Introduction

The concepts of second order semi-elliptic operator, Markov semi-group, diffusion process, diffusion measures on path spaces essentially give different pictures of the same fundamental objects, with related Riemannian or sub-Riemannian geometry. Here we consider a different layer of structure centred around the concepts of sums of squares of vector fields, stochastic differential equations, stochastic flows and Gaussian vector fields; again essentially equivalent, and this time with associated metric linear connections on tangent bundles and subbundles of tangent bundles. The difference between these two levels of structure can be seen from the fact that if a semi-elliptic differential operator on functions on a manifold M is given a representation as a sum of square of vector fields ("Hörmander form") it automatically gets an extension to an operator on differential forms. In exactly the same way representing a diffusion process as the one point motion of a stochastic flow determines a semi-group acting on differential forms (by pulling the form back by the flow and taking expectation.) Given a regularity condition there is an associated linear connection and adjoint 'semi-connection' in terms of which these operators can be simply described (e.g. by a Weitzenbock formula) as can many other important quantities (e.g. existence of moment exponents for stochastic flows). Moreover in the stochastic picture the connections remain relevant in the collapse from this level to the simpler one giving new results and new proofs of results e.g. on path space measures.

In more detail: Chapter 1 is connected with the construction of linear connections of vector bundles as push forwards of connections on trivial bundles. This is a direct analogue of the classical and elementary construction of the covariant derivative of a vector field on a submanifold of Euclidean space, leading to the Levi-Civita connections (Example 1B). Narasimhan & Ramadan's theorem of universal connections is evoked to assure us that all metric connections can be obtained this way (Theorem 1.1.2). We then go on to consider the various forms in which this construction will appear in situations described above. (E.g. how certain Gaussian fields of sections determine a connection.) Homogeneous spaces give a good class of examples described in some detail in §1.1 B. The notion of adjoint connection or semi-connection on a subbundle E of the tangent bundle TM to our underlying manifold M is described in §1.3. A semi-connection allows us to differentiate vector fields on M in E-directions. They play an important role in the theory. One difficulty is that the adjoint of a metric connection may not be metric for any metric (Corollary 1.3.7). In general Hörmander type hypoellipticity conditions on our generator $\mathcal{A}$, or equivalently on E, play little role in this article. However in Theorem 1.3.9 we show how they are related to the behaviour of parallel translations with respect to associated semi-connections.

In chapter 2 we concentrate on a generator $\mathcal{A}$ given in Hörmander form, and its associated stochastic differential equation (s.d.e.). A first result is Theorem 2.1.1 which shows in particular that (for $\dim M > 1$) any elliptic diffusion operator can be written as a sum of squares with no first order term, or equivalently that any elliptic diffusion is given by a Stratonovich equation with no drift term. The extension $\mathcal{A}^q$ of $\mathcal{A}$ to q-forms is shown to have the form $\mathcal{A}^q = -(d\hat{\delta} + \hat{\delta} d)$ for a certain operator $\hat{\delta}$ from q-forms to $q-1$ forms (Proposition 2.3.1) and also a Weitzenbock form $\mathcal{A}^q = \frac{1}{2} trace \hat{\nabla}^2 - \frac{1}{2} R^q$ (if there is no drift term A) (Theorem 2.4.2). Driver's notion of torsion skew symmetry is investigated in §2.2 in order to discuss the operators $\hat{\delta}$, and when they are L^2 adjoints of the exterior derivative d, in §2.3. Later, §3.3.3, the semigroups associated to these operators are used to obtain Böchner type vanishing theorems under positivity conditions on R^q.

The question of the symmetricity of $\mathcal{A}^q$ with respect to some measure on M is discussed in §2.5.2. Theorem 2.5.1 gives a fairly definitive result for $\mathcal{A}^q$ with the zero order terms removed. However conditions under which R^q is symmetric seem not so easy to find if $q > 1$. For $q = 1$ this reduces to symmetricity of the Ricci curvature $\breve{Ric}$ which is shown in Proposition C.6 of the Appendix to hold in the torsion skew symmetric case if and only if the torsion tensor $\breve{T}$ determines a coclosed differential 3-form, c.f. [Dri92]. These sections are not used later in this article.

The main applications in stochastic analysis start with Chapter 3. The basic idea is that the diffusion coefficient of an s.d.e often has a kernel: so that there is "redundant noise" from the point of view of the one point motion. We extend the results from the gradient case in [EY93] to our more general, possibly degenerate, situation giving a canonical decomposition of the noise into its redundant and non-redundant parts. We then show how this can be used to filter out the redundant noise in general situations. (This filtering out corresponds to the collapse in levels of structure mentioned above.) On the way we have to discuss conditional expectations of vector fields along the sample paths of our process, Definition 3.3.2. All this is done in some generality, e.g. allowing for the possibility of explosion. The main application is to the derivative process $T\xi_t$ of a stochastic flow: Theorem 3.3.7 and Theorem 3.3.8. When the redundant noise is filtered out the process becomes a "damped' or Dohrn-Guerra type parallel translation using the associated semi-connection. This procedure works equally for the derivative of the Itô map $\omega \mapsto \xi_t(x_0)(\omega)$ in the sense of Malliavin Calculus from which follow integration by parts theorems for possibly degenerate diffusion measures, Theorem 4.1.1. For gradient systems, using [EY93], this method was used by [EL96] and was suggested by [AE95]. The Levi-Civita connection appears in that case (which is why gradient systems behave so nicely), but in the degenerate case which is allowed here the connections are on arbitrary subbundles of TM and there is no unique particularly well behaved connection to use. Hypoellipticity is not assumed. The "admissible" vector fields are those which satisfy a natural "horizontality" condition, §4.1 B and §4.1 C. Closely related is a Clark-Ocone formula (Theorem 4.1.2) expressing suitably smooth functions on path space as stochastic integrals with respect to the predictable projection of their gradient. From this we use the method given in [CHL] to obtain a Logarithmic Sobolev inequality for our diffusion measures, Theorem 4.2.1. Our "damping" of the parallel translation means that no curvature constants appear: indeed since in general we have no Riemannian metric given on M it would be unnatural to have such constants. Logarithmic Sobolev inequalities automatically imply spectral gap inequalities and the constancy of functionals with vanishing gradient (or equivalently whose derivatives vanish on admissible vector fields), Corollary 4.1.3: a non-trivial result even for Frechet smooth functions on path space for the case of degenerate diffusions. In Theorem 4.1.1 the corresponding results are proved for the measures on paths on the diffeomorphism group Diff$M$ of M coming from stochastic flows, or equivalently from Wiener processes on Diff$M$ [Bax84].

Chapter 5 is concerned with applications to stability properties of stochastic flows. In particular upper and lower bounds for moment exponents are obtained in terms of the Weitzenbock curvatures of the associated connection and a generalization of the second fundamental form to our situations: Theorem 5.0.5. This gives a criterion for moment stability in terms of 'stochastic positivity' of a certain expression in the quantities with consequent topological implications: Corollary 5.0.6.

A weakness of these results is that we usually require the adjoint semi-connection to be metric for some metric. Theorem 5.0.7 shows that the lack of this condition really is reflected in the behaviour of the flow.

Chapter 6 consists of technical appendices. The first gives a detailed description of how the push forward construction of connections we use relates to Narasimhan & Ramanan's pull back of the universal connections. This is needed in the proof of Theorem 1.1.2. The other appendices give the notation of annihilation and creation operators used in the discussion of the Weitzenbock curvatures in section 2.4 and some basic formulae and curvature calculations for connections given in the L-W form.

The connection determined by a non-degenerate stochastic flow first appeared in [LJW84]: for this reason we have called it the LeJan-Watanabe or L-W connection. It was also discovered in the context of quantum flows in [AA96] and for sums of squares of vector fields in [PVB96]. It is used for analysis on loop spaces in [Aid96]. For the non-degenerate case many of the results given here were described in [ELJL97a] with announcements for degenerate situations in [ELJL97b]. They were stimulated by [EY93]. The Chentsov-Amari α-connections in statistics are rather different. They are in general non-metric if $\alpha \neq 0$ and torsion free, see [Ama85], pp42, 46.

Acknowledgement: The authors would like to thank MSRI, Institut Henri Poincaré and Universitaet Bochum (as well as their home institutes) for hospitality during the completion of this project. Support is acknowledged from EU grant ERBF MRX CT 960075 A, NSF grant DMS-9626142, Alexander Humboldt-Stiftung, and the British Council.

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