (df)_x)$, will be non-degenerate and so determines a metric on TM. Raising and lowering indices of Γ^ξusing this metric gives a section k of the bundle $L(TM; TM)$ over $M \times M$,

$$\langle k(x, y)v_1, v_2 \rangle_y = \Gamma^\xi (\langle v_1, - \rangle_x, \langle v_2, - \rangle_y), \qquad v_1 \in T_xM, v_2 \in T_yM. \tag{1.2.3}$$

Following LeJan and Watanabe [LJW84] we can define a connection ∇^ξ on TM by

$$\nabla^\xi_v Z = \frac{d}{dt} k(\sigma(t), x_0) Z(\sigma(t)) \mid_{t=0} \tag{1.2.4}$$

just as in equation (1.1.10). Indeed it was shown in [Bax81] (without assuming ellipticity) that Γ^ξ is the covariance for a Gaussian measure γ on $\Gamma(TM)$, mean $\bar{\gamma}$, and that there is a correspondence between Brownian motions on $\mathcal{D}$ and such Gaussian measures, at least for M compact, see also [Kun90]. Given sufficient regularity the correspondence is obtained via the stochastic differential equation

$$dx_t = \rho_{x_t} \circ dW_t + \bar{\gamma}(x_t)dt$$

where $\{W_t : t \geq 0\}$ is the Wiener process on $\Gamma(TM)$ associated to γ, and ρ_x the evaluation map at $x \in M$, or equivalently for the stochastic differential equation on $\mathcal{D}$ for the flow $\{\xi_t : t \geq 0\}$

$$d\xi_t = (T\mathcal{R}_{\xi_t}) \circ dW_t + T\mathcal{R}_{\xi_t}(\bar{\gamma})dt$$

where $\mathcal{R}_h$ refers to right translation (i.e. composition) by the diffeomorphism h. It was shown in [LJW84] that the generator $\mathcal{A}$ is given by

$$\mathcal{A}(f)(x) = \frac{1}{2}\mathrm{tr}\nabla^\xi(df)(x) + df(\bar{\gamma}(x)).$$

In particular ξ is determined by Γ^ξ and $\bar{\gamma}$. Given our non-degeneracy assumptions we see $\nabla^\xi = \nabla^\gamma$, (of course ellipticity can be replaced by constancy of the rank of the symbol).

Remark:
Recall that $\bar{\gamma}$ vanishes if and only if ξ_t is equal in law to ξ_t^{-1} for any fixed t.

Example C: Stochastic differential equations (s.d.e.). The form in which we will most frequently be using the theory will come from stochastic differential equations

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1.2.5}$$

where $\{B_t\}$ is a R^m-valued Brownian motion, A is a smooth vector field and $\circ$ indicates the integral involved being Stratonovich. For x_0 in M let $\xi_t(x_0) : 0 \le t < \rho(x_0)$ be a maximal solution from x_0, so that $\rho(x_0) \in (0, \infty]$ is the explosion time from x_0. There is the associated semigroup $P_t : t \ge 0$: for bounded measurable functions $P_t f(x) = \mathbb{E} f(\xi_t(x))\chi_{t<\rho(x)}$ with generator $\mathcal{A} = \frac{1}{2}\sum_1^m L_{X^i}L_{X^i} + L_A$, e.g. Example A above. If M is compact the solutions can be chosen (they are only defined up to sets of measure zero) to give a solution flow of diffeomorphisms as described above: though by taking $B.$ to be a finite dimensional Brownian motion we lose some generality.

The equation (1.2.5) will be said to be *regular* (or *non-singular*) if X has constant rank.

Example B: Homogeneous spaces. Suppose we have an action of a Lie group K on M. Let $\underline{\mathfrak{k}}$ be the Lie algebra of K (taken to be the tangent space at the identity) and $\exp : \underline{\mathfrak{k}} \to K$ the exponential map. We have the induced map

$$X : M \times \underline{\mathfrak{k}} \to TM$$

given by

$$X(x)(e) = \frac{d}{dt}(\exp te \cdot x)\,|_{t=0}$$

so that $e \mapsto X^e := X(\cdot)e$ is a Lie algebra homomorphism. For $g \in K$ let L_g denote left multiplication by g acting on K or on M and let R_g be right multiplication by g acting on K. There is the adjoint action

$$ad \equiv ad_K : K \to GL(\underline{\mathfrak{k}})$$

given by

$$ad(g)e = (TR_g)^{-1}\, TL_g(e), \qquad e \in \underline{\mathfrak{k}},\ g \in K.$$

Note that if $x \in M$, $g \in K$ and $e \in \underline{\mathfrak{k}}$ then

$$\begin{aligned} X(gx)e &= \frac{d}{dt}\exp te \cdot gx\,|_{t=0} = TL_g\,\frac{d}{dt}\,g^{-1}(\exp te)\,g \cdot x\,|_{t=0} && (1.2.6)\\ &= TL_g\frac{d}{dt}\exp t\,\mathrm{ad}\,(g^{-1})(e) \cdot x\,|_{t=0} && (1.2.7)\\ &= TL_g X(x)\left(\mathrm{ad}(g^{-1})e\right). && (1.2.8) \end{aligned}$$

Suppose further that the action is transitive, and also fixing $x_0 \in M$ the mapping $K \to M$, $g \mapsto g \cdot x_0$ identifies M with K/H for $H = \{g \in K : g \cdot x_0 = x_0\}$ making M a homogeneous space which is also *reductive* i.e. there is a linear splitting

$$\underline{\mathfrak{k}} = \underline{\mathfrak{h}} + \underline{\mathfrak{m}}, \tag{1.2.9}$$

with $\underline{\mathfrak{h}}\cap\underline{\mathfrak{m}}=0$, where $\underline{\mathfrak{h}}$ is the Lie algebra of H and $\underline{\mathfrak{m}}$ some linear subspace of $\underline{\mathfrak{k}}$ which is ad_H invariant:

$$\mathrm{ad}(g)\,[\underline{\mathfrak{m}}]=\underline{\mathfrak{m}},\qquad \text{all } g\in H.$$

Note that $\underline{\mathfrak{h}}=\ker X(x_0)$, so by (1.2.8), for $g\in K$,

$$\ker X(gx_0)=\mathrm{ad}\,(g)[\underline{\mathfrak{h}}]. \tag{1.2.10}$$

The reductive property allows us to define $\underline{\mathfrak{m}}_x:=\mathrm{ad}\,(g)\,[\underline{\mathfrak{m}}]\subset\underline{\mathfrak{k}}$ for $x=gx_0$ since if $x=ghx_0$, so $h\in H$, we have

$$ad(gh)\,[\underline{\mathfrak{m}}]=\mathrm{ad}\,(g)\mathrm{ad}\,(h)\,[\underline{\mathfrak{m}}]=\mathrm{ad}\,(g)[\underline{\mathfrak{m}}].$$

Thus we have the splitting

$$\underline{\mathfrak{k}}=KerX(x)+\underline{\mathfrak{m}}_x$$

for each $x\in M$ and we can define $Y(x):T_xM\to\underline{\mathfrak{k}}$ to be the inverse of the restriction of $X(x)$ to $\underline{\mathfrak{m}}_x$.

Proposition 1.2.1 *The connection $\breve{\nabla}$ induced on TM by X,Y is K invariant. If $\underline{\mathfrak{k}}$ admits an ad_K-invariant inner product $\langle,\rangle$ for which m is orthogonal to $\underline{\mathfrak{h}}$ then $\breve{\nabla}$ is the L-W connection for $X,\langle,\rangle$.*

Remark: This applies to spheres and Grassmanian manifolds in which cases the connections are the Levi-Civita connections. See Theorem 1.4.8 below.

Proof. First observe from (1.2.8) that

$$\mathrm{ad}\,(g)Y(x)TL_{g^{-1}}:T_{gx}M\to\underline{\mathfrak{k}}$$

has image in $\underline{\mathfrak{m}}_{gx}$ and is a right inverse to $X(gx)$. Thus

$$Y(gx)=\mathrm{ad}\,(g)Y(x)TL_{g^{-1}}. \tag{1.2.11}$$

From this and (1.2.8), for $g\in K$, Z a vector field on M, and $v\in T_xM$ we have

$$\begin{aligned}
\breve{\nabla}_{g_*(v)}g_*(Z) &= X(gx)d\left(Y(\cdot)TL_gZ\left(g^{-1}\cdot\right)\right)TL_g(v)\\
&= TL_gX(x)\ \mathrm{ad}\,(g^{-1})d\left(Y(\cdot)TL_gZ\left(g^{-1}\cdot\right)\right)TL_g(v)\\
&= TL_gX(x)d\left(Y(g^{-1}\cdot)Z\left(g^{-1}\cdot\right)\right)TL_g(v)\\
&= TL_gX(x)d\left(Y(\cdot)Z(\cdot)\right)(v)\\
&= g_*\left(\breve{\nabla}_vZ\right).
\end{aligned}$$

Thus $\breve{\nabla}$ is K-invariant.

When $\langle,\rangle$ is ad_K-invariant, if $\underline{\mathfrak{m}}$ is orthogonal to $\underline{\mathfrak{h}}$ then $\underline{\mathfrak{m}}_x\perp\ker X(x)$ for each x and we see $Y(x)=X(x)^*$, from which it follows that $\breve{\nabla}$ is the L-W connection for $X,\langle\rangle$. ∎

These connections on homogeneous spaces are discussed further in §1.4 below.

1.3 Adjoint connections, torsion skew symmetry, basic formulae

A. From now on in this section we shall assume that E is a subbundle of the tangent bundle TM of M. Then for any linear connection $\tilde{\nabla}$ on E there is an operation which gives a differentiation of arbitrary smooth vector fields V in E-directions: define

$$\tilde{\nabla}'_u V = \tilde{\nabla}_{V(x_0)} U + [U,V](x_0), \qquad u \in E_{x_0} \tag{1.3.1}$$

where U is any section of E with $U(x_0) = u$. In terms of the Lie derivative L_V, mappings sections U of E into vector fields,

$$\tilde{\nabla}' V = \tilde{\nabla}_V - L_V, \tag{1.3.2}$$

c.f. the tensor A_V defined on p.235 of [KN69a]. It is easy to see that $\tilde{\nabla}' V \in \Gamma\mathrm{Hom}(E, TM)$, or see [KN69a], p.235, and also that $\tilde{\nabla}'_U$ is a derivation on $\Gamma(M)$ over $C^\infty(M)$ for each $U \in \Gamma(E)$. We shall call such an operation a *semi-connection* on E and call $\tilde{\nabla}'$ the *adjoint* (semi)connection to $\tilde{\nabla}$. Adjoint connections were introduced in the case $E = TM$ by Driver in [Dri92].

When $E = TM$ it is a genuine connection: indeed, by definition, in this case for vector fields U, V we have

$$[U,V] = \tilde{\nabla}'_U V - \tilde{\nabla}_V U \tag{1.3.3}$$

whence

$$\tilde{\nabla}'_u V = \tilde{\nabla}_u V - \tilde{T}(u, V(x_0)). \tag{1.3.4}$$

Here $\tilde{T} : TM \times TM \to TM$ is the torsion tensor of $\tilde{\nabla}$, defined by

$$-\tilde{T}(u,v) = [U,V](x_0) - \tilde{\nabla}_u V + \tilde{\nabla}_v U, \tag{1.3.5}$$

where $v = V(x_0)$, c.f. Proposition 2.3 of [KN69a].

When E is a genuine subbundle there is still a skew symmetric

$$\tilde{T} : E \oplus E \to TM$$

defined by (1.3.5) with U, V now sections of E, (and so, also vector fields on M). For our semi-connection $\tilde{\nabla}'$ we can again define $\tilde{T}' : E \oplus E \to TM$ by equation (1.3.5). It is immediate from (1.3.3) and (1.3.5) that $\tilde{T}' = -\tilde{T}$.

Coming back to the specific case of a connection $\breve{\nabla}$ as in §1.1 note for any $v, u \in E$, $\breve{\nabla}_v Z^u = \breve{\nabla}_u Z^v \equiv 0$. So

$$\breve{T}(u,v) = -[Z^u, Z^v]. \tag{1.3.6}$$

The canonical example of adjoint connections are given on Lie groups G: the adjoint of the flat right invariant connection on TG is the flat left invariant connection (and conversely). More examples are given in §1.3E below.

The name 'semi-connection' is justified by:

Proposition 1.3.1 *Let $E^\perp$ be a complementary subbundle to E in TM, so $TM = E \oplus E^\perp$. Let $\nabla^\perp$ be any linear connection on $E^\perp$ and let ∇^1 be the direct sum connection induced on TM: $\nabla^1 = \tilde{\nabla} \oplus \nabla^\perp$. Let $\nabla^{1\prime}$ be the adjoint connection to ∇^1. Then*

$$\tilde{\nabla}'_u V = (\nabla^1)'_u V, \qquad u \in E_{x_0}, V \in \Gamma(TM).$$

Proof. This follows from equation (1.3.1) since if $U \in \Gamma(E)$ and $V \in \Gamma(TM)$ then $\tilde{\nabla}_V U = \nabla^1_V U$. ∎

From this proposition we see immediately that it is possible to define operations $\frac{\tilde{D}'}{\partial t}$ on smooth vector fields $\{v_t : t \in [0,T]\}$ along piecewise C^1 curves σ in M with $\dot{\sigma}(t) \in E_{\sigma(t)}$ for each t ("*horizontal*" curves) such that: if $V \in \Gamma(TM)$ and $v_t = V(\sigma(t))$ then

$$\frac{\tilde{D}' v_t}{\partial t} = \tilde{\nabla}'_{\dot{\sigma}(t)} V.$$

For example simply take $\nabla^\perp$ as in the proposition to obtain $(\nabla^1)'$ with corresponding $\frac{D^{1\prime}}{\partial t}$ and then restrict to horizontal curves observing the result is independent of the choice of $E^\perp$ and $\nabla^\perp$. Alternatively there are partially defined "Christoffel symbols", see Remarks in the next subsection. Similarly there are parallel translations

$$\tilde{//}'_t : T_{\sigma(0)} M \to T_{\sigma(t)} M,$$

which are linear isomorphisms, but only defined along horizontal curves and indeed, as usual,

$$\frac{\tilde{D}' v_t}{\partial t} = \tilde{//}'_t \frac{d}{dt} \tilde{//}'^{-1}_t v_t.$$

A vector field $\{v_t : t \in [0,T]\}$ along such a curve is *parallel* if and only if $\frac{\tilde{D}' v_t}{\partial t} = 0$. Proposition 1.3.1 will enable us to be confident in applying the usual rules of connections to $\tilde{\nabla}'$ in consequence. The following extension of (1.3.3) will be of basic importance.

Lemma 1.3.2 *For $S > 0$, $T > 0$, let $\sigma : [0,S] \times [0,T] \to M$ be C^1 with $\sigma(s,\cdot)$ horizontal for each $s \in [0,S]$. Then*

$$\frac{\tilde{D}'}{\partial t} \frac{\partial \sigma}{\partial s} = \frac{\tilde{D}}{\partial s} \frac{\partial \sigma}{\partial t}. \tag{1.3.7}$$

Proof. By Proposition 1.3.1, we can assume $E = TM$. Then

$$\frac{\tilde{D}}{\partial t} \frac{\partial \sigma}{\partial s} = \frac{\tilde{D}}{\partial s} \frac{\partial \sigma}{\partial t} + \tilde{T}\left(\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s}\right).$$

But, from (1.3.4),

$$\frac{\tilde{D}'}{\partial t}\frac{\partial \sigma}{\partial s} = \frac{\tilde{D}}{\partial t}\frac{\partial \sigma}{\partial s} - \tilde{T}(\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial s}).$$

■

In general $\tilde{//}'_t$ does not map fibres of E to fibres of E:

Proposition 1.3.3 *For the adjoint semi-connection $\tilde{\nabla}'$ the following are equivalent:*

1. $\tilde{\nabla}'_u V \in E$ *whenever* $V \in \Gamma(E)$ *and* $u \in E$.
2. $\tilde{//}'_t$ *maps* $E_{\sigma(0)} \to E_{\sigma(t)}$ *whenever* σ *is a horizontal path.*
3. $\tilde{T}$ *maps* $E \oplus E$ *to* E.
4. E *is an integrable foliation and* $\tilde{\nabla}$ *restricts to a connection on each of its leaves.*

If so $\tilde{\nabla}'$ *restricts to a connection on the leaves of* E.

Proof. Certainly (4) implies (1). Also (1) implies (3) by (1.3.4) and if (3) holds then $[U, Z] \in \Gamma(E)$ when $E, Z \in \Gamma(E)$ by (1.3.5) and (1) holds by (1.3.4), so (3) implies (4). To take in (2) observe that (2) implies (3) using (1.3.4) and the formula $\frac{D' v_t}{\partial t} = \tilde{//}'_t \frac{d}{dt} \tilde{//}'^{-1}_t v_t$, while (4) immediately implies (2). ■

Remark: If $P : TM \to TM/E$ is the projection of TM onto the quotient bundle then by (1.3.5)

$$P\left(\tilde{T}(u^1, u^2)\right) = -P\left([U^1, U^2](x)\right) \tag{1.3.8}$$

for $U^1, U^2 \in \Gamma(E)$ with $U^1(x) = u^1$, $U^2(x) = u^2$. Thus $P \circ \tilde{T}$ is independent of the connection on E. It is a well known invariant of the subbundle, e.g. see Strichartz [Str86].

B. Suppose now we have $\underline{H} \overset{Y}{\underset{\longrightarrow}{\longleftarrow}} E \longrightarrow 0$ as in §1.1A, but still with E a subbundle of TM. The adjoint $\breve{\nabla}'$ of the associated connection $\breve{\nabla}$ will be denoted by $\hat{\nabla}$, with parallel translation $\hat{//}$ etc. It takes on a particularly simple form using the sections $\{Z^u : u \in E\}$ of E defined in (1.1.1):

Lemma 1.3.4 *1. For any vector field* V *and* $u \in E_{x_0}$ *we have*

$$\hat{\nabla}_u V = L_{Z^u} V,$$

2. *Let* $\sigma : [0, T] \to M$ *be a horizontal curve with* $\sigma(0) = x_0$. *Then* $\hat{//}_t v_0 = T_{x_0} S^\sigma_t(v_0)$ *all* $v_0 \in T_{x_0} M$, *where* $S^\sigma_t : M \to M, 0 \le t \le T$ *is the flow of the time dependent vector field* $Z^{\dot{\sigma}(t)} = X(\cdot) Y(\sigma(t)) \dot{\sigma}(t)$.

Proof. Part (1) is immediate from (1.3.1) or (1.3.2), the defining property of $\breve{\nabla}$, (that $\breve{\nabla} Z^u$ vanishes at x_0), and the skew symmetry of Lie differentiation of vector fields

$$L_{Z^u} V = [Z^u, V] = -L_V Z^u.$$

For part (2) set $x_t = S_t^\sigma(x_0)$ and $v_t = T_{x_0} S_t^\sigma(v_0)$. Since

$$\begin{cases} \frac{d}{dt} S_t^\sigma(x_0) &= X(S_t^\sigma(x_0)) Y(\sigma(t)) \dot{\sigma}(t) \\ S_0^\sigma(x_0) &= x_0 \end{cases}$$

we see $x_t = \sigma(t)$. Also by Lemma 1.3.2

$$\begin{cases} \frac{\hat{D}}{dt} v_t &= \breve{\nabla}_{v_t} X\left(Y(\sigma(t))\dot{\sigma}(t)\right) = 0 \\ v_t\big|_{t=0} &= v_0 \end{cases}.$$

by the defining property of $\breve{\nabla}$. Thus $\{v_t : 0 \le t \le T\}$ is parallel for $\hat{\nabla}$. ■

Remarks:

(1). Since Lie differentiation with respect to a fixed vector field obeys the usual derivation rules a corollary of Lemma 1.3.4 (1) is that, when $E = TM$, for any smooth tensor field A on M

$$\hat{\nabla}_u A = L_{Z^u} A. \tag{1.3.9}$$

For the general case, for $u \in E$ we could define $\hat{\nabla}_u A$, for example, by using Proposition 1.3.1 and then (1.3.9) will still hold; or more directly we could use (1.3.9) as the definition.

(2). By expanding U over the basis, Lemma 1.3.4 (1) reads

$$\hat{\nabla}_U V = [Z^u, V] = \sum_1^m [X^i, V] \langle X^i, U \rangle, \qquad U \in \Gamma(E), V \in \Gamma(TM); \tag{1.3.10}$$

or in the Gaussian form,

$$\hat{\nabla}_U V = \mathbb{E}[W, V] \langle W, U \rangle. \tag{1.3.11}$$

By (1.3.3),

$$\breve{\nabla}_V U = [V, U] + \mathbb{E}[W, V] \langle W, U \rangle. \tag{1.3.12}$$

(3). For $X : \mathbb{R}^m \to E \subset TM$ as described let $\tilde{\nabla}$ be the Levi-Civita connection for some Riemannian metric on M (or indeed any torsion free connection on TM). Then for $U \in \Gamma(E)$, $V \in \Gamma(TM)$, $V(x_0) = v$, $U(x_0) = u$,

$$\begin{aligned} \breve{\nabla}_v U &= X(x_0) d\left(Y(\cdot) U(\cdot)\right)(v) \\ &= \tilde{\nabla}_v U + X(x_0) \tilde{\nabla}_v Y\left(U(x_0)\right) \\ &= \tilde{\nabla}_v U + \tilde{\Gamma}(v, u), \end{aligned}$$

say. Also by definition, (1.3.1),

$$\begin{aligned}\hat{\nabla}_u V &= \breve{\nabla}_v U + [U,V](x_0)\\ &= \tilde{\nabla}_v U + \tilde{\Gamma}(v,u) + \tilde{\nabla}_u V - \tilde{\nabla}_v U\\ &= \tilde{\nabla}_u V + \tilde{\Gamma}(v,u).\end{aligned}$$

Working in a chart (U,ϕ) of M with X^ϕ the local representation of X

$$X^\phi = T\phi \circ X : \phi(U) \times \mathbb{R}^m \to T\phi[E] \subset \phi(U) \times \mathbb{R}^n$$

etc. and taking $\tilde{\nabla}$ to be the usual differentiation in $\mathbb{R}^n$ this shows that $\breve{\nabla}$ has "Christoffel symbols" given by

$$\Gamma^\phi_{x_0}(\tilde{v},\tilde{u}) = X^\phi(x_0) DY^\phi(x_0)(\tilde{v})(\tilde{u}) \tag{1.3.13}$$

where now $\tilde{v} \in \mathbb{R}^m$ and $\tilde{u} \in T_{x_0}\phi(E_{x_0}), x_0 \in U$. Moreover

$$\hat{\nabla}_u V = DV^\phi\left(\phi(x_0)\right)\tilde{u} + \Gamma^\phi_{x_0}\left(T_{x_0}\phi(v), T_{x_0}\phi(u)\right) \tag{1.3.14}$$

Equivalently in the nondegenerate case

$$\breve{\Gamma}^i_{jk} = -\sum_{r=1}^m \sum_{l=1}^n \frac{\partial X(x_0)^{r,i}}{\partial x^j} X(x_0)^{r,\ell} g_{k\ell}, \tag{1.3.15}$$

where $\{X(x)^{r,i}\}$, $\{1 \le i \le n\}$, $\{1 \le r \le m\}$ is the matrix representing $X(x) : \mathbb{R}^m \to \mathbb{R}$, i.e. $X(x)^{r,i} = \langle X(e_r), f_i\rangle$ for $\{e_i\}$ and $\{f_i\}$ orthonormal bases for $\mathbb{R}^m$ and T_xM respectively, and $\{g_{k\ell}\}$ the metric tensor. This shows that $\breve{\nabla}$ is the L-W connection defined in [LJW84].

C. It will be important to know when $\hat{\nabla}$ is adapted to some Riemannian metric $\langle,\rangle'$ on TM, (see also §2.1) in the sense that

$$d\left\langle Z_1(\cdot), Z_2(\cdot)\right\rangle'(u) = \left\langle \hat{\nabla}_u Z_1(x), Z_2(x)\right\rangle'_x + \left\langle Z_1(x), \hat{\nabla}_u Z_2(x)\right\rangle'_x \tag{1.3.16}$$

for $u \in E_x$ and all $Z_1, Z_2 \in \Gamma(TM)$, or equivalently that parallel translation $\hat{//}_t$ along any smooth horizontal curve preserves $\langle,\rangle'$. In the case that $E = TM$ and $\langle,\rangle' = \langle,\rangle$, when this holds $\breve{\nabla}$ is said to be *torsion skew symmetric*. See §2.2. Let ∇' be the Levi-Civita connection for $\langle,\rangle'$.

Proposition 1.3.5 *The adjoint $\hat{\nabla}$ is adapted to $\langle,\rangle'$ if and only if $\nabla' Z^u$ is skew symmetric on $T_xM, \langle,\rangle'_x$ for all $u \in E_x, x \in M$.*

Proof. By Lemma 1.3.4 (2), $\hat{\nabla}$ is adapted to $\langle,\rangle'$ if and only if

$$\frac{d}{dt}\left\langle TS^\sigma_t(v^1_0), TS^\sigma_t(v^2_0)\right\rangle'_{\sigma(t)}\Big|_{t=0} = 0$$

for all horizontal curves σ and v^1_0, v^2_0 in $T_{\sigma(0)}M$. But this is precisely the condition

$$\left\langle \nabla'_{v^1_0} Z^{\dot{\sigma}(0)}, v^2_0\right\rangle'_{\sigma(0)} + \left\langle v^1_0, \nabla'_{v^2_0} Z^{\dot{\sigma}(0)}\right\rangle'_{\sigma(0)} = 0,$$

since $\frac{D'}{\partial t} TS^\sigma_t(v^1_0) = \nabla' Z^{\dot{\sigma}(0)}$ if D' is differentiation with respect to ∇'. ∎

Corollary 1.3.6 *Suppose $X(x)$ is injective for each $x \in M$. Then $\hat{\nabla}$ is adapted to some Riemannian metric $\langle,\rangle'$ on TM if and only if $X^e \equiv X(\cdot)(e)$ is an infinitesimal isometry for each $e \in H$.*

Proof. Injectivity implies that $X^e = Z^u$ for $u = X(x_0)e$, any $x_0 \in M$. But skew-symmetry of $\nabla' X^e$ is equivalent to X^e being an infinitesimal isometry (e.g. see [KN69a], p.237). ∎

Remark 2C. Note that in the Gaussian form, §1.1C, the injectivity hypothesis becomes the assumption that the vector fields in the reproducing kernel Hilbert space H_γ of γ never vanish. In this case $\breve{\nabla}$ is the trivial connection determined by the trivialization Y of E and so the curvature $\breve{R}$ of $\breve{\nabla}$ vanishes (alternatively $\breve{\nabla} W$ vanishes and $\breve{R}$ is seen to vanish by the expression given in Appendix I of [ELJL97a], see also Proposition C.4 in Appendix C). In the stochastic flow picture, §1.2F, it implies that for any $x_0 \in M$, and $T > 0$ the infinite dimensional process $\{\xi_t : 0 \le t \le T\}$ can be expressed in terms of $\{\xi_t(x_0) : 0 \le t \le T\}$. The standard probabilistic approach to second order elliptic operators on $\mathbb{R}^n$ is to use such X taken to be the positive square root of the symbol of the operator considered as a map of $\mathbb{R}^n$ into the positive definite symmetric matrices.

Corollary 1.3.7 *Suppose X is injective but the Lie algebra generated by $\{X^e : e \in H\}$ has dimension greater than $\frac{1}{2}n(n+1)$ when $n = dimM < \infty$. Then $\hat{\nabla}$ is not adapted to any Riemannian metric on M.*

Proof. The Lie algebra of infinitesimal isometries of a connected Riemannian manifold has dimension at most $\frac{1}{2}n(n+1)$ ([KN69a],Theorem 3.3 on p238). ∎

Example 2C. For $M = \mathbb{R}^2$ define vector fields X^1, X^2 by $X^1(x,y) = \frac{\partial}{\partial x}$, $X^2(x,y) = x^3\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. This gives an injective X as in §1E and the Lie algebra generated by X^1, X^2 is easily seen to be infinite dimensional. Thus the induced adjoint connection $\hat{\nabla}$ is not adapted to any metric. This example could easily be modified outside of a compact set to make it periodic, and so project to a compact surface.

D. Consider the stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt, \tag{1.3.17}$$

where $\{B_t\}$ is a R^m-valued Brownian motion, A is a smooth vector field and $\circ$ indicates the integral involved being Stratonovich. Let $\{\xi_t(x_0)\}$ be the solution with initial value x_0 and $\breve{\nabla}$ the connection constructed from X. Then roughly speaking the parallel translation along the paths of $\{\xi_t\}$ is bounded if and only if $\hat{\nabla}$ is adapted to some Riemannian metric. (This statement is true for smooth paths as easily seen from the proof of the next theorem). See also Theorem 5.0.7 for the corresponding result on the derivative flow.

Theorem 1.3.8 *Assume that X is nondegenerate and the stochastic differential equation (1.3.17) does not explode. For $x_0, y_0 \in M$ and $T > 0$, let $\{\xi_t^{T,y_0}(x_0) : 0 \le t \le T\}$ be the process conditioned to be y_0 at time T of $\{\xi_t(x_0)\}$ and let $\hat{//}_t =: \hat{//}_t^{T,x_0,y_0}$ be parallel translation along $\{\xi_t^{T,y_0}(x_0)\}$. If $\hat\nabla$ is adapted to some Riemannian metric then $\hat{//}_T$ is a bounded $L(T_{x_0}M, T_{y_0}M)$ valued random variable for such x_0, y_0 and $T > 0$. Conversely if just for one set of x_0, y_0 and $T > 0$ the parallel translation process $\hat{//}_T$ along the path of $\{\xi_t^{T,y_0}(x_0)\}$ is bounded then $\hat\nabla$ is adapted to some Riemannian metric on M.*

Proof. The 'if' part is clear since $\hat{//}_T$ would be an isometry.

Suppose that $\hat{//}_T$ is bounded. Let u_0 be a frame at x_0 and $P(u_0)$ the holonomy bundle through u_0, i.e.

$$P(u_0) = \{u \in GL(M) \mid \text{ there exists a horizontal curve from } u_0 \text{ to } u.\},$$

with structure group

$$\Phi(u_0) = \{g \in GL(n) \mid u_0 \cdot g \in P(u_0)\}.$$

We can reduce $\hat\nabla$, now a genuine connection, to a connection on $P(u_0)$. Let (u_t) be the solution to

$$du_t = \tilde X(u_t) \circ dB_t + \tilde A(u_t)dt$$

with initial value u_0. Here $\tilde X$ and $\tilde A$ are the horizontal lifts in $P(u_0)$ of X and A. Then $\{u_t : 0 \le t \le T\}$ is the horizontal lift of $\{\xi_t(x_0) : 0 \le t \le T\}$.

The support of the law $\mu_t(P)$ of u_T is all of $P(u_0)$ by the Stroock-Varadhan support theorem and the definition of $P(u_0)$. Consequently by Carverhill [Car88] the support of u_T when $\{\xi_T(x_0)\}$ is conditioned to have $\xi_T(x_0) = y_0$ is $\pi_0^{-1}(y_0)$, where $\pi_0 : P(u_0) \to M$ is the projection. Thus parallel translations from $T_{x_0}M$ to $T_{y_0}M$ along the paths of the conditioned process $\{\xi_T^{y_0}(x_0) : 0 \le t \le T\}$ are dense in the space of parallel translations along smooth paths. So the latter is a bounded set and $\Phi(u_0)$ is bounded in $GL(n)$. As a consequence there is an inner product on $\mathbb{R}^n$, $\langle , \rangle'$ say, invariant under $\Phi(u_0)$. The required metric at a point z of M is then $\langle v^1, v^2\rangle_z = \langle u^{-1}v^1, u^{-1}v^2\rangle'$ for any $u \in \pi_0^{-1}(z)$. ∎

E. Proposition 1.3.3 was concerned with the case when E is integrable. At the other extreme is the situation where the vector fields $X^1, \ldots, X^m$ together with their iterated brackets span T_xM for each x in M, giving hypoellipticity of the operator $\mathcal{A}$ by Hörmander's theorem. Bismut showed how this hypoellipticity was reflected in the behaviour of the derivative of the associated stochastic flow so that Lemma 1.3.4 (2) makes it not surprising that it is also reflected in the behaviour of parallel translation $\hat{//}_s$ along the paths of the associated diffusion, as we see next. As before we consider the stochastic differential equation (1.3.17) and assume it to be regular with $A(x) \in E_x$ for each x in M. Our discussion is an adaptation of that in [Bel87] which was in turn based on [Bis81].

Let $R_t(\omega) = \operatorname{span}\{\hat{//}_s^{-1} X(x_s)e : e \in \mathbb{R}^m, 0 \le s \le t \wedge \tau\} \in T_{x_0}M$ where τ is a fixed, positive predictable stopping time less than the explosion time for our stochastic differential equation. By Proposition 1.3.3 in the integrable case $R_t(\omega) = E_{x_0}$ for each $t \ge 0$. In general let $\overline{E}_{x_0} \subset T_{x_0}M$ be the linear span of $X^1(x_0), \ldots, X^m(x_0)$ together with all the brackets and iterated brackets of the vector fields $X^1, \ldots, X^m$ evaluated at x_0 (this depends only on $E \hookrightarrow TM$, not on $\breve{\nabla}$ or a choice of $X^1, \ldots, X^m$ determining E):

Theorem 1.3.9 *For each $t > 0$,*

$$\overline{E}_{x_0} \subset R_t(\omega) \qquad \text{almost all } \omega \text{ in } \Omega.$$

Proof. Set $R(\omega) = \cap_{t>0} R_t(\omega)$. By the Blumenthal 0-1 law there exists a non-random $R_0 \subset T_{x_0}M$ with $R_0 = R(\omega)$ almost surely. Moreover there therefore exists a predictable stopping time τ_1 with $0 < \tau_1 < \tau$ such that $R_t(\omega) = R_0$ for $0 \le t \le \tau_1$ almost surely.

Suppose $\ell \in T^*_{x_0}M$ annihilates R_0. Then for $e \in \mathbb{R}^m$, with $X^e := X(\cdot)e$,

$$\ell(\hat{//}_s^{-1} X^e(x_s)) = 0, \qquad 0 \le s \le \tau_1.$$

Now if Z is a vector field with $\ell(\hat{//}_s^{-1} Z(x_s)) = 0$, $0 \le s \le \tau_1$, taking the local martingale part of its canonical decomposition we have

$$\ell(\hat{//}_s^{-1} \hat{\nabla}_{X(x_s)f} Z) = 0$$

all $f \in \mathbb{R}^m$, $0 \le s \le \tau_1$.

By Lemma 1.3.4 (1) this gives

$$\ell(\hat{//}_s^{-1} [Z, X^f](x_s)) = 0, \qquad 0 \le s \le \tau_1, \tag{1.3.18}$$

since $[Z, X^f] = [Z, Z^{X^f}] + \breve{\nabla}_Z X^f$ and $\hat{//}_s^{-1} \breve{\nabla}_Z X^f \in R_0$ because $\breve{\nabla}_Z X^f \in E_{x_s}$. Taking $Z = X^e$ show that $[X^e, X^f](x_0) \in R_0$ each $e, f \in \mathbb{R}^m$. Repeating the argument, using (1.3.18) with $Z = [X^e, X^f]$ gives $[\,[X^e, X^f], X^g\,](x_0) \in R_0$ for all e, f, g in $\mathbb{R}^m$, and the full result follows by induction. ■

1.4 Example: Homogeneous spaces continued

A. For an important class of examples we will go back to situation of a reductive homogeneous space $M = K/H$ described in §1.2 using the notation there. In Proposition 1.2.1 we saw from our construction that the action of K determines a K-invariant connection on M and that given an ad_K-invariant inner product on $\underline{\mathfrak{k}}$ the L-W connection is K-invariant: in particular the metric $\langle\ \rangle^X$ (i.e. induced by $X(x) : \underline{\mathfrak{k}} \to TM$) is K-invariant, as is seen from (1.2.8). In general given an inner product $\langle\ \rangle$ on $\underline{\mathfrak{k}}$ the metric $\langle\ \rangle^X$ induced on TM will not be K-invariant.

Proposition 1.4.1 *A K-invariant connection $\tilde{\nabla}$ on M is uniquely determined by the mapping*

$$\underline{\mathfrak{m}} \to L(T_{x_0}M; T_{x_0}M)$$

given by

$$e \mapsto \tilde{\nabla}' X^e \Big|_{T_{x_0}M},$$

where $\tilde{\nabla}'$ is the adjoint connection.

Proof. Observe that for any vector field Z on M

$$\tilde{A}_Z : TM \to TM$$

defined by $\tilde{A}_Z(v) := -\tilde{\nabla}_v Z - \tilde{T}(Z, v)$ as defined in [[KN69a], p255, [KN69b], p188] is given by $\tilde{A}_Z(v) = \tilde{\nabla}'_v Z$. The result is then a reformulation of Corollary 2.2 of [[KN69b], p191]. ■

B. The K-invariant connection on M corresponding to the identically zero map: $\underline{\mathfrak{m}} \to L(T_{x_0}M; T_{x_0}M)$ is the *canonical connection*

Theorem 1.4.2 *Let $\breve{\nabla}$ be the connection on M determined as in §1.2 by its reductive homogeneous space structure and let $\hat{\nabla}$ be its adjoint. Then $\hat{\nabla}$ is the canonical connection. In particular*

(i). $\hat{R}(u, v)w = -\left[[Z^u, Z^v] - Z^{[Z^u, Z^v]}, Z^w\right], \qquad u, v, w \in T_{x_0}M,$
(ii). $\hat{\nabla}\hat{T} = 0,$
(iii). $\hat{\nabla}\hat{R} = 0.$

Proof. Take $\tilde{\nabla} = \hat{\nabla}$ in Proposition 1.4.1 and use the defining property of $\breve{\nabla}$ to see $\tilde{\nabla}$ is the canonical connection. For (i), (ii) and (iii) see Theorem 2.6 p193 [KN69b] and use the fact that $e \mapsto X^e(\cdot)$ is a homomorphism of Lie algebras. ■

Corollary 1.4.3 *Any K-invariant tensor on M is $\hat{\nabla}$-parallel.*

Proof. See Proposition 2.7 p192 of [KN69b].

From this we immediately have

Corollary 1.4.4 *The connection $\hat{\nabla}$ is metric for any K-invariant metric on M.*

Taking $\tilde{\nabla} = \hat{\nabla}$ we can obtain in this way a class of metric connections whose adjoints are not metric for any metric on M.

C. Suppose next that as well as being reductive there is an ad_H-invariant inner product $\mathcal{B}$ on $\underline{\mathfrak{m}}$. We then obtain a K-invariant Riemannian metric on M which agrees with $\mathcal{B}$ under the isomorphism $\underline{\mathfrak{m}} \to T_{x_0}M$. The space, together with $\mathcal{B}$, is called *naturally reductive* if it has a decomposition as before with also

$$\mathcal{B}\left(\alpha, [\beta, \gamma]_{\underline{\mathfrak{m}}}\right) + \mathcal{B}\left([\beta, \alpha]_{\underline{\mathfrak{m}}}, \gamma\right) = 0, \alpha, \beta, \gamma \in \underline{\mathfrak{m}} \qquad (1.4.1)$$

where the subscript $\underline{\mathfrak{m}}$ refers to the projection in $\underline{\mathfrak{k}}$ onto $\underline{\mathfrak{m}}$.

Proposition 1.4.5 *The decomposition $\underline{\mathfrak{k}} = \underline{\mathfrak{m}} + \underline{\mathfrak{h}}$ together with $\mathcal{B}$ is naturally reductive if and only if $\hat{\nabla}$ is torsion skew symmetric for the induced K-invariant metric on M.*

Proof.

$$\left\langle \hat{T}(u,v), w \right\rangle_{x_0} = - \left\langle [Z^u, Z^v](x_0), w \right\rangle_{x_0}, u, v, w, \in T_{x_0}M$$

by (1.3.6). Set $\alpha = Y(x_0)u$, $\beta = Y(x_0)v$, $\gamma = Y(x_0)w$. Then

$$\begin{aligned} [Z^u, Z^v](x_0) &= \left[X^\alpha, X^\beta\right](x_0) \\ &= X(x_0)([\alpha,\beta]) \\ &= X(x_0)\left([\alpha,\beta]_{\underline{\mathfrak{m}}}\right) \end{aligned}$$

and the result follows from (1.4.1). ■

Corollary 1.4.6 *In the naturally reductive case $\check{\nabla}$ is a torsion skew symmetric connection.*

Note however that the metric involved may not be induced, via X, by an inner product on $\underline{\mathfrak{k}}$. However from [KN69b] p203 Theorem 3.5 we see that if $\underline{\mathfrak{k}}$ has an ad_K-invariant inner product $\langle\ \rangle$, we can let $\underline{\mathfrak{m}} = \ker X(x_0)^\perp = \underline{\mathfrak{h}}^\perp$ to have K/H naturally reductive with $\mathcal{B}(\alpha,\beta) = \langle\alpha,\beta\rangle$, $\alpha, \beta \in \underline{\mathfrak{m}}$. Thus

Theorem 1.4.7 *Let $\underline{\mathfrak{k}}$ have an ad_K-invariant inner product $\langle\ \rangle$. Then the L-W connection on M determined by $X, \langle\rangle$ is torsion skew symmetric and K-invariant. Its adjoint connection is the canonical connection of the corresponding reductive homogeneous space structure.*

D. Specializing further suppose that we have a symmetric space (K, H, σ): so K, H are as before with a reductive decomposition $\underline{\mathfrak{k}} = \underline{\mathfrak{m}} + \underline{\mathfrak{h}}$ such that $\underline{\mathfrak{m}}$ and $\underline{\mathfrak{h}}$ are, respectively, the -1 and $+1$ eigenspaces for the involution σ of $\underline{\mathfrak{k}}$.

Theorem 1.4.8 *For a symmetric space $\check{\nabla}$ and $\hat{\nabla}$ are torsion free. In particular they are the Levi-Civita connection of any K-invariant Riemannian metric on M.*

Proof. That $\check{T} = -\hat{T} = 0$ follows from (1.3.6) and the fact that $[\underline{\mathfrak{m}}, \underline{\mathfrak{m}}] \subset \underline{\mathfrak{h}}$, see Proposition 2.1 p226 [KN69b], or from the corresponding fact for the canonical connection [KN69b] p231 Theorem 3.2. The result follows from Corollary 1.4.4. ■

Corollary 1.4.9 *Let the symmetric space (K, H, σ) be such that $\underline{\mathfrak{k}}$ has an ad_K-invariant inner product $\langle\ \rangle$ invariant under σ. Then the L-W connection for $X, \langle\rangle$ is the Levi-Civita connection for the induced metric (which is K-invariant).*

Proof. In this situation $\underline{\mathfrak{m}} \perp \underline{\mathfrak{k}}$, see [[KN69b], p233], and so $\underline{\mathfrak{m}}_x \perp Ker X(x)$ for each $x \in M$: thus $Y = X^*$ and $\breve{\nabla}$ is the L-W connection and hence metric. Since it is torsion free it is Levi-Civita. ■

Example 1.4.1 *Lie groups as symmetric spaces*

Recall the standard symmetric space structure for a Lie group G. Let $\Delta G = \{(g, g) \in G \times G : g \in G\}$. Let $G \times G$ act on G by

$$(g, h) \cdot x = gxh^{-1}.$$

The stabilizer of e is ΔG and G has the homogeneous space structure $G = G \times G/\Delta G$, and symmetric space structure with involution induced from $\sigma : G \times G \to G \times G$ given by $\sigma(g, h) = (h, g)$. For example the symmetry $s_e : G \to G$ is just $x \mapsto x^{-1}$, see [KN69b] (p228).

The relevant decomposition of $\underline{\mathfrak{k}} = \mathfrak{g} \oplus \mathfrak{g}$ is

$$\underline{\mathfrak{k}} = \Delta\mathfrak{g} \oplus \underline{\mathfrak{m}}$$

where $\Delta\mathfrak{g} = \{(\alpha, \alpha) : \alpha \in \mathfrak{g}\}$, $\underline{\mathfrak{m}} = \{(\alpha, -\alpha) : \alpha \in \mathfrak{g}\}$, see [KN69b] (p198). Now suppose $\mathfrak{g}$ and hence $\mathfrak{g} \times \mathfrak{g}$ has an ad_G-invariant inner product $\langle\ \rangle$.

The stochastic differential equation on G is

$$dx_t = TL_{x_t} \circ dB_t - TR_{x_t} \circ dB'_t$$

for $(B_\cdot)$, $(B'_\cdot)$ independent BM($\mathfrak{g}$). The flow is given by $\xi_t(x) = g_t x {g'_t}^{-1}$ for

$$dg_t = TR_{g_t} \circ dB_t$$

$$dg'_t = TR_{g_t} \circ dB'_t$$

with $g_0 = e = g'_0$.

The flow consists of isometries so that the moment exponents are zero. However for G with bi-invariant metric (as we are considering), if $\mathfrak{g}$ has trivial centre then Ric > 0. Thus we obtain a class of stochastic differential equations such that
(i) $Ri\check{c} > 0$;
(ii) all moment exponents vanish;
(iii) $\xi_\cdot$ consists of isometries;
(iv) $\breve{\nabla}$ is the Levi-Civita connection.
In particular (i) and (ii) can be contrasted with results which says that negative curvature implies first moment exponent positive. They also give an example where the hypotheses of Corollary 6.4.7 in [Li92] hold.

Remark: Among other homogeneous spaces with a Riemannian space structure are the spheres $S^n = SO(n+1)/SO(n)$, oriented Grassmannian manifolds $SO(p+q)/SO(p) \times SO(q)$ and hyperbolic spaces $O(1, n)/SO(n)$.

Chapter 2

The infinitesimal generators and associated operators

For any second order elliptic operator $\mathcal{L}$ with smooth coefficients and $\mathcal{L}1 \equiv 0$ on a manifold M of dimension greater than 1, we construct $X : \underline{\mathbb{R}}^m \to TM$, for some m, such that $\mathcal{L} = \frac{1}{2}\sum_1^m L_{X^j}L_{X^j}$, the differential generator for the solutions of the stochastic differential equation without drift $dx_t = X(x_t) \circ dB_t$. This result is, in fact, proved for a class of semi-elliptic operators. In section 2.4 the Hölmander form operator $\mathcal{A}^q = \frac{1}{2}\sum_j L_{X^j}L_{X^j} + L_A$ on differential q-forms is analysed. Section §2.2 discusses a special class of connections on TM and also the natural generalization $\hat{\delta}$ of δ to our situation. We show in §2.4 that $\mathcal{A}^q = -\frac{1}{2}(d\hat{\delta} + \hat{\delta}d) + L_A$ so that when the L-W connection associated to X is the Levi-Civita connection and $A = 0$ then $\mathcal{A}^q$ is the De Rham-Hodge Laplacian. In the regular case there is the Weitzenbock formula:

$$\mathcal{A}^q\phi \;=\; \frac{1}{2}\mathrm{tr}_E\hat{\nabla}.(\hat{\nabla}.\phi) + L_A\phi - \frac{1}{2}\breve{R}^q\phi$$

for $\breve{R}^q$ a zero order term, the 'Weitzenbock curvature' related to $\breve{\nabla}$. In the last section we give conditions for the leading order terms of $\mathcal{A}^q$ to be symmetrizable as an operator on a suitable L^2 space. In particular we show that this holds if $\mathcal{A}^0$ is symmetrizable and $\hat{\nabla}$ is metric for some metric on TM.

2.1 The irrelevance of drift in dimension greater than 1

A. The first application of the construction in section §1.1 is that any elliptic differential operator can be considered as an infinitesimal generator to some stochastic differential equation with zero drift: $dx_t = X(x_t) \circ dB_t$. Let E be a subbundle of TM with fibre dimension p. Let Z be a E-valued vector field, and $V : \mathbb{R}^m \to E \subset TM$ a C^∞ surjection. Consider the operator $\mathcal{L}$ on M:

$$\mathcal{L} = \frac{1}{2}\sum_1^m L_{V^i}L_{V^i} + L_Z, \tag{2.1.1}$$

where $V^i = V(e^i)$ for an orthonormal base of $\mathbb{R}^m$.

Theorem 2.1.1 *Assume $p > 1$. For $\mathcal{L}$ as given above, there is a map $X : M \times \mathbb{R}^m \to TM$ linear in the second variable such that the solution to $dx_t = X(x_t) \circ dB_t$ has $\mathcal{L}$ as infinitesimal generator, i.e. $\mathcal{L} = \frac{1}{2}\sum_j L_{X^j} L_{X^j}$.*

Given E the Riemannian metric induced by V. Recall that a diffusion $\{\xi_t(x) : 0 \le t < \rho(x)\}$ generator $\mathcal{L}$ is a $\breve{\nabla}$ martingale if $f(\xi_t(x)) - \text{trace}_E \breve{\nabla} df(\xi_t(x))$ is a local martingale for any smooth $f : M \to \mathbb{R}$. (e.g. [Eme89]). An immediate corollary of the Theorem is that a diffusion $\{\xi_t(x) : 0 \le t < \rho(x), x \in M\}$, whose generator satisfies the conditions of Theorem 2.1.1 is a $\breve{\nabla}$-martingale for some metric connection $\breve{\nabla}$ on E. In fact (as will be seen explicitly in §3.3), $\xi_t(x_0)$ will be the stochastic development of a Brownian motion on E_{x_0} using $\breve{\nabla}$. The existence of a connection on TM for which this is true for a non-degenerate diffusion was shown by Ikeda-Watanabe [IW89] and a modification of their construction for the case $E = TM$ is one of the main ingredients of our proof. The other key ingredient is Narasimhan & Ramanan's theorem on universal connections used via Theorem 1.1.2.

The proof is given in §§C below. To adapt Ikeda & Watanabe's construction we need the modification of the classical result on connections given in Proposition 2.1.2 below. Its proof taken up §§B. Some of the notation in §§B will be used later.

B. Let E be a subbundle of TM with a given metric. Two connections ∇^a and ∇^b on E are associated with a bilinear map $D^{ab} : TM \times E \to E$ such that

$$\nabla^a_V U = \nabla^b_V U + D^{ab}(V, U), \quad V \in TM, U \in E.$$

Let T^a and T^b be respectively the torsions, defined by (1.3.5), for the two connections, then

$$T^a(u, v)) = T^b(u, v) + D^{ab}(u, v) - D^{ab}(v, u), \qquad u, v \in E. \tag{2.1.2}$$

In this section we use uppercase letters for vector fields and lowercase letters for tangent vectors.

It will be convenient to have a class of connections on E with which to relate metric connections. In the non-degenerate case the obvious base connection is the Levi-Civita connection ∇ for the given metric. In the degenerate, regular, case let $\langle,\rangle^0$ be an extension to TM of the metric $\langle,\rangle$ on E and let $P_E : TM \to E$ be the corresponding orthogonal projection. Let ∇^0 be the connection on E which is the push forward by P_E of the Levi-Civita connection ∇ of $\langle,\rangle$:

$$\nabla^0_V U = P_E \nabla_V U, \qquad U \in \Gamma(E),\ V \in \Gamma(TM). \tag{2.1.3}$$

In fact ∇^0 is a metric connection. Note that if $T^0 : T \times E \to TM$ is the torsion for ∇^0, defined by (1.3.5) or rather its modification for connections on the subbundle E,

$$T^0(U, V) = P_E(\nabla_U V - \nabla_V U) - [U, V] \;=\; -(I - P_E)[U, V]. \tag{2.1.4}$$

Let $\tilde{T}$ be the torsion for a connection $\tilde{\nabla}$ on E and take $(I - P_E)$ of (1.3.5) to see:

$$(I - P_E)\tilde{T}(U,V) = -(I - P_E)[U,V] = T^0(U,V), \tag{2.1.5}$$

and so

$$\tilde{T}(U,V) = P_E\tilde{T}(U,V) - (I - P_E)[U,V]. \tag{2.1.6}$$

The converse also holds:

Proposition 2.1.2 *Let $\tilde{T} : E \times E \to TM$ be a skew symmetric map satisfying (2.1.5). There is a metric connection $\tilde{\nabla}$ on E with $\tilde{T}$ as its torsion.*

The proof of Proposition 2.1.2 will be given after Lemma 2.1.3. First we introduce the tensors $\tilde{D}$ and S. Define $\tilde{D} : TM \times E \to E$ by

$$\tilde{\nabla}_V U = \nabla^0_V U + \tilde{D}(V,U). \tag{2.1.7}$$

For $U, V \in \Gamma E$, write $\tilde{D}$ as the sum of its symmetric part S and antisymmetric part A:

$$\tilde{D}(u,v) = A(u,v) + S(u,v).$$

By (2.1.2), $\tilde{T}(u,v) = T^0(u,v) + 2A(u,v)$. So

$$A(u,v) = \frac{1}{2}P_E\tilde{T}(u,v),$$

so that

$$\tilde{\nabla}_v U = \nabla^0_v U + \frac{1}{2}P_E\tilde{T}(u,v) + S(u,v), \quad u, v \in \Gamma E. \tag{2.1.8}$$

Let Cyl denote cyclic sum.

Lemma 2.1.3 *A connection $\tilde{\nabla}$ on E is metric if and only if the map $\tilde{D}(v,\cdot) : E \to E$ is skew symmetric for each $v \in TM$,*

$$\left\langle \tilde{D}(v,u_1), u_2 \right\rangle + \left\langle \tilde{D}(v,u_2), u_1 \right\rangle = 0, \qquad u_1, u_2 \in E. \tag{2.1.9}$$

For $V \in \Gamma E$, (2.1.9) is equivalent to

$$\langle S(u_1,u_2), v \rangle = \frac{1}{2}\left\langle P_E\tilde{T}(v,u_1), u_2 \right\rangle + \frac{1}{2}\left\langle P_E\tilde{T}(v,u_2), u_1 \right\rangle \tag{2.1.10}$$

and (2.1.10) implies

$$Cyl\,\langle S(\cdot,\cdot), \cdot \rangle = 0. \tag{2.1.11}$$

Consequently for U_1, U_2, and V in ΓE,

$$\begin{aligned} &\left\langle \tilde{D}(V,U_1), U_2 \right\rangle \\ &= \tfrac{1}{2}\left\{ \left\langle P_E\tilde{T}(V,U_1), U_2 \right\rangle + \left\langle P_E\tilde{T}(U_2,V), U_1 \right\rangle + \left\langle P_E\tilde{T}(U_2,U_1), V \right\rangle \right\} \end{aligned} \tag{2.1.12}$$

Proof. First take $V \in \Gamma(TM)$ and $U_i \in \Gamma(E)$. The equivalence of $\tilde{\nabla}$ being metric and the skew symmetricity of $\tilde{D}(-,\cdot)$ follows from differentiating $\langle U_1, U_2\rangle$:

$$\begin{aligned} & d_V \langle U_1, U_2\rangle \\ = \ & \left\langle \nabla^0_V U_1, U_2 \right\rangle + \left\langle U_1, \nabla^0_V U_2 \right\rangle \\ = \ & \left\langle \tilde{\nabla}_V U_1, U_2 \right\rangle + \left\langle U_1, \tilde{\nabla}_V U_2 \right\rangle - \left\langle \tilde{D}(V, U_1), U_2 \right\rangle - \left\langle U_1, \tilde{D}(V, U_2) \right\rangle. \end{aligned}$$

So $\tilde{\nabla}$ is metric if and only if

$$\left\langle \tilde{D}(V, U_1), U_2 \right\rangle + \left\langle U_1, \tilde{D}(V, U_2) \right\rangle = 0.$$

Next suppose V and $U_i \in \Gamma E$, writing $\tilde{D} = A + S$ to get:

$$\begin{aligned} & \langle A(V, U_1), U_2\rangle + \langle A(V, U_2), U_1\rangle \\ & = -\langle S(V, U_1), U_2\rangle - \langle S(V, U_2), U_1\rangle \end{aligned} \tag{2.1.13}$$

Suppose (2.1.13) holds. Observe for an alternating bilinear map $L : E \times E \to E$:

$$\mathrm{Cyl}\left[\langle L(v, u_1), u_2\rangle + \langle L(v, u_2), u_1\rangle\right] = 0. \tag{2.1.14}$$

Take the cyclic sums of equation (2.1.13) and apply (2.1.14) to A to obtain

$$\mathrm{Cyl}\,\langle S(V, U_1), U_2\rangle = 0.$$

Substitute the above back to (2.1.13) to see:

$$\langle A(V, U_1), U_2\rangle + \langle A(V, U_2), U_1\rangle = \langle S(U_1, U_2), V\rangle\,.$$

So (2.1.13), and therefore (2.1.9), implies (2.1.10). On the other hand (2.1.11) clearly follows from (2.1.10) and the two give (2.1.9). ■

Proof of Proposition 2.1.2. Define $S : E \times E \to TM$ by (2.1.10). Set $\tilde{D}(v, u) = 0$ for $v \in (\Gamma E)^{\perp}$, and $\tilde{D}(v, u) = \frac{1}{2} P_E \tilde{T}(v, u) + S(v, u)$ for $v \in \Gamma E$. Define $\tilde{\nabla}$ by

$$\tilde{\nabla}_V U = \nabla^0_V U + \tilde{D}(V, U).$$

Then the equality (2.1.9) holds for $v \in \Gamma E$ and extends to ΓTM since by the construction $\tilde{D}(v, u) = \tilde{D}(P_E v, u)$. The connection $\tilde{\nabla}$ is the required connection. ■

C. Proof of Theorem 2.1.1. Let ∇ be the Levi-Civita connection for a metric extending the metric induced on E by the map V and ∇^0 its projection to E as before. Set $\bar{Z} = \frac{1}{2}\sum_1^m \nabla^0_{V^i} V^i + Z$ and choose $\tilde{T}$ such that

$$\mathrm{trace}_E \left\langle P_E \tilde{T}(u, -), - \right\rangle = -2\left\langle \bar{Z}(x), u \right\rangle, \qquad u \in E_x. \tag{2.1.15}$$

One choice of such $\tilde{T}$ is given by

$$P_E\tilde{T}(v,u) = \frac{2}{p-1}(u\wedge v)\bar{Z}(x), \qquad u,v\in E_x.$$

Recall $(u\wedge v)\bar{Z}(x) = \left\langle \bar{Z}(x), u\right\rangle v - \left\langle \bar{Z}(x), v\right\rangle u$. Let $\tilde{\nabla}$ be the metric connection on E with torsion $\tilde{T} : E\times E \to TM$ as constructed in Proposition 2.1.2. We show the associated stochastic differential equation has $\mathcal{L}$ as infinitesimal generator.

Let $X : \mathbb{R}^m \to E \subset TM$ be a bundle map, as in Theorem 1.1.2, which gives rise to the metric connection $\tilde{\nabla}$. The solution to the following stochastic differential equation

$$dx_t = X(x_t)\circ dB_t \tag{2.1.16}$$

has generator

$$\mathcal{A}^0 = \mathrm{trace}_E[\nabla^0]^2 + \frac{1}{2}\sum_{i=1}^m \nabla^0 X^i(X^i),$$

On the other hand from (2.1.1)

$$\mathcal{L} = \frac{1}{2}\mathrm{trace}_E[\nabla^0]^2 + (\frac{1}{2}\sum_1^m \nabla^0_{V^i}V^i + Z) = \frac{1}{2}\mathrm{trace}_E[\nabla^0]^2 + \bar{Z}.$$

The required result then follows after we show

$$\sum_{i=1}^m \nabla^0 X^i(X^i) = -\sum_1^m \tilde{D}(X^i, X^i) = tr_E\tilde{D}(-,-)$$

equals $\bar{Z}$. For this note that for all $v\in E$,

$$\begin{aligned}\left\langle \sum \tilde{D}(X^i,X^i), v\right\rangle &= \left\langle \sum S(X^i,X^i), v\right\rangle \\ = -\sum_i \left\langle P_E\tilde{T}(v,X^i), X^i\right\rangle &= -2\left\langle \bar{Z}, v\right\rangle.\end{aligned}$$

Consequently

$$\mathrm{trace}_E[\nabla^0]^2 + \frac{1}{2}\sum_{i=1}^m \nabla^0 X^i(X^i) = \mathrm{trace}_E[\nabla^0]^2 + \bar{Z}$$

and the X so constructed is the required map. ■

2.2 Torsion Skew Symmetry

A metric connection $\breve{\nabla}$ on the tangent bundle TM can be expanded in terms of another connection $\tilde{\nabla}$ and its defining map X or W as introduced in Theorem 1.1.2. More precisely by (C.2) in the appendix,

$$\breve{\nabla}_v U = \tilde{\nabla}_v U + \sum_j \tilde{\nabla}_v X^j \left\langle u, X^j \right\rangle.$$

However this process is not reversible: we do not seem to be able to write the second term of the right hand side in terms of $\breve{\nabla}$ and X. For example if $\tilde{\nabla}$ is taken to be the Levi-Civita connection, the special case when

$$\sum_j \nabla_v X^j \left\langle U, X^j \right\rangle_{x_0} = -\frac{1}{2}\breve{T}(v, u),$$

turns out to be particularly interesting. This is just the torsion skew symmetric case. See Proposition 2.2.2 below. In this section we shall explore this situation.

A metric connection $\breve{\nabla}$ on TM is *torsion skew symmetric* if $\hat{\nabla}$ is adapted to the same metric. Here is a corollary of Lemma 2.1.3.

Corollary 2.2.1 *Suppose $\tilde{\nabla}$ is a metric connection on TM. The following are equivalent: (a). $\tilde{\nabla}$ is torsion skew symmetric; (b). the symmetric part of $\tilde{D}(\cdot,\cdot)$ vanishes; (c). $\left\langle \tilde{T}(u,\cdot), \cdot \right\rangle$ is skew symmetric for each $u \in TM$.*

In the following we put together the equivalent conditions for a connection to be torsion skew symmetric, in terms of X.

Proposition 2.2.2 *Let X be a defining map for a metric connection $\breve{\nabla}$ on TM as in Theorem 1.1.2. Then $\breve{\nabla}$ is torsion skew symmetric if and only if*
$\sum_{i=1}^m \left\langle X^i(x), v \right\rangle \nabla_u X^i + \sum_{i=1}^m \left\langle X^i(x), u \right\rangle \nabla_v X^i = 0,$ *or*
$\sum_{i=1}^m X^i(x) \left\langle v, \nabla_u X^i \right\rangle + \sum_{i=1}^m X^i(x) \left\langle u, \nabla_v X^i \right\rangle = 0,$
equivalently $\breve{T}(v, u) = 2\sum_{i=1}^m X^i(x) \left\langle \nabla_v X^i, u \right\rangle$. *In this case,*

$$\nabla_v U = \breve{\nabla}_v U - \frac{1}{2}\breve{T}(v, u). \tag{2.2.1}$$

In particular, $\nabla_{X^i} X^i = 0$ for each i.

Proof. The first identity comes from (C.4) with $\tilde{\nabla}$ replaced by ∇:

$$\breve{T}(v,u) = \sum_{i=1}^{m} X^i(x)\,\langle u, \nabla_v X^i\rangle - \sum_{i=1}^{m} X^i(x)\,\langle v, \nabla_u X^i\rangle. \tag{2.2.2}$$

The second comes from the first and (C.3). The third is a consequence of (2.2.2) and the second. Finally (2.2.1) follows from (C.2) and the third identity. ■

Recall the definition of Z^u: $Z^u(x) = X(x)Y(\pi(u))u$ as in §1.1.

Proposition 2.2.3 *Let $\breve{\nabla}$ be a metric connection on TM with defining map X. In terms of the adjoint Y of X,*

$$\breve{T}(v_1,v_2) = X(x_0)dY(v_1,v_2), \qquad v_i \in T_{x_0}M. \tag{2.2.3}$$

Furthermore the connection $\breve{\nabla}$ is

- *the Levi-Civita connection if and only if $X(x)dY(u,v) = 0$ for all $u,v \in T_xM$, all $x \in M$, or ∇Z^v vanishes at x_0 for all $v \in T_{x_0}M$.*
- *torsion skew symmetric if $\nabla Z^w|_{T_{x_0}M} : T_{x_0}M \to T_{x_0}M$ is skew symmetric for all $w \in TM$, or $\nabla_v Z^u + \nabla_u Z^v = 0$ for any $u,v \in TM$, equivalently $\breve{\nabla}_U V + \breve{\nabla}_V U = \nabla_U V + \nabla_V U$ for all vector fields U and V.*

Proof. Recall $Z^{v_i} = X(x)Y(x_0)v_i$ as in (1.1.1). By Proposition 1.1.1, the definition of $\breve{\nabla}$,

$$\begin{aligned}
\breve{T}(v_1,v_2) &= \breve{\nabla}_{v_1}Z^{v_2} - \breve{\nabla}_{v_2}Z^{v_1} - [Z^{v_1}, Z^{v_2}] \\
&= X(x_0)\nabla_{v_1}Y(v_2) + \nabla Z^{v_2}(v_1) - X(x_0)\nabla_{v_2}Y(v_1) + \nabla Z^{v_1}(v_2) - [Z^{v_1}, Z^{v_2}] \\
&= X(x_0)\nabla_{v_1}Y(v_2) - X(x_0)\nabla_{v_2}Y(v_1) = X(x_0)dY(v_1,v_2).
\end{aligned}$$

That ∇Z^u vanishes for all u iff $\breve{\nabla}$ is the Levi-Civita connection follows from (1.1.2), the defining property of the connection.

For the equivalent conditions of torsion skew symmetricity: the first is exactly Proposition 1.3.5: $\breve{\nabla}$ is torsion skew symmetric if and only if

$$\langle v, \nabla_u Z^w\rangle + \langle u, \nabla_v Z^w\rangle = 0 \qquad u,v \in TM.$$

From the equivalence of the first two identities in Proposition 2.2.2:

$$\langle w, \nabla_u Z^v\rangle + \langle w, \nabla_v Z^u\rangle = 0.$$

Adding $\nabla_v U$ with $\nabla_u V$ and use (C.2) to obtain the last equivalence. ■

Finally if $\tilde{\nabla}$ is a metric connection on TM we can define a differential 3-form $D^\#$ by

$$D^\#(-,-,-) := \frac{1}{3}\mathrm{Alt}(\langle D(-,-),-\rangle) \equiv \frac{2}{3}\mathrm{Cyl}(\langle D(-,-),-\rangle) \equiv \frac{1}{3}\mathrm{Cyl}\,\langle T(-,-),-\rangle,$$

where *Alt* is the alternating mapping. In the torsion skew symmetric case, there is a differential 3-form $T^\#$ from the torsion tensor:

$$T^\#(u,v,w) = \langle T(u,v), w\rangle . \tag{2.2.4}$$

Indeed there is a bijection between torsion skew symmetric connection for a given metric and 3-forms given by $T \mapsto T^\#$ using Lemma 2.1.3. In Appendix C we see that $dT^\#$ and $\delta T^\#$ appear in curvature identities.

2.3 The 'divergence operator' $\hat{\delta}$

A. Let $X : \mathbb{R}^m \to TM$ be a smooth bundle map (not necessarily of constant rank). Let $\wedge^* T^* M$ be the space of differential forms on M. Define $\hat{\delta} : \wedge^q T^* M \to \wedge^{q-1} T^* M$ by

$$\hat{\delta}\phi = -\sum_{j=1}^{m} \iota_{X^j} L_{X^j}\phi. \tag{2.3.1}$$

Here $\iota_Y\phi$ is the interior product of ϕ by Y: $\iota_Y\phi(-) = \phi(Y,-)$. On smooth functions $\iota_Y = 0$. In the case that X comes from an isometric immersion of M to $\mathbb{R}^m$, $\hat{\delta}$ is the usual divergence operator δ.

Let A be a smooth vector field on M. Consider our operator $\mathcal{A}$ in Hörmander form

$$\mathcal{A} = \frac{1}{2}\sum L_{X^j} L_{X^j} + L_A. \tag{2.3.2}$$

Since Lie differentiations also act on forms we can also extend $\mathcal{A}$ to operators on forms and will use $\mathcal{A}^q$ when we want to emphasize that we consider it acting on q-forms.

One of the observations which demonstrates the role of $\hat{\delta}$ is the following proposition:

Proposition 2.3.1

$$\mathcal{A}\phi = \frac{1}{2}\sum_{j=1}^{m} L_{X^j} L_{X^j}\phi + L_A\phi = -\frac{1}{2}\left(\hat{\delta} d + d\hat{\delta}\right)\phi + L_A\phi. \tag{2.3.3}$$

Proof. Just observe that the Lie differentiation L_{X^j} is given by

$$L_{X^j}\phi = \iota_{X^j} d\phi + d\left(\iota_{X^j}\phi\right)$$

and d commutes with the differentiation. ∎

B. Assume that X has constant rank. The covariant derivative $\hat{\nabla}.\phi$ of a q-form is a linear map

$$\psi = \hat{\nabla}.\phi : E \to \wedge^q T^* M$$

over M, i.e. a section of $L(E; \wedge^q T^* M)$. It is not obvious in the degenerate case how to apply $\hat{\nabla}$ to it again: we would want

$$\hat{\nabla}_w(\psi(U)) = \hat{\nabla}_w\psi(U(x)) + \psi\left(\hat{\nabla}_w U\right) \quad \in \wedge^q T_x^* M \tag{2.3.4}$$

for $U \in \Gamma(E)$ and $w \in E_x$, but in general $\hat{\nabla}_w U$ will not lie in E_x. However we can use (2.3.4) to define "$\mathrm{tr}_E \hat{\nabla}.\psi(\cdot)$" by

$$\mathrm{tr}_E \hat{\nabla}.\psi(\cdot) := \sum_{j=1}^m \hat{\nabla}_{X^j}\left(\psi(X^j)\right) \tag{2.3.5}$$

since $\sum_{j=1}^m \hat{\nabla}_{X^j} X^j = 0$.

Since this agrees with (2.3.4) it will coincide with the result obtained by taking any extension $\tilde{\psi} : TM \to \wedge^q T^*M$ and using (2.3.4) as the definition of $\tilde{\nabla}\tilde{\psi}$, or extending $\breve{\nabla}$ as in Proposition 1.3.1 to some ∇^1 on TM and using $(\nabla^1)'$ in the usual way.

Proposition 2.3.2 *Let $\breve{\nabla}$ be a metric connection on a subbundle E of TM and X its defining map as in Theorem 1.1.2. Then $\hat{\delta}$, defined by (2.3.1), does not depend on the choice of X. In fact,*

$$\hat{\delta}\phi(\cdot) = -tr_E \hat{\nabla}.\phi(\cdot,\cdot).$$

Proof. Let $E^\perp$ be a complementary bundle to E so that $TM = E \oplus E^\perp$. Let $\nabla^\perp$ be a connection on $E^\perp$ and set $\nabla^1 \equiv \breve{\nabla} \oplus \nabla^\perp$. Observe that for a connection $\tilde{\nabla}$ on TM with adjoint connection $\tilde{\nabla}'$,

$$L_Y\phi(v_1,\dots,v_q) = \left(\tilde{\nabla}_Y\phi\right)(v_1,\dots,v_q) + \sum_{j=1}^q \phi(v_1,v_2,\dots,\tilde{\nabla}'_{v_j}Y,\dots,v_q) \tag{2.3.6}$$

for $v = (v_1,\dots,v_q) \in \wedge^q TM$. Take $\tilde{\nabla}'$ to be ∇^1. By Proposition 1.3.1,

$$L_{X^p}\phi(v) = \hat{\nabla}_{X^p}\phi(v) + \sum_{j=1}^q \phi(v_1,\dots,\breve{\nabla}_{v_j}X^p,\dots,v_q), \quad 1 \le p \le m, \tag{2.3.7}$$

and $\nabla^\perp$ is actually not involved. By the defining property of the L-W connection $\breve{\nabla}$,

$$\hat{\delta}\phi(\cdot) = -\sum_1^m \iota_{X^i}\hat{\nabla}_{X^i}\phi(\cdot) = -\mathrm{tr}_E\hat{\nabla}_-\phi(-,\cdot). \tag{2.3.8}$$

■

In the Gaussian field formulation of §1.1C we have

$$\mathcal{A} = \frac{1}{2}\mathbb{E} L_W L_W + L_A \tag{2.3.9}$$

with $\hat{\delta}$ defined accordingly. The extension of (2.3.3) to this case holds in the same way as does Proposition 2.3.2. An important consequence which follows using Theorem 2.1.1 is:

Corollary 2.3.3 *The operator $\mathcal{A}$ on forms defined by (2.3.2) or (2.3.9) depend on X or the field W only through the associated connection $\breve{\nabla}$ and the induced metric on E. In particular an operator $\mathcal{A}$ given by (2.3.9) can always be written in the form (2.3.2) using a finite set of vector fields $X^1, \dots, X^m$.*

C. The rest of this section will be on the comparison of $\hat{\delta}$ and the usual divergence δ.

Remark 2.3.1 *Let $\breve{\nabla}$ be a metric connection on TM with defining map X.*

1. *Assume $\sum \nabla_{X^j} X^j = 0$. Then $\hat{\delta} = \delta$, the usual divergence, on differential 1-forms. (E.g. this holds in the gradient Brownian system case, or if $\breve{\nabla}$ is torsion skew symmetric).*
2. *Assume $\sum_j \nabla_{X^j} X^j = \nabla h$ for some smooth $h : M \to \mathbb{R}$. Then $\sum_i L_{X^i} L_{X^i} = \Delta + L_{\nabla h}$ on differential 1-forms. Here Δ is the Laplace-Beltrami operator.*
3. *$\sum_1^m X^i \wedge \nabla X^i = 0$ if and only if $\breve{\nabla} = \nabla$. In particular for 2-forms, $\hat{\delta} = \delta - \iota_{\sum_i \nabla_{X^i} X^i}$ if and only if $\breve{\nabla}$ is the Levi-Civita connection.*

Proof. Remarks 1 and 2 are readily seen by

$$\hat{\delta}\phi(V) = -\sum_1^m \iota_{X^j} \nabla_{X^j} \phi - \phi(\sum_1^m \nabla_{X^j} X^j). \tag{2.3.10}$$

(i.e. (2.3.6) using ∇). Similarly on differential q (> 1) forms,

$$\begin{aligned} \hat{\delta}\phi(V) \;&= -\textstyle\sum_1^m \iota_{X^j} \nabla_{X^j} \phi(v_1, \dots, v_{q-1}) - \phi(\sum_1^m \nabla_{X^j} X^j, v_1, \dots, v_{q-1}) \\ &\quad - \textstyle\sum_{j=1}^m \left[\sum_{k=1}^{q-1} \phi(X^j, v_1, \dots, \nabla_{v_k} X^j, \dots, v_{q-1}) \right], \end{aligned} \tag{2.3.11}$$

for any $V = (v_1, \dots, v_{q-1}) \in \wedge^{q-1} TM$.

For $q = 2$,

$$\hat{\delta}\phi(V) = \sum_1^m \iota_{X^j} \nabla_{X^j} \phi(v_1) - \sum \phi(X^i, \nabla_{v_1} X^i) - \sum \phi(\nabla_{X^i} X^i, v_1).$$

This leads to the second statement of Remark 3 (assuming the first). Now we show the first. Firstly $\breve{\nabla} = \nabla$ implies the vanishing of $\sum X^i \wedge \nabla X^i$ by the characterization of $\breve{\nabla}$. Now assume $\sum X^i \wedge \nabla X^i = 0$, i.e. for any vectors w_1, w_2,

$$\left\langle \sum X^i \wedge \nabla X^i, w_1 \wedge w_2 \right\rangle = 0.$$

Since $\sum_j \nabla_v X^j \langle w_1, X^j \rangle = -\sum_j \langle \nabla_v X^j, w_1 \rangle X^j$ by (C.3), We see that

$$\sum_j \left\langle w_1, X^j \right\rangle \left\langle w_2, \nabla_v X^j \right\rangle = 0$$

and therefore $\langle w_2, \nabla_v X^j\rangle = 0$ if $X^j \neq 0$. This is exactly the characteristic property of the connection $\breve{\nabla}$ associated to X. ∎

D. Let $\breve{\nabla}$ be a metric connection on E. Using the Levi-Civita connection ∇^0 for some metric on TM extending the metric of E as in section 1.3, set

$$\delta^0 = -\sum_{j=1}^m \iota_{X^j} \nabla^0_{X^j}$$

acting on q-forms, $1 \leq q \leq n$, and annihilating smooth functions. So $\delta^0 = \delta$ when $E = TM$. Let $K_0^q : \wedge^q T^*M$ to $\wedge^{q-1}T^*M$ be defined by $K_0^1\phi \equiv 0$ and for $q > 1$,

$$K_0^q\phi = \hat{\delta}\phi - \delta^0\phi + \iota_{\sum_1^m \nabla^0_{X^j} X^j}(\phi). \tag{2.3.12}$$

Then for $\hat{D}$ defined by $\hat{D}(V,U) = \hat{\nabla}_V U - \nabla^0_V U$,

$$K_0^q\phi(-) = -\phi\left((d\Lambda)^q \hat{D}(\cdot, X^i)\left(X^i \wedge -\right)\right).$$

Set

$$A^0 = \frac{1}{2}\sum_{j=1}^m \nabla^0_{X^j} X^j.$$

It follows from (2.3.6) and (2.3.3), for $q > 1$,

$$\sum_{j=1}^m L_{X^j} L_{X^j}\phi = -\frac{1}{2}(d\delta^0 + \delta^0 d)\phi + L_{A^0}\phi - \frac{1}{2}d\left(K_0^q\phi\right) - \frac{1}{2}K_0^{q+1}(d\phi). \tag{2.3.13}$$

Proposition 2.3.4 *For nondegenerate X, $\mathcal{A}^q = \frac{1}{2}\Delta^q + L_{A^0}$ if and only if $dK_0^q + K_0^{q+1}d = 0$. Also $\mathcal{A}^1 = \frac{1}{2}\Delta + L_{A^0}$ only when $\breve{\nabla} = \nabla$.*

Proof. The first statement of the theorem is clear. The second statement follows from part 3 of Remark 2.3.1 and $K_0^2(d\phi) = -\sum_{j=1}^m d\phi(X^j, \nabla_v X^j)$. ∎

Note also

$$\begin{aligned}
&\sum_j L_{X^j} L_{X^j}\phi(v_1, v_2) \\
&= \frac{1}{2}\Delta^q\phi(v_1,v_2) + L_{A^0}\phi(v_1,v_2) + + \sum_1^m \phi(X^i, R(v_1,v_2)X^i) \\
&+2\sum_1^m \nabla_{X^i}\phi(D(v_1,X^i),v_2) + 2\sum_1^m \nabla_{X^i}\phi(v_1, D(v_2,X^i)) + 2\phi(\nabla_{v_1}X^i, \nabla_{v_2}X^i).
\end{aligned}$$

Finally as an example we calculate the divergence and $\mathcal{A}^q$ for the connection used in the proof of Theorem 2.1.1:

Example 2.3.5. Consider the connection constructed in the proof of Theorem 2.1.1 of §2.1. Set $W=\bar{Z}$. Let $\tilde{\nabla}$ be the associated connection. Recall

$$\tilde{D}(v,u)=\frac{2}{p-1}\left[\langle W,u\rangle v-\langle v,u\rangle W\right].$$

Then $-\sum_{i=1}^m \nabla^0 X^i(X^i)=\sum_1^m \tilde{D}(X^i,X^i)=-2W$ and for each j,

$$\begin{aligned}
&\sum_{i=1}^m \phi(X^i,v_1,\dots,\tilde{D}(v_j,X^i),\dots,v_{q-1})\\
=\;& \frac{2}{p-1}\sum_1^m \phi\left(X^i,v_1,\dots,\left\langle W,X^i\right\rangle v_j-\left\langle v_j,X^i\right\rangle W,\dots,v_{q-1}\right)\\
=\;& \frac{2}{p-1}\phi\left(W,v_1,\dots,v_j,\dots,v_{q-1}\right)-\frac{2}{p-1}\phi\left(v_j,v_1,\dots,W,\dots,v_{q-1}\right)\\
=\;& \frac{4}{p-1}\phi\left(W,v_1,\dots,v_j,\dots,v_{q-1}\right).
\end{aligned}$$

Consequently by (2.3.12), for a q-form ϕ,

$$\hat{\delta}\phi=\delta^0\phi-\iota_{\sum_i\nabla^0_{X^i}X^i}\phi-\frac{4(q-1)}{p-1}\iota_W\phi=\delta^0\phi-[\frac{4(q-1)}{p-1}+2]\iota_W\phi$$

and in the nondegenerate case

$$\sum_1^m L_{X^j}L_{X^j}=\frac{1}{2}\Delta+L_{[\frac{4(q-1)}{n-1}+2]W}-\frac{4}{n-1}\iota_W d.$$

Thus if W is a gradient, $W=\nabla h$ say, in the non-degenerate case $\mathcal{A}^q$ restricted to closed forms is the Bismut-Witten Laplacian corresponding to the measure $\exp\left([\frac{8(q-1)}{n-1}+4]h(x)\right)\,dx$

2.4 Hörmander form generators on differential forms

We will treat in detail the case of finite dimensional noise (or equivalently a finite sum of squares of vector fields); the infinite dimensional situation can be reduced to this by Corollary 2.3.3.

Let $\{\xi_t(x)\}$ be the solution flow to the s.d.e. (1.2.5) and P_t the induced semigroup on measurable forms defined by

$$P_t\phi=\mathbb{E}(\xi_t)^*(\phi)\chi_{t<\zeta}$$

when the expectation exists.

Its differential generator $\mathcal{A}$ is given by (2.3.2) and hence (2.3.3), as seen by Itô's formula. See e.g. [Elw92].

Proposition 2.4.1 *Let M be a Riemannian manifold. Assume non-explosion and $\mathbb{E}\sup_{s\le t}|T_x\xi_s|<\infty$ for each x in M. Let ϕ be a closed bounded C^2 q-form with both $\hat\delta\phi$ and $\mathcal{A}\phi$ bounded. Then $P_t\phi$ differs from ϕ by an exact form provided $dP_s\hat\delta\phi = P_s d\hat\delta\phi$, $0\le s\le t$.*

Proof. The conditions allow us to apply Itô's formula and take expectations to obtain

$$P_t\phi = \phi + \int_0^t P_s\mathcal{A}\phi ds.$$

However by Theorem 2.3.1

$$\begin{aligned}\int_0^t P_s\mathcal{A}\phi ds &= \int_0^t P_s(d\hat\delta\phi)ds\\ &= d\int_0^t P_s(\hat\delta\phi)ds\end{aligned}$$

as required. ■

Remark: The condition $dP_s = P_s d$ holds by differentiation under the expectation sign if M is compact and somewhat more generally see [Li92], [Li94b], and [EL94]. The fact that $P_t\phi$ is cohomologous to ϕ for closed q-forms on compact M was noted in [Elw92] and on non-compact M under the related hypothesis that the flow is strongly q-complete in [Li94b]Theorem 2.4, using the fact that $\int_\sigma \xi_t^*\phi = \int_{(\xi_t)_*\sigma}\phi$ for any q-simplex σ.

Now assume X has constant rank. Then $\mathcal{A}^0$ is given by

$$\mathcal{A}^0(f) - L_A f = tr\breve\nabla \mathrm{grad}_E f = tr\breve\nabla df = tr\hat\nabla df,$$

but not $\hat\nabla\mathrm{grad}_E f$ in general. Here $\mathrm{grad}_E f$ is the gradient of f with respect to the metric on E induced by X:

$$\langle \mathrm{grad}_E f, u\rangle = df(u), \qquad \text{any } u\in E. \tag{2.4.1}$$

Set

$$A^X = \frac{1}{2}\sum_1^m \nabla_{X^j}X^j + A.$$

In the nondegenerate case $\mathcal{A}^0 = \frac12\Delta + L_{A^X}$ and for gradient Brownian systems $\mathcal{A}^q = \frac12\Delta + L_A$, as can be seen in section 2.3, see also [Elw92], [Kus88]. In fact $\mathcal{A}^q = \frac12\Delta + L_A$, each q, for gradient Brownian systems.

The main theorem of this section is the following Weitzenbock formula (we use the notation in Appendix B for the linear operators $d\Lambda$ and $\delta^2\Lambda$):

Theorem 2.4.2 (Weitzenbock formula) *Suppose X has constant rank. Let $\breve{\nabla}$ be the associated connection on its image bundle E with adjoint $\hat{\nabla}$, $\breve{R}$ its curvature tensor and*

$$\breve{R}^q\phi = -\phi\left((d\Lambda)^q\left(\sum_{p=1}^m \breve{R}(X^p,-)(X^p)\right) + \sum_{p=1}^m \delta^2\Lambda(\breve{\nabla}.X^p)(-)\right) \tag{2.4.2}$$

the so called Weitzenbock term. Then for $q > 1$,

$$\begin{aligned} \mathcal{A}^q\phi &= \frac{1}{2}tr_E\hat{\nabla}.(\hat{\nabla}.\phi) + L_A\phi - \frac{1}{2}\breve{R}^q\phi & (2.4.3)\\ &= \frac{1}{2}tr_E\hat{\nabla}.(\hat{\nabla}.\phi) + \hat{\nabla}_A\phi + \phi\left((d\Lambda)^q(\breve{\nabla}A)\right) - \frac{1}{2}\breve{R}^q\phi & (2.4.4)\end{aligned}$$

and

$$\breve{R}^q\phi = \phi\left((d\Lambda)^q\left(\breve{Ric}^{\#}\right)\right) + 2\sum_{1\le i<k\le n, 1\le j<l\le dim(E)} \breve{R}_{ikjl}(a^i)^*(a^k)^*a^j a^l\,\phi. \tag{2.4.5}$$

Here $\breve{R}$ is the curvature tensor for $\breve{\nabla}$ as below, with

$$\breve{R}_{ikjl} = \left\langle \breve{R}(e_i,e_k)e_l, e_j\right\rangle, \qquad 1\le i,k\le n, 1\le j,\ell\le dim(E)$$

Also $\breve{Ric}^{\#}: TM \to E$ is defined by

$$\breve{Ric}^{\#}(v) = \sum_{j=1}^m \breve{R}(v, X^j(x))X^j(x), \qquad v\in T_xM,$$

and a^i, $(a^i)^*$, $1\le i\le n$ are the annihilation and creation operators corresponding to some base $e_1,\dots,e_n$ of T_xM which extends an orthonormal base $e_1,\dots,e_{\dim(E)}$ of E_x.

In the nondegenerate case $\left\langle \breve{Ric}^{\#}(v_1), v_2\right\rangle$ is the Ricci curvature $\breve{Ric}(v_1,v_2)$ of $\breve{\nabla}$.

A. Let $\breve{R}: TM\times TM \to L(E;E)$ be the curvature tensor for $\breve{\nabla}$ on E given by

$$\breve{R}(U,V)W = \breve{\nabla}_U\left(\breve{\nabla}_V W\right) - \breve{\nabla}_V\left(\breve{\nabla}_U W\right) - \breve{\nabla}_{[U,V]}W$$

for all vector fields U, V and E-valued vector fields W. Then we have an expression for $\breve{R}$ in terms of X:

$$\breve{R}(u,v;z,w) := \left\langle \breve{R}(u,v)w, z\right\rangle = -\sum_1^m\left\langle \breve{\nabla}_u X^i\wedge\breve{\nabla}_v X^i, w\wedge z\right\rangle, \tag{2.4.6}$$

See Proposition C.4.

Lemma 2.4.3 *Suppose X has constant rank. For a differential q-form ϕ,*

$$\sum_{p=1}^{m} L_{X^p} L_{X^p}\phi$$
$$= tr_E \hat{\nabla}.(\hat{\nabla}.\phi) + \sum_{p=1}^{m} \phi\left((d\Lambda)^q\left(\breve{R}(X^p,-)(X^p)\right)\right) + \sum_{p=1}^{m} \phi\left(\delta^2\Lambda(\breve{\nabla}X^p(-))\right),$$

with the convention $\phi\left(\delta^2\Lambda(\breve{\nabla}X^p(-))\right)$ vanishes for $q = 1$ and so the infinitesimal generator is given by

$$\begin{aligned} \mathcal{A}^q\phi &= \tfrac{1}{2} tr_E \hat{\nabla}.(\hat{\nabla}.\phi) + L_A\phi + \tfrac{1}{2}\textstyle\sum_{p=1}^{m} \phi\left((d\Lambda)^q\left(\breve{R}(X^p,-)(X^p)\right)\right) \\ &+ \tfrac{1}{2}\textstyle\sum_{p=1}^{m} \phi\left(\delta^2\Lambda(\breve{\nabla}X^p(-))\right). \end{aligned} \tag{2.4.7}$$

Proof. Let ϕ be a q-form, and v a q-vector. By (2.3.7),

$$\begin{aligned} L_{X^p}(L_{X^p}\phi)(v) &= \hat{\nabla}_{X^p}\left(\hat{\nabla}_{X^p}\phi\right) + \hat{\nabla}_{X^p}\phi\left((d\Lambda)^*(\breve{\nabla}X^p)(v)\right) \\ &+ \phi\left((d\Lambda^*(\hat{\nabla}_{X^p}(\breve{\nabla}X^p))(v)\right) + \hat{\nabla}_{X^p}\phi\left((d\Lambda)^*(\breve{\nabla}X^p)(v)\right) \\ &+ \phi(d\Lambda^*(\nabla X^p) \circ d\Lambda^*(\breve{\nabla}X^p)(v)). \end{aligned}$$

Summing up from 1 to m, the second, and the fourth term disappear by the defining property. In Appendix B setting $A(-) = \breve{\nabla}_{-}X^p$ so that $A^2(-) = \breve{\nabla}_{\breve{\nabla}_{-}X^p}X^p$ to obtain,

$$\begin{aligned} \sum_{p=1}^{m} L_{X^p}(L_{X^p}\phi)(v) &= \mathrm{tr}_E\hat{\nabla}.\hat{\nabla}.\phi(v) + \sum_{p=1}^{m} \phi\left((d\Lambda)^*(\hat{\nabla}_{X^p}(\breve{\nabla}X^p))(v)\right) \\ &+ \sum_{p=1}^{m} \phi\left((\delta^2\Lambda)^*(\breve{\nabla}X^p)(v)\right) + \sum_{p=1}^{m} \phi\left((d\Lambda)^*\left(\breve{\nabla}_{\breve{\nabla}X^p}X^p\right)(v)\right) \end{aligned}$$

Observe $\sum_{p=1}^{m} \hat{\nabla}_{X^p}(\breve{\nabla}.X^p)(v) = \sum_{p=1}^{m} \breve{\nabla}_{X^p}(\breve{\nabla}.X^p)(v)$, again by the defining property of the connection $\breve{\nabla}$.

Let $U \in \Gamma(E)$. Then since $\sum_{p=1}^{m} \breve{\nabla}_U\left(\breve{\nabla}_{X^p}X^p\right) = 0$,

$$\begin{aligned} \sum_{p=1}^{m}[\breve{\nabla}_{\breve{\nabla}_U X^p}X^p + \hat{\nabla}_{X^p}(\breve{\nabla}_U X^p)] &= \sum_{p=1}^{m}[\breve{\nabla}_{\breve{\nabla}_U X^p}X^p + \breve{\nabla}_{X^p}(\breve{\nabla}_U X^p)] \\ &= \sum_{p=1}^{m}\left[-\breve{\nabla}_{[X^p,U]}X^p + \breve{\nabla}_{\breve{\nabla}_{X^p}U}X^p - \breve{\nabla}_{T(X^p,U)}X^p\right] \end{aligned}$$

$$+\sum_{p=1}^{m}\breve{\nabla}_{X^p}(\breve{\nabla}_U X^p)-\sum_{p=1}^{m}\breve{\nabla}_U\left(\breve{\nabla}_{X^p}X^p\right)$$
$$=\sum_{p=1}^{m}\breve{R}(X^p,U)X^p.$$

The stated result now follows. ■

Proof of Theorem 2.4.2. Let a^i and $(a^i)^*$ be respectively the annihilation and creation operators as given in Appendix B. By Corollary B.3 in the Appendix,

$$\phi\left(\delta^2\Lambda(\breve{\nabla}.X^p(-)\right)=-2\sum_{1\leq i<k\leq n,1\leq j<l\leq dim(E)}\langle\breve{\nabla}_{e_i}X^p\wedge\breve{\nabla}_{e_k}X^p,e_j\wedge e_l\rangle(a^i)^*(a^k)^*a^ja^l\phi$$

and by (2.4.6),

$$\breve{R}_{ikjl}=\sum_{p=1}^{m}\left\langle\breve{\nabla}_{e_i}X^p\wedge\breve{\nabla}_{e_k}X^p,e_j\wedge e_l\right\rangle,$$

giving

$$\sum_{p=1}^{m}\phi\left(\delta^2\Lambda\left(\breve{\nabla}.X^p(-)\right)\right)=-2\sum_{1\leq i<k\leq n,1\leq j<l\leq dim(E)}\breve{R}_{ikjl}(a^i)^*(a^k)^*a^ja^l\phi.$$

The theorem follows. ■

Remark: The last term agrees with the term $\tilde{R}_{(4)}$ in [CFKS87] (on page 260) if $\breve{\nabla}$ is the Levi-Civita connection, after applying Bianchi's identity. Indeed: in the Levi-Civita case:

$$\begin{aligned}\tilde{R}_{(4)}&\equiv&-\sum_{i,j,k,l=1}^{n}R_{klji}(a^i)^*(a^k)^*a^ja^l\\&=&-\sum_{i,j,k,l=1}^{n}R_{ijkl}(a^i)^*(a^k)^*a^ja^l\\&=&-\sum_{1\leq i<k}^{n}\sum_{j,l=1}^{n}[R_{ijkl}-R_{kjil}(a^i)^*(a^k)^*a^ja^l\\&=&-\sum_{i<k}\sum_{j<l}[R_{ijkl}-R_{kjil}-R_{ilkj}+R_{klij}](a^i)^*(a^k)^*a^ja^l.\end{aligned}$$

However by Bianchi's identity:

$$R_{ijkl}-R_{kjil}=-R_{kijl}=R_{ikjl},$$
$$-R_{ilkj}+R_{klij}=-R_{iklj}=R_{ikjl}$$

and thus

$$\tilde{R}_{(4)}=-2\sum_{i<k}\sum_{j<l}R_{ikjl}(a^i)^*(a^k)^*a^ja^l.$$

Note 3B In terms of the Gaussian field in the non-degenerate case the second term in $\breve{R}^q$ can be written

$$2\mathbb{E}\left\langle \breve{R}(W^{(1)}, W^{(2)})W^{(3)}, W^{(4)}\right\rangle (a^{W(1)})^*(a^{W(2)})^* a^{W(3)} a^{W(4)}$$

where $W^{(1)}, \ldots, a^{W(4)}$ are independent copies of W and $(a^{W(1)})^*$ etc. are the corresponding creation and annihilation operators, e.g. $(a^{W(1)})^*\phi(v) = \phi(a^{W(1)} \wedge v)$.

B. Finally we give a formula for the adjoint of $\breve{R}^q$ in terms of the vector fields $X^1, \ldots, X^m$, extending those given in [Elw92]. This result is to be used in section 5 for analyzing the dynamical properties of the stochastic flows. Let $(\breve{R}^q)^* : \wedge^q T_x M \to \wedge^q T_x M$ be given by

$$\phi\left((\breve{R}^q)^*(v)\right) = (\breve{R}^q\phi)(v). \tag{2.4.8}$$

Lemma 2.4.4 *Let $S_t^i : M \to M$ be the solution flow of the vector fields X^i, $i = 0, 1, \ldots, m$. Here X^0 is taken to be A. If $v_0 \in \wedge^q T_{x_0} M$, then*

$$\sum_1^m \frac{\hat{D}^2}{\partial r^2} \wedge^q \left(TS_r^i\right)(v_0)\Big|_{r=0} = -\left(\breve{R}_x^q\right)^*(v_0). \tag{2.4.9}$$

Proof. In fact this follows from Itô's formula for the flow of

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt,$$

c.f. [Elw88], if ϕ is a q-form, then

$$\begin{aligned}
\phi(v_T) &= \phi(v_0) + \int_0^T \sum_1^m \frac{d}{dt}\left[\phi_{S_t^i(x_r)}\left(\wedge^q TS_t^i(v_r)\right)\right]\Big|_{t=0} dB_r^i \\
&\quad + \frac{1}{2}\int_0^T \sum_1^m \frac{d^2}{dt^2}\left[\phi_{S_t^i(x_r)}\left(\wedge^q TS_t^i(v_r)\right)\right]\Big|_{t=0} dr \\
&\quad + \int_0^T \sum_1^m \frac{d}{dt}\left[\phi_{S_t^0(x_r)}\left(\wedge^q TS_t^0(v_r)\right)\right]\Big|_{t=0} dr.
\end{aligned}$$

From here we conclude the infinitesimal generator on q forms in terms of $\hat{\nabla}$ is given by:

$\mathcal{A}\phi(v_0) = \frac{1}{2}\phi\left(\sum_i \frac{\hat{D}^2}{\partial r^2} \wedge^q TS_r^i(v_0)\Big|_{r=0}\right) + \phi\left(\frac{\hat{D}}{\partial r} \wedge^q TS_r^0(v_0)\Big|_{r=0}\right)$ plus first order terms and second order terms, since

$$\frac{d}{dt}\left[\phi_{S_t^i(x_r)}\left(\wedge^q TS_t^i(v_r)\right)\right] = \hat{\nabla}_{X^i(S_t^i)}\phi\left(\wedge^q TS_t^i(v_r)\right) + \phi\left(\frac{\hat{D}}{\partial t} \wedge^q TS_t^i(v_r)\right).$$

Compare this with (2.3.4) to obtain the required (2.4.9). ∎

2.5 On the infinitesimal generator

It is well known that the Laplacian is a symmetric operator. We consider the question of the self-adjointness of the operators $\mathcal{A}^q$, coming from a general s.d.e. rather than a gradient Brownian system. We conclude, in the case of a gradient drift, $\mathcal{A}^q$ minus a zero order term is symmetric provided the associated L-W connection is torsion skew symmetric (Corollary 2.5.6).

2.5.1 Example

There is an important class of examples for which $\mathcal{A}^q$ is self adjoint although not the de Rham-Hodge (or Witten) Laplacian and here we do not even need to assume regularity of our stochastic differential equation.

Let Γ_0 refer to C^∞ sections with compact support.

Example 2.5.2A. *Suppose the X^e are Killing fields for a complete Riemannian metric $\langle,\rangle^\sim$ on M. Assume $A \equiv 0$. Let μ be the volume element for $\langle,\rangle^\sim$. Then $\mathcal{A}^q$ with domain $\Gamma_0(\wedge^q T^*M)$ using the measure μ and inner product induced by $\langle,\rangle^\sim$ is symmetric.*

Proof. Let $\phi, \psi \in \Gamma_0 \wedge^q T^*M$. Then

$$\begin{aligned}&\int_M \langle P_t\phi, \psi\rangle^\sim \, d\mu = \int_M \langle E\xi_t^*\phi(-), \psi\rangle^\sim \\ &= \mathbb{E}\int_M \left\langle \phi_{\xi_t(x)}(T_x\xi_t-, \dots, T_x\xi_t-), \psi_x(-,\dots,-)\right\rangle^\sim \mu(dx).\end{aligned}$$

Let $f_1, \dots, f_n$ be measurable vector fields forming an orthonormal base for T_xM at each x of M. Our hypothesis implies that

$$\tilde{f}_i(x) := T_{\xi_t^{-1}(x)}\xi_t\left(f_i\left(\xi_t^{-1}(x)\right)\right), \qquad i = 1 \text{ to } n,$$

also gives an orthonormal basis. (It also implies that we have a flow $\xi.$ of diffeomorphisms either from the fact that we can lift our equation to the, finite dimensional, isometry group of M, $\langle,\rangle^\sim$, see [Kun80], or by [Li94b] since $T\xi_t$ consists of isometries.)

$$\begin{aligned}&\int_M \langle P_t\phi, \psi\rangle^\sim \, d\mu \\ &= \mathbb{E}\int_M \sum \phi_{\xi_t(x)}\left(T_x\xi_t(f_{\ell_1}(x)), \dots, T_x\xi_t(f_{\ell_q}(x))\right)\cdot \\ &\quad \psi_x\left(f_{\ell_1}(x), \dots, f_{\ell_q}(x)\right)\,\mu(dx), \qquad \text{summed over all } \ell_1 < \ell_2 < \ldots < \ell_q, \\ &= \mathbb{E}\int_M \sum \phi_y\left(T_{\xi_t^{-1}(y)}\xi_t(f_{\ell_1}(\xi_t^{-1}(y))), \dots, T_{\xi_t^{-1}(y)}\xi_t(f_{\ell_q}(\xi_t^{-1}(y)))\right) \\ &\quad \psi_{\xi_t^{-1}(y)}\left(f_{\ell_1}(\xi_t^{-1}(y)), \dots, f_{\ell_q}(\xi_t^{-1}(y))\right)\,\mu(dy)\end{aligned}$$

$$= \mathbb{E}\int_M \sum \phi_y\left(\tilde{f}_{\ell_1}(y),\ldots,\tilde{f}_{\ell_q}(y)\right)\cdot$$
$$\psi_{\xi_t^{-1}(y)}\left(T\xi_t^{-1}\tilde{f}_{\ell_1}(y),\ldots,T\xi_t^{-1}\tilde{f}_{\ell_q}(y)\right)\mu(dy),$$

since ξ_t preserves μ. Now, since $A \equiv 0$, the law of $\{\xi_t^{-1} : 0 \leq t < \infty\}$ on the group of diffeomorphisms of M is the same as that of $\{\xi_t : 0 \leq t < \infty\}$, [LJW84], [Kun80], [CE83]. Thus we have

$$\begin{aligned}
&\int_M \langle P_t\phi,\psi\rangle^{\sim}\, d\mu \\
= &\ \mathbb{E}\int_M \sum \phi_y\left(\tilde{f}_{\ell_1}(y),\ldots,\tilde{f}_{\ell_q}(y)\right)\ \psi_{\xi_t(y)}\left(T\xi_t\tilde{f}_{\ell_1}(y),\ldots,T\xi_t\tilde{f}_{\ell_q}(y)\right)\mu(dy) \\
= &\ \int_M \langle\phi,P_t\psi\rangle^{\sim}\, d\mu
\end{aligned}$$

as before. Thus P_t is symmetric on $\Gamma_0(\wedge^q TM)$, consequently so is its generator $\mathcal{A}^q$, [RS80].

2.5.2 Symmetricity of the generator $\mathcal{A}^q$

A. Consider the regular case $X : \underline{\mathbb{R}}^m \to E \subset TM$ for E a subbundle with $A \in \Gamma(E)$. Recall the definition of $\mathrm{tr}_E\hat{\nabla}.\psi(\cdot)$ in §2.4 C. Set

$$\Psi\phi = \phi\left((d\Lambda)^q(\breve{\nabla}A)\right) - \frac{1}{2}\breve{R}^q\phi,$$

where $\breve{R}^q$ is as given in (2.4.5). In the following which will allow us to deduce symmetricity of $\mathcal{A}^q - \Psi^q$ from that of $\mathcal{A}^0$, part IV is essentially a finite dimensional specialization of results by Bogachev &Roeckner [BR95] and by H. Long [Lon].

Theorem 2.5.1 *Let $\Psi(x) \in (\wedge^q T^*M;\wedge^q T^*M)$ be the zero order term in (2.4.4) and μ be a Borel measure on M. Assume*
(i) the s.d.e. (1.2.5) is regular with $A(x) \in E_x$ for each x,
(ii) the adjoint semi-connection $\hat{\nabla}$ for (1.2.5) is adapted to a Riemannian metric $\langle,\rangle^{\tilde{}}$ on TM.

Then the following are equivalent:
I. The generator $\mathcal{A}^0$ with domain restricted to $C_0^\infty(M;R)$ is symmetric on $L^2(TM)$.
II. For a given $q \in \{1,2,\ldots,n-1\}$, $\mathcal{A}^q - \Psi$ with domain $\Gamma_0(\wedge^q T^*M)$ is symmetric on the space of L^2 q-forms using the measure μ and the inner product induced on $\wedge^q T^*M$ by $\langle,\rangle^{\tilde{}}$.

Proposition 2.5.2 *Under the conditions of Theorem 2.5.1, statements I and II are equivalent to each of the following:*
III. The adjoint of $grad_E$ with domain $C_0^\infty(M;R)$ as an operator from $L^2(M,\mu;R)$ to L^2 sections of E using the metric product $\langle,\rangle^X$ is given by

$$(grad_E)^*(U) = -tr_E\breve{\nabla}.U - 2\langle A,U\rangle^X,$$

μ almost all $x \in M$, for $U \in \Gamma_0(E)$.
III'. For $f \in C_0^\infty(M;R)$ let $d_E f$ be the restriction of df to E. Then as an operator from $L^2(M,\mu;R)$ to L^2 section of E^* using $\langle , \rangle^X$ the adjoint of d_E is given by

$$d_E^* \phi = \hat{\delta}\phi - 2\phi(A(\cdot)).$$

IV. There exists a 'logarithmic derivative' for μ

$$\alpha : R^m \to L^0(M;R),$$

a linear map of R^m into the space of measurable functions on M such that for each $e \in R^m$,

$$\int_M f L_{X^e} g \; d\mu = -\int_M \left(L_{X^e} f + \alpha(e) f\right) \, g \; d\mu \tag{2.5.1}$$

all f, g smooth with compact support for fg, and $A = \sum_{j=1}^m \alpha(e_j) X^j$.

Remarks: (i). If $\nu = \mathrm{e}^{2h}\mu$ is an equivalent measure to a Borel measure μ, then the adjoint of $\hat{\nabla}$ with respect to ν is given by: $\hat{\nabla}\psi + 2\psi(\mathrm{grad}_E h)$ where $\hat{\nabla}$ is the adjoint using μ.

(ii). Example 2.3.5 gives an example when $\mathcal{A}^q$ is symmetric when restricted to closed q-forms, using the given metric on TM and the measure $\exp\left([\frac{8(q-1)}{n-1} + 4]h(x)\right) dx$. In many geometrical situations it is the behaviour on closed forms which is important.

B. Here we give the proof of Theorem 2.5.1 and Proposition 2.5.2. We first give three lemmas.

Lemma 2.5.3 *Let A_1 be a section of E and μ be a Borel measure on M. The equation*

$$\int_M \left(tr_E \hat{\nabla}.\phi(\cdot) + 2\phi(A_1)\right) \mu(dx) = 0, \tag{2.5.2}$$

holds for all $\phi \in \Gamma_0(E^*)$ if and only if for all compactly supported q-forms θ and ψ in $\Gamma_0 L(E, \wedge^q T^*M)$.

$$\int_M \left(tr_E \hat{\nabla}. \left(\langle \psi(\cdot), \theta\rangle^\sim\right)(x) + 2\left\langle \psi(A_1)(\cdot), \theta\right\rangle_x^\sim \right) \mu(dx) = 0 \tag{2.5.3}$$

In particular if the equation (2.5.3) holds for some $q \in \{1, \ldots, n-1\}$ it holds for all such q.

Proof. Clearly (2.5.3) implies (2.5.2). Now assume that (2.5.3) holds for some q. Let $\phi \in \Gamma_0(E^*)$. Take $X^1, \ldots, X^q$ to obtain the elements of $\Gamma_0 L(E; \wedge^q T^*M)$

$$\psi(\cdot)(-) = \phi(\cdot)\left\langle X^1 \wedge \ldots \wedge X^q, -\right\rangle^\sim.$$

Set $\theta(-) = \langle X^1 \wedge \ldots \wedge X^q, -\rangle^\sim$. Then (2.5.3) reduces to (2.5.2) which in turn implies (2.5.3) for all q. ∎

Lemma 2.5.4 *Let q belong to $\{1,\dots,n-1\}$ and A_1 be a section of E. Assume $\hat{\nabla}$ is adapted to a Riemannian metric $\langle,\rangle^\sim$ on TM. Let μ be a Borel measure on M. Then the adjojnt*

$$\hat{\nabla}^*:\Gamma_0 L(E;\wedge^q T^*M)\to\Gamma_0\Lambda^q T^*M$$

of $\hat{\nabla}$, in the sense of the L^2 spaces using μ, the inner product $\langle,\rangle^\sim$ on TM, $\langle,\rangle$ on E and corresponding inner product on $\wedge^q T^*M$ and $L(E;\wedge^q T^*M)$, is given by

$$\hat{\nabla}^*\psi(-)=-tr_E\hat{\nabla}.\psi(\cdot)(-)-2\psi(A_1)(-),\tag{2.5.4}$$

if and only if for all $\phi\in\Gamma_0(E^*)$

$$\int_M\left(tr_E\hat{\nabla}.\phi(\cdot)+2\phi(A_1)\right)\mu(dx)=0.\tag{2.5.5}$$

Proof. Let $\tilde{\psi}:TM\to\wedge^q T^*M$ restrict to ψ on E. For $\theta\in\Gamma\wedge^q T^*M$, consider the 1-form ψ^θ defined by

$$\psi^\theta(v)=\left\langle\tilde{\psi}(v)(-),\theta(-)\right\rangle_x^\sim,\qquad v\in T_xM.$$

Let S_t^j denote the (possibly partial) flow of X^j, for some basis $e_1,\dots,e_m$ of $\mathbb{R}^m$ which we shall choose to be *adapted* at a point x_0 of M in the sense that $\{e_1,\dots,e_p\}$ span $KerX(x_0)^\perp$. Recall from Lemma 1.3.4 that $T_{x_0}S^j$ is the parallel translation $\hat{//}_\cdot$ along $S_\cdot^j(x_0)$. Therefore for $v\in T_{x_0}M$,

$$\begin{aligned}
&\left\langle\tilde{\psi}_{S_t^j(x_0)}\left(TS_t^j(v)\right)(-),\theta_{S_t^j(x_0)}(-)\right\rangle_{S_t^j(x_0)}^\sim\\
=\;&\left\langle\tilde{\psi}_{S_t^j(x_0)}\left(TS_t^j(v)\right)(TS_t^j(-)),\theta_{S_t^j(x_0)}(TS_t^j)(-)\right\rangle_{S_t^j(x_0)}^\sim\\
=\;&\left\langle\left(S_t^j\right)^*(\tilde{\psi})(v)(-),\left(S_t^j\right)^*\theta(-)\right\rangle_{x_0}^\sim
\end{aligned}$$

Thus by (2.3.1), the definition of $\hat{\delta}$:

$$\begin{aligned}
-\hat{\delta}\psi^\theta(x_0)&=\sum_j\iota_{X^j(x_0)}L_{X^j}\left\langle\tilde{\psi}(\cdot)(-),\theta(-)\right\rangle^\sim\\
&=\sum_j\iota_{X^j(x_0)}\frac{d}{dt}\left\langle\left(S_t^j\right)^*(\tilde{\psi})(\cdot)(-),\left(S_t^j\right)^*\theta(-)\right\rangle_{x_0}^\sim\Bigg|_{t=0}\\
&=\sum_j\iota_{X^j(x_0)}\left\{\left\langle L_{X^j}(\tilde{\psi})(\cdot)(-),\theta(-)\right\rangle_{x_0}^\sim+\left\langle\tilde{\psi}(\cdot)(-),L_{X^j}\theta(-)\right\rangle_{x_0}^\sim\right\}\\
&=\left\langle\sum_j\hat{\nabla}_{X^j(x_0)}(\tilde{\psi})(X^j(x_0))(-),\theta(-)\right\rangle_{x_0}^\sim
\end{aligned}$$

$$+\sum_j \left\langle \tilde{\psi}(X^j(x_0))(-), \hat{\nabla}_{X^j(x_0)}\theta(-)\right\rangle^{\sim}_{x_0}$$
$$= \left\langle \mathrm{tr}_E\hat{\nabla}.\tilde{\psi}(\cdot)(-), \theta(-)\right\rangle^{\sim}_{x_0} + \left\langle \psi_{x_0}, (\hat{\nabla}\theta)_{x_0}\right\rangle^{\sim}_{x_0}.$$

Then for any vector field A_1,

$$\begin{aligned}\int_M \left\langle \psi, \hat{\nabla}\theta\right\rangle^{\sim} d\mu &= \int_M \left\{-\hat{\delta}\psi^\theta(x) - 2\psi^\theta(A_1(x))\right\}\mu(dx) \\ &\quad - \left\langle \mathrm{tr}_E\hat{\nabla}.\psi(\cdot)(-) - 2\psi(A_1(x))(-), \theta\right\rangle^{\sim}_x \mu(dx).\end{aligned}$$

Thus

$$\hat{\nabla}^*\psi = -\mathrm{tr}_E\hat{\nabla}.\psi(\cdot) - 2\psi(A_1(\cdot))$$

if and only if for all θ,

$$\int_M \left(\hat{\delta}\psi^\theta + 2\psi^\theta(A_1)\right)\mu(dx) = 0. \tag{2.5.6}$$

This is (2.5.3). But, by Lemma 2.5.3, (2.5.3) is equivalent to (2.5.5). ∎

Remark: In Lemma 2.5.4 we used two metrics and we now show this is essential unless we have torsion skew symmetricity:

Let $\tilde{\nabla}'$ be the adjoint of a metric connection $\tilde{\nabla}$. Let θ be a one form, then

$$\tilde{\nabla}'_v\theta = \tilde{\nabla}_v\theta + \theta(\tilde{T}(v,-)), \tag{2.5.7}$$

and

$$\mathrm{trace}\tilde{\nabla}'.\theta(\cdot) = \mathrm{trace}\tilde{\nabla}.\theta(\cdot).$$

So condition (2.5.5) does not change if we use the adjoint $\tilde{\nabla}'$ replacing $\tilde{\nabla}$. Consequently if both connections are metric connections (for the same metric) and the adjoint of one of the covariant derivatives is given in the form of (2.5.4) then so is the other. The converse is also true as seen in the following lemma:

Lemma 2.5.5 *Let $\tilde{\nabla}$ be a metric connection with $\tilde{\nabla}^*$ given by*

$$\tilde{\nabla}^*\psi = -\mathit{trace}\tilde{\nabla}.\psi(\cdot)(-) - 2\psi(A_1)(-)$$

for 1-forms. Then the adjoint connection has $(\tilde{\nabla}')^*$ given by

$$(\tilde{\nabla}')^*\psi = -\mathit{trace}\tilde{\nabla}'.\psi(\cdot)(-) - 2\psi(A_1)(-)$$

if and only if $\tilde{\nabla}$ is torsion skew symmetric.

Proof. We only need to prove that if both $\tilde{\nabla}^*$ and $(\tilde{\nabla}')^*$ are given in the prescribed form, then $\tilde{\nabla}'$ must be a metric connection.

Without loss of generality assume $A \equiv 0$. Take $\psi \in \Gamma L(TM, T^*M)$. Then

$$\mathrm{tr}\tilde{\nabla}'\psi = \sum_i \tilde{\nabla}_{X^i}\psi(X^i) + \sum_i \psi(X^i)(\tilde{T}(X^i, -)).$$

Thus for θ a 1-form,

$$\begin{aligned}\left\langle \tilde{\nabla}'\theta, \psi \right\rangle &= -\left\langle \theta, \mathrm{tr}\tilde{\nabla}'_{\cdot}\psi(\cdot) \right\rangle \\ &= -\left\langle \theta, \mathrm{tr}\tilde{\nabla}_{\cdot}\psi(\cdot) \right\rangle - \left\langle \theta, \sum_i \psi(X^i)(\tilde{T}(X^i, -)) \right\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}\left\langle \tilde{\nabla}'_{\cdot}\theta, \psi \right\rangle &= \left\langle \tilde{\nabla}_{\cdot}\theta, \psi \right\rangle + \left\langle \theta(\tilde{T}(\cdot, -)), \psi \right\rangle \\ &= -\left\langle \theta, \mathrm{tr}\tilde{\nabla}_{\cdot}\psi(\cdot) \right\rangle + \left\langle \theta(\tilde{T}(\cdot, -)), \psi \right\rangle,\end{aligned}$$

giving that for all such θ and ψ,

$$\left\langle \theta(\tilde{T}(\cdot, -)), \psi \right\rangle + \left\langle \theta, \sum_i \psi(X^i)(\tilde{T}(X^i, -)) \right\rangle = 0. \tag{2.5.8}$$

Take $\theta = \langle U, - \rangle$ and $\psi(\cdot)(-) = \langle V, . \rangle \langle W, - \rangle$ for vector fields U, V and W. Then

$$\begin{aligned}\left\langle \theta(\tilde{T}(\cdot, -)), \psi \right\rangle &= \int_M \sum_{i,j} \left\langle \theta(\tilde{T}(X^i, X^j), \psi(X^i)(X^j) \right\rangle \mu(dx) \\ &= \int_M \left\langle U, \tilde{T}(V, W) \right\rangle \mu(dx),\end{aligned}$$

and

$$\begin{aligned}&\left\langle \theta, \sum_i \psi(X^i)(\tilde{T}(X^i, -)) \right\rangle = \int_M \sum_{i,j} \theta(X^j)\psi(X^i)(\tilde{T}(X^i, X^j))\, \mu(dx) \\ &= \int_M \sum_{i,j} \langle U, X^j \rangle \langle V, X^i \rangle \left\langle W, \tilde{T}(X^i, X^j) \right\rangle \mu(dx) \\ &= \int_M \left\langle W, \tilde{T}(V, U) \right\rangle.\end{aligned}$$

So

$$\int_M \left(\left\langle U, \tilde{T}(V, W) \right\rangle + \left\langle W, \tilde{T}(V, U) \right\rangle \right) \mu(dx) = 0$$

and $\tilde{\nabla}$ is torsion skew symmetric. ∎

Proof of Theorem 2.5.1 and Proposition 2.5.2.

(1). Assume I. Let $f \in C_0^\infty(M;R)$. Then taking a suitable sequence $\{g_n\}$ in $C_0^\infty(M;[0,1])$ converging to 1 we have

$$\int_M \mathcal{A}(f)d\mu = 0.$$

i.e.

$$\int_M \left(\frac{1}{2}tr_E\breve{\nabla}.(\text{grad}_E f) + \langle A, \text{grad}_E f\rangle^X\right) d\mu = 0. \tag{2.5.9}$$

For any $\lambda \in C_0^\infty(M;R)$ we see

$$\begin{aligned}
&\int_M \left\{\frac{1}{2}tr_E\breve{\nabla}.(\lambda\text{grad}_E f) + \langle A, \lambda\text{grad}_E f\rangle^X d\mu\right\}\\
= &\int_M \lambda A^0(f)d\mu + \int_M \frac{1}{2}\langle \text{grad}_E\lambda, \text{grad}_E f\rangle^X d\mu
\end{aligned}$$

which is symmetric in λ and f. Since $\int_M \mathcal{A}^0(\lambda f)d\mu = 0$. this gives

$$\int_M \left(\frac{1}{2}tr_E\breve{\nabla}.(\lambda \ \text{grad}_E f) + \langle A, \lambda \ \text{grad}_E f\rangle^X\right) d\mu = 0.$$

Now any $U \in \Gamma_0(E)$ has the form

$$U(x) = \sum_{i=1}^N \lambda^i(x) \ \text{grad}_E f^i(x)$$

where the λ^i and the f^i are in $C_0^\infty(M;R)$; this is true when $E = TM$ using local coordinates and a partition of unity and follows for general E by applying any projection $P_E : TM \to E$. Thus (2.5.9) holds with $\text{grad}_E f$ replaced by an arbitrary element of $\Gamma_0(E)$. Replacing it by λU where $\lambda \in C_0^\infty(M)$ and $U \in \Gamma_0(E)$ yields III. Thus I implies III and similarly, using $\mathcal{A}^0 = -\frac{1}{2}\hat{\delta}d + L_A$, we see I gives III'.

(2). Next assume III and apply it to

$$U(x) = f(x)X^e(x)$$

for $e \in R^m$ and $f \in C_0^\infty(M;R)$ to obtain, for any $g \in C_0^\infty(M;R)$:

$$\begin{aligned}
\int_M dg(fX^e)d\mu = &-\int_M g \left(tr_E\breve{\nabla}.(fX^e) + 2\langle A, fX^e\rangle^X d\mu\right)\\
-&\int_M g(x)\left(df(X^e(x)) + 2\langle Y(x)A(x), e\rangle_{R^m} f(x)\right) d\mu(x)
\end{aligned}$$

giving (2.5.1) with

$$\alpha(e)x = 2\langle Y(x)A(x), e\rangle_{R^m}$$

and hence

$$A(x) = \frac{1}{2}\sum_j \alpha(e_j)X^j(x).$$

(3). Thus III gives rise to IV. Similarly III' gives IV by applying III' to the section of E^* given by

$$v \mapsto f(x)\,\langle X^e(x), v\rangle :$$

or observe that since $\hat{\delta} = -tr\breve{\nabla}$. on 1-forms III and III' are essentially the same.

(4). Next assume IV. Let $\phi \in \Gamma_0(E^*)$. Then for each j, if $\tilde{\phi}$ is a 1-form extending ϕ, (taking $g \equiv 1$)

$$\begin{aligned} 0 &= \int_M \left\{L_{X^j}\left(\phi(X^j)\right) + \alpha(e_j)\phi(X^j)\right\} d\mu \\ &= \int_M \left(\hat{\nabla}_{X^j}\tilde{\phi}(X^j) + \phi\left(\hat{\nabla}_{X^j}X^j\right) + \alpha(e_j)\phi(X^j)\right\} d\mu \end{aligned}$$

giving (2.5.5) with

$$A = A_1 = \sum_{j=1}^m \alpha(e_j)X^j.$$

By Lemma 2.5.4 and the Weitzenbock formula (2.4.4) we see IV implies II.

(4). To show II implies I, take $f, g \in C_0^\infty(M;R)$ and $\phi \in \Gamma_0(\wedge^q T^*M)$. Using $\mathcal{A} = \frac{1}{2}\sum L_{X^j}L_{X^j} + L_A$ we have

$$\begin{aligned} \langle \mathcal{A}(f\phi), g\phi\rangle_{L^2} = \quad & \int_M \mathcal{A}(f)\,g\,\left(|\phi|\tilde{}\right)^2\,d\mu + \int_M fg\,\langle \mathcal{A}\phi, \phi\rangle\tilde{}\,d\mu \\ & +2\int_M \sum_j g\,L_{X^j}f\,\langle L_{X^j}\phi, \phi\rangle\tilde{}\,d\mu. \end{aligned}$$

Let

$$\Psi(x) \in L\left(\wedge^q T^*M, \wedge^q T^*M\right)$$

be the zero order term of $\mathcal{A}^q$. Since $\langle \Psi(f\phi), g\phi\rangle\tilde{}$ is symmetric in f, g statement II implies that this is also symmetric in f, g. If $(|\phi|\tilde{})^2 = 1$ for $x \in$ Supp g then

$$g\sum_j L_{X^j}f\,\langle L_{X^j}\phi, \phi\rangle\tilde{}$$

vanishes identically using (2.3.6), the adaptness of $\hat{\nabla}$ to $\langle,\rangle\tilde{}$ and the defining property of $\hat{\nabla}$. In this case we therefore do have

$$\langle \mathcal{A}f, g\rangle_{L^2} = \langle f, \mathcal{A}g\rangle_{L^2.}$$

However we can always find an open cover $\{U_i\}_{i=0}^\infty$ of M with a partition of unity $\{\lambda_i\}_{i=0}^\infty$ subordinate to an open $\{U_i\}_{i=0}^\infty$ and with a q-form ϕ_i in $\Gamma_0(\wedge^q T^*M)$ with $|\phi_i\lceil_x = 1$ for $x \in U_i$. Then for any $g \in C_0^\infty(M;\mathbb{R})$,

$$\langle \mathcal{A}f, g\rangle_{L^2} = \sum_i \langle \mathcal{A}f, \lambda_i g\rangle_{L^2} = \sum_i \langle f, \mathcal{A}(\lambda_i g)\rangle_{L^2} = \langle f, \mathcal{A}g\rangle_{L^2}$$

proving I, and completing the proof. ∎

C. As an application we return to our stochastic differential equation assuming nondegeneracy. Let $\langle , \rangle$ be the induced metric. Denote by dx the Riemannian volume measure and ∇ the Levi-Civita connection. Recall that the infinitesimal generator $\mathcal{A}^0$ for the s.d.e. on functions is symmetric if and only if the drift term $A^X = \frac{1}{2}\sum \nabla_{X^j} X^j + A$ is gradient, e.g. see [IW89]. Thus in the torsion skew symmetric case, since then $A^X = A$, we only need to consider A of the form ∇h.

Corollary 2.5.6 *Let h be a smooth function. Consider a nondegenerate s.d.e.:*

$$dx_t = X(x_t) \circ dB_t + \nabla h(x_t)dt.$$

Suppose the associated connection $\breve{\nabla}$ is torsion skew symmetric. Then $\mathcal{A}^q - \Psi$ is symmetric with respect to the measure $e^{2h}dx$ for dx the volume element of the Riemannian metric induced on TM.

Example 2.5.2C. Let $M = G$ be a Lie group and (1.2.5) a left invariant s.d.e. as in §1.3, with μ_R right invariant Haar measure. Then the hypotheses of the theorem are satisfied if $A \equiv 0$ with $\tilde{\langle\ \rangle}$ any right invariant metric which agrees with that given on E_{id} by $X(id)$. (This is a special case of Example 2.5.2A.) If left invariant Haar measure μ_L is used then we must take A to be the logarithmic E-derivative of the Radon-Nikodym derivative $\frac{d\mu_L}{d\mu_R}$.

Finally we observe that if $A = grad\, h$. *Then $(d\Lambda)^q\breve{\nabla}A$ is symmetric for all q if and only if $\left\langle \breve{T}(u,v), \nabla h \right\rangle = 0$, i.e. ∇h is orthogonal to the image of T.* However the $\breve{R}^q$ term is in general not symmetric. In Appendix §C the symmetricity of $\breve{R}^1$, the Ricci curvature, is discussed.

Chapter 3

Decomposition of noise and filtering

Throughout this chapter we will assume that we are in the non-singular situation with a subbundle E of TM and surjective vector bundle homomorphism $X : \underline{\mathbb{R}}^m \to E$ over M or a mean zero Gaussian field W of sections of E with the non-degeneracy assumption as in §1.1C, (which in particular allows infinite dimensional noise). In the first case, given a smooth vector field A on M there is the S.D.E.

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{3.0.1}$$

as Example C in §1.2. We will be interested here in the case where $X(x)$ is not injective for each x, in which case some of the noise given by $\{B_t : t \geq 0\}$ is 'redundant' from the point of view of a solution $x_t = \xi_t(x_0); t \geq 0$: we will make this notion precise, giving a decomposition of $\{B_t : t \geq 0\}$ into redundant and relevant noise (see Theorem 3.1.2). This will enable redundant noise to be filtered out in a variety of situations, see §3.3 below. We follow very closely the method used in [EY93] for gradient systems. We also describe the corresponding decomposition of $\{W_t : t \geq 0\}$ in the Gaussian field picture. As an application we show that the adjoint connection is metric with respect to some Riemannian metric for which $\breve{Ric}^{\#}$ is bounded if and only if the derivative process and its inverse are uniformly bounded in t when conditioned on the end point.

3.1 A canonical decomposition of the noise driving a stochastic differential equation

The trivial bundle $\underline{R}^m$ is decomposed into the sum of the subbundles $kerX$ and $kerX^{\perp}$ with total spaces $\bigsqcup\{kerX(x) : x \in M\}$ and $\bigsqcup\{(kerX(x))^{\perp} : x \in M\}$ where $\bigsqcup$ refers to disjoint unions. Define the connection $\tilde{\nabla}$ on $\underline{R}^m$ to be the direct sum connection of the push forward connections induced on $kerX$ and on $(kerX)^{\perp}$ by the orthogonal projections $K : \underline{R}^m \to kerX$ and $K^{\perp} : \underline{R}^m \to (kerX)^{\perp}$ respectively as in §1B. Given any suitable process or path σ in M the corresponding parallel translation $\tilde{//}_t$ will preserve the decomposition of $\mathbb{R}^m$ and if $e \in [KerX\,(\sigma(0))]^{\perp}$ then

$$X(\sigma(t))\tilde{//}_t(e) = \breve{//}_t X\,(\sigma(0))\,(e). \tag{3.1.1}$$

for the L-W connection $\breve{\nabla}$ on E.

Fix x_0 in M, and define the following processes:

$$
\begin{aligned}
\beta_t &:= \int_0^{t\wedge\rho} \tilde{//}_s^{-1} K(x_s) dB_s \\
\tilde{B}_t &:= \int_0^{t\wedge\rho} \tilde{//}_s^{-1} K^\perp(x_s) dB_s \\
\breve{B}_t &:= \int_0^{t\wedge\rho} \breve{//}_s^{-1} X(x_s) dB_s
\end{aligned}
$$

where the parallel translations are along the solution $\{x_s := \xi_s(x_0) : 0 \le s < \rho\}$ of (3.0.1) defined up to explosion time $\rho := \rho(x_0)$. These processes are Brownian motions on $kerX(x_0)$, $(kerX(x_0))^\perp$, and E_{x_0}, respectively, stopped at $\rho(x_0)$, by Levy's characterization. Note that

$$\breve{B}_t = X(x_0)\tilde{B}_t. \tag{3.1.2}$$

We first show that on $[0,\rho)$ it is the martingale part of the stochastic anti-development of $\{x_t : 0 \le t < \rho\}$ using $\breve{\nabla}$, (or, in case A does not have image in E, to be more precise using any connection ∇^1 on TM extending $\breve{\nabla}$ as in Proposition 2A):

Lemma 3.1.1 *Take a Riemannian metric on TM which extends the given one on E, giving the decomposition $TM = E \oplus E^\perp$. Take any metric connection $\nabla^\perp$, for this metric, on $E^\perp$ and let ∇^1 be the direct sum connection $\breve{\nabla} + \nabla^\perp$ on TM, with parallel translations denoted by $//^1_\cdot$. The corresponding stochastic anti-development*

$$z_t := \int_0^t (//^1_s)^{-1} \circ dx_s, \quad 0 \le t < \rho \tag{3.1.3}$$

of $x_\cdot := \xi_\cdot(x_0)$ is then given by

$$z_t := \breve{B}_t + \int_0^t (//^1_s)^{-1} A(x_s) ds, \quad 0 \le t < \rho. \tag{3.1.4}$$

Proof. Since the stochastic differentials along $(x_s : 0 \le s < \rho)$ satisfy

$$
\begin{aligned}
\circ d\left((//^1_s)^{-1} X(x_s)\right) &= (//^1_s)^{-1} \circ D^1 X(x_s) \\
&= (//^1_s)^{-1} \nabla^1 X(\circ dx_s)(-)
\end{aligned}
$$

by definition of the covariant differential $\circ D^1 X(x_s)(-)$,see [Elw88], we see that for $0 \le t < \rho$

$$z_t = \int_0^t (//^1_s)^{-1} \circ dx_s = \int_0^t (//^1_s)^{-1} X(x_s) \circ dB_s + \int_0^t (//^1_s)^{-1} A(x_s) ds$$

$$
\begin{aligned}
&= \int_0^t (//_s^1)^{-1} X(x_s) dB_s + \frac{1}{2}\text{trace} \int_0^t (//_s^1)^{-1} \nabla^1 X\,(X(x_s)-)\,(-) ds \\
&\quad + \int_0^t (//_s^1)^{-1} A(x_s) ds \\
&= \int_0^t (//_s^1)^{-1} X(x_s) dB_s + \int_0^t (//_s^1)^{-1} A(x_s) ds \\
&= \breve{B}_t + \int_0^t (//_s^1)^{-1} A(x_s) ds,
\end{aligned}
$$

since $\nabla^1 X(X(x)-) = \breve{\nabla} X(X(x)-$, and by the defining properties of $\breve{\nabla}$ and the definition of $\breve{B}_t$. ∎

Recall that for any stopping time τ the σ-algebra $\mathcal{F}_{\tau-}$ is defined to be that generated by $\mathcal{F}_0$ together with the sets $\{t < \tau\} \cap A$ for $A \in \mathcal{F}_t$. See [RY91] (4.18) (p.44). For any process $\{y_s : 0 \le s < \rho\}$ let $\mathcal{F}_t^{y_\cdot}$ be the σ-algebra generated by $\{t < \rho\}$ together with the elements of the σ-algebra on $\{t < \rho\}$ generated by $y_s, s \le t$, augmented as usual. Note that then $\mathcal{F}_{(t\wedge\rho)-}^{y_\cdot} = \mathcal{F}_{t-}^{y}$. Furthermore if $\{\tau_n\}_{n=1}^{\infty}$ is any increasing sequence of $\mathcal{F}_\cdot^{y_\cdot}$ -stopping times converging to ρ, then ([RY91], Ex 4.18)

$$
\mathcal{F}_{\rho-}^{y_\cdot} = \bigvee_n \mathcal{F}_{\tau_n}^{y_\cdot}. \tag{3.1.5}
$$

When $y_s = \xi_s(x_0)$, we set $\mathcal{F}_t^{y_\cdot} = \mathcal{F}_t^{x_0}$, etc. In the situations we will be considering we will have left continuity with $\mathcal{F}_{\tau-}^{y} = \mathcal{F}_{\tau}^{y}$. However we will not need to use this result and will keep to the more intuitively correct notation with the minus sign.

The following gives direct analogues of the corresponding decomposition in [EY93].

Theorem 3.1.2 *(1).* $\mathcal{F}_{t\wedge\rho(x_0)-}^{\breve{B}_\cdot} = \mathcal{F}_{t\wedge\rho(x_0)-}^{x_0}, \qquad 0 \le t < \infty$
and $\mathcal{F}_{\rho(x_0)-}^{\breve{B}} = \mathcal{F}_{\rho(x_0)-}^{x_0}$.
(2). If $\bar{B}_t = \beta_t + \tilde{B}_t$ then $\{\bar{B}_t : t \ge 0\}$ is a Brownian motion on $\mathbb{R}^m$ stopped at $\rho(x_0)$ with

$$
B_{t\wedge\rho(x_0)} = \int_0^{t\wedge\rho(x_0)} \tilde{//}_s d\bar{B}_s. \tag{3.1.6}
$$

In particular $\{\beta_t : t \ge 0\}$ and $\{\breve{B}_t : t \ge 0\}$ are orthogonal martingales (and independent Brownian motions when there is no explosion).

Proof. For part (1), it is clear from (3.1.3) and (3.1.4) that $\mathcal{F}^{\breve{B}_\cdot} \subset \mathcal{F}^{\breve{x}_0}$. The opposite inclusion comes from the fact that the stochastic anti-development is the inverse of the stochastic development and so is essentially known. In detail let OM denote the orthonormal frame bundle for M, with $u_0 \in OM$ a frame at x_0 adapted to the splitting $T_{x_0}M = E_{x_0} + E_{x_0}^{\perp}$ with u_0 restricted to $\mathbb{R}^p \times \{0\} \subset \mathbb{R}^n$ identified with the restriction of $X(x_0)$ to $(ker X(x_0))^{\perp}$. For u a frame at x let $H_u : T_xM \to T_uOM$ be the horizontal lift (using ∇^1). Let $(\tilde{x}_s : 0 \le s < \rho\}$ be the horizontal lift of $x.$ starting from u_0, so

$$d\tilde{x}_s = H_{\tilde{x}_s} X(x_s) \circ dB_s + H_{\tilde{x}_s} A(x_s) ds. \tag{3.1.7}$$

Using (3.1.3) and (3.1.4) and parallel translation along $\{x_s : 0 \le s < \rho\}$:

$$\begin{aligned} \tilde{x}_s u_0^{-1} \circ d\breve{B}_s &= //_s^1 \circ d\breve{B}_s = //_s^1 \circ dz_s - A(x_s) ds & (3.1.8)\\ &= \circ dx_s - A(x_s) ds = X(x_s) \circ dB_s. & (3.1.9) \end{aligned}$$

Thus, if $\pi : OM \to M$ denotes the projection, $\tilde{x}.$ satisfies the SDE on M

$$d\tilde{x}_s = H_{\tilde{x}_s} \tilde{x}_s u_0^{-1} \circ d\breve{B}_s + H_{\tilde{x}_s} A\left(\pi(\tilde{x}_s)\right) ds \tag{3.1.10}$$

which is driven by $\{\breve{B}_s : 0 \le s < \infty\}$. Since the explosion time for $\tilde{x}.$ is $\rho(x_0)$, e.g. see [Elw82] we have that $\rho(x_0)$ is an $\mathcal{F}_{\cdot}^{\breve{B}.}$-stopping time. To complete the proof of (1) it is enough to observe that, for any s.d.e. with smooth coefficients driven by a continuous martingale $M.$, if $\{y_t : 0 \le t < \tau(y_0)\}$ is a solution with explosion time $\tau(y_0)$ then $\mathcal{F}^{y.}_{t\wedge\tau(y_0)} \subset \mathcal{F}^{M.}_{t\wedge\tau(y_0)}$. (This is easily seen by choosing τ_n to be the first exit time of $y.$ from a ball of radius n about y_0; then $\tau_n \nearrow \tau(y_0)$ and $\mathcal{F}^{y.}_{t\wedge\tau_n} \subset \mathcal{F}^{M.}_{t\wedge\tau_n}$.)

Part (2) is immediate. ∎

3.2 Canonical decomposition of the Gaussian field W_t

For the corresponding results in the Gaussian form we use the notation of §1.3C, §1.3F. We have a splitting $\underline{H} = \ker\rho \oplus (\ker\rho)^\perp$ of the trivial H-bundle of M with $\ker\rho = \bigsqcup\{\ker\rho_x : x \in M\}$ for ρ_x the evaluation map at x. The projections $K(\cdot)$ and $K^\perp(\cdot)$ of $\underline{H}$ onto these subbundles determine connections on the subbundles as in §1.1A and hence a direct sum connection $\tilde{\nabla}$ on $\underline{H}$. The restriction of $\tilde{\nabla}$ to $(\ker\rho)^\perp$ and the connection $\tilde{\nabla}$ on E are intertwined by ρ, as therefore are the corresponding parallel translations.

There are two main complications when H is infinite dimensional. The first is that the driving process $\{W_t : t \ge 0\}$ is a process on $\Gamma(E)$ not on H, whereas our connections is on $\underline{H}$. To construct the analogues of the processes $\beta.$, $\tilde{B}.$ to decompose $W.$ we will therefore either have to extend the parallel translation in some sense over some space on which $W.$ lies (with no obvious choice for non-compact M) or consider the *cylindrical Wiener process* $\{W^c_t : t \ge 0\}$ on H, corresponding to $W.$ in the sense that if $i : H \to \Gamma(E)$ is the inclusion, then (as cylindrical processes) $W = iW^c, t \ge 0$. The theory of these generalized processes and stochastic integrals with respect to them is now standard, e.g. see [Ű95] and [DPZ92] and we will use this approach. The second point is that in general solutions of linear stochastic differential equations in infinite dimensions, such as the equation for parallel transport given by a connection are not known to have versions which are almost surely linear in their initial conditions. This is more an aesthetic than a serious obstacle. However both this and the potential problems arising from the use of cylindrical processes disappear because of the finite codimensionality of $\ker \rho$.

Indeed if $u_0 \in \ker \rho(x_0)$ and $\sigma : [0, \infty) \to M$ is a smooth curve with $\sigma(0) = x_0$ let $u_t = \tilde{//}_t u_0$ for $\tilde{//}_t := \tilde{//}_t(\sigma)$ parallel translation along $\sigma(t) : t \geq 0$. By definition of the connection

$$0 = \frac{\tilde{D} u_t}{\partial t} = K(\sigma(t)) \frac{d}{dt} u_t$$

and so, since $u_t = K(\sigma(t)) u_t$,

$$\frac{du_t}{dt} = \left(\frac{d}{dt} K(\sigma(t)) \right) u_t = -\left(\frac{d}{dt} K^{\perp}(\sigma(t)) \right) u_t. \tag{3.2.1}$$

The corresponding equation for $u_0^{\perp} \in (\ker \rho)^{\perp}$ is

$$\frac{du_t^{\perp}}{dt} = \left(\frac{d}{dt} K^{\perp}(\sigma(t)) \right) u_t^{\perp}. \tag{3.2.2}$$

Now $\frac{d}{dt} K^{\perp}(\sigma(t))$ has finite rank for each t and hence is Hilbert-Schmidt, as an operator on H. From this we see that $\tilde{//}_t$ considered as a map from H to H will lie in the group $O_2(H)$ which is the intersection of the orthogonal group of H with the 'Fredholm group' $GL_2(H)$:

$$GL_2(H) = \{ T \in GL(H) : Tv = v + \alpha v \text{ where } \alpha \in \mathbb{L}(H; H) \text{ is Hilbert Schmidt} \}.$$

Our diffusion on M is given by

$$dx_t = \rho_{x_t} \circ dW_t + \bar{\gamma}(x_t) dt$$

as in §1.1F. The parallel translation $(u_0, u_0^{+}) \in \ker \rho(x_0) + (\ker \rho(x_0))^{\perp} = H$ along $\{x_t : 0 \leq t < \rho\}$ is obtained by solving the analogues of (3.2.1), (3.2.2) in their Stratonovich form. However the evolution $\tilde{//}_t$ can be obtained now as the solution of an equation on the separable Hilbert Lie group $O_2(H)$, (or in the standard way by taking the horizontal lift of $\{x_t : 0 \leq t < \rho\}$ to the principal bundle of $\underline{H}$, taken to be the trivial $O_2(H)$ bundle). Thus $\tilde{//}_t(x_\cdot(\omega)) \in O_2(H)$ and parallel translation on $\underline{H}$, and hence on the subbundle, is almost surely linear.

We can follow §3.1 above to define

$$\begin{aligned} \beta_t^c \ &: = K(x_0) \int_0^{t \wedge \rho} \tilde{//}_s^{-1} dW_s^c \\ \tilde{B}_t \ &: = \int_0^{t \wedge \rho} \tilde{//}_s^{-1} K^{\perp}(x_s) dW_s^c \\ \breve{B}_t \ &: = \int_0^{t \wedge \rho} \breve{//}_s^{-1} \rho_{x_s} dW_s. \end{aligned}$$

Here $\beta^c_\cdot$ is a cylindrical Wiener process on $\ker\rho_{x_0}$ with $\beta_\cdot = i\,\beta^c$ a genuine Wiener process on $\Gamma(E)$ since

$$\int_0^{t\wedge\rho} \tilde{//}_s^{-1}\, dW_s^c = W_{t\wedge\rho}^c + \int_0^{t\wedge\rho} \lambda_s dW_s^c$$

where $\lambda_\cdot$ is a continuous adapted Hilbert-Schmidt operator valued process, so that the stochastic integral gives a genuine continuous process in H. Note that we could also write

$$\beta_t^c = \int_0^{t\wedge\rho} \tilde{//}_s^{-1} K(x_s) dW_s^c$$

to agree with §3.1 above. Since $\tilde{B}_\cdot$ and $\breve{B}_\cdot$ take values in $(\ker\rho_{x_0})^\perp$ and E_{x_0} respectively they are finite dimensional. As before we can, and, will write

$$dW_t^c = \tilde{//}_t(d\beta_t^c + d\tilde{B}_t) \tag{3.2.3}$$

observing that $\beta_t^c + \tilde{B}_t$ is a cylindrical Wiener process on H and $\beta_\cdot$ and $\tilde{B}_\cdot$ are orthogonal martingales. Thus the analogue of Theorem 3.1.2 holds true in this case, and (3.2.3) is our canonical decomposition.

Remark: The real point behind this discussion is that since $\ker\rho$ has finite codimension in $\underline{H}$ it has an induced Fredholm structure: a reduction of its structure group to $GL_2(H^{\infty-p})$ where $H = H^{\infty-p} \oplus H^p$ is a fixed orthogonal splitting of H with $dim H^p = p = dim E$, [ET70]; and the connection on $\ker\rho$ is compatible with this structure.

To see this, first use the fact that $\ker\rho$ is a subbundle of $\underline{H}$ to take an open covering $\{u_j : j \in T\}$, say, of M with subbundle trivializations

$$\theta_j : u_j \to GL(H)$$

such that $\theta_j(x)[\ker\rho_x] = H^{\infty-p}$ and $\theta_j(x)[(\ker\rho_x)^\perp] = H^p$. A simple argument using the fact that $GL(H^{\infty-p})$ is open in $\mathbb{L}(H^{\infty-p}; H^{\infty-p})$ allows us to modify the θ_j so as to choose them so that $\theta_j(x)$ differs from the identity map of H by a finite rank operator; see [ET70]. In particular $\theta_j(x) \in GL_2(H)$. The restrictions of the $\theta_j(x)$ to $\ker\rho$ give the Fredholm structure. Because of the special form of these trivializations the fact that the connection is compatible with this Fredholm structure is immediate from (3.2.1) which, as we have observed shows that parallel translates of $u_0 \in \ker\rho_{x_0}$ differ from u_0 by the action of Hilbert-Schmidt operators. The frame bundle of $\ker\rho$ can be taken to be an $O_2(H^{\infty-p})$ bundle: in particular a separable Hilbert manifold, so that the horizontal lift of $\{x_t : 0 \le t < \rho\}$ to it can be defined as usual, to give a direct version of parallel translation in $\ker\rho$ (which will of course agree with that given by (3.2.1)). From the point of view of the connection the structure group of $\underline{H}$ is most naturally taken to be $O_2(H^{\infty-p}) \times O(H^p) \subset O_2(H)$, but it will not be trivial as such a bundle in general.

3.3 Filtering out redundant noise

The following is an abstraction of the method used in [EY93] to take conditional expectations of solutions to certain linear stochastic differential equations with random coefficients. We first prove a general result: Lemma 3.3.1 on filtering of linear stochastic differential equations. It will be used in the proof of the main theorem.

By 'filtration' we shall mean a filtration satisfying the usual conditions. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, and $\{N_t, 0 \leq t < \infty\}$ an R^p-valued continuous local $\mathcal{F}_*$-martingale. Denote by $\mathcal{F}_.^N$ the filtration generated by $(N_t, t < \infty)$ and let ζ be an $\mathcal{F}_.^{N.}$-stopping time.

Let $M.$ be an R^p-valued continuous local $\mathcal{F}_*$-martingale which can be decomposed as

$$M_t = \bar{M}_t + N_t^{\perp} \tag{3.3.1}$$

where $\bar{M}.$ is an $\mathcal{F}_.^N$-local martingale and $N_.^{\perp}$ is orthogonal to $N.$ (i.e. the quadratic variation $dN. \otimes dN_.^{\perp} = 0$). Assume $(v_s, 0 \leq s < \zeta)$ is a solution, starting from $v_0 \in \mathbb{R}^n$, of the equation

$$dv_s = P_s(v_s)dM_s + Q_s(v_s)ds + R_s dN_s + r_s ds, \tag{3.3.2}$$

where

$$\begin{aligned} P : [0,\zeta) \times \Omega &\to L(R^n, R^p; R^n), \\ Q : [0,\zeta) \times \Omega &\to L(R^p; R^n), \\ R : [0,\zeta) \times \Omega &\to L(R^p; R^n) \\ \textit{and} \qquad r : [0,\zeta) \times \Omega &\to L(R^p; R^n) \end{aligned}$$

are respectively $\mathcal{F}_*^N$, $\mathcal{F}_*^N$, $\mathcal{F}_*$, and $\mathcal{F}_*$ progressively measurable processes.

It turns out that the equation which governs the conditional expectation of $v.$ is of the same type as (3.3.2):

Lemma 3.3.1 *Assume $\{N_t\}$ has the predictable representation property. Let τ be a $\mathcal{F}_.^N$ stopping time with $0 \leq \tau < \zeta$ such that the stopped processes P^τ, Q^τ, R^τ, r^τ, and v^τ satisfy*

1. *P^τ and Q^τ are bounded,*
2. *R^τ belongs to $\mathcal{L}^2(N^\tau)$, (so $\mathbb{E}\int_0^\tau |R_s^\tau|^2 d < N, N >_s < \infty$.*
3. *r^τ belongs to $\mathcal{L}^1([0,\tau) \times \Omega)$,*
4. *v^τ belongs to $\mathcal{L}^2(M^\tau) \cap \mathcal{L}^2(N^\tau) \cap \mathcal{L}^2([0,\tau) \times \Omega)$.*

Set

$$\bar{v}_{\tau\wedge s} = \mathbb{E}\{v_{\tau\wedge s} | \mathcal{F}^{N.}\}.$$

Then $\{\bar{v}_{\tau\wedge s}\}$ is $\mathcal{F}_.^{N.}$ adapted and satisfies the equation up to time τ

$$
\begin{cases} d\bar{v}_s &= P_s(\bar{v}_s)d\bar{M}_s + Q_s(\bar{v}_s)ds + \bar{R}_s dN_s + \bar{r}_s ds, \\ \bar{v}_0 &= v_0, \end{cases} \tag{3.3.3}
$$

where $\bar{R}_{\tau\wedge s} = \mathbb{E}\{R_{\tau\wedge s}|\mathcal{F}_s^{N.}\}$, $\bar{r}_{\tau\wedge s} = \mathbb{E}\{r_{\tau\wedge s}|\mathcal{F}_s^{N.}\}$, *and* $\bar{M}$ *is defined in (3.3.1).*

Proof. Take $\phi \in L^\infty(\Omega, \mathcal{F}^{N.}, P)$. The representation property gives an $\mathcal{F}_\cdot^{N\cdot}$ predictable $\Phi : [0,\infty) \times \Omega \to L(R^p; R)$ with

$$
\phi = \mathbb{E}\phi + \int_0^\infty \Phi_s(dN_s).
$$

Set $\phi_t = \mathbb{E}\phi + \int_0^t \Phi_s(dN_s)$. Let $\bar{\bar{v}}_{\tau\wedge s} = \mathbb{E}\left\{v_{\tau\wedge s}|\mathcal{F}_s^{N\cdot}\right\}$. Note that since $\phi_\cdot$ is both an $\mathcal{F}_\cdot$-martingale and an $\mathcal{F}_\cdot^N$ martingale

$$
\mathbb{E}\phi v_{\tau\wedge t} = \mathbb{E}\phi_{\tau\wedge t} v_{\tau\wedge t} = \mathbb{E}\phi_{\tau\wedge t}\bar{\bar{v}}_{t\wedge\tau} = \mathbb{E}\phi\bar{\bar{v}}_{\tau\wedge t}
$$

so that $\bar{\bar{v}}_{\tau\wedge t} = \bar{v}_{\tau\wedge t}$.

Next by Itô's formula, using the orthogonality of $N^\perp$ to N

$$
\begin{aligned}
\phi_{t\wedge\tau} v_{t\wedge\tau} \;=\;& (\mathbb{E}\phi)v_0 + v_0(\phi_{t\wedge\tau} - \mathbb{E}\phi) + (\mathbb{E}\phi)(v_{t\wedge\tau} - v_0) \\
&+ \int_0^{t\wedge\tau} \Phi_s(dN_s)v_s + \int_0^{t\wedge\tau} \phi_s(P_s(v_s)dM_s + R_s dN_s) \\
&+ \int_0^{t\wedge\tau} \phi_s\left(Q_s(v_s) + r_s\right) ds + \frac{1}{2}\int_0^{t\wedge\tau} P_s(v_s)(d\bar{M}_s)\Phi_s(dN_s) \\
&+ \frac{1}{2}\int_0^{t\wedge\tau} R_s(dN_s)\Phi_s(dN_s).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\phi_{t\wedge\tau} v_{t\wedge\tau} \;=\;& -(\mathbb{E}\phi)v_0 + v_0\mathbb{E}\phi_{t\wedge\tau} + (\mathbb{E}\phi)\mathbb{E}v_{t\wedge\tau} \\
&+ \mathbb{E}\int_0^{t\wedge\tau} \phi_s\left(Q_s(v_s) + \bar{r}_s\right) ds \\
&+ \tfrac{1}{2}\mathbb{E}\int_0^{t\wedge\tau} P_s(v_s)(d\bar{M}_s)\Phi_s(dN_s) + \tfrac{1}{2}\mathbb{E}\int_0^{t\wedge\tau} \bar{R}_s(dN_s)\Phi_s(dN_s)
\end{aligned} \tag{3.3.4}
$$

using (1), (2), (3), (4). Using (1), (2), (3), (4) again, the boundedness of ϕ, and the Kunita-Watanabe inequality we obtain from (3.3.4) that

$$
\begin{aligned}
\mathbb{E}\phi_{t\wedge\tau}\tilde{v}_{t\wedge\tau} =\;& -(E\phi)v_0 + v_0\mathbb{E}\phi_{t\wedge\tau} + (\mathbb{E}\phi)\mathbb{E}v_{t\wedge\tau} + \mathbb{E}\int_0^{t\wedge\tau} \phi_s\left(Q_s(\tilde{v}_s) + \bar{r}_s\right) ds \\
&+ \tfrac{1}{2}\mathbb{E}\int_0^{t\wedge\tau} P_s(\tilde{v}_s)(d\bar{M}_s)\Phi_s(dN_s) + \tfrac{1}{2}\mathbb{E}\int_0^{t\wedge\tau} \bar{R}_s(dN_s)\Phi_s(dN_s), \\
=\;& \mathbb{E}\phi_{t\wedge\tau}\left\{v_0 + \int_0^{t\wedge\tau} P_s(\tilde{v}_s)d\bar{M}_s + \int_0^{t\wedge\tau} Q_s(\tilde{v}_s)ds + \int_0^{t\wedge\tau} \bar{R}_s dN_s + \int_0^{t\wedge\tau} \bar{r}_s ds,\right\}
\end{aligned} \tag{3.3.5}
$$

whence

$$
\tilde{v}_{t\wedge\tau} = v_0 + \int_0^{t\wedge\tau} P_s(\tilde{v}_s)d\bar{M}_s + \int_0^{t\wedge\tau} Q_s(\tilde{v}_s)ds + \int_0^{t\wedge\tau} \bar{R}_s dN_s + \int_0^{t\wedge\tau} \bar{r}_s ds.
$$

■

To apply the previous results we will first make a general definition. Note first that if $p : K \to M$ is a vector bundle over M with possibly infinite dimensional, but separable, Banach spaces as fibres, then there are measurable trivializations $\theta : K \to M \times E$ where E is a linear space, with $\theta_x \equiv \theta|_{p^{-1}(x)}$ a continuous linear isomorphism from $p^{-1}(x)$ to E. Any two such, θ_1, θ_2 say, will have $\theta_1 \circ \theta_2 : M \times E \to M \times E$ measurable. Let $v : \Omega \to K$ be $\mathcal{F}$-measurable. Set $p \circ v = y : \Omega \to M$ and let $\mathcal{G}$ be some σ sub-algebra of $\mathcal{F}$ containing that generated by y. We can define the *conditional expectation of v given $\mathcal{G}$* by

$$\mathbb{E}\{v|\mathcal{G}\} = \theta^{-1}\left(y(\omega), \alpha(\omega)^{-1}\mathbb{E}\left\{\alpha(\cdot)\theta_{y(\cdot)}(v(\cdot))|\mathcal{G}\right\}\right)$$

whenever there exists a measurable trivialization θ and a $\mathcal{G}$-measurable $\alpha : \Omega \to (0, \infty)$ such that $\omega \mapsto \alpha(\omega)\theta_{y(\omega)}(v(\omega)) : \Omega \to E$ is integrable. From the $\mathcal{G}$-measurability of $y.$ this definition is independent of the choice of suitable trivialization; from standard results it does not depend on the choice of α: see also the proof of the next lemma. The introduction of α is helpful particularly because the trivialization does not necessarily have any relationship to any norm on E (and in practice we will want to use ones which do not).

When we have a continuous process $\{V_t : 0 \le t \le T\}$ in K over $y_t := pV_t$ in M we can consider ourselves to have a random variable with values in the total space of the vector bundle

$$C([0,T]; K) \xrightarrow{p\circ} C([0,T]; M)$$

of continuous paths with values in K over these with values in M. When p is smooth and $y.$ is a semi-martingale (starting from a fixed $y_0 \in M$ for simplicity) any connection on p gives a parallel translation $//_t$ along the paths of $y.$ and hence a measurable trivialization, almost surely defined for the law of $y.$:

$$C([0,T]; K) \xrightarrow{p\circ} C([0,T]; M) \times C\left([0,T]; p^{-1}(y_0)\right)$$

$$V_t(\omega) \mapsto \left(y_t(\omega), //_t^{-1} V_t(\omega)\right).$$

When discussing conditional expectations for such processes as $v.$ it will be particularly useful to use such a trivialization, especially since in this context we will often want to take a predictable projection for some filtration $\{\mathcal{G}_t : 0 \le t < \infty\}$. In this case we can also use localization in time to further extend the flexibility of this procedure. Moreover we also want to include processes defined only up to some stopping time:

Let $\{y_t : 0 \le t < \tau\}$ be a continuous process starting from a point y_0. Let $p : K \to M$ be a smooth vector bundle over M, possibly infinite dimensional, but separable. Let $\{V_t : 0 \le t < \tau\}$ be a process in K over $y.$, i.e. $pV_t = y_t$. Assume τ is an $\mathcal{F}^{y.}_{\cdot}$-predictable stopping time. Let $\mathcal{G}$ be a σ-subalgebra of $\mathcal{F}$ containing $\mathcal{F}^{y.}_{\tau-}$.

Definition 3.3.2 *We say that $V.$ has a local conditional expectation with respect to $\mathcal{G}$, denoted by $\bar{V}.$, a process in K along $\{y.\}$, if there exist*

1. *an affine connection on K (or semi-connection) with parallel translation $\tilde{//}_t : T_{y_0}M \to T_{y_t}M, 0 \le t < \tau$ defined along $\{y_t : 0 \le t < \tau\}$,*
2. *a $\mathcal{G}$-measurable process $\alpha_\cdot : [0,\tau) \times \Omega \to R(>0)$,*
3. *$\mathcal{F}^{y_\cdot}_{\cdot\wedge\tau-}$-stopping times $\{\tau_n : n \ge 0\}$ increasing to τ such that*

$$\tilde{//}^{-1}_{t\wedge\tau_n} \alpha_{t\wedge\tau_n} V_{t\wedge\tau_n} \in L^1\left(\Omega, \mathcal{F}, P; p^{-1}(y_0)\right), \tag{3.3.6}$$

and

$$\tilde{//}_{t\wedge\tau_n} \mathbb{E}\{\tilde{//}^{-1}_{t\wedge\tau_n} \alpha_{t\wedge\tau_n} V_{t\wedge\tau_n} | \mathcal{G}\} = \alpha_{t\wedge\tau_n} \bar{V}_{t\wedge\tau_n} \tag{3.3.7}$$

for all $t \ge 0, n = 1, 2, \ldots$.

We will see the local conditional expectation is well defined in the following lemma, in the sense that if there are another set of parallel translations $\tilde{//}$, function α' and stopping times τ'_n increasing to τ such that the relevant random variables are integrable, then the corresponding conditional expectations must be the same:

Proposition 3.3.3 *If (3.3.6) holds there is (up to modification) a unique $\bar{V}_\cdot$ over $y_\cdot$ satisfying (3.3.7). Moreover $\bar{V}_\cdot$ is independent of the choice of the α, the τ_n, and the connection satisfying (3.3.6).*

Proof. For fixed n set

$$\bar{V}^n_t = (\alpha_{t\wedge\tau_n})^{-1} \tilde{//}_{t\wedge\tau_n} \mathbb{E}\left\{\tilde{//}^{-1}_{t\wedge\tau_n} \alpha_{t\wedge\tau_n} V_{t\wedge\tau_n} | \mathcal{G}\right\}, \qquad t \ge 0.$$

For $m < n$ we see $\bar{V}^n_{t\wedge\tau_m} = \bar{V}^m_t$.

Then $\bar{V}_t$ is well defined up to equivalence and is similarly seen to be independent of the choice of stopping times $\{\tau_n\}_{n=1}^\infty$ satisfying (3.3.6). Suppose now we have another set up $\tilde{//}'_t$, α'_t, τ'_n, $n = 1, 2, \ldots$ as in the definition and defining $\bar{V}'_\cdot$ by the analogue of (3.3.7). Set $\tau''_n = \tau_n \wedge \tau'_n$. Since $\left(\tilde{//}'_{t\wedge\tau''_n}\right)^{-1} \tilde{//}_{t\wedge\tau''_n}$ is $\mathcal{F}^y_{\tau-}$ measurable, we see, from above,

$$\begin{aligned}
\alpha'_{t\wedge\tau''_n} \bar{V}'_{t\wedge\tau''_n} &= \tilde{//}'_{t\wedge\tau''_n} \mathbb{E}\left\{\left(\tilde{//}'\right)^{-1}_{t\wedge\tau''_n} \alpha'_{t\wedge\tau''_n} V_{t\wedge\tau''_n} | \mathcal{G}\right\} \\
&= \tilde{//}'_{t\wedge\tau''_n}\, \mathbb{E}\left\{\left(\tilde{//}'_{t\wedge\tau''_n}\right)^{-1} \tilde{//}_{t\wedge\tau''_n}\, \tilde{//}^{-1}_{t\wedge\tau''_n} \alpha'_{t\wedge\tau''_n} V_{t\wedge\tau''_n} | \mathcal{G}\right\} \\
&= \frac{\alpha'_{t\wedge\tau''_n}}{\alpha_{t\wedge\tau''_n}}\, \tilde{//}_{t\wedge\tau''_n} \mathbb{E}\{//^{-1}_{t\wedge\tau''_n} \alpha_{t\wedge\tau''_n} V_{t\wedge\tau''_n} | \mathcal{G}\} \\
&= \alpha'_{t\wedge\tau''_n} \bar{V}_{t\wedge\tau''_n}.
\end{aligned}$$

from above and (3.3.7). Thus $\bar{V}'_\cdot = \bar{V}_\cdot$ as required. ∎

Corollary 3.3.4 *Suppose $\tau = \infty$. If for some Riemannian or Finsler metric on K, $||V_t||_{y_t} \in L^1$ each t, then the local conditional expectation exists and is just the conditional expectation in the sense of (3.3.7).*

Corollary 3.3.5 *With the notation above, suppose that $V_\cdot$, as above, has a local conditional expectation, $\bar{V}_\cdot$ over $y_\cdot$. Let $\phi_t : 0 \le t < \tau$ be a $\mathcal{G}$-measurable process over $\{y_\cdot\}$ in the dual bundle K^*. If $\phi_t(V_t)\chi_{t<\tau}$ is integrable then so is $\phi_t(\bar{V}_t)\chi_{t<\tau}$ and*

$$\mathbb{E}\,\phi_t(V_t)\chi_{t<\tau} = \mathbb{E}\,\phi_t(\bar{V}_t)\chi_{t<\tau}.$$

Proof. With the notation of (3.3.6),

$$\begin{aligned}
\mathbb{E}\,\{\phi_t(V_t)\chi_{t<\tau_n}|\mathcal{G}\} &= \mathbb{E}\,\left\{\chi_{t<\tau_n}\alpha_{t\wedge\tau_n}^{-1}\phi_{t\wedge\tau_n}\left(\tilde{//}_{t\wedge\tau_n}\tilde{//}_{t\wedge\tau_n}^{-1}\alpha_{t\wedge\tau_n}V_{t\wedge\tau_n}\right)|\mathcal{G}\right\}\\
&= \alpha_{t\wedge\tau_n}^{-1}\phi_{t\wedge\tau_n}\tilde{//}_{t\wedge\tau_n}\mathbb{E}\,\left\{\tilde{//}_{t\wedge\tau_n}^{-1}\alpha_{t\wedge\tau_n}V_{t\wedge\tau_n}|\mathcal{G}\right\}\chi_{t<\tau_n}\\
&= \phi_t(\bar{V}_t)\chi_{t<\tau_n}.
\end{aligned}$$

Thus

$$\mathbb{E}\,sgn(\phi_t(\bar{V}_t))\phi_t(V_t)\chi_{t<\tau_n} = \mathbb{E}\,|\phi_t(\bar{V}_t)|\chi_{t<\tau_n}.$$

Hence by the monotone convergence theorem $\phi_t(\bar{V}_t)\chi_{t<\tau}$ is integrable. Since

$$\mathbb{E}\phi_t(V_t)\chi_{t<\tau_n} = \mathbb{E}\phi_t(\bar{V}_t)\chi_{t<\tau_n}$$

for each n the result follows. ∎

Remarks:
(i). The independence from the choice of connections was originally pointed out by Emery.
(ii). We are not assuming any Riemannian structure, and the connections are not necessarily metric for any metric.

C. The following generalizes the main result of [EY93]: we consider first a s.d.e. (3.0.1) with solution (partial) flow $\{\xi_t(x_0) : 0 \le t < \rho(x_0), x_0 \in M\}$, explosion time $\rho : M \times \Omega \to (0, \infty]$, and derivative (partial) flow $T_{x_0}\xi_t : T_{x_0}M \to TM$, $x_0 \in M$. with the same explosion time. Assume the non-singularity condition that we have the subbundle E of TM. Recall the definition (2.4.2), (2.4.5) of the Weitzenbock curvature for the L-W connection, $\breve{R}_x^q : \wedge^q T_xM \to \wedge^q T_xM$ from §2.4. First we have a crucial lemma; for the case A is not in $\Gamma(E)$ see (3.3.14) below.

Lemma 3.3.6 *Let $1 \le q \le n$. Assume $A(x) \in E_x$ for each $x \in M$. Let $V_0 \in \wedge^q T_{x_0}M$ and $V_t = \wedge^q T\xi_t(V_0)$. Then V_t is a solution of the following equation (in Itô form):*

$$\hat{D}V_t = d\Lambda^q\left(\breve{\nabla}X(-)dB_t\right)(V_t) - \frac{1}{2}(\breve{R}_{x_t}^q)^*(V_t)dt + d\Lambda^q\breve{\nabla}A(V_t)dt, \quad 0 \le t < \rho(x_0). \tag{3.3.8}$$

Proof. The process $V_\cdot$ satisfies the equation

$$\hat{D}V_t = d\Lambda^q\left(\breve{\nabla}X(-)\circ dB_t\right)(V_t) + d\Lambda^q\breve{\nabla}A(V_t)dt, \quad 0 \le t < \rho(x_0), \tag{3.3.9}$$

which in Itô form (using $\hat\nabla$) is

$$\hat D V_t = d\Lambda^q\left(\breve\nabla X(-)dB_t\right)(V_t) - \frac{1}{2}(\breve R^q_{x_t})^*(V_t)dt + d\Lambda^q\breve\nabla A(V_t)dt, \qquad 0\le t<\rho(x_0), \tag{3.3.10}$$

since the Itô correction term for $d\Lambda^q\left(\breve\nabla X(-)\circ dB_t\right)(V_t)$ is:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^m\Big[d\Lambda^q\left(\hat\nabla_{X^i}(\breve\nabla X^i(-))(V_t)\right)dt + d\Lambda^q\left(\breve\nabla X^i(\breve\nabla X^i)\right)(V_t)dt\\
&\qquad + \delta^2\Lambda^q\left(\breve\nabla.X^i\right)(V_t)dt\Big]\\
=\;&\frac{1}{2}\sum_1^m\Big[d\Lambda^q\left(\breve R(X^i,\cdot)X^i\right)(V_t)dt + \delta^2\Lambda^q\left(\breve\nabla.X^i(V_t))\right)dt\Big]\\
=\;&-\frac{1}{2}(\breve R^q)^*(V_t)dt,
\end{aligned}$$

Here we used the notation from Appendix B, (2.4.7) in §2.4D and the calculations before (2.4.7). ■

Theorem 3.3.7 *Let $1\le q\le n$. Assume $A(x)\in E_x$ for each $x\in M$. Then the local conditional expectation of $\{V_t: 0\le t<\rho(x_0)\}$ with respect to $\mathcal{F}^{x_0}_{\rho(x_0)-}$ exists and is equal to the solution $\{\bar V_t: 0\le t<\rho(x_0)\}$ to*

$$\frac{\hat D\bar V_t}{\partial t} = -\frac{1}{2}(\breve R^q_{x_t})^*(\bar V_t) + d\Lambda^q\breve\nabla A(\bar V_t), \qquad 0\le t<\rho(x_0) \tag{3.3.11}$$

where $x_t=\xi_t(x_0)$.

Proof. Using any complete metric on M let τ_n be the first exit time of $x.$ from the ball radius n about x_0, $n=1,2\ldots$. Since these balls are compact (3.3.6) holds with $\alpha_t\equiv 1$ for any $\tilde{//}.$.

Now, with the notation of §1B,

$$\begin{aligned}
\breve\nabla X(\cdot)dB_{s\wedge\rho(\omega)} &= \breve\nabla X(\cdot)\tilde{//}_s d\bar B_s = \breve\nabla X(\cdot)\tilde{//}_s(d\beta_s + d\tilde B_s)\\
&= \breve\nabla X(\cdot)\tilde{//}d\beta_s
\end{aligned}$$

by the defining property of $\breve\nabla$, since $\tilde{//}_s\beta_s$ is orthogonal to $Ker X(x_s)$ for each s. Use $\hat{//}_t^{-1}$ to pull (3.3.8) back to $\wedge^q T_{y_0}M$. We then have an equation of the form (3.3.2) for $v_s=\tilde{//}_s^{-1}V_s$, with $R\equiv 0$ and $M_s=\beta_s$. Taking $N=\breve B.$ and $\tau=\tau_n$ in Lemma 3.3.1, since a stopped Brownian motion has the predictable representation property and $\mathcal{F}^{\breve B.}_{\rho(x_0)-}=\mathcal{F}^{x.}_{\rho(x_0)-}$ by Theorem 3.1.2, we see

$$\mathbb{E}\{\hat{//}_t^{-1}V_{t\wedge\tau_n}|\mathcal{F}^{x.}_{\rho(x_0)-}\} = \hat{//}_t^{-1}\bar V_{t\wedge\tau_n}$$

for $\bar V_t$ given by (3.3.11), as required. ■

3.3.1 When A does not belong to the image of X

In the next result we give a version of (3.3.9) without the assumption that $A(x) \in E_x$ and also translate (3.3.9) into an equation using a Levi-Civita connection. In it we keep the non-singularity condition, and the notation, of §3.3C above. We take any Riemannian metric on M and let ∇ be its Levi-Civita connection, with corresponding covariant differentiation D along $x_t = \xi_t(x_0)$, $0 \le t < \zeta(x_0)$. We also let $E^\perp$ be the orthogonal bundle to E in TM with $\nabla^\perp$ a metric connection on $E^\perp$, and as in §1.3 set $\nabla^1 = \breve{\nabla} + \nabla^\perp$ to get a connection on TM with adjoint $\nabla^{1\prime}$ say.

Theorem 3.3.8 *For $V_t = \wedge^q T\xi_t(V_0)$ as in Theorem 3.3.7, the local conditional expectation $\bar{V}_t$, $0 \le t < \rho(x_0)$, with respect to $\mathcal{F}^{x_0}_{\rho(x_0)}$, is adapted to $\mathcal{F}^{x_0}_{\cdot}$ and satisfies*

$$\frac{D^{1\prime}\bar{V}_t}{\partial t} = -\frac{1}{2}(\breve{R}^q_{x_t})^*(\bar{V}_t) + d\Lambda^q(\nabla^1_\cdot A)(\bar{V}_t) \tag{3.3.12}$$

or equivalently

$$D\bar{V}_t = d\Lambda^q\left((\nabla_\cdot X)\left(Y(x_t)\tilde{/\!/}_t \circ d\tilde{B}_t\right)\right)(\bar{V}_t) - \frac{1}{2}(\breve{R}^q_{x_t})^*(\bar{V}_t)dt + d\Lambda^q\left(\nabla_\cdot A\right)\bar{V}_t dt \tag{3.3.13}$$

for $\tilde{/\!/}_t$ and $\tilde{B}_t$ as in §3.1B.

Proof. We have

$$\begin{aligned}
D^{1\prime}V_t &= d\Lambda^q\left(\nabla^1_\cdot X \circ dB_t\right)(V_t) + d\Lambda^q\left(\nabla^1_\cdot A\right)V_t dt \\
&= d\Lambda^q\left(\breve{\nabla}_\cdot X \circ dB_t\right)(V_t) + d\Lambda^q(\nabla^1_\cdot A)V_t dt \\
&= d\Lambda^q\left(\breve{\nabla}_\cdot X dB_t\right)(V_t) - \frac{1}{2}(\breve{R}^q_{x_t})^*(V_t)dt + d\Lambda^q(\nabla^1_\cdot A)V_t dt
\end{aligned}$$

by the calculations in the proof of Lemma 3.3.6, where the Itô equation is with respect to ∇^1. As in the proof of Theorem 3.3.7 this gives

$$D^{1\prime}V_t = d\Lambda^q\left(\breve{\nabla}_\cdot X\tilde{/\!/}_t d\beta_t\right)(V_t) - \frac{1}{2}(\breve{R}^q_{x_t})^*(V_t)dt + d\Lambda^q(\nabla^1_\cdot A)V_t dt \tag{3.3.14}$$

from which the local conditional expectation $\{\bar{V}_t : 0 \le t < \rho(x_0)\}$ exists and satisfies (3.3.12). To obtain this result in terms of ∇ it is convenient to take an $X^\perp : \underline{R}^\ell \to E^\perp$ which induces $\nabla^\perp$ and set $X^1 = X^\perp + X : \underline{R}^\ell \oplus \underline{R}^m \to TM$, so that X^1 induces ∇^1. By (C.8) in Proposition C.3, in Appendix C, if $Y^1 : TM \to \underline{R}^\ell \oplus \underline{R}^m$ is the adjoint of X^1 we have

$$D^{1\prime}\bar{V}_t = D\bar{V}_t - d\Lambda^q\left(\nabla Z^{\circ dx_t}\right)(\bar{V}_t)$$

with $Z^{\circ dx_t} = X^1(\cdot)Y^1(x_t)(\circ dx_t)$.

By (3.1.4)

$$\begin{aligned} Z^{\circ dx_t} &= X^1(\cdot)Y^1(x_t)\left(//_t^1 \circ d\breve{B}_t + A(x_t)dt\right) \\ &= X(\cdot)Y(x_t)\breve{//}_t \circ d\breve{B}_t + Z^{A(x_t)}dt \end{aligned}$$

Thus $\nabla Z^{\circ dx_t} = \nabla.X\left(Y(x_t)\breve{//}_t \circ d\breve{B}_t\right) + \nabla Z^{A(x_t)}dt$. Since, by §C

$$\nabla_v^1 A = \nabla_v A - \nabla_v Z^{A(y)}, \qquad \text{if } v \in T_yM,\ y \in M$$

equation (3.3.13) follows. ■

We will let

$$\breve{W}_{t,x_0}^{q,A} : \wedge^q T_{x_0}M \to \wedge^q T_{x_t}M, \qquad 0 \le t < \rho(x_0) \tag{3.3.15}$$

be the solution map determined by (3.3.11); and that given by (3.3.12) will be written $\overset{1}{W}{}_{t,x_0}^{q,A}$. We will also let $W_{t,x_0}^{q,\xi}$ denote the local conditional expectation of $\wedge^q T_{x_0}\xi_t$ with respect to $\mathcal{F}_{\rho(x_0)-}^{x_0}$.

For our regular s.d.e. define

$$\frac{\mathbb{D}}{\partial t} \equiv \frac{\breve{\mathbb{D}}^{A,q}}{\partial t}$$

to be the operator on q-vector fields $V_t : 0 \le t < \zeta(x_0)$ along $\{\xi_t(x_0) : 0 \le t < \zeta(x_0)\}$ defined by

$$\frac{\mathbb{D}}{\partial t}V_t = \frac{D^{1'}V_t}{\partial t} + \frac{1}{2}(\breve{R}_{x_t}^q)^* V_t - d\Lambda^q(\nabla_{\cdot}^1 A)(V_t) \tag{3.3.16}$$

for ∇^1 as above.

Proposition 3.3.9 *The operator $\frac{\mathbb{D}}{\partial t}$ and its related modified parallel transport operator $\overset{1}{W}{}_{t,x_0}^{q,A}$ are independent of the choice of $E^\perp$ and $\nabla^\perp$ used to define ∇^1.*

Proof. Clearly $\overset{1}{W}{}_{t,x_0}^{q,A}$ depends only on our s.d.e. since it is just $W_{t,x_0}^{q,\xi}$. However

$$\frac{\mathbb{D}}{\partial t}V_t = \overset{1}{W}{}_{t,x_0}^{q,A} \frac{d}{dt}\left([\overset{1}{W}{}_{t,x_0}^{q,A}]^{-1} V_t\right).$$

■

In general $\frac{\mathbb{D}}{\partial t}$ will depend on $\nabla^\perp$ if considered as an operator on arbitrary smooth paths in M. For example if $\sigma(t)$ satisfies $\dot{\sigma}(t) = X(\sigma(t))\alpha(t)$ for some continuous paths α and if V is a vector field on M, setting $V(t) = V(\sigma(t))$

$$\frac{D^{1'}}{\partial t}V(t) - \nabla_{V(t)}^1 A = \hat{\nabla}_{\dot{\sigma}(t)}V - \nabla_{V(t)}^1 A,$$

by Proposition 1.3.1, which will depend on $\nabla^1_{V(t)}A$. (Since $\breve{Ric}$ depends only on $\breve{\nabla}$ it cannot help to make $\frac{\mathbb{D}}{\partial t}$ intrinsic.) However if $\dot{\sigma}(t) = X(\sigma(t))\alpha(t) + A(\sigma(t))$ and $V(t) = V(\sigma(t))$ then

$$\begin{aligned}
\frac{D^{1'}}{\partial t}V(t) - \nabla^1{}_{V(t)}A &= \hat{\nabla}_{X(\sigma(t))\alpha(t)}V + \nabla^{1'}{}_{A(\sigma(t))}V - \nabla^1_{V(t)}A \\
&= \hat{\nabla}_{X(\sigma(t))\alpha(t)}V + [A, V](\sigma(t))
\end{aligned}$$

which is intrinsic.

3.3.2 The inverse derivative flow

In the regular case the local conditional expectation $\mathbb{E}\{\Lambda^q(T_{x_0}\xi_t)^{-1}|\mathcal{F}^{x_0}\}$ is just $(W^{q,\xi}_{t,x_0})^{-1}$. Indeed using the notation of Theorem 3.3.8 to allow $A(x) \notin E_x$.

$$D^{1'}\Lambda^q(T\xi_t)^{-1} = -\Lambda^q(T\xi_t)^{-1} \circ D^{1'}\left(\Lambda^q(T\xi_t)\right)\Lambda^q(T\xi_t)^{-1}$$

whence, by (3.3.14)

$$\begin{aligned}
D^{1'}\Lambda^q(T_{x_0}\xi_t)^{-1} &= -\Lambda^q(T_{x_0}\xi_t)^{-1}d\Lambda^q\left(\breve{\nabla}.X\tilde{//}d\tilde{B}_t\right) \\
&\quad +\frac{1}{2}\Lambda^q(T_{x_0}\xi_t)^{-1}(\breve{R}^q_{x_t})^*dt - \Lambda^q\left(T\xi_t\right)^{-1}d\Lambda^q(\nabla^1_. A)dt
\end{aligned}$$

from which, as before, the local conditional expectation $Z_t : 0 \leq t < \rho(x_0)$ is defined and satisfies

$$\frac{D^{1'}Z_t}{\partial t} = \frac{1}{2}Z_t(\breve{R}^q_{x_t})^* - Z_t\, d\Lambda^q(\nabla^1_. A) \tag{3.3.17}$$

which is just the equation for $(\overset{1}{W}{}^{q,A}_t)^{-1}$.

3.3.3 Integrability of certain C^r norms for compact M

For further reference we quote the following result of Kifer with a corollary obtained as an application of the filtering result given above. Kifer's proof was an elegant and quick application of Baxendale's integrability theorem [Bax84]. The results could also be obtained by the direct inductive proof in [Nor86] given for certain non-compact situations. For $r = 0, 1, \ldots,$ and a Riemannian metric on M define

$$|T\xi_t|_{C^r} = \sup_{x\in M} |(\nabla^r T\xi_t)|_x,$$

where ∇ is a connection on TM and $T\xi_t$ is treated as a section of the bundle $L(TM; \xi_t^*(TM))$ over M given its induced connection. By compactness different metrics and connections will give equivalent norms and equivalent norms would also be obtained using local coordinates systems as in [Kif88].

Proposition 3.3.10 *[Kif88] For compact M and $r = 0, 1, \ldots$, both* $\sup_{0\le t\le T} |T\xi_t|_{C^r}$ *and* $\sup_{0\le t\le T} |T\xi_t^{-1}|_{C^r}$ *lie in $\mathcal{L}^p$ for $1 \le p < \infty$, all $T > 0$.*

Proposition 3.3.11 *For a regular stochastic differential equation on a compact Riemannian manifold M let $\overset{1}{W}{}^{q,A}_{t,x_0}$ be given by (3.3.12), $x_0 \in M$. Then both*

$$\sup_{0\le s\le T}\sup_{x_0}\left|\overset{1}{W}{}^{q,A}_{t,x_0}\right| \qquad \textit{and} \qquad \sup_{0\le s\le T}\sup_{x_0}\left|\left(\overset{1}{W}{}^{q,A}_{t,x_0}\right)^{-1}\right|$$

lie in $\mathcal{L}^p$ for $1 \le p < \infty$, all $T > 0$. If $\hat\nabla$ is adapted to some metric on M they both lie in $\mathcal{L}^\infty$ for each $T > 0$.

Proof. The case of $1 \le p < \infty$ follows from the previous proposition and §§3.3.1, 3.3.2 above together with Jensen's inequality for conditional expectations. When $\hat\nabla$ is adapted to a Riemannian metric $\langle , \rangle'$ choose such a metric in Theorem 3.3.8. Then by Proposition 1.3.5 the Levi-Civita covariant derivative $\nabla X(e)$ satisfies

$$\langle \nabla_u X(e), v\rangle' = -\langle \nabla_v X(e), u\rangle'_x, \qquad \text{any } u, v \in T_xM, x \in M, e \in Ker^\perp X(x).$$

If we apply Itô's formula to $|\bar V_t|^2_{x_t}$ in (3.3.13) we see that the Itô stochastic integral vanishes and an $\mathcal{L}^\infty$ bound for $\overset{1}{W}{}^{q,A}$ follows by compactness of M. The same argument shows that there exists $c \in \mathbb{R}$ such that for all $v_0 \in \wedge^q T_{x_0}M$, all $x_0 \in M$ and $0 \le t \le T$

$$\left|\overset{1}{W}{}^{q,A}_{t,x_0}(V_0)\right|'^2_{x_t} \ge \mathrm{e}^{\,cT}|V_0|^2_{x_0} \qquad \text{a.s..}$$

From this we obtain a uniform bound on $\left(\overset{1}{W}{}^{q,A}_{t,x_0}\right)^{-1}$, $0 \le t \le T$, $x \in M$. ∎

A more precise result, obtained by the argument above, gives the key estimate for semigroup domination:

Proposition 3.3.12 *For general M with a regular s.d.e. suppose $\hat\nabla$ is adapted to a metric $\langle , \rangle'$. Set*

$$\rho^q(x) = \inf\{\left\langle \breve R^q(V) - 2d\Lambda^q(\nabla A)(V), V\right\rangle'_x : V \in \wedge^q T_xM, |V|' \le 1\}$$

where ∇ is the Levi-Civita connection for $\langle , \rangle'$. Then

$$|\overset{1}{W}{}^{q,A}_{t,x_0}(V_0)|'_{x_t} \le \mathrm{e}^{\,-\frac12\int_0^t \rho^q(x_s)ds}|V_0|'_{x_0}$$

all $V_0 \in \Lambda^q T_{x_0}M$.

∎

3.3.4 The semigroup on forms: Bochner type vanishing theorems

For any stochastic flow the 'semi-group' on q-forms $P_t\phi = \mathbb{E}\xi_t^*(\phi)\chi_{t<\zeta}$ was considered in §2.4. From Corollary 3.3.5 and Theorem 3.3.7 we know that

$$P_t\phi(V_0) = \mathbb{E}\phi\left(\overset{1}{W}{}^{q,A}_{t,x_0}(V_0)\right)\chi_{t<\zeta(x_0)} \tag{3.3.18}$$

whenever ϕ is a q-form for which $\mathbb{E}\xi_t^*(\phi)(V_0)\chi_{t<\zeta(x_0)}$ exists, $V_0 \in T_{x_0}M$. In case $\hat{\nabla}$ is adapted to some metric and $\rho^q : M \to \mathbb{R}$, defined in Proposition 3.3.12, is bounded below (i.e. $R^q - \frac{1}{2}d\Lambda^q(\nabla A)$ is bounded below) this shows that $P_t\phi$ has an 'extension' to a semigroup on bounded measurable forms defined by the right hand side of (3.3.18): which we will also denote by P_t. For future reference note Itô's formula for $\phi(\bar{V}_t)$ when $\bar{V}_t = \overset{1}{W}{}^{q,A}_{t,x_0}(V_0)$:

$$\begin{aligned}\phi(\bar{V}_t) &= \phi(V_0) + \textstyle\int_0^t \nabla\phi(V_s)(\breve{/\!/}_s d\breve{B}_s)(\bar{V}_s)\\ &+ \textstyle\int_0^t \phi\left(d\Lambda^q\left((\nabla.X)(Y(x_s)\breve{/\!/}_s d\breve{B}_s)\right)(\bar{V}_s)\right)\\ &+ \textstyle\int_0^t \mathcal{A}^q(\phi)(\bar{V}_s)ds\end{aligned} \tag{3.3.19}$$

for ∇ a Levi-Civita connection.

Proposition 3.3.13 *Assume nonexplosion. Suppose $\hat{\nabla}$ is adapted to some metric $\langle,\rangle'$ and the corresponding $\rho^q : M \to \mathbb{R}$ is bounded below and satisfies*

$$\lim_{t\to\infty}\sup\sup_{x\in U}\mathbb{E}\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^q(x_s)ds}\chi_{t<\zeta(x_0)}\chi_K(x_t) = 0 \tag{3.3.20}$$

for all compact sets U and K of M. Then $|P_t\phi|'_x \to 0$ as $t \to \infty$ for all $|-|'$-bounded q-forms ϕ, uniformly on compact subsets of M. In particular, if also there is no explosion, there are no non-zero bounded C^2 forms ϕ with $\mathcal{A}^q\phi = 0$. If further ρ^{q-1} is bounded below and $dP_t\theta = P_t d\theta$ for all compactly supported $(q-1)$-forms then every closed compactly supported q-form vanishes in De Rham cohomology.

Proof. The convergence to 0 is immediate from (3.3.18) and Proposition 3.3.12 which gives the semigroup domination

$$\begin{aligned}|P_t\phi|'_{x_0} &\leq \mathbb{E}\,|\phi|'_{x_t}\,\chi_{t<\zeta(x_0)}\,\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^q(x_s)ds}\\ &\leq \sup_{x\in M}|\phi|'_x \cdot \mathbb{E}\;\mathrm{e}^{-\frac{1}{2}\int_0^t \rho^q(x_s)ds}\,\chi_{t<\zeta(x_0)}\,\chi_{supp(\phi)}(x_t).\end{aligned} \tag{3.3.21}$$

The vanishing of $\mathcal{A}^q$-harmonic bounded forms then follows from the fact (which comes from Itô's formula (3.3.19) and (3.3.21)) that $\mathcal{A}^q\phi = 0$ for ϕ bounded implies $P_t\phi = \phi$. Just as in the proof of Proposition 2.4.1 the commutativity of d and P_t implies that $P_t\phi$ is cohomologous to ϕ when $d\phi = 0$ using (3.3.19). However if σ is a closed q-simplex and ϕ a closed compactly supported q-form the decay of $P_t\phi$ implies the decay of $\int_\sigma P_t\phi$. Thus $\int_\sigma \phi = 0$ and so by De Rham's theorem $[\phi] = 0$ in $H^q(M;\mathbb{R})$. ∎

Corollary 3.3.14 *Suppose M is compact and $\hat{\nabla}$ is adapted to a metric for which $\breve{\mathcal{R}}^q$ satisfies (3.3.20). Then $H^q(M;\mathbb{R}) = 0$. In particular $H^q(M;\mathbb{R}) = 0$ if there is a subbundle E of TM and a metric connection $\breve{\nabla}$ on E with $\hat{\nabla}$ adapted to some metric for which $\breve{R}^q$ is positive.*

Proof. For M compact $P_t\phi = \mathbb{E}\xi_t^*\phi$ for bounded ϕ and $dP_t = P_t d$ follows by differentiating under the expectation using Theorem 3.3.8. The first part then follows from the proposition and the second also using Theorem 1.1.2 to know that $\breve{\nabla}$ is the L-W connection for some s.d.e.. ■

For various versions of such results and their consequences when $\breve{\nabla}$ is the Levi-Civita connection on TM see [ER88] and [ER91]. For relationships between Ric and $\breve{Ric}$ see Corollary C.7 of the Appendix and Remark (ii) following it.

3.3.5 Bismut formulae

The simplest application of the filtering procedure is to obtain Bismut formulae (in fact [EY93] originated from considering (3.3.23) below). We briefly describe a simple case, assuming that M is compact and that we have a regular s.d.e. (1.2.5) such that E is integrable and $A \in \Gamma(E)$. Then E is preserved by $T\xi_t$. If $f : M \to \mathbb{R}$ is C^2 and $T > 0$, Itô's formula for $P_{T-t}f(\xi_t(x_0))$ gives

$$f(x_T) = P_T f(x_0) + \int_0^T d(P_{T-s}f)X(x_s)dB_s. \tag{3.3.22}$$

Multiply both sides by $\int_\alpha^\beta \langle Y(x_s)T\xi_s(v_0), X(x_s)dB_s\rangle_{x_s}$ where $v_0 \in E_{x_0}$ and $0 \le \alpha < \beta \le T$, and taking expectations giving

$$\mathbb{E}f(x_T)\int_\alpha^\beta \langle Y(x_s)T\xi_s(v_0), X(x_s)dB_s\rangle_{x_s} = \mathbb{E}\int_\alpha^\beta d(P_{T-s}f)(T\xi_s(v_0))ds$$
$$= \int_\alpha^\beta d(P_s(P_{T-s}f))(v_0)ds$$

by differentiating under the expectation sign. Thus if $v_0 \in E_{x_0}$

$$d(P_T f)(v_0) = \frac{1}{\beta-\alpha}\mathbb{E}f(x_T)\int_\alpha^\beta \langle T\xi_s(v_0), X(x_s)dB_s\rangle \tag{3.3.23}$$

and so by Theorem 3.3.7,

$$d(P_T f)(v_0) = \frac{1}{\beta-\alpha}\mathbb{E}f(x_T)\int_\alpha^\beta \left\langle W_s^1(v_0), \breve{/\!/}_s d\breve{B}_s\right\rangle_{x_s} \tag{3.3.24}$$

since $X(x_s)dB_s = \breve{/\!/}_s d\breve{B}_s$.

Formulae (3.3.24) now extends by continuity to continuous $f : M \to \mathbb{R}$ and exhibits the smoothing properties of P_T along the leaves of our foliation. From it come formulae for the logarithmic gradient of the heat kernel, proved for Brownian motions and the Levi-Civita connection by Bismut [Bis84]. For this, variations, and non-compact cases, see [EL94], [TW96] and [SZ96]. It is a primitive form of integration by parts formula like (4.1.2) below and can be proved from it (and implies it in the integrable case, as in [EL96]). Similarly (3.3.22) is an explicit form of the Clark-Ocone formula (4.1.3) below.

Chapter 4

Application: Analysis on spaces of paths

A. For our manifold M consider $C_{x_0} = C_{x_0}([0,T], M)$, the space of continuous $\sigma : [0,T] \to M$ with $\sigma(0) = x_0$, equipped with the law $\mu = \mu_{x_0}$ given by our stochastic differential equation (3.0.1).

Since C_{x_0} has a C^∞ Banach manifold structure we can consider C^1 functions $F : C_{x_0} \to \mathbb{R}$. Smooth cylindrical functions are a subclass of these. There is then the (Fréchet) derivative map

$$dF : TC_{x_0} \to \mathbb{R}$$

with

$$(dF)_\sigma : T_\sigma C_{x_0} \to \mathbb{R}$$

a bounded linear map for each $\sigma \in C_{x_0}$.

Any Riemannian metric on M, or Finsler metric $\{|\ |_x : x \in M\}$, determines a Finsler metric on C_{x_0} with norm on $T_\sigma C_{x_0}$:

$$\|V.\|_\sigma^{Fins} = \sup_{0 \leq t \leq T} \|V_t\|_{\sigma(t)}.$$

The norms on $T_\sigma C_{x_0}$ which arise this way all determine the underlying Banachable structure of $T_\sigma C_{x_0}$ and are all equivalent: though not uniformly so in σ if M is not compact. We say F is BC^1 if both F and $dF \in L(TC_{x_0}; \mathbb{R})$ are continuous and bounded, using such a given Finsler norm.

B. Consider first the regular case with $A(x) \in E_x$ for each $x \in M$. Assume that there is no explosion. We shall define the 'tangent space' for μ_{x_0}, relative to $\breve{\nabla}$, at a path σ, to be the subspace $H_\sigma = \breve{H}_\sigma^\mu$ of $T_\sigma C_{x_0}([0,T]; M)$ defined for μ almost all σ by

$$H_\sigma := \left\{ V. \in T_\sigma C_{x_0} \,\middle|\, V_t = W_t^A \int_0^t \left(W_s^A\right)^{-1} \breve{//}_s \dot{h}_s ds, \quad h \in L_0^{2,1}(E_{x_0}) \right\} \tag{4.0.1}$$

where the translations W_t^A, $\breve{//}_s$ are along σ and (W_t^A) is the solution map of (3.3.11).

Give it the Hilbert space structure inherited from $L_0^{2,1}(E_{x_0})$, so it has inner product

$$\langle V_\cdot^1, V_\cdot^2\rangle_\sigma = \int_0^T \left\langle W_t^A \frac{d}{dt}\left[\left(W_t^A\right)^{-1} V_t^1\right], W_t^A \frac{d}{dt}\left[\left(W_t^A\right)^{-1} V_t^2\right]\right\rangle_{\sigma_t} dt \tag{4.0.2}$$

$$= \int_0^T dt \left\langle \frac{\hat{D}}{\partial t}V_t^1 + \frac{1}{2}\breve{\mathrm{Ric}}^\#\left(V_t^1\right) - \breve{\nabla}A\left(V_t^1\right), \frac{\hat{D}}{\partial t}V_t^2 + \frac{1}{2}\breve{\mathrm{Ric}}^\#\left(V_t^2\right) - \breve{\nabla}A\left(V_t^2\right)\right\rangle$$

by (3.3.11). Note that if $V \in T_\sigma C_{x_0}$ then, almost surely, $V_\cdot \in H_\sigma$ if and only if $\left\{\hat{//}_t^{-1} V_t : 0 \le t \le T\right\}$ is absolutely continuous and

$$\frac{\hat{D}}{\partial t}V_t + \frac{1}{2}\breve{\mathrm{Ric}}^\#\left(V_t\right) - \breve{\nabla}A\left(V_t\right) \in E_{\sigma(t)} \tag{4.0.3}$$

for almost all $t \in [0,T]$ with $|V_\cdot|_\sigma$ finite, for $|\ |_\sigma$ defined by (4.0.2). Since $\breve{\mathrm{Ric}}^\#$ and $\breve{\nabla}A$ both map TM to E, (4.0.3) can equivalently be expressed as

$$\frac{\hat{D}}{\partial t}V_t \in E_{\sigma(t)}, \quad 0 \le t \le T, \tag{4.0.4}$$

which is a direct analogue for vector fields of the usual notion of 'horizontality' for paths.

For a BC^1 function $F : C_{x_0} \to \mathbb{R}$ the gradient $\nabla F := \breve{\nabla} F := \nabla_H F$ is then defined as the measurable vector field, defined μ-almost surely, by $\nabla F(\sigma) \in H \equiv \breve{H}_\sigma^\mu$

$$\langle \nabla F(\sigma), V\rangle_\sigma = dF(V\cdot), V \in H.$$

C. For the regular case with A not necessarily a section of E we can define H_σ as in (4.0.1) and (4.0.2) with $\overset{1}{W}{}_t^A$ replacing W_t^A, using the notation of §3.3.2. The analogue of (4.0.3) will hold in the form

$$\frac{\mathbb{D}}{\partial t}V_t \equiv \frac{D^{1'}}{\partial t}V_t + \frac{1}{2}\breve{\mathrm{Ric}}^\#\left(V_t\right) - \nabla^1 A\left(V_t\right) \in E_{\sigma(t)} \tag{4.0.5}$$

and

$$\langle V_\cdot^1, V_\cdot^2\rangle_\sigma = \int_0^T \left\langle \frac{\mathbb{D}}{\partial t}V_t^1, \frac{\mathbb{D}}{\partial t}V_t^2\right\rangle_{\sigma(t)} dt.$$

But in general (4.0.4) will no longer be true. Thus (4.0.5) expresses the "horizontality' of these vector fields. Note that (4.0.5) is intrinsic by Proposition 3.3.9. The spaces H_σ are determined by the measure μ_{x_0} (i.e. by the generator $\mathcal{A}$) and the choice of any metric connection on E with Riemannian metric induced by the principal symbol of $\mathcal{A}$.

Our basic assumptions in this section will be

A(i) M is compact, possibly with boundary ∂M but if so A, X vanish on ∂M.
A(ii) The stochastic differential equation (1.2.5) is regular.

From Proposition 3.3.10 we have immediately

Proposition 4.0.15 *Under assumptions A(i) and A(ii) the gradient ∇F of a BC^1 function $F : C_{x_0} \to \mathbb{R}$ lies in $\mathcal{L}^p$ for $1 \le p < \infty$, i.e.*

$$\int_{C_{x_0}} |\nabla F|_\sigma^p \, d\mu(\sigma) < \infty.$$

If $\hat{\nabla}$ is adapted to some metric on M then $|\nabla F|$ is in $\mathcal{L}^\infty$.

4.1 Integration by parts and Clark-Ocone formulae

Clark-Ocone formulae and integration by parts formulae are closely connected e.g. see [Ű95], [AM95] and [Hsu] and it will be efficient to prove them together.

A vector field V on C_{x_0} with $V(\sigma) \in \breve{H}^\mu_\sigma$ for almost all σ will be said to be *adapted* if there is a version of $\{\frac{\mathbb{D}}{\partial s} V(\sigma)_s : 0 \le s \le T\}$ adapted to $\{\mathcal{F}_t^{x_0} : 0 \le t \le T\}$. If so by $\frac{\mathbb{D}}{\partial s} V(\sigma)_s$ we will always mean such a version.

Theorem 4.1.1 (Integration by parts) *Under assumptions A(i), A(ii) let $F : C_{x_0} \to \mathbb{R}$ be BC^1 and let V be a vector field on C_{x_0} with $V(\sigma) \in \breve{H}^\mu_\sigma$ almost surely which is adapted and has*

$$\int_{C_{x_0}} |V(\sigma)|_\sigma^{1+\epsilon} \, d\mu_{x_0}(\sigma) < \infty. \tag{4.1.1}$$

for some $\epsilon > 0$. Then

$$\int_{C_{x_0}} dF(V(\sigma)) \, d\mu_{x_0}(\sigma) = -\int_{C_{x_0}} F(\sigma) \, div_\mu V(\sigma) \, d\mu_{x_0}(\sigma) \tag{4.1.2}$$

where $div_\mu V : C_{x_0} \to \mathbb{R}$ is given by

$$\begin{aligned} div_\mu V(\sigma) &= \int_0^T \left\langle \overset{1}{W_t^A} \frac{d}{dt} \left[(\overset{1}{W_t^A})^{-1} V(\sigma)_t \right], \ \breve{//}_t d\breve{B}_t \right\rangle \\ &= \int_0^T \left\langle \frac{\mathbb{D}}{\partial t} V(\sigma)_t, \breve{//}_t d\breve{B}_t \right\rangle_{\sigma(t)} \end{aligned}$$

where $\{\breve{B}_t(\sigma), 0 \le t \le T, \sigma \in C_{x_0}\}$ is the martingale part of the stochastic anti-development of the canonical process given by μ on C_{x_0}, using $\breve{\nabla}$. If $\hat{\nabla}$ is adapted to some Riemannian metric on M we can take $\epsilon = 0$ in (4.1.1).

Theorem 4.1.2 (Clark-Ocone formula for possibly degenerate diffusions) *Let $F : C_{x_0} \to \mathbb{R}$ be BC^1. Under assumptions A(i), A(ii) for μ_{x_0} almost all $\sigma \in C_{x_0}$*

$$F(\sigma) = \int_{C_{x_0}} F(\sigma)\, d\mu_{x_0}(\sigma) + \int_0^T \left\langle \mathbb{E}\left\{ \frac{\mathbb{D}}{\partial t}[\nabla_H F]_t \,|\mathcal{F}_t^{x_0}\right\}, \breve{//}_t d\breve{B}_t \right\rangle_\sigma \tag{4.1.3}$$

for $\frac{\mathbb{D}}{\partial t}$ as in (3.3.16).

Proof of Theorem 4.1.1 and 4.1.2. First we will prove Theorem 4.1.1 for a special class of $V_\cdot$. Let $0 \leq t_j < t_{j+1} \leq T$ and let α_j be bounded $\mathbb{R}^m$-valued and $\mathcal{F}_{t_j}^{x_0}$-measurable. Set $k_t = k_t^j$ for

$$k_t^j = (t \wedge t_{j+1} - t \wedge t_j)\alpha_j$$

to give a bounded, $\mathcal{F}_t^{x_0}$-adapted process, with paths in the Cameron-Martin space $L_0^{2,1}([0,T];\mathbb{R}^m)$. Let $\{\xi_t^\tau : 0 \leq t \leq T\}$ be the solution to the stochastic flow of the perturbed stochastic differential equation

$$dy_t = X(y_t) \circ dB_t + A(y_t)dt + \tau X(y_t)\dot{k}_t\, dt \tag{4.1.4}$$

obtained by replacing $B_\cdot$ by $B_\cdot + \tau k_\cdot$ in (1.2.5), $\tau \in \mathbb{R}$. Set $x_t^\tau = \xi_t^\tau(x_0)$. By the Cameron-Martin theorem and Markov property of Brownian motion (or by the Girsanov-Maruyama theorem) the law of $x_\cdot^\tau$ is equivalent to μ_{x_0} and

$$\mathbb{E}\, F(x_\cdot) = \mathbb{E}F(x_\cdot^\tau) \exp\left(-\tau \int_0^T \left\langle \dot{k}_s, dB_s \right\rangle - \frac{1}{2}\tau^2 \int_0^T |\dot{k}_s|^2 ds \right).$$

Differentiating for τ at $\tau = 0$ under the expectation gives

$$\mathbb{E}\, dF(v_\cdot) = \mathbb{E}F(x_\cdot) \int_0^T \left\langle \dot{k}_s, dB_s \right\rangle \tag{4.1.5}$$

where $v_t = \frac{\partial}{\partial \tau} x_t^\tau|_{\tau=0}$, $0 \leq t \leq T$, and so satisfies

$$D^{1'} v_t = \breve{\nabla}_{v_t} X \circ dB_t + \nabla^1_{v_t} A\, dt + X(x_t)\dot{k}_t dt. \tag{4.1.6}$$

with $v_0 = 0$. This goes back to Bismut's approach to Malliavin's calculus [Bis81]; see also [Nor86] where the differentiation under the expectation is carefully justified in a more general case with M not compact.

Let $\bar{v}_t = \mathbb{E}\{v_t\, |\mathcal{F}_t^{x_0}\}$. Just as in the proof of Theorem 3.3.8 we see

$$\frac{D^{1'}\bar{v}_t}{\partial t} = -\frac{1}{2}(\breve{\mathrm{Ric}}_{x_t})(\bar{v}_t) + \nabla^1_{\bar{v}_t} A(x_t) + X(x_t)(\dot{k}_t) \tag{4.1.7}$$

with $\bar{v}_0 = 0$, from which we see by 'variation of parameters' that

$$\bar{v}_t = \overset{1}{W}{}_t^A \int_0^t (\overset{1}{W}{}_s^A)^{-1} X(x_s)\dot{k}_s\, ds. \tag{4.1.8}$$

Set $V(\sigma)_t = \mathbb{E}\{v_t \,|x. = \sigma.\}$ so $V(\sigma)$ is given by (4.1.8) with $x.$ replaced by σ. The left hand side of (4.1.5) therefore reduces to $\int_{C_{x_0}} dF(V(\sigma))\, d\mu_{x_0}(\sigma)$. For the right hand side, by Theorem 3.1.2,

$$\begin{aligned}
\mathbb{E}\left\{ \int_0^T \left\langle \dot{k}_s, dB_s \right\rangle \middle| \mathcal{F}_\cdot^{x_0} \right\} &= \int_0^T \left\langle \dot{k}_s, \tilde{//}_s d\tilde{B}_s \right\rangle \\
&= \int_0^T \left\langle \dot{k}_s, Y(x_s)\breve{//}_s d\breve{B}_s \right\rangle \\
&= \int_0^T \left\langle X(x_s)\dot{k}_s, \breve{//}_s d\breve{B}_s \right\rangle . \qquad (4.1.9)
\end{aligned}$$

Thus (4.1.2) holds for V of the form

$$V(\sigma)_t = \overset{1}{W}{}_t^A \int_0^t (\overset{1}{W}{}_s^A)^{-1} X(x_s)\dot{k}_s ds$$

with $k = k^j$. By linearity it holds when k is any bounded elementary $\mathbb{R}^m$-valued process, adapted to $\mathcal{F}_\cdot^{x_0}$.

Before completing the proof of Theorem 4.1.1 we will prove Theorem 4.1.2. We can assume

$$\int_{C_{x_0}} F(\sigma)\, d\mu_x(\sigma) = 0.$$

For this let g be a bounded elementary process with values in $\mathbb{R}^m$, adapted to $\{\mathcal{F}_t : t \geq 0\}$. For some $c \in \mathbb{R}^m$ set

$$G = c + \int_0^T \langle g_s, dB_s \rangle_{\mathbb{R}^m} .$$

By the martingale representation theorem such G are dense in $L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$. As in (4.1.9)

$$\mathbb{E}\{G \,|\mathcal{F}_\cdot^{x_0}\} = c + \int_0^T \left\langle X(x_s)\dot{k}_s, \, \breve{//}_s \breve{B}_s \right\rangle$$

where $\dot{k}_s = \mathbb{E}\{g_s \,|\mathcal{F}_t^{x_0}\}$ is again bounded and simple. It follows from our special case of Theorem 4.1.1 that

$$\begin{aligned}
&\mathbb{E}F(x.)G = \mathbb{E}\,F(x.) \int_0^T \left\langle X(x_s)\dot{k}_s, \breve{//}_s d\breve{B}_s \right\rangle \\
= \;& \mathbb{E}\, dF \left(\overset{1}{W}{}_\cdot^A \int_0^\cdot (\overset{1}{W}{}_s^A)^{-1} X(x_s)\dot{k}_s ds \right) \\
= \;& \mathbb{E} \left\langle \nabla_H F, \overset{1}{W}{}_\cdot^A \int_0^\cdot (\overset{1}{W}{}_s^A)^{-1} X(x_s)\dot{k}_s ds \right\rangle_{x.} \\
= \;& \mathbb{E} \int_0^T \left\langle \mathbb{E}\left\{ \frac{\mathbb{D}}{\partial t}(\nabla_H F)_t \middle| \mathcal{F}_t^{x_0} \right\}, X(x_t)\dot{k}_t \right\rangle_{x_t} dt
\end{aligned}$$

$$= \mathbb{E}\int_0^T \left\langle \mathbb{E}\{\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t \,|\mathcal{F}_t^{x_0}\}, \breve{/\!/}_t d\breve{B}_t \right\rangle_{x_t} \int_0^T \left\langle X(x_t)\dot{k}_t, \breve{/\!/}_t d\breve{B}_t \right\rangle_{x_t} dt$$
$$= \mathbb{E}\int_0^T \left\langle \mathbb{E}\left\{ \frac{\mathbb{D}}{\partial t}(\nabla_H F)_t \,|\mathcal{F}_t^{x_0} \right\}, \breve{/\!/}_t d\breve{B}_t \right\rangle_{x_t} G,$$

proving (4.1.3), and Theorem 4.1.2.

To complete the proof of Theorem 4.1.1, simply multiply both sides of (4.1.3) by $\mathrm{div}_\mu V$ and take expectations using Proposition 4.0.15 and (4.1.1). ■

Remarks:
For Brownian motion measures μ these integration by parts results go back to Driver [Dri92] in the torsion skew symmetric case. As pointed out in [EL96] in the nondegenerate case our vector fields V are all "tangent processes" in the sense of Driver, for which integration by parts formulae are known see [Dri95], [CM], [AM95], and [Aid97], [Dri97b], and the monograph [Mal91] which gives further references. In the degenerate case a formula for a special class of hypoelliptic diffusions is given in [Lea].

It is shown in [EM] that from Theorem 4.1.1 follows the closability of the form

$$\mathcal{E}(F,G) := \int_{C_{x_0}} \langle \nabla_H F(\sigma), \nabla_H G(\sigma)\rangle_\sigma \, d\mu_{x_0}(\sigma)$$

with domain the BC^1 functions and the result that its closure, $\bar{\mathcal{E}}(F,G)$ say, is a quasi-regular local Dirichlet form on C_{x_0}. In particular there is an associated sample continuous process on C_{x_0}: the generalized Ornstein-Uhlenbeck process determined by the μ_{x_0} and the connection $\breve{\nabla}$ on the Riemannian subbundle $E, \langle,\rangle$ of TM determined by μ_{x_0}. The general results in [EM] give an automatic extension of the integration by parts formula to a class of non-adapted vector fields with values in $\{H_\sigma, \sigma \in C_{x_0}\}$ with an extended definition of $\mathrm{div}_{\mu_{x_0}}$.

The Clark-Ocone formula extends to F in the domain $D(\bar{\mathcal{E}})$ of $\bar{\mathcal{E}}$ and immediately gives "uniqueness of the ground states" for $\bar{\mathcal{E}}$.

Corollary 4.1.3 *If $F \in D(\bar{\mathcal{E}})$ and $\bar{\mathcal{E}}(F,F) = 0$ then F is almost surely constant. In particular if F is BC^1 and $\nabla_H F$ vanishes (or equivalently dF vanishes on H_σ^μ for almost all σ) then F is almost surely constant.*

We have described a family of Hilbert spaces $\breve{H}_\sigma^\mu$ for each metric connection $\breve{\nabla}$ (and vector field A when A is not a section of E). The corollary shows that each family is sufficiently large to give at least the beginning of a Sobolev space theory. It would be interesting to know if each family is in any sense minimal with respect to the property that dF vanishes on $\breve{H}_\sigma^\mu$ for almost all σ implies F almost surely constant. In [EM] the set of *all* tangent processes is shown to be too big to give a gradient and hence a Dirichlet form theory in any obvious way.

4.2 Logarithmic Sobolev Inequality

We can now follow the path mapped out by Capitaine-Hsu-Ledoux [CHL] for the non-degenerate case to obtain the Logarithmic Sobolev inequality for our degenerate diffusions from the Clark-Ocone formula. We include the details, based on [CHL], for completeness.

Theorem 4.2.1 *Under assumptions A(i) and A(ii), the logarithmic Sobolev inequality*

$$\int_{C_{x_0}} F^2(\sigma)\log F^2(\sigma)\, d\mu(\sigma) - \int_{C_{x_0}} F^2(\sigma)d\mu(\sigma)\log\int_{C_{x_0}} F^2(\sigma)d\mu(\sigma) \le 2\int_{C_{x_0}} |\nabla_H F|^2_\sigma d\mu(\sigma)$$

holds for $F \in D(\bar{\mathcal{E}})$.

Proof. It is enough to prove it for a BC^1 function F. Following Capitaine-Hsu-Ledoux [CHL] set

$$\begin{aligned} F_t &:= \mathbb{E}\{F(\xi_\cdot(x_0)) \mid \mathcal{F}_t^{x_0}\} \\ &= \mathbb{E}F + \int_0^t \left\langle \mathbb{E}\{\frac{\mathbb{D}}{\partial s}(\nabla_H F)_s \,|\mathcal{F}_s^{x_0}\}\,,\ \breve{//}_s d\breve{B}_s \right\rangle. \end{aligned}$$

Suppose first that $F > \epsilon > 0$. Then Itô's formula applied to $F\log F$ gives:

$$\mathbb{E}(F\log F) - \mathbb{E}F\log\mathbb{E}F = \frac{1}{2}\mathbb{E}\int_0^T dt\, \frac{|\mathbb{E}\{\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t\,|\mathcal{F}_t^{x_0}\}|^2_{\xi_t(x_0)}}{F_t}.$$

Replace F by F^2 in the above and use the Cauchy-Schwartz inequality to estimate the right hand side:

$$\begin{aligned} |\mathbb{E}\{\frac{\mathbb{D}}{\partial t}(\nabla_H F^2)_t\,|\mathcal{F}_t^{x_0}\}|^2_{\xi_t(x_0)} &= 4\,|\,\mathbb{E}\{\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t F\,|\mathcal{F}_t^{x_0}\}\,|^2_{\xi_t(x_0)} \\ &\le 4\,\mathbb{E}\{F^2\,|\mathcal{F}_t^{x_0}\}\,\mathbb{E}\{\,|\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t|^2\ |\mathcal{F}_t^{x_0}\}. \end{aligned}$$

Consequently there is the logarithmic Sobolev inequality:

$$\begin{aligned} &\int_{C_{x_0}} F^2(\sigma)\log F^2(\sigma)\, d\mu(\sigma) - \int_{C_{x_0}} F^2(\sigma)d\mu(\sigma)\log\int_{C_{x_0}} F^2(\sigma)d\mu(\sigma) \\ \le\ & 2\,\mathbb{E}\int_0^T dt\ \mathbb{E}\{\ |\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t|^2_{\sigma(t)}\ |\mathcal{F}_t^{x_0}\} \\ =\ & 2\,\mathbb{E}\int_0^T |\frac{\mathbb{D}}{\partial t}(\nabla_H F)_t|^2_{\sigma(t)}\, dt. = 2\int_{C_{x_0}} |\nabla_H F|^2_\sigma\, d\mu(\sigma). \end{aligned}$$

For general $F \ge 0$ this holds by using $(F+\epsilon)^2$ instead of F^2 etc and taking the limit. ■

An immediate corollary of the Logarithmic Sobolev inequality is the spectral gap inequality (e.g. see [Bak97]).

Corollary 4.2.2 *For $F \in D(\bar{\mathcal{E}})$,*

$$\int_{C_{x_0}} F^2\, d\mu - \left(\int_{C_{x_0}} F\, d\mu\right)^2 \le \frac{1}{2}\bar{\mathcal{E}}(F,F). \tag{4.2.1}$$

■

Note that the curvature constants which have appeared in the nondegenerate case do not appear here. This is because we use a different inner product on our spaces of admissible tangent vectors, and in this case it is easy to compare these inner products when $\breve{\nabla}$ is metric for some metric with respect to which $\breve{R}$ is bounded. However in the degenerate case we have no given Riemannian metric on M and so no canonical way of estimating curvatures, e,g, $\breve{\mathrm{Ric}}^{\#} : TM \to E$ and $\breve{W}_t$ does not preserve E. The definition of ∇_H used here appears to be the most natural in the degenerate case, and so probably in the non-degenerate case.

4.3 Analysis on $C_{id}(\mathbf{Diff}M)$

A. It was pointed out in [ELJL97b] that the integration by parts formula (4.1.2) was really derived from a 'mother formula' on the space of paths on the diffeomorphism group of M. Here we give that formula together with the resulting 'mothers' for the Clark-Ocone formula and logarithmic Sobolev inequality. As observed in [ELJL97b] the method and formulae are equally valid when the induced stochastic differential equation we use on Diff$M$ is replaced by any right invariant systems on a Hilbert manifold with sufficiently regular group structure. We consider the Gaussian form of Proposition 1.1.3, §3.2, but use $\mathcal{H}$ to denote the reproducing kernel Hilbert space H_γ of sections of E. Recall $\{W_t : 0 \le t < \infty\}$ is the Wiener process on $\Gamma(E)$ which has law γ at time 1.

We assume M is compact, possibly with smooth boundary. Since the connections used here are the right and left invariant connections on Diff$M$ no conditions on our basic stochastic differential equation on M (i.e. on the Gaussian measure γ) are assumed apart from the smoothness of the fields in $\mathcal{H}$ and their vanishing on ∂M.

Let $K.(-) \in L_0^{2,1}([0,T];\mathcal{H})$. Let $\mathcal{D}^s$ be the Hilbert manifold of diffeomorphisms of M of Sobolev class H^s, $s > n/2+3$ which are the identity on ∂M. Consider the random time dependent ordinary differential equation on $\mathcal{D}^s$, parameterized by $\tau \in R$:

$$\begin{aligned} \frac{d}{dt}H_t^\tau &= \tau\ ad(\xi_t^{-1})\frac{dK_t}{dt}, \qquad 0 \le t \le T \\ H_0^\tau &= id, \end{aligned}$$

where $ad(\theta)$ denotes the adjoint action of $\theta \in \mathrm{Diff}M$ on $\Gamma(TM)$, i.e. $ad(\theta)(V) = T\theta(V(\theta^{-1}(\cdot)))$, the push forward $\theta_*(V)$. The solution exists and we can perturb our flow by it to obtain $\xi_t^\tau := \xi_t \circ H_t^\tau$, $0 \le t \le T$. This satisfies the analogue of (4.1.4):

$$d\xi_t^\tau = TR_{\xi_t^\tau} \circ dW_t + TR_{\xi_t^\tau}(A)dt + \tau TR_{\xi_t^\tau}(\dot{K}_t)dt.$$

As for x_t^τ in §4.1 the flows of $\xi_\cdot^\tau$ on $C_{id}([0,T];\mathrm{Diff}M)$ are equivalent and if $F : C_{id}([0,T];\mathrm{Diff}M) \to \mathbb{R}$ is bounded and measurable

$$\mathbb{E}\,F(\xi_\cdot) = \mathbb{E}F(\xi_\cdot^\tau)M^\tau \tag{4.3.1}$$

where

$$M^\tau = \exp\left\{-\tau\int_0^T \left\langle \dot{K}_s, dW_s\right\rangle_{\mathcal{H}} - \frac{1}{2}\tau^2\int_0^T |\dot{K}_s|^2 ds\right\}.$$

To stay flexible we will say that $F : C_{id}([0,T];\mathrm{Diff}M) \to \mathbb{R}$ is C^1 with *suitably bounded derivative* DF if one of the following holds:

Case (i), F is the restriction of a C^1 map $F : C_{id}([0,T];\mathcal{D}^s) \to \mathbb{R}$ where the (Fréchet) derivative $dF : TC_{Id}([0,T];\mathcal{D}^s) \to \mathbb{R}$ is uniformly bounded using the Finsler metric on the tangent space $T_{\theta.}C_{id}([0,T];\mathcal{D}^s)$ at $\theta.$ given by

$$||V||_{\theta.}^{Fins} := \sup_{0\le t\le T} |V_t|_{\theta_t}$$

where $|\cdot|_{\theta_t}$ is the value at θ_t of the left invariant Riemannian structure on $\mathcal{D}^s$ determined by a standard H^s inner-product on $\Gamma(TM)$.

Case (ii), The same as case (i) but with the Finsler metric given by

$$||V||_{\theta.}^{Fins} := \sup_{0\le t\le T} |V_t|_{H^s(\theta_t)}$$

where $H^s(\theta_t)$ is a standard H^s norm on the space of H^s vector fields over the diffeomorphism θ_t, i.e. on H^s sections of $\theta_t^*(TM)$.

Case (iii), The analogue of case (i) with $C^r\mathrm{Diff}M$ replacing $\mathcal{D}^s$ for $1 \le r < \infty$.

Case (iv), The analogue of case (ii) with $C^r(M,M)$ replacing $\mathcal{D}^s$, $0 \le r < \infty$ or $C^r\mathrm{Diff}M$ if $r \ge 1$.

We need two lemmas. The proof of the first is by straightforward calculus, since $\mathcal{H}$ is continuously included in $C^r\Gamma(TM)$ for each r.

Lemma 4.3.1 *Let $\theta : M \to M$ be a C^r diffeomorphism, some $r \ge 1$. Then the adjoint action $ad(\theta)$ of θ on $\mathcal{H}$ is continuous linear as a map $ad(\theta) : \mathcal{H} \to C^{r-1}\Gamma(TM)$ with norm bounded by the C^{r-1} norm of $T\theta$.*

Lemma 4.3.2 *For each of the Finsler norms described above there exists $\Phi^\tau : C_{id}(DiffM) \to \mathbb{R}(\ge 0)$ with $\sup_{0\le\tau\le 1}\Phi^\tau(\xi^\tau)$ in L^p for $1 \le p < \infty$ such that*

$$||T\xi_\cdot^\tau \int_0^\cdot ad(\xi_s^\tau)^{-1}\dot{K}_s ds||_{\xi_\cdot^\tau}^{Fins} \le \Phi^\tau(\xi^\tau)\,|K|_{L_0^{2,1}(\mathcal{H})} \tag{4.3.2}$$

for all $K. \in L_0^{2,1}(\mathcal{H})$, almost surely.

Proof. This follows from the previous lemma and Proposition 3.3.10 using the Sobolev embedding theorem to switch between C^r and H_s norms. ∎

If F is C^1, bounded and with suitably bounded derivative, we can therefore differentiate (4.3.1) at $\tau = 0$ to obtain

$$\mathbb{E}\, dF\left(T\xi_\cdot \int_0^\cdot ad(\xi_s^{-1})\dot{K}_s ds\right) = \mathbb{E}F(\xi_\cdot)\int_0^T \left\langle \dot{K}_s, dW_s\right\rangle_{\mathcal{H}}$$

which can immediately be written in terms of the law $\mu_{\mathcal{D}}$ of $\xi_\cdot$ on Diff$M$, (since $W_t = \int_0^t TR_{\xi_s}^{-1} \circ d\xi_s - \int_0^t TR_{\xi_s}^{-1} A\, ds)$)

$$\begin{aligned}&\textstyle\int_{C_{id}(\mathrm{Diff}M)} dF\left(T\theta_\cdot \int_0^\cdot ad(\theta_s^{-1}\dot{K}_s ds\right) d\mu_{\mathcal{D}}(\theta_\cdot)\\ &= \textstyle\int_{C_{id}(\mathrm{Diff}M)} F(\theta_\cdot) \int_0^T \left\langle \dot{K}_s, dW_s\right\rangle_{\mathcal{H}} d\mu_{\mathcal{D}}(\theta_\cdot).\end{aligned} \tag{4.3.3}$$

As we saw for $C_{x_0}(M)$ this holds true when K is an adapted process with sample paths in $L_0^{2,1}([0,T];\mathcal{H})$ provided that $\mathbb{E}\left(\int_0^T |\dot{K}_s|^2_{\mathcal{H}} ds\right)^{\frac{1+\alpha}{2}} < \infty$ for some $\epsilon > 0$.

C. From (4.3.3) we see that the "tangent space" we obtained for $\mu_{\mathcal{D}}$ at $\theta_\cdot$ is the Hilbert space $H_\theta = H_{\theta_\cdot}^{\gamma,A}$ of $V_\cdot \in T_\theta C_{id}(\mathrm{Diff}M)$ with $ad(\theta_t)\frac{d}{dt}\left[(T\theta_t)^{-1}V_t\right] \in \mathcal{H}$ for almost all $0 \le t \le T$ and having

$$|V_\cdot|_{H_\theta} := \int_0^T \left| ad(\theta_t)\frac{d}{dt}\left[(T\theta_t)^{-1}V_t\right]^2_{\mathcal{H}}\right| dt < \infty.$$

Note that the first condition can be written as

$$\frac{\hat{D}}{dt}V_t \in T\mathcal{R}_{\theta_t}(\mathcal{H})$$

where now we define

$$\frac{\hat{D}}{\partial t}V_t = T\theta_t \frac{d}{dt}\left[(T\theta_t)^{-1}V_t\right]$$

i.e. using the left invariant connection on Diff$M$, in complete analogy with the case of paths on M when M is a Lie group.

For our bounded C^1 function F with suitably bounded derivative, by (4.3.2) we obtain $\nabla_H F(\theta_\cdot) \in H_\theta$ for each θ, satisfying

$$dF(V_\cdot) = \left\langle \nabla_H F(\theta_\cdot), V_\cdot\right\rangle_{H_\theta}, \qquad \text{all } V_\cdot \in H_\theta.$$

By (4.3.2),

$$|\nabla_H F(\theta)|_{H(\theta)} \le \Phi(\theta_\cdot)\, ||dF||_\theta^{Fins}$$

for $\Phi = \Phi^0$. In particular $\nabla_H F$ lies in L^p for $1 \le p < \infty$.

D. Just as in Theorem4.1.2 there is the Clark-Ocone formula

Theorem 4.3.3 *Let $F : C_{id}(\mathit{Diff}M) \to \mathbb{R}$ be bounded and C^1 with suitably bounded derivative. Then*

$$F(\theta.) = \int_{C_{id}(\mathit{Diff}_M)} F(\theta.)d\mu_{\mathcal{D}}(\theta) + \int_0^T \left\langle ad(\theta_t)\mathbb{E}\left\{ \frac{d}{dt}\left[(T\theta_t)^{-1}\nabla_H F_t\right] | \mathcal{F}_t \right\}, dW_t \right\rangle_{\mathcal{H}}$$

almost surely, where $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$.

From this follows, as before, the logarithmic Sobolev inequality:

Theorem 4.3.4 *For F bounded, C^1, and with suitably bounded derivative*

$$\int_{C_{id}(\mathit{Diff}_M)} F^2 \log F^2 d\mu_{\mathcal{D}}(\theta) - \int_{C_{id}(\mathit{Diff}_M)} F^2 \log \int_{C_{id}(\mathit{Diff}_M)} F^2 d\mu_{\mathcal{D}}(\theta)$$
$$\leq 2\int_{C_{id}(\mathit{Diff}_M)} |\nabla_H F(\theta)|^2_{H(\theta)} d\mu_{\mathcal{D}}(\theta).$$

Chapter 5

Stability of stochastic dynamical systems

A. Consider SDE (3.0.1). Let $\{\xi_t\}$ be the solution flow and $T_{x_0}\xi_t : T_{x_0}M \to T_{x_0}M$ the derivative flow for $\xi_t(x_0)$. For $v_0 \in T_{x_0}M$, the almost sure limit $\lim_{t\to\infty} \log |T\xi_t(v_0)|$, called the sample Lyapunov exponent, describes the rates of convergence or divergence of solutions initiated from nearby points. We are also interested in the moment stability determined by the moment exponents:

$$\mu_K(p) = \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} \mathbb{E}|T_x\xi_t|^p \tag{5.0.1}$$

for a subset K of M. The system (3.0.1) is *strongly pth-moment stable* if $\mu_K(p) < \infty$ for all compact sets K. It is *pth moment stable* if $\mu_x(p) \equiv \mu_{\{x\}}(p) < 0$ for all x, *pth moment unstable* if $\mu_x(p) \geq 0$ for every x. Under suitable hypoelliptic conditions for compact manifolds, $\mu_x(p)$ is independent of x [BS88]. See e.g. [Elw88].

There are generalizations of the moment exponents to q-vectors:

$$\mu_K^q(p) = \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in K} \mathbb{E}| \wedge^q T_x\xi_t|^p \tag{5.0.2}$$

with the related concept of *(q,p)-moment stability.*

We shall apply the technique of filtering to obtain estimates on the moment exponents, extending that in Li [Li94a] for gradient systems, (with corresponding homotopy vanishing result extending Elworthy-Rosenberg [ER96]). We also use the L-W connection to give a neat form to a Carverhill's version of Khasminskii's formula, and show that in certain situations an L^∞ condition on the derivative flow implies that $\hat\nabla$ is metric form some metric on M.

B. Assume that X has constant rank and image $E \subset TM$ with $\breve\nabla$ the associated L-W connection for X. Write $E = Im(X)$ and define $H_p^q : \wedge^q E \to \wedge^q E$ by

$$\begin{aligned} H_p^q(V,V) = \;& \textstyle\sum_{i=1}^m \frac{1}{|V|'^2} |d\Lambda^q(\breve\nabla X^i)(V)|'^2 \\ & +(p-2)\textstyle\sum_1^m \frac{1}{|V|'^4} \left\langle V, d\Lambda^q(\breve\nabla X^i)(V)\right\rangle'^2 \\ & +\frac{1}{|V_s|'^2}\left\langle V, -(\breve R_x^q)^*(V)\right\rangle' + 2\int_0^t \frac{\left\langle (d\Lambda^q \breve\nabla A)(V), V\right\rangle'^2}{|V|'^2} ds. \end{aligned} \tag{5.0.3}$$

Let $\mathcal{P}^q_x$ be the set of primitive vectors in $\wedge^q E_x$ and set

$$\underline{h}^q_p(x) = \inf\{\, p\, H^q_p(V,V) : V \in \mathcal{P}^q_x, |V|' = 1\}$$

and

$$h^q_p(x) = \sup\{\, p\, H^q_p(V,V) : V \in \mathcal{P}^q_x, |V|' = 1\}.$$

Then,

Theorem 5.0.5 *Assume the stochastic differential equation does not explode, $A(x) \in E_x$ for each x, and that $\hat{\nabla}$ is metric with respect to a metric $\langle , \rangle'$ on TM.*

Then for $p \in R$, $V_0 \in \mathcal{P}^q_{x_0}$ and $V_t = \wedge^q T\xi_t(V_0)$,

$$|V_0|'^p\, \mathbb{E} \exp\left(\frac{1}{2}\int_0^t \underline{h}^q_p(x_s) ds\right) \le \mathbb{E}|V_t|'^p \le |V_0|'^p\, \mathbb{E} \exp\left(\frac{1}{2}\int_0^t h^q_p(x_s) ds\right). \tag{5.0.4}$$

Here the norm $|-|'$ corresponds to the metric $\langle , \rangle'$.

Proof. First note that by (3.3.9) $\hat{D}V_t \in \wedge^q E$. Also $\wedge^q T\xi_t$ maps primitive vectors to primitive vectors. By Itô's formula and Lemma 3.3.6,

$$\begin{aligned} |V_T|'^p = \ & |V_0|'^p + \textstyle\sum_1^m p \int_0^T |V_s|'^{p-2} \left\langle V_s, d\Lambda^q \left(\breve{\nabla} X(-) dB_s\right)(V_s)\right\rangle' \\ & + \tfrac{p}{2}\textstyle\int_0^T |V_s|'^p \bar{H}^q_p(V_s,V_s) ds, \end{aligned} \tag{5.0.5}$$

where

$$\begin{aligned} \bar{H}^q_p(V,V) = \ & \textstyle\sum_{i=1}^m \frac{1}{|V|'^2} |d\Lambda^q(\breve{\nabla} X^i)(V)|'^2 \\ & + (p-2)\textstyle\sum_1^m \frac{1}{|V|'^4} \left\langle V, d\Lambda^q(\breve{\nabla} X^i)(V)\right\rangle'^2 \\ & + \tfrac{1}{|V|'^2}\left\langle V, -(\breve{R}^q_x)^*(V)\right\rangle'. \end{aligned} \tag{5.0.6}$$

Then

$$H^q_p(V,V) = \bar{H}^q_p(V,V) + 2\int_0^t \frac{\left\langle (d\Lambda^q \breve{\nabla} A)(V), V\right\rangle'}{|V|'^2} ds. \tag{5.0.7}$$

Let $W_s = \frac{V_s}{|V_s|'^2}$ be the process projected on the unit sphere $S(\Lambda^q TM)$ and set $M_t = p\sum_{i=1}^m \int_0^t \left\langle W_s, (d\Lambda^q \breve{\nabla} X^i)(W_s)\right\rangle' dB^i_s$. Then

$$|V_t|'^p = |V_0|'^p \exp\left(M_t - \frac{1}{2}\langle M\rangle_t + \frac{p}{2}\int_0^t H^q_p(W_s,W_s) ds\right). \tag{5.0.8}$$

Let $(\tilde{x}_t, \tilde{V}_t)$ be the solution and the derivative process of the stochastic differential equation

$$d\tilde{x}_t = X(\tilde{x}_t) \circ d\tilde{B}_t + A(\tilde{x}_t) dt$$

where

$$\tilde{B}^i_t = B^i_t - p\int_0^t \left\langle W_s, (d\Lambda^q)\breve{\nabla} X^i(-))(W_s)\right\rangle' ds.$$

Then $x.$ and $\tilde{x}.$ have the same distribution from the defining property of $\breve{\nabla}$. By the Girsanov-Cameron-Martin theorem, if $\tilde{W}_s = \frac{\tilde{V}_s}{|\tilde{V}_s|'}$,

$$\mathbb{E}|V_t|'^p = |V_0|'^p \, \mathbb{E} exp\left(\frac{p}{2}\int_0^t H_p^q(\tilde{W}_s, \tilde{W}_s)ds\right)$$

and so it follows that

$$\mathbb{E}\, \exp\left(\frac{1}{2}\int_0^t \underline{h}_p^q(x_s)ds\right) \le \mathbb{E}[\frac{|V_t|'^p}{|V_0|'^p}] \le \mathbb{E}\, \exp\left(\frac{1}{2}\int_0^t h_p^q(x_s)ds\right).$$

■

Remarks :

(i). The quantity $\breve{\nabla}X^i$ which appears here and below is a generalization of the shape operator of a submanifold of $\mathbb{R}^m$. Indeed in the gradient system case

$$\left\langle \breve{\nabla}_v X^i, u\right\rangle_x = \langle \alpha(u,v), e_i\rangle_{\mathbb{R}^m}$$

where α is the second fundamental form of the embedding determining X. See [Li94a], [ER96] for geometric implications of Corollary 5.0.6 in the gradient case.

(ii). Fix $x \in M$ and let $\langle , \rangle$ be an inner product on T_xM extending that given by our s.d.e. on E_x. Let $\{e_1, \ldots, e_n\}$ be an orthonormal base for $T_{x_0}M$, $\langle , \rangle$. As observed for the Levi-Civita connection in [ER96], see also [Ros97], the Weitzenbock curvatures acting on primitive vectors $V = e_1 \wedge \ldots \wedge e_q$, say, satisfy

$$\left\langle (\breve{R}^q)^* V, V\right\rangle = \sum_{j=1}^{q}\sum_{l\ge q+1}^{\lambda} \breve{K}(e_j, e_l')$$

where $\breve{K}$ is the sectional curvature defined by

$$\begin{aligned}\left\langle \breve{\mathcal{R}}(v\wedge w), v\wedge w\right\rangle &= \breve{K}(v,w)\\ &(= 0 \quad \text{unless } v, w \in E_x)\end{aligned}$$

for $\breve{\mathcal{R}}_x : \wedge^q TM \to \wedge^q E_x$ the curvature operator and $\{e_1', \ldots, e_\lambda'\}$ together with $E_x \cap \{e_1, \ldots, e_q\}$ gives an orthonormal base for E_x.

Proof. First by Corollaries B.2 and C.5 of the Appendix,

$$\begin{aligned}&\sum_{r=1}^{m}\left\langle \delta^2\Lambda(\breve{\nabla}.X^r)(V), V\right\rangle\\ &= -2\sum_{r=1}^{m}\sum_{i<k,j<k}\left\langle \breve{\nabla}_{e_j}X^r \wedge \breve{\nabla}_{e_k}X^r, e_i\wedge e_l\right\rangle \langle a_j a_k V, a_l a_i V\rangle\\ &= 2\sum{}^{E}_{1\le j<k\le q}\left\langle \breve{\mathcal{R}}_x(e_j\wedge e_k), e_j\wedge e_k\right\rangle\end{aligned}$$

where $\sum^E$ refers to summation only over these j, k with e_j, e_k in E_x.

On the other hand

$$\sum_{r=1}^{m}\left\langle (d\Lambda^q)\breve{R}(X^r,-)X^rV,V\right\rangle = \sum_{r=1}^{m}\sum_{j=1}^{q}\left\langle \breve{R}(X^r,e_j)(X^r),e_j\right\rangle.$$

The result follows by choose $\{X^r(x)\}_{r=1}^{dim(E)}$ to be the given orthonormal base for E_x.
■

The following corollary is immediate using [ER96] for the last part.

Corollary 5.0.6 *Assume the stochastic differential equation has no explosion and Suppose that $\hat{\nabla}$ is metric with respect to a metric $\langle,\rangle'$ on TM. Then the stochastic dynamical system is strongly (q,p)-moment stable, with respect to the metric $\langle,\rangle'$, if*

$$\limsup_{t\to\infty}\frac{1}{t}\sup_{x\in K}\log\mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_p^q(\xi_s(x))ds\right) < 0, \qquad \textit{all } K \textit{ compact.}$$

It is not strongly (q,p)-moment stable if $\underline{h}_p^q \geq 0$. In particular if

$$\limsup_{t\to\infty}\sup_{x\in M}\frac{1}{t}\log\mathbb{E}\exp\left(\frac{1}{2}\int_0^t h_1^q(\xi_s(x))ds\right) < 0$$

and M is compact then the homology group $H_q(M,Z)$ vanishes.

Remarks: (i). In fact we have a more general formula than (5.0.5): Let $\tilde{\nabla}$ be a semi-connection, metric with respect to $\langle,\rangle'$. Write $X^0 \equiv A$ for simplicity and denote by $S^i(t,x)$ the flow of the vector fields X^i, and $TS^i(t,x)(v)$ the derivative flow. Then

$$\begin{aligned} |V_T|'^p = \; & |V_0|'^p + \textstyle\sum_1^m p\int_0^T |V_s|'^{p-2}\left\langle V_s, \frac{\tilde{D}}{\partial t}\wedge^q TS^i(t,x_s)(V_s)|_{t=0}\right\rangle' dB_s^i \\ & +\frac{p}{2}\int_0^T |V_s|'^p \bar{H}_p^q(V_s,V_s)ds, \end{aligned} \tag{5.0.9}$$

where

$$\begin{aligned} \bar{H}_p^q(V,V) = \; & \textstyle\sum_{i=1}^m \frac{1}{|V|'^2}\left|\frac{\tilde{D}}{\partial t}\Lambda^q TS_t^i(V)|_{t=0}\right|'^2 \\ & +(p-2)\textstyle\sum_1^m \frac{1}{|V|'^4}\left\langle V, \frac{\tilde{D}}{\partial t}\Lambda^q TS_t^i(V)|_{t=0}\right\rangle'^2 \\ & +\textstyle\sum_1^m \frac{1}{|V|'^2}\left\langle V, \frac{\tilde{D}^2}{\partial t^2}\Lambda^q TS^i(t,x)(V)|_{t=0}\right\rangle'. \end{aligned} \tag{5.0.10}$$

This is deduced by an Itô's formula in Elworthy [Elw88], see also Elworthy-Rosenberg [ER96] for gradient systems. Now take $\tilde{\nabla} = \hat{\nabla}$ and observe that

$$\begin{aligned} \frac{\hat{D}}{\partial t}TS^i(t,x)(v) &= \hat{\nabla}X^i(S^i(t,x))\left(TS^i(t,x)(v)\right) - \breve{T}\left(X^i(S^i(t,x)), TS^i(t,x)(v)\right) \\ &= \breve{\nabla}X^i\left(S^i(t,x)\right)\left(TS^i(t,x)(v)\right), \end{aligned}$$

and similarly

$$\frac{\hat{D}}{\partial t}(\Lambda^q TS_t^i(t,x)(V)) = (d\Lambda^q \breve{\nabla} X^i)\left(\Lambda^q TS^i(t,x)(V)\right). \tag{5.0.11}$$

Formula (5.0.5) now follows from Lemma 2.4.4, and (5.0.11).

(ii). If $A \notin E$, let ∇^1 be any extension of $\breve{\nabla}$ as in §1.3B and $\nabla^{1\prime}$ be its adjoint. Suppose $\nabla^{1\prime}$ is metric with respect to a metric $\langle , \rangle\ '$on TM. Then the above result holds with H_p^q replaced by

$$H_p^q(V,V) = \bar{H}_p^q(V,V) + 2\int_0^t \frac{\langle ((d\Lambda^q)\nabla^1 A)(V), V\rangle'}{|V|'^2} ds.$$

C. From Theorem 3.3.8 and (3.3.17) we see that in the nonsingular case with M compact if $\hat{\nabla}$ is adapted to some Riemannian metric on M then the conditional expectations $\mathbb{E}\{T_{x_0}\xi_t | \mathcal{F}^{x_0}\}$ and $\mathbb{E}\{T_{x_0}\xi_t^{-1} | \mathcal{F}^{x_0}\}$ are bounded processes in $(t,\omega) \in [0,T]\times\Omega$, any $T > 0$. This boundedness can be with respect to any Riemannian metric on M, by compactness. In particular in the nondegenerate case these two processes conditioned on $\xi_T(x_0) = y_0$, will also be almost surely bounded on $[0,T]\times\Omega$, any $y_0 \in M$. (As described in the proof of theorem below we can make sense of this for all, not just almost all y_0.) The following partial converse to this shows that the existence or not of a metric to which $\hat{\nabla}$ is adapted is reflected rather drastically in the behaviour of the flow.

Theorem 5.0.7 *Suppose our s.d.e (3.0.1) is nondegenerate and has no explosion. Fix $x_0 \in M$ and $T > 0$. Let $\{\xi_t(x_0) : 0 \le t < \infty\}$ be the solution from x_0, with $\{T_{x_0}\xi_t : t \ge 0\}$ the derivative process at x_0. Suppose from $y_0 \in M$ the local conditional expectations*

$$\mathbb{E}\{T_{x_0}\xi_t | \mathcal{F}^{x_0}\} \quad \textit{and} \quad \mathbb{E}\{T_{x_0}\xi_t^{-1} | \mathcal{F}^{x_0}\}, \quad 0 \le t \le T,$$

when conditioned on $\xi_T(x_0) = y_0$ are in L^∞ uniformly in $t \in [0,T]$, for some Riemannian metric on M for which $\breve{Ric}^{\#} - \breve{\nabla}A$ is bounded. Then $\hat{\nabla}$ is a metric connection for some metric on M.

Note: The conditioned process are the same in law as the conditional expectations, given the bridge process, of the processes $T_{x_0}\xi_t$ and $T_{x_0}\xi_t^{-1}$, respectively, conditioned on $\xi_T(x_0) = y_0$. (Their expectations at time T are $\mathbb{E}\{T_{x_0}\xi_t | \xi_T(x_0) = x_0\}$ and $\mathbb{E}\{T_{x_0}\xi_t^{-1} | \xi_T(x_0) = x_0\}$.)

Proof. By Theorem 3.3.7 and (3.3.17) the local conditional expectations are given by W_t and W_t^{-1} respectively. From [Car88], for example, the corresponding conditioned processes $\{W_{y_0,t} : 0 \le t \le T\}$, $\{W_{y_0,t}^{-1} : 0 \le t \le T\}$ are given by the equations (3.3.11) and (3.3.17), for $q = 0$, but with $x_t = \xi_t(x_0)$ replaced by the bridge, $\xi_{y_0,t}(x_0)$, i.e. $\xi_\cdot(x_0)$ conditioned to be at y_0 at time T. From these equations they are seen to be mutually inverse. Let $\{\hat{//}_t : 0 \le t \le T\}$ be parallel translation along the bridge using $\hat{\nabla}$. We have

$$\frac{d}{dt}\left(W_{y_0,t}^{-1}\hat{//}_t\right) = \frac{1}{2}W_{y_0,t}^{-1}\breve{Ric}^{\#}\hat{//}_t - W_{y_0,t}^{-1}\breve{\nabla}A\hat{//}_t$$

Whence $W_{y_0,t}^{-1}\hat{//}_t$ is in L^∞ uniformly in $t \in [0,T]$ for the given metric. From this our assumption on W_t implies that $\{\hat{//}_t : 0 \le t \le T\}$ is in L^∞. The result follows from Theorem 1.3.8. ■

D. Finally we give the following Khasminski formula; c.f. [Car85], [Elw88],

Proposition 5.0.8 *Suppose M is compact and our s.d.e. is nondegenerate and $\hat{\nabla}$ is metric with respect to a metric $\langle , \rangle'$. Let $S^q(TM)$ be the unit sphere subbundle of $\wedge^q TM$, ν an ergodic invariant measure for the process induced on $S^q(TM)$, and $\lambda^n \le \ldots \le \lambda^1$ the corresponding Lyapunov exponents of the solution flow. Then for some choice of ν,*

$$\lambda^1 + \ldots + \lambda^q = -\int_{S(TM)} \sum_1^m \left\langle V, (d\Lambda)^q(\breve{\nabla}X^i)(V)\right\rangle'^2 d\nu$$
$$+\frac{1}{2}\int_{S(TM)} \sum_{i=1}^m |(d\Lambda)^q(\breve{\nabla}X^i)(V)|'^2 d\nu$$
$$-\frac{1}{2}\int_{S(TM)} \left\langle V, (\breve{R}^q)^*(V)\right\rangle' d\nu.$$

Proof. From [Car85], there exists an ergodic ν such that $\lambda^1 + \ldots + \lambda^q = \lim_{t\to\infty} \frac{1}{t}\log|V_t(\omega)|$ for $\nu * P$ almost all (V_0, ω) in $S^q(TM) \times \Omega$. For such V_0 we apply Itô's formula to $\log|V_t|'$ and use (5.0.3)

$$\log|V_t|' = \log|V_0|' + \int_0^t \frac{d|V_s|'}{|V_s|'} - \frac{1}{2}\int_0^t \frac{d|V_s|' d|V_s|'}{|V_s|'^2}$$
$$= \log|V_0| + \int_0^t \frac{(|V_s|')^{-1}\sum_1^m \left\langle V_s, (d\Lambda^q\breve{\nabla}X^i)V_s\right\rangle' dB_s^i}{|V_s|'}$$
$$-\frac{1}{2}\int_0^t \frac{(|V_s|')^{-2}\sum_1^m \left\langle V_s, (d\Lambda^q\breve{\nabla}X^i)V_s\right\rangle'^2}{|V_s|'^2} ds$$
$$+\frac{1}{2}\int_0^t \sum_{i=1}^m \frac{1}{|V_s|'^2}|d\Lambda^q(\breve{\nabla}X^i)(V_s)|'^2 ds$$
$$-\frac{1}{2}\int_0^t \sum_{i=1}^m \frac{1}{|V_s|'^4}\left\langle V, d\Lambda^q(\breve{\nabla}X^i)(V)\right\rangle'^2 ds$$
$$+\frac{1}{2}\int_0^t \sum_{i=1}^m \frac{1}{|V_s|'^2}\left\langle V_s, -(\breve{R}_x^q)^*(V_s)\right\rangle' ds$$

$$
\begin{aligned}
= \;& \log|V_0| + \int_0^t \sum_1^m \left\langle W_s, (d\Lambda^q \breve{\nabla} X^i) W_s \right\rangle' dB_s^i \\
& - \int_0^t \sum_1^m \left\langle W_s, (d\Lambda^q \breve{\nabla} X^i) W_s \right\rangle'^2 ds + \frac{1}{2}\int_0^t \sum_{i=1}^m |d\Lambda^q(\breve{\nabla} X^i)(W_s)|'^2 ds \\
& + \frac{1}{2}\int_0^t \sum_{i=1}^m \left\langle W_s, -(\breve{R}_x^q)^*(W_s) \right\rangle' ds.
\end{aligned}
$$

Taking the limit and by the standard ergodic theorem we have the result. ■

Proposition 5.0.9 *Suppose X is injective, and $\hat{\nabla}$ is adapted to some metric on M and $A(x) \in E_x$ for each x with $\breve{\nabla} A \equiv 0$ (or more generally A is a Killing field for that metric). Then all the Lyapunov exponents vanish.*

Proof. In the non-degenerate case with $\breve{\nabla} A \equiv 0$ this is immediate from Proposition 5.0.8 since all terms in the integrand there vanish. The general case comes from Corollary 1.3.6 which implies that the flows ξ_t will consist of isometries. Note that $\breve{\nabla} A \equiv 0$ implies that $A(x) = \sum_{j=1}^m \alpha_j X^j(x)$ some $\alpha_j \in \mathbb{R}$, and hence that A is a Killing field if each X^j is. ■

Remarks (i). Under certain hypoellipticity condition on the stochastic differential equation and its derivative flow Baxendale showed in [Bax86] that vanishing of all the Lyapunov exponents implies that the flow consists of isometries for some Riemannian metric on M. He also showed that equality of the exponents holds if and only if there is a metric on M for which the flow consists of conformal transformations. See also [BS88].

(ii). We do not have examples satisfying the hypothesis of the proposition other than left or right invariant systems on Lie groups.

Chapter 6

Appendices

A Universal Connections as L-W connection

A. Let $\pi : E \to M$ be a vector bundle over a manifold M with a surjective bundle map X from the trivial bundle $\pi_0 : M \times \mathbb{R}^m \to M$, to π. The adjoint of the bundle map is denoted by $Y : E^* \to M \times \mathbb{R}^m$. We give E the induced metric so $XY : E^* \to E$ is an isometry, and use the metric to identify E^* with E, so π can be considered as a subbundle of π_0 via Y. The fibre of E above x will be written E_x, and similarly for other bundles.

Let $O(\pi)$ be the bundle of orthonormal frames of π with structure group $O(p)$ for $p =$dim E_x, $O(\pi_0)$ the principal bundle $O(\pi_0) : M \times O(m) \to M$ with structure group $O(m)$. The adapted frame bundle $O(\pi_0, \pi) : O(M; M) \to M$ is the bundle of adapted frames in $O(\pi_0)$, i.e. $O(M; M)_x$ consists of frames (x, u), $u \in O(m)$ such that $u\,(\mathbb{R}^p \times 0) = Y(E_x)$.

$$\begin{tikzcd}
M \times O(m) \arrow[d, "O(m)"] & O(M, M) \arrow[l] \arrow[r, "h"] \arrow[d, "O(p) \times O(m-p)"] & O(E) \arrow[d, "O(p)"] \\
M \arrow[r, leftrightarrow, "\mathrm{id}"] & M \arrow[r, leftrightarrow, "\mathrm{id}"] & M
\end{tikzcd}$$

B. Denote by ω_0 the trivial connection form on $O(\pi_0)$, i.e. $\omega_0(w, v) = TL_a^{-1}v = a^{-1}v$ for each $(w, v) \in T_{(x,a)}(M \times O(m))$. The induced connection ω_{00} on $O(\pi, \pi_0)$ is the restriction of ω_0 to $O(\pi, \pi_0)$ followed by the projection onto the Lie algebra $o(p) \times o(m-p)$. By Proposition 1.2 of volume II and Proposition 6.4 of volume I of Kobayashi and Nomizu ([KN69a],[KN69b]) we see that ω_{00} is a connection because ad$(O(p) \times O(m-p)$ sends the complement $o(p, m-p)$ of $o(p) + o(m-p)$ to itself.

Define

$$h : O(\pi_0, \pi) \to O(\pi), \qquad \text{by}$$
$$u \mapsto X(x) \circ u\,|_{\mathbb{R}^p} .$$

Here we are using u for both the frame and its principal part. Let ω be the unique connection on $O(\pi)$ satisfying (c.f. p.79 of Kobayashi-Nomizu [KN69a]):

1. It is the only connection such that the horizontal subspaces of ω_{00} are mapped into the horizontal subspaces of ω by h.

2. $h^*\omega = h \cdot \omega_{00}$, where $h \cdot \omega_{00}$ is the $o(p)$ component of ω_{00}.

This unique connection ω on $O(\pi)$ (related to X) is the universal connection in the sense of Narisimhan and Ramanan when M is the Grassmann manifold and π the universal bundle, as is discussed later.

C. With no loss of generality we assume $Y(E_x) = \mathbb{R}^p \times 0$. Let σ be a curve with $\sigma(0) = x$ and $\dot\sigma(0) = v$, with horizontal lifts in $O(\pi_0)$, $O(\pi_0, \pi)$ and $O(\pi)$ respectively $\tilde\sigma_0(t) = (\sigma(t), \tilde\sigma_0^1(t))$, $\tilde\sigma_{00}(t) = (\sigma(t), \tilde\sigma_{00}^1(t))$, and $\tilde\sigma(t) = (\sigma(t), \tilde\sigma^1(t))$, for $\tilde\sigma_{00}^1(0) = \tilde\sigma_0^1(0) = Id$, and $\tilde\sigma^1(0) = h(\tilde\sigma_{00}(0))$. The covariant differentiation coming from ω will be denoted by ∇. We shall show this is in fact the L-W connection corresponding to X.

Recall that if ξ is a section of E and $v \in T_xM$, the L-W connection determined by X is given by

$$\breve\nabla_v \xi(x) = X(x) d\left[Y(\cdot)\xi(\cdot)\right](v) \tag{A.1}$$

and is the unique metric connection such that

$$\text{if } e \in [\text{Ker} X(x)]^\perp, \quad \text{then } (\nabla X^e)_x = 0.$$

Here $X^e(\cdot) = X(\cdot)(e)$.

Proposition A.1 *The L-W connection $\breve\nabla$ induced by X is the unique connection ω on $O(\pi)$ related to X as in §B.*

Proof. Fix $x \in M$. Take $e \in \text{Im}(Y(x))$. We only need to show that

$$(\nabla X^e)_x = 0.$$

Here $X(x)^e = X(x)(e)$. By definition,

$$(\nabla X^e)_x (v) = \tilde\sigma(0) \frac{d}{dt} (\tilde\sigma(t))^* X^e(\sigma(t))_{|t=0} .$$

But by the definition of the induced connection ω,

$$\tilde\sigma(t) = h(\tilde\sigma_{00}(t)) = X(\sigma(t))\left(\tilde\sigma_{00}(t)|_{\mathbb{R}^p}\right).$$

So

$$(\tilde{\sigma}(t))^* = P_{\mathbb{R}^p}\left(\tilde{\sigma}_{00}(t)\right)^* Y\left(\sigma(t)\right),$$

where $P_{\mathbb{R}^p}$ is the projection from $\mathbb{R}^m$ to $\mathbb{R}^p$. Consequently

$$\begin{aligned}
\left(\nabla X^e\right)_x(v) &= \tilde{\sigma}(0)\frac{d}{dt}P_{\mathbb{R}^p}\left(\tilde{\sigma}_{00}(t)\right)^* Y\left(\sigma(t)\right) X^e\left(\sigma(t)\right)\Big|_{t=0} \\
&= \tilde{\sigma}(0)P_{\mathbb{R}^p}\left[\left(\tilde{\sigma}_{00}(0)\right)^*\right]\frac{d}{dt}Y(\sigma(t))X\left(\sigma(t)\right)(e)\Big|_{t=0} \\
&\quad +\sigma(0)P_{\mathbb{R}^p}\left[\frac{d}{dt}\left(\tilde{\sigma}_{00}(t)\right)^*\right]|_{t=0}(e), \\
&= \sigma(0)P_{\mathbb{R}^p}\left[\frac{d}{dt}\left(\tilde{\sigma}_{00}(t)\right)^*\right]|_{t=0}(e).
\end{aligned}$$

The last step comes from the fact that $Y(x)X(x)$ is the projection P to $Y(E_x)$ and $P(0)\dot{P}(0)P(0) = 0$.

The required conclusion will now follow from the following lemma:

Lemma A.2

$$\sigma(0)P_{\mathbb{R}^p}\left[\frac{d}{dt}\left(\tilde{\sigma}_{00}(t)\right)^*\right]|_{t=0}Y(\sigma) = 0.$$

Proof. By skew adjointness,

$$\frac{d}{dt}\left[\tilde{\sigma}_{00}(t)\right]^*|_{t=0} = -\dot{\sigma}^1_{00}(0).$$

Since $\tilde{\sigma}_{00}(t)$ is horizontal,

$$\begin{aligned}
0 = \omega_{00}\left(\dot{\tilde{\sigma}}^1_{00}(t)\right) &= o(p)\times o(m-p) - \text{component of } \omega_0\left(\dot{\tilde{\sigma}}^1_{00}(t)\right) \\
&= o(p)\times o(m-p) - \text{component of } \left(\tilde{\sigma}^1_{00}(t)\right)^{-1}\dot{\tilde{\sigma}}^1_{00}(t).
\end{aligned}$$

So $\left(\tilde{\sigma}^1_{00}(t)\right)^{-1}\dot{\tilde{\sigma}}^1_{00}(t)$ belongs to $o(p, m-p)$. In particular so does $\dot{\tilde{\sigma}}^1_{00}(0)$. The result follows. ■

D. Let $M = G(p,q)$ be the Grassmann manifold of p-planes in $\mathbb{R}^m$ for $m = p+q$, and $\pi : B \to M$ the vector bundle over M where the fibre B_L at $L \in G(n,p)$ is the set of points in L. It is a Riemannian bundle. Let Y^U be the vector bundle map from π to the trivial bundle $\underline{\mathbb{R}}^m$ defined by $Y^U(L)(x) = i_L(x)$ for each $x \in B_L$. Here i_L is the inclusion of L onto $\mathbb{R}^m$. Its adjoint map X^U is given by $X^U(L) = P_L$, the orthogonal projection. Thus B is identified with a subbundle of $\underline{\mathbb{R}}^m$.

Let $V(p,q)$ be the set of p-frames in $\mathbb{R}^m$,i.e.

$$V(p,q) = \left\{(e_1,\ldots,e_p)\,|e_i \in \mathbb{R}^m, \langle e_i, e_j\rangle = \delta_{i,j}\right\}.$$

This is the Stiefel manifold of p-frames in $\mathbb{R}^m$. With structure group $O(p)$ it is the Stiefel bundle over $G(p,q)$. It is in fact the associated principal bundle of B.

Consider the principal bundle $\pi : O(m) \to G(p,q)$ with structure group $O(p) \times O(q)$, where

$$\pi(T) \equiv T\left[\mathbb{R}^p \times 0\right] \subset \mathbb{R}^m$$

for each $T \in O(m)$. It turns out to be the bundle of adapted frames of $\pi : V(p,q) \to G(p,q)$ in $\pi_0 : G(p,q) \times O(m) \to G(p,q)$. This is because $T \in O(m)$ is adapted if and only if $T\left[\mathbb{R}^p \times 0\right] = L = B_L$. We are now in the picture of earlier discussions.

$$\begin{tikzcd}
M \times O(m) \arrow[d, "O(m)"] & O(m) \arrow[l] \arrow[r, "h"] \arrow[d, "O(p) \times O(q)"] & O(B) \arrow[d, "O(p)"] \\
M \arrow[r, leftrightarrow, "\mathrm{id}"] & M \arrow[r, leftrightarrow, "\mathrm{id}"] & M
\end{tikzcd}$$

Now $h : O(m) \to V(p,q)$ is given by $h(T) = T(\mathbb{R}^n)$. As before there is the connection ω on $O(\pi)$.

Now $V(p,q) \to G(p,q)$ has a canonical connection ω_U as described by Narasimhan and Ramanan [NR61] as "the universal connection" via the connection form $S^T dS$, where S maps $V(p,q)$ to m rows and p columns matrices in the following way: if $\alpha \in V(p,q)$ with $\alpha_i = \sum_{j=1}^m v_{i,j} e_j$, where (e_j) is an orthonormal basis of $\mathbb{R}^m$, $S(\alpha) = (v_{j,i})$. Denote by $S(\alpha)^T$ its transpose. Set

$$\omega_U = S^T dS.$$

Then ω_U is a $o(p)$-valued 1-form (since $S(\alpha)^T S(\alpha) = Id_{p\times p}$).

Lemma A.3 *The L-W connection ω induced by $X^U : \underline{\mathbb{R}}^m \to B$ is given by the universal connection $S^T dS$ on the Stiefel bundle.*

Proof. We only need to show that the connection ω is given by $S^T dS$, i.e. that $h^*(S^T dS)$ is the $o(p)$-component of ω_{00}. This is clear since for $A \in o(m)$, $h^*(S^T dS)(A) = (S\circ)^T d(S \circ h)(A) = P_{\mathbb{R}^p} A i_{\mathbb{R}^p}$. ∎

Given our surjection $X : \underline{\mathbb{R}}^m \to E$ with adjoint Y so that $Y(x) : E_x \to \mathbb{R}^m$, there is a bundle homomorphism Φ from $O(E)$ to the Stiefel bundle,

$$\begin{tikzcd}
\Phi\colon \; O(E) \arrow[r] \arrow[d] & V(p,q) \arrow[d] \\
M \arrow[r, "\Phi_0"] & G(p,q)
\end{tikzcd}$$

defined by:

$$\begin{aligned}
\Phi_0(x) &= \text{Image } Y(x), \\
\Phi(u) &= \left(Y(x)u(e_1), \ldots, Y(x)u(e_p)\right),
\end{aligned}$$

where (e_i) is an orthonormal basis for $\mathbb{R}^p$.

Conversely any such bundle homomorphism $\Phi : O(E) \to V(p,q)$ over $\Phi_0 : M \to G(p,q)$ comes from a surjective map: indeed for $x \in M$, take a frame $u \in O(E)$ at x, and set

$$Y(x) = \Phi(u) \circ u^{-1} : T_x M \to \mathbb{R}^m.$$

This is independent of the choice of u. Here we have used $\Phi(u)$ for the induced transformation: $\mathbb{R}^p \to \mathbb{R}^m$. Set $X(x) = Y(x)^*$.

On the vector bundle level let $E^1 \to M_1$ and $E^2 \to M_2$ be two vector bundles over manifolds and let f be a vector bundle map which is a 1-1 map on fibres over $f_0 : M_1 \to M_2$. Given a vector bundle map $X_2 : \underline{\mathbb{R}}^m \to E^2$ there is an induced map $X_1 : \underline{\mathbb{R}}^m \to E^1$ given by

$$X_1(x)(e) = (f)^{-1}_{f_0(x)} X_2\left(f_0(x)\right)(e).$$

$$\begin{tikzcd}
E^1 \arrow[r, "f"] \arrow[d] & E^2 \arrow[d] & \underline{\mathbb{R}}^m \arrow[l, "X_2"'] \arrow[dl] \\
M_1 \arrow[r, "f_0"] & M_2 &
\end{tikzcd}$$

This gives E^1 an induced metric and makes f an isometry on fibres. Let Γ^{X_2} be the L-W connection on $O(E^2)$ from X_2 and Γ^{X_1} be the L-W connection on $O(E^1)$ from X_1. Let $f^*\left(\Gamma^{X_2}\right)$ be the pull back of Γ^{X_2}.

Lemma A.4

$$f^*\left(\Gamma^{X_2}\right) = \left(\Gamma^{X_1}\right).$$

Proof. Let Y_1 and Y_2 be respectively the adjoint of X_1 and X_2. Take $e \in Image(Y_1(x))$, say $e = Y_1(v_e)$. We need only to show

$$\left(\nabla X_1^e\right)_x = 0.$$

Here ∇ is the covariant differentiation corresponding to the pulled back connection. Take a curve $\sigma_1 \in M_1$ with $\sigma_1(0) = x$. Let $\tilde{\sigma}_1$ be the horizontal lift of $\sigma_1(\cdot)$ in the frame bundle of E^1, and $\tilde{\sigma}_2$ the horizontal lift of $f_0\left(\sigma_1(\cdot)\right)$. Then

$$f\left(\tilde{\sigma}_1(t)\right) = \tilde{\sigma}_2(t).$$

By definition,

$$
\begin{aligned}
(\nabla X_1^e)_x &= \tilde{\sigma}_1(0)\frac{d}{dt}\left(\tilde{\sigma}_*(t)\right)^{-1} X_1^e\left(\sigma(t)\right)|_{t=0} \\
&= f^{-1}\left(\tilde{\sigma}_2(0)\right)\frac{d}{dt}\left(\tilde{\sigma}_2(t)\right)^{-1} X_2^e\left(f_0(\sigma(t))\right)|_{t=0}
\end{aligned}
$$

The right hand side is zero from the characterization of L-W connections and the observation that e is in the image of Y_2 since $e = Y_1(x)(v_e) = Y_2(f_0(x))\,(f(v_e))$. ■

Theorem A.5 *Every metric connection on E is a L-W connection for some $X : \underline{\mathbb{R}}^m \to E$, some finite m.*

Proof. By Narasimhan and Ramanan, any metric connection form ω_g is given by $\omega_g = \Phi^*(\omega_U)$ from some bundle homomorphism $\Phi : O(E) \to V(p,q)$. Define X, Y from Φ as before.

$$
\begin{tikzcd}
E \arrow[r, "f"] \arrow[d] & B \arrow[d] & \underline{\mathbb{R}}^m \arrow[l, "X_U"'] \arrow[dl] \\
M \arrow[r] & G(p,q) &
\end{tikzcd}
$$

Set $f = (X^U Y)$. Then Φ is induced by f and so, by lemma A.3, $\Phi^*(\omega_U)$ is the connection form for $f^*(\Gamma^{X^U})$. By Lemma A.4 $f^*\left(\Gamma^{X^U}\right) = \Gamma^X$. So ω_g is the connection form for the L-W connection Γ^X. ■

B Creation and Annihilation operators (notation for section 2.4)

Let $A : V \to V$ be a linear map on an inner product space V and $(e_1, \ldots, e_n)$ an orthonormal base for V. There are the operators $(d\Lambda)^*(A)$ on the space of tensor products of $\wedge^* V$, which restricts to $\wedge^p V$ to give $(d\Lambda)^p(A)$:

$$(d\Lambda)^p(A)\,(u_1 \wedge \ldots \wedge u_p) = \sum_1^p u_1 \wedge \ldots \wedge u_{j-1} \wedge Au_j \wedge u_{j+1} \wedge \ldots \wedge u_p,$$

and $(\delta^2\Lambda)^*$ defined by:

$$(\delta^2\Lambda)^* A = (d\Lambda)^* A \circ (d\Lambda)^* A - (d\Lambda)^* A^2.$$

If ϕ is a linear map on $\wedge^p V$, we define $\widehat{A}\phi(v) = \phi(Av)$ and so the last linear transformations we defined on $\wedge^p V$ give:

$$\widehat{(d\Lambda)}^p(A)(\phi)(u_1, \ldots, u_p) = \sum_{j=1}^{p} \phi(u_1, \ldots, Au_j, \ldots, u_p),$$

and for $p > 1$

$$(\widehat{\delta^2\Lambda^*})A(\phi)(u_1, \ldots, u_p) = \sum_{i \neq j} \phi(u_1, \ldots, Au_i, \ldots, Au_j, \ldots, u_p).$$

And so

$$(\widehat{\delta^2\Lambda^*})A(\phi) = (\widehat{d\Lambda^*}) \circ (\widehat{d\Lambda^*})A(\phi) - (\widehat{d\Lambda^*})A^2(\phi),$$

Let a_j^* be the "creation operator" on $\wedge^* V$, $a_j^* v = e_j \wedge v$ if $(e_1, \ldots, e_n)$ is an orthonormal basis for $\wedge^* V$, and a_j its adjoint, the "annihilation operator". For linear forms we have the corresponding operators: $(a^j)^*\phi(v) = \phi(a_j v)$ and $(a^j\phi)(v) = \phi(a_j^* v)$. In particular $a^j\phi(v) = \phi(e_j \wedge v)$ and $(a^j)^*\phi(v) = e_j^* \wedge \phi$.

Then

$$d\Lambda^*(A) = \sum_{i,j} \langle Ae_j, e_i \rangle \, a_i^* a_j. \tag{B.1}$$

See [CFKS87].

Lemma B.1

$$(\delta^2\Lambda)^* A = -\sum_{i,j,k,l} \langle Ae_j, e_i \rangle \langle Ae_k, e_l \rangle \, a_i^* a_l^* a_j a_k.$$

Proof. By (B.1),

$$\begin{aligned}
(d\Lambda)^* \circ (d\Lambda)^*(A)(-) &= \sum_{i,j} \langle Ae_j, e_i \rangle \, a_i^* a_j \left((d\Lambda)^*(A)(-)\right) \\
&= \sum_{i,j,k,l} \langle Ae_j, e_i \rangle \langle Ae_k, e_l \rangle \, a_i^* a_j a_l^* a_k(-) \\
&= -\sum_{i,j,k,l} \langle Ae_j, e_i \rangle \langle Ae_k, e_l \rangle \, a_i^* a_l^* a_j a_k(-) \\
&\qquad + \sum_{i,j,k} \langle Ae_j, e_i \rangle \langle Ae_k, e_j \rangle \, a_i^* a_k(-)
\end{aligned}$$

The last step comes from the identity $a_j a_l^* = -a_l^* a_j + \delta_{jl}$, as used in [CFKS87].

However

$$\sum_{i,j,k} \langle Ae_j, e_i \rangle \langle Ae_k, e_j \rangle \, a_i^* a_k(-) = \sum_{i,j,k} \langle A^2 e_k, e_i \rangle \, a_i^* a_k(-) = (d\Lambda^*)(A^2).$$

Consequently,

$$
\begin{aligned}
&(d\Lambda)^* \circ (d\Lambda)^*(A)(-) \\
&= -\sum_{i,j,k,l} \langle Ae_j, e_i\rangle \langle Ae_k, e_l\rangle a_i^* a_l^* a_j a_k(-) + (d\Lambda)^*(A^2)(-),
\end{aligned}
$$

■

Corollary B.2 *The map $(\delta^2\Lambda)^* A : V \to V$ can be written as:*

$$(\delta^2\Lambda)^* A = -2 \sum_{i<l, j<k} \langle Ae_j \wedge Ae_k, e_i \wedge e_l\rangle a_i^* a_l^* a_j a_k.$$

Proof. Note $a_j^* a_j^* = 0$. We split the sum in $(\delta^2\Lambda)^* A$ into two parts: $i < l$ and $i > l$. After rearrangements, we have:

$$
\begin{aligned}
&(\delta^2\Lambda)^*(A) \\
&= -\sum_{i<l, 1\le k,j\le m} \left[\langle Ae_j, e_i\rangle \langle Ae_k, e_l\rangle - \langle Ae_j, e_l\rangle \langle Ae_k, e_i\rangle\right] a_i^* a_l^* a_j a_k \\
&= -\sum_{i<l, 1\le k,j\le m} \langle Ae_j \wedge Ae_k, e_i \wedge e_l\rangle a_i^* a_l^* a_j a_k \\
&= -2 \sum_{i<l, k<j} \langle Ae_j \wedge Ae_k, e_i \wedge e_l\rangle a_i^* a_l^* a_j a_k.
\end{aligned}
$$

■

There is a similar expression for linear forms from

$$(\widehat{d\Lambda}^* A)(\phi) = \sum_{i,j} \phi(\langle Ae_i, e_j\rangle a_i^* a_j) = \sum_{i,j} \langle Ae_j, e_i\rangle (a^i)^* a^j \phi :$$

Corollary B.3 *The map $(\widehat{\delta^2\Lambda}^*) A : \wedge^* V^* \to \wedge^p V^*$ is given by:*

$$
\begin{aligned}
\left(\widehat{\delta^2\Lambda}\right)^* A(\phi) &= -\sum_{i,j,k,l} \langle Ae_i, e_j\rangle \langle Ae_k, e_l\rangle (a^i)^*(a^k)^* a^j a^l \ \phi \\
&= -2 \sum_{i<k, j<l} \langle Ae_i \wedge Ae_k, e_j \wedge e_l\rangle (a^i)^*(a^k)^* a^j a^l \ \phi.
\end{aligned}
$$

C Basic formulae

In this section we give some basic formulae for $\breve{\nabla}$ and $\breve{T}$ in terms of the defining map X or the Gaussian field W.

Let W be a mean zero Gaussian field of sections $\Gamma(E)$ of a vector bundle E, $\breve{\nabla}$ the associated L-W connection, and let $\langle , \rangle_x$ be the metric on E induced from the Gaussian structure, as given in section 1.1C. Recall (1.1.5):

$$\breve{\nabla}_v U = \mathbb{E} W(x) \frac{d}{dt} \langle U(\sigma(t)), W(\sigma(t)) \rangle_{\sigma(t)} \Big|_{t=0} \qquad v \in T_x M, U \in \Gamma E, \tag{C.1}$$

where σ is any C^1 smooth curve with $\dot{\sigma}(0) = v$, and also recall the expansion $U = \mathbb{E} W \langle u, W \rangle$ for $U \in \Gamma(E)$.

Proposition C.1 *1. For any connection $\tilde{\nabla}$ on E,*

$$\tilde{\nabla}_v U = \mathbb{E} \, \tilde{\nabla}_v W \langle U, W \rangle_{x_0} + \breve{\nabla}_v U. \tag{C.2}$$

2. $\mathbb{E} \, \breve{\nabla}_W W = 0$.

3. A connection $\tilde{\nabla}$ on E is adapted to $\langle , \rangle_x$ if and only for any $U \in \Gamma(E)$ and tangent vector v,

$$\mathbb{E} \, \tilde{\nabla}_v W \langle U, W \rangle + \mathbb{E} W \left\langle U, \tilde{\nabla}_v W \right\rangle = 0. \tag{C.3}$$

Proof. Take $v \in T_{x_0} M$ and let σ be a C^1 smooth curve with $\dot{\sigma}(0) = v$. Expanding $U(\sigma(t))$ in W we see:

$$\begin{aligned}
\tilde{\nabla}_v U &= \frac{\tilde{D}}{dt} \mathbb{E} W(\sigma(t)) \langle U(\sigma(t)), W(\sigma(t)) \rangle_{\sigma(t)} \Big|_{t=0} \\
&= \mathbb{E} \, \tilde{\nabla}_v W \langle U, W \rangle_{x_0} + \mathbb{E} W(x_0) \frac{d}{dt} \langle U(\sigma(t)), W(\sigma(t)) \rangle \Big|_{t=0} \\
&= \mathbb{E} \, \tilde{\nabla}_v W \langle U, W \rangle_{x_0} + \breve{\nabla}_v U.
\end{aligned}$$

Thus (C.2). Take $\tilde{\nabla} = \breve{\nabla}$ in (C.2)) to see part (2) (or use the defining property of Proposition 1.1.3).

Next suppose that $\tilde{\nabla}$ is adapted to the metric. Applying $\tilde{\nabla}_v$ in the second line of the earlier calculation, we have

$$\tilde{\nabla}_v U = \mathbb{E} \, \tilde{\nabla}_v W \langle U, W \rangle + \mathbb{E} W \left\langle U, \tilde{\nabla}_v W \right\rangle + \tilde{\nabla}_v U$$

and thus (C.3).

On the other hand (C.3) implies

$$\mathbb{E} \left\langle \tilde{\nabla}_v W, U \right\rangle \langle U, W \rangle = 0.$$

Consequently,

$$\begin{aligned}
d \langle U, U \rangle (v) &= 2 \left\langle \breve{\nabla}_v U, U \right\rangle (v) \\
&= 2 \left\langle \tilde{\nabla}_v U, U \right\rangle - 2\mathbb{E} \left\langle \tilde{\nabla}_v W, U \right\rangle \langle U, W \rangle \\
&= 2 \left\langle \tilde{\nabla}_v U, U \right\rangle,
\end{aligned}$$

i.e. $\tilde{\nabla}$ is metric. ■

Corollary C.2 *Let $\tilde{\nabla}$ be a connection on a subbundle E of TM. Denote by $\breve{T}$ and $\tilde{T}$ respectively the torsions for $\breve{\nabla}$ and $\tilde{\nabla}$. Then for sections U, Z of E with $U(x_0) = u$ and $Z(x_0) = z$,*

$$\begin{aligned} \breve{T}(z,u) - \tilde{T}(z,u) &= \mathbb{E} W(x_0) \left\langle u, \tilde{\nabla}_z W \right\rangle - \mathbb{E} W(x_0) \left\langle z, \tilde{\nabla}_u W \right\rangle && \text{(C.4)} \\ &= -\mathbb{E} \left\langle W(x_0), u \right\rangle \tilde{\nabla}_z W + \mathbb{E} \left\langle W(x_0), z \right\rangle \tilde{\nabla}_u W && \text{(C.5)} \end{aligned}$$

and

$$\breve{T}(z,u) = [Z,U](x_0) + \mathbb{E}[W,Z](x_0) \left\langle W, u \right\rangle_{x_0} - \mathbb{E}[W,U](x_0) \left\langle W, z \right\rangle_{x_0}. \qquad \text{(C.6)}$$

Proof. The first formula follows from (1.3.12) and the second from Proposition C.1 and (C.5). ■

Finally we write $\breve{\nabla}$ in terms of the vector fields $Z^{\cdot}$ as defined by (1.1.1):

Proposition C.3 *Let $\tilde{\nabla}$ be a connection on a subbundle $\tilde{E}$ of TM containing E. Then for $U \in \Gamma(E)$ and $V \in \Gamma(TM)$ with $U(x_0) = u$ and $V(x_0) = v$,*

$$\begin{aligned} \breve{\nabla}_v U &= \tilde{\nabla}_v U - \tilde{\nabla}_v Z^u, && \text{(C.7)} \\ \hat{\nabla}_u V &= \tilde{\nabla}'_u V - \tilde{\nabla}_v Z^u. && \text{(C.8)} \end{aligned}$$

Proof. The first identity is just (1.1.3). The second follows from (1.3.1) and (C.7):

$$\hat{\nabla}_u V = \breve{\nabla}_v U + [U,V](x_0) = \tilde{\nabla}_v U - \tilde{\nabla}_v Z^u + [U,V](x_0) = \tilde{\nabla}'_u V - \tilde{\nabla}_v Z^u.$$

■

Formulae related to curvatures

First we give a formula for the curvature tensor $\breve{R}$ in terms of X or W. From this a series of identities and inequalities relating the quantities of $\breve{\nabla}$ with those of $\hat{\nabla}$ are observed. In particular we give an interpretation of the form $\breve{T}^{\#}$, derived from torsion $\breve{T}$, being closed or co-closed.

A. Let E be a subbundle of TM, $\breve{R}: TM \times TM \to L(E;E)$ the curvature tensor for a metric connection $\breve{\nabla}$, and W a Gaussian vector field which induces $\breve{\nabla}$ in the sense of section 1.1.

Proposition C.4 *For $u \in E_{x_0}$, and $v_1, v_2 \in T_{x_0}M$,*

$$\breve{R}(v_1,v_2)(u) = \mathbb{E} \breve{\nabla}_{v_1} W \left\langle \breve{\nabla}_{v_2} W, u \right\rangle - \mathbb{E} \breve{\nabla}_{v_2} W \left\langle \breve{\nabla}_{v_1} W, u \right\rangle. \qquad \text{(C.9)}$$

In the metric form,

$$\breve{R}(v_1,v_2)(u) = \sum_i \breve{\nabla}_{v_1} X^i \left\langle \breve{\nabla}_{v_2} X^i, u \right\rangle - \sum_i \breve{\nabla}_{v_2} X^i \left\langle \breve{\nabla}_{v_1} X^i, u \right\rangle. \qquad \text{(C.10)}$$

Proof. First recall formula (1.3.12) from section 1.3:

$$\breve{\nabla}_v U = [V, U] + \mathbb{E}[W, V] \langle W, U \rangle \tag{C.11}$$

Let U be a horizontal vector field and V, Z vector fields,then

$$\begin{aligned}
\breve{\nabla}_Z[\breve{\nabla}_V U] &= [Z, \breve{\nabla}_V U] + \mathbb{E}\,[W, Z] \left\langle W, \breve{\nabla}_V U \right\rangle \\
&= [Z, [V, U]] + \mathbb{E}\,[Z, [W, V]] \langle W, U \rangle \\
&\quad + \mathbb{E}\,[W, V]\, d \langle W, U \rangle (Z(\cdot)) + \mathbb{E}[W, Z] \left\langle W, \breve{\nabla}_V U \right\rangle .
\end{aligned}$$

Use Jacobi's identity twice to obtain

$$\begin{aligned}
\breve{R}(Z, V)U := &\quad \breve{\nabla}_Z \breve{\nabla}_V U - \breve{\nabla}_V \breve{\nabla}_Z U - \breve{\nabla}_{[Z,V]} U \\
= &\quad \mathbb{E}\{ \, [W, V]\, d \langle W, U \rangle (Z(\cdot)) \\
&\quad - [W, Z]\, d \langle W, U \rangle (V(\cdot)) \, \} \\
&\quad + \mathbb{E}\left\{ [W, Z] \left\langle W, \breve{\nabla}_V U \right\rangle - [W, V] \left\langle W, \breve{\nabla}_Z U \right\rangle \right\} .
\end{aligned}$$

Take $U = Z^u$ and V, Z with $V(x_0) = v$, $Z(x_0) = z$ for $z, v \in T_{x_0} M$, and $u \in E_{x_0}$. Then

$$\begin{aligned}
\breve{R}(z, v)u &= \mathbb{E}\left\{ [W, V] \left\langle \breve{\nabla}_z W, u \right\rangle - [W, Z] \left\langle \breve{\nabla}_v W, u \right\rangle \right\} \\
&= \mathbb{E}\left\{ \breve{\nabla}_z W \langle \breve{\nabla}_v W, u \rangle - \breve{\nabla}_v W \langle \breve{\nabla}_z W, u \rangle \right\} ,
\end{aligned}$$

using (1.3.1) and the independence of $W(x)$ and $\breve{\nabla}.W|_{T_{x_0} M}$. ■

Consequently for $v_i \in T_x M$, $u_i \in E_x$,

$$\left\langle \breve{R}(v_1, v_2)u_1, u_2 \right\rangle = -\mathbb{E} \left\langle \breve{\nabla}_{v_1} W \wedge \breve{\nabla}_{v_2} W, \; u_1 \wedge u_2 \right\rangle_{\Lambda^2 E_x} .$$

Corollary C.5 *Let $\breve{\mathcal{R}} : \Lambda^2 TM \to \Lambda^2 E$ be the curvature operator defined by*

$$\langle \breve{\mathcal{R}}(v_1 \wedge v_2), u_1 \wedge u_2) \rangle = \langle \breve{R}(v_1, v_2)u_2, u_1 \rangle.$$

Then

$$\breve{\mathcal{R}} = \mathbb{E}\, \breve{\nabla}.W \wedge \breve{\nabla}.W.$$

In the metric form,

$$\breve{\mathcal{R}} = \sum_{i=1}^{m} \breve{\nabla}.X^i \wedge \breve{\nabla}.X^i.$$

Remark: For $A : TM \oplus KerX \to E$ the 'shape operator', defined by

$$A(u,e) = \breve{\nabla} X(e)(u),$$

the proposition gives

$$\breve{R}(u,v)w = \text{ trace } \{A(u,-)\langle A(v,-),w\rangle - A(v,-)\langle A(u,-),w\rangle\}$$

which reduces in the gradient case to Gauss's equation for the curvature of a submanifold in $\mathbb{R}^m$ (e.g. p. 23 [KN69b]).

B. Let $\tilde{\nabla}$ and ∇^0 be two connections on E with $\tilde{D} : TM \times E \to E$ the bilinear map defined by :

$$\tilde{\nabla}_V U = \nabla^0_V U + \tilde{D}(V,U). \tag{C.12}$$

Their curvature tensors are related by the following formula: for v_1, v_2 and u in E_x,

$$\begin{aligned}\tilde{R}(v_1,v_2)u &= R^0(v_1,v_2)u + \left(\nabla^0_{v_1}\tilde{D}\right)(v_2,u) - \left(\nabla^0_{v_2}\tilde{D}\right)(v_1,u) \\ &+\tilde{D}(v_1,\tilde{D}(v_2,u)) - \tilde{D}(v_2,\tilde{D}(v_1,u)) + \tilde{D}\left(T^0(v_1,v_2),u\right),\end{aligned} \tag{C.13}$$

and the two covariant derivatives of $\tilde{D}$ restricted to $E \times E$ are related by

$$\begin{aligned}&\left(\tilde{\nabla}_{v_1}\tilde{D}\right)(v_2,u) - \left(\nabla^0_{v_1}\tilde{D}\right)(v_2,u) \\ =\ & \tilde{D}\left(v_1,\tilde{D}(v_2,u)\right) - \tilde{D}\left(v_2,\tilde{D}(v_1,u)\right) - \tilde{D}\left(\tilde{D}(v_1,v_2),u\right), \quad v_1,v_2,u \in E_{x_0}\end{aligned}$$

In the nondegenerate case, we shall take $\nabla^0 = \nabla$, the Levi-Civita connection for the induced metric. Recall the differential 3-form $T^{\#}$ defined in section 2.2.

Proposition C.6 *Let $\breve{\nabla}$ be a torsion skew symmetric connection on TM. If R, $\breve{R}$ and $\hat{R}$ denote respectively the curvature tensors for ∇, $\breve{\nabla}$ and $\hat{\nabla}$ then:*

$$\begin{aligned}\breve{R}(v_1,v_2)u = \quad & R(v_1,v_2)u + \tfrac{1}{2}\left(\nabla_{v_1}\breve{T}\right)(v_2,u) - \tfrac{1}{2}\left(\nabla_{v_2}\breve{T}\right)(v_1,u) \\ & +\tfrac{1}{4}\breve{T}(v_1,\breve{T}(v_2,u)) - \tfrac{1}{4}\breve{T}(v_2,\breve{T}(v_1,u)).\end{aligned} \tag{C.14}$$

In particular,

1. $$\breve{Ric}(u,v) = Ric(u,v) - \frac{1}{2}\delta\breve{T}^{\#}(u,v) - \frac{1}{4}tr\left\langle \breve{T}(-,u),\breve{T}(-,v)\right\rangle, \tag{C.15}$$

   so $\breve{Ric}(-,-)$ is symmetric if and only if $\delta\breve{T}^{\#} = 0$.

2. $$\breve{Ric}(u,v) - \hat{Ric}(u,v) = -\delta\breve{T}^{\#}(u,v). \tag{C.16}$$

3. $$\breve{R}(v_1,v_2)u = \hat{R}(v_1,v_2)u - \left(\breve{\nabla}_{v_1}\breve{T}\right)(v_2,u) + \left(\breve{\nabla}_{v_2}\breve{T}\right)(v_1,u). \tag{C.17}$$

4.
$$\left\langle \breve{R}(v_1,v_2)u,w\right\rangle - \left\langle \hat{R}(u,w)v_1,v_2\right\rangle = \frac{1}{2}d\breve{T}^{\#}(v_1,v_2,u,w) \tag{C.18}$$
and
$$\breve{Ric}(u,v) = \hat{Ric}(v,u), \tag{C.19}$$

Remark: See also Lemma 3.5 in [Dri97a] for symmetricity of $\breve{Ric}$ in the torsion skew symmetric case.

Proof. First note that $\breve{\nabla} = \nabla + \frac{1}{2}\breve{T}$ in the torsion skew symmetric case. Equation (C.14) follows straightaway from (C.13). Let $X : \mathbb{R}^m \to TM$ be a map which has $\breve{\nabla}$ as its L-W connection. Recall $\breve{Ric}(u,v) = \sum \left\langle \breve{R}(X^i,u)v, X^i\right\rangle$ so that

$$\begin{aligned} \breve{Ric}(u,v) - Ric(u,v) = \quad & \frac{1}{2}\left\langle \nabla_{X^i}\breve{T}(u,v), X^i\right\rangle - \frac{1}{2}\left\langle (\nabla_u\breve{T})(X^i,v), X^i\right\rangle \\ & -\frac{1}{4}\left\langle \breve{T}(u,\breve{T}(X^i,v), X^i\right\rangle. \end{aligned}$$

The first term of the right hand side is $\frac{1}{2}\delta\breve{T}^{\#}$. On the other hand the torsion skew symmetry gives
$$\left(\breve{\nabla}_z\breve{T}\right)(u,v) = \left(\nabla_z\breve{T}\right)(u,v) + Cyl\,\breve{T}\left(z, \breve{T}(u,v)\right) \tag{C.20}$$
and thus
$$\left(\nabla_u\breve{T}\right)(v,v) = \left(\breve{\nabla}_u\breve{T}\right)(v,v) = 0.$$
It follows that the second term $-\frac{1}{2}\left\langle (\nabla_u\breve{T})(X^i,X^i), v\right\rangle$ vanishes. The last terms is now
$$\begin{aligned} -\frac{1}{4}\left\langle \breve{T}(X^i,\breve{T}(u,X^i)), v\right\rangle &= \frac{1}{4}\left\langle \breve{T}(X^i,v), \breve{T}(u,X^i)\right\rangle \\ &= -\frac{1}{4}tr\left\langle \breve{T}(-,u), \breve{T}(-,v)\right\rangle. \end{aligned}$$

We have proved (C.15). Apply (C.15), and (C.14) respectively to both $\breve{\nabla}$ and $\hat{\nabla}$ to obtain (C.16) and (C.17). Equation (C.18) follows from
$$\left\langle (\breve{\nabla}_u\breve{T})(w,v_1), v_2\right\rangle = \left\langle (\breve{\nabla}_u\breve{T})(v_1,v_2), w\right\rangle,$$
and (C.19) from (C.18). ■

Corollary C.7 *If $\breve{\nabla}$ is a torsion skew symmetric connection on TM,*
$$\breve{Ric}(u,u) \le Ric(u,u), \tag{C.21}$$
Furthermore if $\breve{k}, \hat{k}$, and k are respectively the scaler curvature of the connections $\breve{\nabla}$, $\hat{\nabla}$ and ∇. Then
$$\breve{k} = \hat{k} = k - \frac{1}{4}|\breve{T}(-,-)|^2.$$

Remarks:

(i) When M is a Lie group with $\breve{\nabla}$ the left invariant connection and $\langle , \rangle$ is bi-invariant then $\breve{R} \equiv 0$, $\nabla\breve{T} \equiv 0$ and (C.14) reduces to the standard formula for the curvature

$$R(v_1, v_2)u = -\frac{1}{4}[\,[Z^{v_1},\, Z^{v_2}], Z^u\,]$$

by (1.3.6) and the Jacobi identity.

In this case (C.18) shows that $\breve{T}^{\#}$ is a closed form, as is well known: for non-Abelian compact Lie groups it represents a non-trivial class in $H^3(G)$, [Car36], which is clear since (C.16) shows it is harmonic. Indeed we obtain the following:

if M is compact with $\dim M \geq 3$ and admits a torsion skew symmetric connection with nonzero torsion which is flat together with its adjoint connection then $H^3(M;\mathbb{R}) \neq 0$.

This would be an extension of Cartan's result if the existence of such a connection were known on any manifolds other than Lie groups.

(ii) The inequality (C.21) does not hold in general without the assumption of torsion skew symmetry, even when $\hat{\nabla}$ is adapted to some metric. A class of counter examples is provided by Lie groups with left invariant metrics having negative curvature, e.g. see [Mil76].

D List of notation

M		basic manifold,
H		Hilbert space, $=\mathbb{R}^m$ if finite dimensional
$X(x)$		bundle homomorphism between the trivial bundle $M\times H$ and E over M,
$Y(x)$	$=$	$X(x)^*: T_xM \to H$,
$N(x)$	$=$	$\mathrm{Ker}X(x)$ (often aasume of constant rank);
Z^v	$=$	$X(\cdot)Y(x_0)v, \ \ v\in T_{x_0}M$
$\langle,\rangle$ or $\langle,\rangle^X$		metric induced on E from X,
$\langle,\rangle^1$		a Riemannian metric on $TM=E\oplus E^\perp$, extending the metric induced on E by X, having $E^\perp$ orthogonal to E
$\langle,\rangle'$		a Riemannian metric not necessarily coming from X;
$\breve\nabla$,		the connection associated to X, $\breve R$, $\breve{Ric}$, $\breve R^q$ its curvature, Ricci curvature and Weitzenbock terms, and $\breve T$ its torsion tensor
$\hat\nabla$		the adjoint semi-connection of $\breve\nabla$, $\hat R$, $\hat{Ric}$, $\hat R^q$, its curvature, Ricci curvature, Weitzenbock terms and $\hat T$ its torsion tensor,
∇		Levi-Civita connection, R, Ric, R^q, its curvature, Ricci curvature, and Weitzenbock terms
∇^1	$=$	$\breve\nabla\oplus\nabla^\perp$, the direct sum connection of $\breve\nabla$ on E and a connection $\nabla^\perp$ on $E^\perp$, T^1 its torsion
$\tilde\nabla$		any connection, $\tilde R$ its curvature tensor, $\tilde{Ric}$ its Ricci curvature, $\tilde R^q$ its Weitzenbock terms, $\tilde{//}_\cdot$ the corresponding parallel translation, $\tilde T$ its torsion tensor, $\tilde R_{ijkl}$ the associated curvatures, $\tilde{\mathrm{Ric}}^\#(v)=\sum_1^m \tilde{\mathrm{Ric}}(v,X^i(x))X^i(x)$,
$\tilde T^\#$		the 3-form related to $\tilde T$,
W		Gaussian field of sections of E,
D		(2,0)-tensor, the difference between two linear connections,
$\mathcal{A}, \mathcal{A}^q$		infinitesimal generator associated with a given sde and its restriction on q-forms,
P_t		semigroup associated with stochastic flows;
A^X	$=$	$\frac{1}{2}\sum_j^m \nabla X^j(X^j)+A$, where A the drift coefficient of the s.d.e. involved,
$\hat\delta$		'divergence' operator associated to $\breve\nabla$,
$a^i, (a^i)^*$		annihilation and creation operator,
$\xi_t(x_0), \rho(x_0)$		solution to sde with initial point x_0 and life time $\rho(x_0)$,
x_t	$=$	$\xi_t(x_0)$,

$T\xi_t$		the derivative process associated to ξ_t,
v_t		often $T_{x_0}\xi_t(v_0)$; for $v_0 \in T_{x_0}M$,
$\breve{W}_t^{q,A}$		the solution to covariant equation (3.3.11) involving $\breve{R}^q$
$\beta_t, \tilde{B}_t$		orthogonal decomposition of the Brownian motion B_t on $\mathbb{R}^m$ using $\breve{\nabla}\tilde{//}_t^{-1} dB_t = d\beta_t + \tilde{B}_t$,
$\breve{B}_t$	$=$	$X(x_0)\tilde{B}_t$, the martingale part of the stochastic anti-development of $\xi_t(x_0)$
$d_E f$		the restriction of df to E
Diff$M$		C^∞ diffeomorphisms of M.

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