Martin Boundary and Integral Representation for Harmonic Functions of Symmetric Stable Processes

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Abstract

Martin boundaries and integral representations of positive functions which are harmonic in a bounded domain D with respect to Brownian motion are well understood. Unlike the Brownian case, there are two different kinds of harmonicity with respect to a discontinuous symmetric stable process. One kind are functions harmonic in D with respect to the whole process X, and the other are functions harmonic in D with respect to the process X^D killed upon leaving D. In this paper we show that for bounded Lipschitz domains, the Martin boundary with respect to the killed stable process X^D can be identified with the Euclidean boundary. We further give integral representations for both kinds of positive harmonic functions. Also given is the conditional gauge theorem conditioned according to Martin kernels and the limiting behaviors of the h-conditional stable process, where h is a positive harmonic function of X^D. In the case when D is a bounded $C^{1,1}$ domain, sharp estimate on the Martin kernel of D is obtained.

Keywords and phrases: Symmetric stable processes, harmonic functions, conditional stable processes, and Martin boundaries.

Running Title: Martin Boundary for Stable Processes

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1 Introduction

Martin boundary and integral representation for harmonic functions of diffusions processes (or of elliptic differential operators) are well studied. However there is little detailed analysis of these for Markov processes with jumps (or for integro-differential operators). In this paper we take a closer look at an important class of discontinuous Markov processes—symmetric α-stable processes with $0 < \alpha < 2$, and study the notion and integral representation of harmonic functions for these processes, where some new phenomena arise. We hope that this paper can shed some new light on the potential theory of general Markov processes.

Symmetric stable processes constitute an important subfamily of Lévy processes. A symmetric α-stable process X on $\mathbf{R}^n$ is a Lévy process whose transition density $p(t, x-y)$ relative to the Lebesgue measure is uniquely determined by its Fourier transform $\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t,x)dx = e^{-t|\xi|^\alpha}$. Here α must be in the interval $(0, 2]$. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. In this paper, symmetric stable processes are referred to the case when $0 < \alpha < 2$, unless otherwise specified.

Unlike the Brownian case, there are two different kinds of harmonicity with respect to symmetric stable processes, one kind are functions harmonic in D with respect to the whole process X, and the other are functions harmonic in D with respect to the process X^D killed upon leaving D. The theory of Martin kernel and Martin boundary for the killed process X^D is known from the general theory. This Martin boundary gives an integral representation for positive functions harmonic in a domain D with respect to the killed process X^D. We show that when D is a bounded Lipschitz domain, the Martin boundary with respect to killed symmetric stable process X^D in D coincide with the Euclidean boundary. It seems that integral representations of positive functions harmonic in a domain D with respect to the whole processes X have not been studied in the literature. In this paper, we present an integral representation for positive functions harmonic in a domain D with respect to the whole processes and this representation is shown to be unique. In particular, this implies that any harmonic function with respect to the whole process is uniquely determined by its values in D. In the case when D is a bounded $C^{1,1}$ domain, sharp estimates on the Martin kernel are given. As a consequence of these estimates, we prove a conditional gauge theorem conditioned according to Martin kernel. We also study the limiting behavior of the h–conditioned symmetric stable process in D when h is a positive harmonic function of X^D, and the the limiting behavior of the h–conditioned symmetric stable process will provide a probabilistic interpretation to positive harmonic functions of X^D.

This paper is organized as follows. The definitions of harmonic and superharmonic functions with respect to symmetric stable processes are given in section 2. Some important facts about those harmonic functions are also given in section 2. Section 3 contains results on Martin boundary and conditional gauge theorem. Integral representations of positive functions harmonic in a domain D with respect to the whole processes are given in section 4.

In the sequel, we will use v^+ and v^- to denote the positive and negative part of a real-valued Borel measurable function v, i.e., $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$.

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2 Definitions and Preliminaries

In sections 2–4 of this paper, we always assume $n \geq 2$. Let $X = (\Omega, \mathcal{F}, X_t, \mathcal{F}_t, P_x)$ be a symmetric α-stable process on $\mathbf{R}^n$ with $0 < \alpha < 2$, where $\{\mathcal{F}_t, t \geq 0\}$ is the minimal admissible σ-fields generated by X. The process X is transient and we are going to use G to denote the potential of X. We know that the Green function of X is given by

$$G(x, y) = 2^{-\alpha}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)^{-1}|x-y|^{\alpha-n}.$$

For a domain D in $\mathbf{R}^n$, let $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Adjoin a cemetery point ∂ to D and set

$$X_t^D(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau_D, \\ \partial & \text{if } t \geq \tau_D. \end{cases}$$

X^D is a strong Markov process with state space $D_\partial = D \cup \{\partial\}$, which is called the subprocess of the symmetric α–stable process X killed upon leaving D, or simply the symmetric α–stable process in D. We are going to use G_D to denote the Green function of X^D.

For Brownian motion or other diffusion processes, there is only one kind of harmonicity on a domain D. However, for symmetric stable processes, there are two kinds of harmonic functions on D: functions which are harmonic in D with respect to the killed process X^D and functions which are harmonic in D with respect to the process X. The precise definitions of these two kinds of harmonic functions are as follows.

Definition 2.1 *Let D be a domain in $\mathbf{R}^n$. A locally integrable function f defined on D taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}\cap D} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be*

1) harmonic with respect to X^D if f is continuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) = E_x[f(X_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \tau_D];$$

2) superharmonic respect to X^D if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) \geq E_x[f(X_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \tau_D].$$

The next definition is taken from Landkof [17].

Definition 2.2 *Let D be a domain in $\mathbf{R}^n$. A locally integrable function f defined on $\mathbf{R}^n$ taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be*

1) harmonic in D with respect to X if f is continuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) = E_x[f(X_{\tau_{B(x,r)}})];$$

2) superharmonic in D with respect to X if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) \geq E_x[f(X_{\tau_{B(x,r)}})].$$

Remark 2.1 (1) If f is a lower semicontinuous function defined on D taking values in $(-\infty, \infty]$, then f is bounded from below on any subdomain whose closure is contained in D. Thus for such kind of function f which is locally integrable and satisfying $\int_{\{|x|>1\}} |f(x)||x|^{-(n+\alpha)}dx < \infty$, it follows from estimate (2.2) below that $E_x[f^-(X_{\tau_{B(x,r)}})] < \infty$ for any ball $B(x,r)$ with $\overline{B(x,r)} \subset D$. Therefore the expections in Definitions 2.1 and 2.2 are well defined.

(2) For a function f which is (super)harmonic with respect to X^D, if we extend it to be zero off the domain D, then the resulting function is (super)harmonic in D with respect to X.

(3) Conversely, if f is a non-negative superharmonic in D with respect to X, then clearly it is a superharmonic with respect to X^D.

We now record some facts, which will be used later, concerning bounded Lipschitz domains and the exit distributions of X from a domain U. Recall that a bounded domain D in $\mathbf{R}^n$ is said to be a bounded Lipschitz domain with Lipschitz characteristic constants (r_0, A_0) if for every $z \in \partial D$, there is a local coordinate system $(\xi_1, \xi^{(1)}) \in \mathbf{R} \times \mathbf{R}^{n-1}$ with origin sitting at z and there is a Lipschitz function f defined on $\mathbf{R}^{n-1}$ with Lipschitz constant A_0 such that $D \cap B(z, r_0) = B(z, r_0) \cap \{\xi = (\xi_1, \xi^{(1)}) : \xi_1 > f(\xi^{(1)})\}$. A domain U in $\mathbf{R}^n$ is said to satisfy the *uniform exterior cone condition* if there exist constants $\eta > 0$, $r > 0$ and a cone $\mathcal{C} = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : 0 < x_n, (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < \eta x_n\}$ such that for every $z \in \partial U$, there is a cone $\mathcal{C}_z$ with vertex z, isometric to $\mathcal{C}$ and satisfying $\mathcal{C}_z \cap B(z, r) \subset U^c$. It is well known that bounded Lipschitz domains satisfy the uniform exterior cone condition. Bogdan showed in [5] that for a bounded domain U satisfying the uniform exterior cone condition,

$$P_x(X_{\tau_U} \in \partial U) = 0 \quad \text{for all } x \in U. \tag{2.1}$$

In [8], Chen and Song showed that if U is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$, there is a Poisson kernel $K_U(x, z)$ defined on $U \times (\mathbf{R}^n \setminus \overline{U})$ such that for any bounded Borel measurable function ϕ, $E_x[\phi(X_{\tau_U})] = \int_{U^c} \phi(z) K_U(x, z) dz$, where dz is the Lebesgue measure on $\mathbf{R}^n$. Furthermore there exists a $C = C(U, \alpha) > 1$ such that for $x \in U$ and $z \in \overline{U}^c$,

$$\frac{\delta(x)^{\alpha/2}}{C\,\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}} \, \frac{1}{|x-z|^n} \le K_U(x, z) \le \frac{C\,\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}} \, \frac{1}{|x-z|^n}, \tag{2.2}$$

where $\delta(y) = \text{dist}(y, \partial U)$ is the Euclidean distance from point y to the set ∂U. Here a domain U is $C^{1,1}$ means that for every $z \in \partial D$, there exists a $r > 0$ such that $B(z, r) \cap \partial D$ is the graph of a function whose first derivatives are Lipschitz.

It is well known that for any domain D, there exists an increasing sequence of bounded C^∞-smooth domains $\{D_k\}_{k \ge 1}$ such that $\overline{D_k} \subset D_{k+1}$ for $k \ge 1$ and that $\bigcup_{k=1}^\infty D_k = D$ (see, for example, Lemma 2.4 of [7]).

Theorem 2.1 *Suppose that D is a bounded domain in $\mathbf{R}^n$. If h is superharmonic in D with respect to X, then for any domain $D_1 \subset \overline{D_1} \subset D$, $E_x[h^-(X_{\tau_{D_1}})] < \infty$ and*

$$h(x) \ge E_x[h(X_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

Proof. For a fixed $\epsilon > 0$ and each $x \in D_1$ we put

$$r(x) = \frac{1}{2} \delta(x, \partial D_1) \wedge \epsilon, \quad B(x) = B(x, r(x)),$$

where $\delta(x, \partial D_1)$ denotes the Euclidean distance between x and ∂D_1. Define a sequence of stopping times $\{T_m, m \geq 1\}$ as follows:

$$T_1 = \inf\{t > 0 : X(t) \notin B(X_0)\},$$

and for $m \geq 2$,

$$T_m = \begin{cases} T_{m-1} + T_{B(X_{T_{m-1}})} \circ \theta_{T_{m-1}} & \text{if } X_{T_{m-1}}) \in D_1, \\ \tau_{D_1} & \text{otherwise.} \end{cases}$$

The superharmonicity of h and the strong Markov property imply that

$$h(X_{T_{m-1}}) \geq E_x[h(X_{T_m})|\mathcal{F}_{T_{m-1}}].$$

Thus $\{h(X_{T_m}), m \geq 1\}$ is a supermartingale under P_x.

We claim that for each $x \in D_1$,

$$P_x(\lim_{m\to\infty} T_m = \tau_{D_1}) = 1.$$

It is clear that P_x–a.s., $T_m \uparrow$ and $T_m \leq \tau_{D_1} < \infty$ because D is bounded. Let $T_\infty = \lim_{m\to\infty} T_m$. Then $X_{T_\infty} = \lim_{m\to\infty} X_{T_m}$ by quasi left continuity. On the set $\{T_\infty < \tau_{D_1}\}$, we have $X_{T_\infty} \in D_1$. Then for all sufficiently large values of m, we have $\delta(X_{T_{m-1}}, \partial D_1) > \frac{1}{2}\delta(X_{T_\infty}, \partial D_1) > 0$ and $|X_{T_{m-1}} - X_{T_m}| < \frac{1}{4}\delta(X_{T_\infty}, \partial D_1) \wedge \epsilon$. But by the definition of T_m, $|X_{T_{m-1}} - X_{T_m}| > \frac{1}{2}\delta(X_{T_{m-1}}, \partial D_1) \wedge \epsilon$. These inequalities are incompatible. Hence $P_x(T_\infty < \tau_{D_1}) = 0$.

Put $A = \{\tau_{D_1} = T_m \text{ for some } t \geq 1\}$. Since h is bounded from below on $\overline{D_1}$, we have by Fatou's lemma

$$\liminf_{m\to\infty} E_x[h(X_{T_m}); T_m < \tau_{D_1}] \geq E_x[\liminf_{m\to\infty} h(X_{T_m}); A^c] \geq E_x[h(X_{\tau_{D_1}}); A^c].$$

Take two smooth domains D_2 and D_3 such that $\overline{D_1} \subset D_2 \subset \overline{D_2} \subset D_3 \subset \overline{D_3} \subset D$, then h is bounded from below on $\overline{D_3}$. Since

$$\begin{aligned} h(X_{\tau_{D_1}}) &= h(X_{\tau_D})1_{\{X_{\tau_{D_1}} \in D_3\}} + h(X_{\tau_{D_1}})1_{\{X_{\tau_{D_1}} \notin D_3\}} \\ &= h(X_{\tau_{D_1}})1_{\{X_{\tau_{D_1}} \in D_3\}} + h(X_{\tau_{D_2}})1_{\{X_{\tau_{D_1}} \notin D_3\}}, \end{aligned}$$

we have by estimate (2.2) with D_2 in place of U and the integrability assumption about h in Definition 2.2 that

$$E_x[h^-(X_{\tau_{D_1}})] < \infty.$$

Thus by Fatou's Lemma

$$\begin{aligned} h(x) &\geq \liminf_{m\to\infty} E_x[h(X_{T_m})] \\ &\geq \liminf_{m\to\infty} E_x[h(X_{\tau_{D_1}}), T_m = \tau_{D_1}] + \liminf_{m\to\infty} E_x[h(X_{T_m}), T_m < \tau_{D_1}] \\ &\geq E_x[h(X_{\tau_{D_1}}); A] + E_x[h(X_{\tau_{D_1}}); A^c] \\ &= E_x[h(X_{\tau_{D_1}})]. \end{aligned}$$

This completes the proof. ∎

Theorem 2.2 *Suppose that D is a bounded domain in $\mathbf{R}^n$. If h is a harmonic in D with respect to X, then for any domain $D_1 \subset \overline{D_1} \subset D$, $h(X_{\tau_{D_1}})$ is P_x-integrable and*

$$h(x) = E_x[h(X_{\tau_{D_1}})] \quad \textit{for every } x \in D_1. \tag{2.3}$$

Proof. We can always take a smooth domain D_2 such that $D_1 \subset \overline{D_1} \subset D_2 \subset \overline{D_2} \subset D$. If we could prove that for any $x \in D_2$, $h(X_{\tau_{D_2}})$ is P_x-integrable and

$$h(x) = E_x[h(X_{\tau_{D_2}})].$$

then by strong Markov property we immediately get $h(X_{\tau_{D_1}})$ is P_x-integrable and

$$h(x) = E_x[h(X_{\tau_{D_1}})], \qquad x \in D_1.$$

Therefore we can assume, without loss of generality, that D_1 is a smooth domain.

Define T_m as in the proof of the previous theorem, then in this case $\{h(X_{T_m}), m \geq 1\}$ is a martingale under P_x for any $x \in D_1$. By (2.1) with D_1 in place of U, we have $P_x(\tau_{D_1} = T_m \text{ for some } m \geq 1) = 1$. Since h is bounded on D_1, we have

$$|E_x[h(X_{T_m}), T_m < \tau_{D_1}]| \leq C P_x(T_m < \tau_{D_1}) \to 0.$$

Take a domain D_2 such that $\overline{D_1} \subset D_2 \subset \overline{D_2} \subset D$, then h is continuous and therefore bounded on $\overline{D_2}$. By the estimate (2.2) with D_1 in place of U and the integrability assumption about h in Definition 2.2 we have $E_x[|h|(X_{\tau_{D_1}})] < \infty$. Thus by the dominated convergence theorem

$$\lim_{m\to\infty} E_x[h(X_{\tau_{D_1}}), T_m = \tau_{D_1}] = E_x[h(X_{\tau_{D_1}})].$$

Therefore

$$\begin{aligned} h(x) &= \lim_{m\to\infty} E_x[h(X_{T_m})] \\ &= \lim_{m\to\infty} E_x[h(X_{\tau_{D_1}}), T_m = \tau_{D_1}] + \lim_{m\to\infty} E_x[h(X_{T_m}), T_m < \tau_{D_1}] \\ &= E_x[h(X_{\tau_{D_1}})]. \end{aligned}$$

∎

Similarly, we have the following result for functions harmonic with respect to X^D.

Theorem 2.3 *Suppose that D is a bounded domain in $\mathbf{R}^n$. If h is superharmonic in D with respect to X^D, then for any domain $D_1 \subset \overline{D_1} \subset D$, $E_x[h^-(X^D_{\tau_{D_1}})] < \infty$ and*

$$h(x) \geq E_x[h(X^D_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

If h is harmonic in D with respect to X^D, then for any domain $D_1 \subset \overline{D_1} \subset D$, $h(X^D_{\tau_{D_1}})$ is P_x-integrable and

$$h(x) = E_x[h(X^D_{\tau_{D_1}})] \quad \textit{for every } x \in D_1.$$

Theorem 2.4 *Suppose that D is a bounded domain and h is harmonic in D with respect to X and continuous on $\overline{D}$, then $h(X_{\tau_D})$ is P_x-integrable and*

$$h(x) = E_x[h(X_{\tau_D})], \qquad \text{for each } x \in D.$$

Proof. Take an increasing sequence of smooth domains $\{D_m\}_{m\geq 1}$ such that $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. Set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$ and $\lim_{m\to\infty} X_{\tau_m} = X_{\tau_D}$ by quasi-left continuity of X. Set

$$A = \{\tau_m = \tau_D \text{ for some } m \geq 1\}. \tag{2.4}$$

From Theorem 2.2 we know that for any $m \geq 1$,

$$h(x) = E_x[h(X_{\tau_m})], \qquad x \in D_m.$$

Since h is continuous on $\overline{D}$, we have by dominated convergence theorem that

$$\lim_{m\to\infty} E_x[h(X_{\tau_m}), \tau_m < \tau_D] = E_x[h(X_{\tau_D}), A^c]. \tag{2.5}$$

Since h is continuous on $\overline{D}$, we can find two smooth domains U_1 and U_2 such that $\overline{D} \subset U_1 \subset \overline{U_1} \subset U_2$ and that h is bounded on $\overline{U_2}$. Since

$$\begin{aligned} |h|(X_{\tau_D}) &= |h|(X_{\tau_D}) 1_{\{X_{\tau_D} \in U_2\}} + |h|(X_{\tau_D}) 1_{\{X_{\tau_D} \notin U_2\}} \\ &= |h|(X_{\tau_D}) 1_{\{X_{\tau_D} \in U_2\}} + |h|(X_{\tau_{U_1}}) 1_{\{X_{\tau_{U_1}} \notin U_2\}}, \end{aligned}$$

we have by estimate (2.2) with U_1 in place of U that

$$E_x[|h|(X_{\tau_D})] < \infty.$$

Thus by the dominated convergence theorem

$$\begin{aligned} h(x) &= \lim_{m\to\infty} E_x[h(X_{\tau_m})] \\ &= \lim_{m\to\infty} E_x[h(X_{\tau_D}), \tau_m = \tau_D] + \lim_{m\to\infty} E_x[h(X_{\tau_m}), \tau_m < \tau_D] \\ &= E_x[h(X_{\tau_D})]. \end{aligned}$$

∎

Remark 2.2 If D is a bounded domain satisfying the uniform exterior cone condition, then the conclusion of Theorem 2.4 holds for any harmonic function h in D with respect to X that is bounded in a neighborhood of $\overline{D}$. This is because in this case by (2.1) $P_x(A) = 1$ for $x \in D$, where A is the set defined in (2.4) and the term in (2.5) vanishes. The rest of the argument goes through without the continuous assumption on h up to the boundary ∂D.

Obviously there are plenty of bounded functions which are harmonic in D with respect to the whole processes X. The following results says that, when D is a bounded domain satisfying the uniform exterior cone condition, the only bounded function which is harmonic in D with respect to X^D is constant zero.

Theorem 2.5 *Suppose that D is a bounded domain in $\mathbf{R}^n$ satisfying the uniform exterior cone condition. If h is a bounded function harmonic in D with respect to X^D, then h must be identically zero.*

Proof. Take an increasing sequence of smooth domains D_m such that $D_m \subset \overline{D_m} \subset D_{m+1} \subset \overline{D_{m+1}} \subset D$ and set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$. By (2.1), we know that $P_x(\tau_D = \tau_m$ for some $m \geq 1) = 1$ for $x \in D$. From Theorem 2.2 we get that

$$\begin{aligned} |h(x)| &= |E_x[h(X^D_{\tau_m})]| \\ &\leq C\, P_x(\tau_m < \tau_D) \to 0. \end{aligned}$$

The proof is now complete. ∎

3 Martin Boundary

Superharmonic and harmonic functions with respect to X^D have been studied in the context of general theory of Markov processes and their potential theory (see, for instance, Kunita-Watanabe [15]). From the general theory, we know that positive harmonic functions with respect to X^D admit Martin representations. However, no particular attention was paid to the special case of harmonic functions with respect to the killed stable process. For instance, the relationship between the Martin boundary of X^D and the Euclidean boundary ∂D of D has not been studied.

In this section we assume that D is a bounded Lipschitz domain. In the first part of this section we are going to show that the Martin boundary of X^D and the Euclidean boundary ∂D coincide. Our proof of the identification between the Martin boundary and the Euclidean boundary is similar to the argument of Bass–Burdzy [2] in the Brownian motion case.

Fix $x_0 \in D$ and set

$$M_D(x,y) = \frac{G_D(x,y)}{G_D(x_0,y)}, \qquad x, y \in D.$$

The Martin boundary is the set $\partial_M D = D^* \setminus D$, where D^* is the smallest compact set for which $M_D(x,y)$ is continuous in y in the extended sense.

Lemma 3.1 *Suppose that D is a bounded Lipschitz domain. Then for any $\gamma > 0$,*

$$\lim_{x \to \partial D} G_D(x,y) = 0$$

uniformly on $D_\gamma = \{y \in D : \delta(y, \partial D) \geq \gamma\}$.

Proof. Suppose that $\mathcal{C} = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : 0 < x_n, (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < \eta x_n\}$ is a cone with vertex at the origin O. For any $r > 0$, set

$$\mathcal{C}_r = \mathcal{C} \cap B(O, r)$$

and

$$T_{\mathcal{C}_r} = \inf\{t > 0, X_t \in \overline{\mathcal{C}_r}\}.$$

One can easily show (similar to the proof of Proposition 1.19 of [11]) that for any $t > 0$, the function

$$x \mapsto P_x(t < T_{\mathcal{C}_r})$$

is upper semi–continuous in $\mathbf{R}^n$. Thus for any $s > 0$, we have

$$\limsup_{x \to O} P_x(T_{\mathcal{C}_r} > s) \leq P^O(T_{\mathcal{C}_r} > s) = 0$$

since $P^O(T_{\mathcal{C}_r} = 0) = 1$. Now use the fact that D satisfies the uniform exterior cone condition we can easily see that

$$\lim_{x \to z \in \partial D} P_x(\tau_D > s) = 0$$

uniformly in $z \in \partial D$, i.e., for any $\epsilon > 0$ there exists $\delta' > 0$ such that

$$P_x(\tau_D > s) < \epsilon, \qquad \text{if } \delta(x, \partial D) < \delta'. \tag{3.1}$$

We know that

$$G_D(x, y) = G(x, y) - E_x[G(X_{\tau_D}, y)]$$

and that $G(x, y)$ is bounded $D^c \times D_\gamma$. Now use the fact (3.1) and argue along the line of the proof of Theorem 1.23 of [11] we easily arrive at our conclusion. ∎

Take a positive number $\epsilon < \delta(x_0, \partial D)/4$.

Lemma 3.2 *Suppose $x \in D$ with $|x - x_0| > 4\epsilon$. There exists a constant $c_1 = c_1(\epsilon, D, x, x_0)$ such that*

$$M_D(x, y) \leq c_1 \quad \textit{for } y \in D \setminus \left(\overline{B(x_0, \epsilon)} \cup \overline{B(x, \epsilon)}\right).$$

Proof. Pick $y_0 \in \partial B(x_0, 2\epsilon)$. By the explicit formula for the Green function of balls (see [3] for instance) we know that

$$G_D(x_0, y_0) \geq G_{B(x_0, 3\epsilon)}(x_0, y_0) \geq \delta(\epsilon) > 0.$$

On the other hand, we know that $G_D(x, y_0) \leq 2^{-\alpha}\pi^{-n/2}\Gamma((n-\alpha)/2)\Gamma(\alpha/2)^{-1}|x - y_0|^{\alpha - n}$. Therefore $G_D(x, y_0)$ is bounded above by a constant depending on ϵ in $x \in D$ with $|x - x_0| > 4\epsilon$. Thus $M_D(x, y_0)$ is bounded above in $x \in D$ with $|x - x_0| > 4\epsilon$. But from the boundary Harnack principle (see [5]) we get that $M_D(x, y)$ is comparable to $M_D(x, y_0)$ for all points y in $D \setminus (B(x, \epsilon) \cup B(x_0, \epsilon))$. The lemma follows. ∎

Lemma 3.3 *Let x, x_0, ϵ be as above. Then $M_D(x,y)$ is a Hölder continuous function of y in $D \setminus \left(\overline{B(x_0,\epsilon)} \cup \overline{B(x,\epsilon)}\right)$ with Hölder exponent and coefficient depending only on x, x_0, ϵ and D.*

Proof. For any set A, we define that

$$\mathrm{Osc}_A f = \sup_{y \in A} f(y) - \inf_{y \in A} f(y).$$

Let $f(y) = M_D(x,y)$. Let $y_0 \in D_\epsilon = D \setminus \left(\overline{B(x_0,\epsilon)} \cup \overline{B(x,\epsilon)}\right)$. Since by Lemma 3.2 f is bounded by c_1 on D_ϵ, $\mathrm{Osc}_{D_\epsilon} f \le c_1$. So it suffices to show that there exists $\rho = \rho(\epsilon, D, x, x_0) < 1$ such that

$$\mathrm{Osc}_{D \cap B(y_0,r)} f \le \rho \, \mathrm{Osc}_{D \cap B(y_0,2r)} f \quad \text{for } r < \epsilon/4. \tag{3.2}$$

Suppose $r < \epsilon/4$, and let g be the ratio of any two positive harmonic functions on $D_{\epsilon/4}$ vanishing continuously on D^c. By considering $ag+b$ for suitable a and b, we may assume

$$\sup_{D \cap B(y_0,2r)} g = 1, \quad \inf_{D \cap B(y_0,2r)} g = 0.$$

If $\sup_{D \cap B(y_0,r)} g \le 1/2$, then since $\inf_{D \cap B(y_0,2r)} g \ge 0$, we have

$$\mathrm{Osc}_{D \cap B(y_0,r)} g \le \frac{1}{2} = \frac{1}{2} \mathrm{Osc}_{D \cap B(y_0,2r)} g.$$

If $\sup_{D \cap B(y_0,r)} g \ge 1/2$, there exists a point y_1 in $D \cap B(y_0,r)$ with $g(y_1) \ge 1/2$. But then by the boundary Harnack principle with $V = \{x : \delta(x,D) < \epsilon\} \setminus (\overline{B}(x_0,r) \cup \overline{B}(x,r))$ and $K = \overline{D} \setminus (\overline{B}(x_0,r) \cup \overline{B}(x,r))$, there exists a constant $c_2 = c_2(\epsilon, D, x, x_0) \in (0,1)$ such that

$$\inf_{D \cap B(y_0,r)} g \ge c_2 g(y_1).$$

Since $\sup_{D \cap B(y_0,r)} g \le 1$, in this case we have

$$\mathrm{Osc}_{D \cap B(y_0,r)} g \le 1 - \frac{c_2}{2} = \left(1 - \frac{c_2}{2}\right) \mathrm{Osc}_{D \cap B(y_0,2r)} g.$$

So we have (3.1) with $\rho = \max\left\{\frac{1}{2}, 1 - \frac{c_2}{2}\right\}$. Therefore $M_D(x,y)$ is a (globlly) Hölder continuous in $y \in D \setminus \left(\overline{B(x_0,\epsilon)} \cup \overline{B(x,\epsilon)}\right)$. ∎

A direct consequence of Lemma 3.3 is that $M_D(x,y) = G_D(x,y)/G_D(x_0,y)$ converges when $y \to z \in \partial D$. Let the limit be denoted as $M_D(x,z)$. This implies that the Martin boundary of D can identified with a subset of ∂D.

It is also well known that for a bounded Lipschitz domain D with Lipschitz characteristic constants (r_0, A_0), there exists $\kappa = \kappa(A_0) \in (0,1)$ such that for every $\epsilon \in (0, r_0)$ and $z \in \partial D$, there is a point $A_\epsilon(z) \in D \cap B(z,r)$ such that $B(A_\epsilon(z), \kappa r) \subset D \cap B(z,r)$. It is not difficult to show the following (cf. Lemma 6 of Bogdan [6]).

Lemma 3.4 *For any $z \in \partial D$, $M_D(\cdot, z)$ is harmonic with respect to X^D.*

Proof. Clearly any fixed $x \in D$ and and $r < \delta(x, \partial D)$,

$$M_D(x,y) = E_x\left[M_D(X_{\tau_{B(x,r)}}, y);\ \tau_{B(x,r)} < \tau_D\right] \quad \text{for } y \in D \setminus \overline{B(x,r)}.$$

In particular,

$$M_D(x, A_\epsilon(z)) = E_x\left[M_D(X_{\tau_{B(x,r)}}, A_\epsilon(z));\ \tau_{B(x,r)} < \tau_D\right] \tag{3.3}$$

for any $0 < \epsilon < \min\{r,\ r_0\}$. By Fatou's lemma,

$$M_D(x,z) \ge E_x\left[M_D(X_{\tau_{B(x,r)}}, z);\ \tau_{B(x,r)} < \tau_D\right].$$

Therefore $M_D(X_{\tau_{B(x,r)}}, z)$ is P_x-integrable. Put

$$\epsilon_0 = \min\left\{\frac{\delta(x_0, \partial D)}{4}, \frac{r_0}{2}, \frac{r}{4}\right\},$$

Then by Lemma 13 of [5] we get that there exists $C_1 = C_1(D) > 0$ such that for any $y \in D \cap B(z, \epsilon_0)$ and $\epsilon \in (0, \epsilon_0)$,

$$M_D(w, A_\epsilon(z)) \le C_1 M_D(w, y) \quad \text{for } w \in D \setminus B(z, 2\epsilon).$$

Letting $y \to z$ we get that for any $\epsilon \in (0, \epsilon_0)$,

$$M_D(w, A_\epsilon(z)) \le C_1 M_D(w, z) \quad \text{for } w \in D \setminus B(z, 2\epsilon). \tag{3.4}$$

For $w \in D \cap B(z, 2\epsilon)$, $|w - x| > 3r/2$ and thus by the explicit formula for $K_{B(x,r)}$ we know that there is a constant $C_2 = C_2(r) > 0$ such that $K_{B(x,r)}(x, w) \le C_2$ for $w \in D \cap B(z, 2\epsilon)$. Hence for any $w \in D \cap B(z, 2\epsilon)$,

$$\begin{aligned}
& E_x\left[M_D(X_{B(x,r)},\ A_\epsilon(z));\ X_{B(x,r)} \in D \cap B(z, 2\epsilon)\right] \\
\le\ & \frac{C_2}{G_D(x_0, A_\epsilon(z))} \int_{D \cap B(z,2\epsilon)} G(w, A_\epsilon(z))\, dw \\
\le\ & \frac{C_3}{G_D(x_0, A_\epsilon(z))} \int_{D \cap B(z,2\epsilon)} |w - A_\epsilon(z)|^{\alpha - n}\, dw \\
\le\ & \frac{C_4}{G_D(x_0, A_\epsilon(z))} \epsilon^\alpha.
\end{aligned}$$

From Lemma 5 of [5] we know that there exists a constant $C_5 = C_5(D, x_0) > 0$ and positive number $\gamma = \gamma(D) < \alpha$ such that $G_D(x_0, A_\epsilon(z)) \ge C_5 \epsilon^\gamma$. Therefore

$$E_x\left[M_D(X_{B(x,r)},\ A_\epsilon(z));\ X_{B(x,r)} \in D \cap B(z, 2\epsilon)\right] \le C_4 C_5^{-1} \epsilon^{\alpha - \gamma}. \tag{3.5}$$

Now combine (3.4) and (3.5) we see that the family of functions $\{M_D(X_{B(x,r)}),\ A_\epsilon(z)) : 0 < \epsilon < \epsilon_0\}$ is P_x-uniformly integrable. Letting $\epsilon \to 0$ in (3.3) yields

$$M_D(x,z) = E_x\left[M_D(X_{\tau_{B(x,r)}}, z);\ \tau_{B(x,r)} < \tau_D\right] \qquad \text{for any } r < \delta(x, \partial D).$$

Thus $M_D(\cdot, z)$ is harmonic with respect to X^D. ∎

The next result tells that each Euclidean boundary point corresponds to a different non-negative harmonic function. Hence the Martin boundary can not be identified with a proper subset of the Euclidean boundary.

Lemma 3.5 *If $M_D(\cdot, z_1) \equiv M_D(\cdot, z_2)$ for $z_1,\ z_2 \in \partial D$, then $z_1 = z_2$.*

Proof. Let $\epsilon > 0$ be such that

$$\epsilon < \min\left\{\frac{\delta(x_0,\, \partial D)}{4},\ \frac{r_0}{2}\right\},$$

where r_0 comes from the Lipschitz characteristic constants (r_0, A_0) of D. First we are going to show that $M_D(x,w) \to 0$ uniformly in $w \in \partial D$ as $\delta(x, \partial D \setminus B(w, 3\epsilon)) \to 0$. In fact, for any given $\eta > 0$, by Lemma 3.1 there is a $\beta = \beta(\eta, A_0, \epsilon) > 0$ such that $G_D(x, A_\epsilon(w)) < \eta$ for $w \in \partial D$ and $x \in D$ with $\delta(x, \partial D) < \beta$. Let D_0 be a smooth domain such that

$$\left\{x \in D; \delta(x, \partial D) \geq \frac{\kappa\epsilon}{2}\right\} \subset D_0 \subset \overline{D_0} \subset D.$$

Then for all $w \in \partial D$, $\delta(A_\epsilon(w), \partial D_0) > \kappa\epsilon/2$ and so by Theorem 1.1 of [8]

$$\begin{aligned} G_D(x_0, A_\epsilon(w)) &\geq G_{D_0}(x_0, A_\epsilon(w)) \\ &\geq C(D_0) \min\left\{\frac{1}{|x_0 - A_\epsilon(w)|^{n-\alpha}},\ \frac{\delta(x_0, \partial D_0)^{\alpha/2}\delta(A_\epsilon(w), \partial D_0)^{\alpha/2}}{|x_0 - A_\epsilon(w)|^n}\right\} \\ &\geq C(D_0) \min\left\{\frac{1}{d_D^{n-\alpha}},\ \frac{(3\epsilon)^{\alpha/2}(\kappa\epsilon/2)^{\alpha/2}}{d_D^n}\right\} \\ &:= C_1 > 0, \end{aligned}$$

where d_D is the diameter of D. Therefore

$$M_D(x, A_\epsilon(w)) < \eta / C_1, \qquad \forall w \in \partial D,$$

whenever $\delta(x, \partial D) < \beta$. Fix $w \in \partial D$. Clearly $|x - w| > 2\epsilon$ for any $x \in D$ with $\delta(x, \partial D \setminus B(w, 3\epsilon)) < \epsilon$. Now by Lemma 13 of [5] with $r = \epsilon$, we get that there is a $C_2 = C_2(A_0) > 0$ such that

$$M_D(x,y) \leq C_2 M_D(x, A_\epsilon(w)) \quad \text{for } y \in D \cap B(w, \epsilon)$$

whenever $x \in D$ satisfies $\delta(x, \partial D \setminus B(w, 3\epsilon)) < \epsilon$. Therefore

$$M_D(x, y) \leq \frac{C_2}{C_1}\eta \quad \text{for } y \in D \cap B(w, \epsilon)$$

whenever $x \in D$ satisfies $\delta(x, \partial D \setminus B(w, 3\epsilon)) < \epsilon \wedge \beta$. Letting $y \to w$ we get that

$$M_D(x, w) \leq \frac{C_2}{C_1}\eta$$

for $x \in D$ with $\delta(x, \partial D \setminus B(w, 3\epsilon)) < \epsilon \wedge \beta$. Therefore

$$M_D(x, w) \to 0 \text{ uniformly in } w \in \partial D \text{ as } \delta(x, \partial D \setminus B(w, 3\epsilon)) \to 0. \tag{3.6}$$

Suppose that $M_D(\cdot, w) = M_D(\cdot, z)$ for some w, $z \in \partial D$, $w \neq z$, and let $\epsilon < |w - z|/8$. By the above argument, $M_D(x, w) \to 0$ uniformly when $\delta(x, \partial D \setminus B(w, 2\epsilon)) \to 0$ or when $\delta(x, \partial D \setminus B(z, 2\epsilon)) \to 0$. Therefore $M_D(x, w) \to 0$ uniformly as $\delta(x, \partial D) \to 0$. Since $M_D(\cdot, w)$ is a non-negative harmonic function with respect to X^D which continuously vanishes on ∂D, it must be identically zero by Theorem 2.5. This contradicts the fact that $M_D(x_0, w) = 1$. The proof is now complete. ∎

Combining the lemmas above we get the following result.

Theorem 3.6 *The Martin boundary of D can be identified with its Euclidean boundary ∂D.*

Theorem 3.7 *For each $z \in \partial D$, $M_D(x, z)$ is minimal harmonic with respect to X^D.*

Proof. Fix $z \in \partial D$ and suppose $h \leq M_D(\cdot, z)$, where h is a positive harmonic function with respect to X^D. By Theorem 3.6 we know that there is a measure μ on ∂D such that

$$h(\cdot) = \int_{\partial D} M_D(\cdot, w)\mu(dw).$$

If μ is not a multiple of the point mass at z, then there is a finite measure $\nu \leq \mu$ such that $\delta(z, \text{supp}(\nu)) > 0$. Let

$$u(\cdot) = \int_{\partial D} M_D(\cdot, w)\nu(dw).$$

Then u is a positive harmonic function with respect to X^D bounded by $M_D(\cdot, z)$.

Recall from (3.6 in the proof of Lemma 3.5 that $M_D(x, z) \to 0$ uniformly as $\delta(x, \partial D \setminus B(z, \epsilon) \to 0$. So the same is true of u. But for each $w \in \text{supp}(\nu)$, we can see that $M_D(x, w) \to 0$ uniformly as $\delta(x, \partial D \cap B(z, 2\epsilon) \to 0$ provided $2\epsilon < \delta(z, \text{supp}(\nu))$. So it follows by the dominated convergence theorem that $u(x) \to 0$ as $\delta(x, \partial D \cap B(z, 2\epsilon) \to 0$. But then u is a positive harmonic function of X^D which continuously vanishes on ∂D. This implies that ν is 0, or that $\mu = c\delta_z$ for some c. ∎

From Theorem 3.6 and the general theory of Martin representation (cf. [15]), we have

Theorem 3.8 *If D is a bounded Lipschitz domain in $\mathbf{R}^n$, then the restriction to D of any positive superharmonic function f with respect to X^D can be written uniquely as*

$$f(x) = \int_D G_D(x,y)\nu(dy) + \int_{\partial D} M_D(x,z)\mu(dz), \tag{3.7}$$

where ν and μ are finite measures on D and ∂D respectively.

When D is a bounded $C^{1,1}$ domain, we can say more about the Martin kernel of D.

Theorem 3.9 *Suppose that D is a bounded $C^{1,1}$ domain, then $M_D(\cdot,\cdot)$ is continuous function on $D \times \partial D$. Furthermore, there exists a constant $C = C(D,\alpha) > 0$ such that for any $x \in D$ and $z \in \partial D$,*

$$\frac{\delta(x,\partial D)^{\alpha/2}}{C|x-z|^n} \le M_D(x,z) \le \frac{C\delta(x,\partial D)^{\alpha/2}}{|x-z|^n}.$$

Proof. The joint continuity of M_D follows from the definition of Martin kernel and Lemma 3.3. The estimates on M_D follows easily from Theorems 1.1 and 1.2 of Chen and Song [8]. ∎

From Theorems 1.1–1.2 of Chen and Song [8] and Theorem 3.9 above we have

Theorem 3.10 *(3G Theorem). Suppose that D is a bounded $C^{1,1}$ domain, then there exists $C = C(D,\alpha) > 0$ such that*

$$\frac{G_D(x,y)M_D(y,z)}{M_D(x,z)} \le C\frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}, \qquad x,y \in D, z \in \partial D.$$

Proof. The proof is the same as the proof of Theorem 1.6 of Chen and Song [8] and we omit it here. ∎

Using Theorems 3.9 and 3.10, we can prove a conditional gauge theorem, which complements the two conditional gauge theorems established in Chen and Song [9]. Before we state and prove the conditional gauge theorem, we need to do some preparations first.

Definition 3.1 *A Borel measurable function q on $\mathbf{R}^n$ is said to be in the Kato class $\mathbf{K}_{n,\alpha}$ if*

$$\lim_{r\downarrow 0}\sup_{x\in\mathbf{R}^n}\int_{|x-y|\le r}\frac{|q(y)|}{|x-y|^{n-\alpha}}dy = 0. \tag{3.8}$$

For $q \in \mathbf{K}_{n,\alpha}$, set

$$e_q(t) = \exp\left(\int_0^t q(X_s)ds\right).$$

From Chen and Song [9], we know that the following semigroup

$$T_t f(x) = E_x[e_q(t)f(X_t);\, t < \tau_D], \qquad x \in D,$$

admits an integral kernel $k_q(t,x,y)$. The function

$$g(x) := E_x[e_q(\tau_D)]$$

is called the gauge function of (D,q). It is shown in [12] that either g is identically identically infinite or g is bounded on D. In the latter case, (D,q) is said to be *gaugeable*. When (D,q) is gaugeable, it can be showb (see [9]) that

$$V_q(x,y) = \int_0^\infty k_q(t,x,y)dt, \qquad x,y \in D,$$

is well defined and is continuous off the diagonal.

Suppose that $h > 0$ is a positive superharmonic function with respect to X^D. Note that by Theorem 2.3 above, we have (see, e.g., page 11 of Dynkin [14]) that

$$h(x) \geq E^x[h(X_t^D)].$$

We define

$$p_D^h(t,x,y) = h(x)^{-1} p_D(t,x,y)h(y), \qquad t > 0, x,y \in D,$$

where p_D is the transition density function of killed symmetric stable process X^D in D. It is easy to check that p_D^h is a transition density and it determines a Markov process on the state space $D_\partial = D \cup \{\partial\}$. This process is called the h–conditioned symmetric stable process. Similar to Propositions 5.2–5.4 of Chung and Zhao [13], we have the following

Lemma 3.11 *For $t > 0$, if $\Phi \geq 0$ is an $\mathcal{F}_t$–measurable function, then*

$$E_x^h[\Phi; t < \tau_D] = h(x)^{-1} E_x[\Phi \cdot h(X_t); t < \tau_D], \qquad x \in D.$$

Recall that $\{\mathcal{F}_t,\, t \geq 0\}$ be the minimal admissible σ-fields generated by X. For any stopping time T of $\{\mathcal{F}_t,\, t \geq 0\}$, $\mathcal{F}_{T+}$ is the class of subsets Λ of $\mathcal{F}$ such that

$$\Lambda \cap \{T \leq t\} \in \mathcal{F}_{t+}, \qquad t \geq 0.$$

$\mathcal{F}_{T-}$ is the σ–field generated by $\mathcal{F}_{0+}$ and the class of sets

$$\{t < T\} \cap \Lambda, \qquad t \geq 0, \lambda \in \mathcal{F}_t.$$

Lemma 3.12 *For any stopping time T and any $\mathcal{F}_{T+}$–measurable function $\Phi \geq 0$,*

$$E_x^h[\Phi; T < \tau_D] = h(x)^{-1} E_x[\Phi \cdot h(X_T); T < \tau_D].$$

Lemma 3.13 *For any stopping time T, $A \in \mathcal{F}_{T+}$ and any $\mathcal{F}_{\tau_{D-}}$–measurable variable $\Phi \geq 0$,*

$$E_x^h[A \cap (T < \tau_D); \Phi \circ \theta_T] = E_x^h[A \cap (T < \tau_D); E_h^{X_T}(\Phi)],$$

where θ_t is the shift operator for process X.

Now let D be a bounded Lipschitz domain. For each $z \in \partial D$, the $M_D(\cdot, z)$–conditioned symmetric stable process will be called the z-symmetric stable process, and the associated probability and expectation will be denoted by P_x^z and E_x^z, respectively.

For any $y \in D$, $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$ with respect to $X^{D\setminus\{y\}}$. Hence we can define the $G_D(\cdot, y)$–conditioned symmetric stable process on the state space $(D \setminus \{y\}) \cup \{\partial\}$, with lifetime $\tau_{D\setminus\{y\}}$. It will be referred to as the y–conditioned symmetric stable process, and the associated probability and expectation will be denoted by P_x^y and E_x^y respectively. The following result immediately follows from Theorem 3.10.

Corollary 3.14 *(Conditional Lifetime) Suppose that D is a bounded $C^{1,1}$ domain. Then*

$$\sup_{x\in D,\, z\in\partial D} E_x^z[\tau_D] < \infty.$$

Theorem 3.15 *(Conditional Gauge Theorem). Suppose that D is a bounded $C^{1,1}$ domain and $q \in \mathbf{K}_{n,\alpha}$. If (D, q) is gaugeable, then there exists $c > 1$ such that*

$$c^{-1} \le \inf_{x\in D, z\in\partial D} E_x^z[e_q(\tau_D)] \le \sup_{x\in D, z\in\partial D} E_x^z[e_q(\tau_D)] \le c.$$

Proof. Suppose $x, y \in D$ and $z \in \partial D$. For any $w \in D$, by it follows from Lemma 6.5 of Chen and Song [9] that

$$\begin{aligned} \lim_{y\to z} \frac{1}{G_D(x,y)} V_q(x,w)q(w)G_D(w,y) &= \lim_{y\to z} V_q(x,w)q(w)\frac{G_D(w,y)/G_D(x_0,y)}{G_D(x,y)/G_D(x_0,y)} \\ &= V_q(x,w)q(w)\frac{M_D(w,z)}{M_D(x,z)}. \end{aligned}$$

Now from Theorem 1.6 (3G Theorem) of Chen and Song [8] and Theorem 5.2 of Chen and Song [9] we have that

$$\{V_q(x,\cdot)G_D(\cdot,y)|q(\cdot)|/G_D(x,y) : x, y \in D\}$$

is uniformly integrable. Hence it follows from Theorem 5.4 of Chen and Song [9] that

$$\lim_{y\to z} E_x^y[e_q(\tau_{D\setminus\{y\}})] = 1 + \frac{1}{M_D(x,z)} \int_D V_q(x,u)q(u)M_D(u,z)du.$$

However, one can show, by using an argument similar to the proof of Theorem 5.4 of Chen and Song [9], that

$$E_x^z[e_q(\tau_D)] = 1 + \frac{1}{M_D(x,z)} \int_D V_q(x,u)q(u)M_D(u,z)du.$$

Therefore

$$\lim_{y\to z} E_x^y[e_q(\tau_{D\setminus\{y\}})] = E_x^z[e_q(\tau_D)] \tag{3.9}$$

and the theorem now follows from Theorem 5.6 of Chen and Song [9]. ∎.

Theorem 3.16 *Suppose that D is a bounded $C^{1,1}$ domain and $q \in \mathbf{K}_{n,\alpha}$. If (D,q) is gaugeable, then for any fixed point $x_0 \in D$ and $z \in \partial D$,*

$$\lim_{D \ni y \to z} \frac{V_q(x,y)}{V_q(x_0,y)} = \frac{E_x^z[e_q(\tau_D)]}{E_z^{x_0}[e_q(\tau_D)]} M_D(x,z). \tag{3.10}$$

Furthermore,

$$\lim_{D \ni y \to z} \frac{V_q(x,y)}{\delta(y,\partial D)^{\alpha/2}} = E_x^z[e_q(\tau_D)] \lim_{D \ni y \to z \in \partial D} \frac{G_D(x,y)}{\delta(y,\partial D)^{\alpha/2}}. \tag{3.11}$$

Proof. Since

$$E_x^y[e_q(\tau_{D\setminus\{y\}})] = V_q(x,y)/G_D(x,y) \tag{3.12}$$

for $x, y \in D$ by Theorem 5.5 of Chen and Song [9] and so (3.10) follows immediately from it. Identity (3.11) follows from (3.12) and from Lemma 6.5 of Chen and Song [8] which asserts that the limit

$$\lim_{D \ni y \to z \in \partial D} \frac{G_D(x,y)}{\delta(y,\partial D)^{\alpha/2}}$$

exists and forms a positive and continuous function in $(x,z) \in D \times \partial D$. ∎

As a consequence of Theorems 3.15 and 3.16 we get that, for a bounded $C^{1,1}$ domain D and a $q \in \mathbf{K}_{n,\alpha}$, if (D,q) is gaugeable, then the Martin kernel of the generalized Schrödinger operator $-(-\Delta)^{\alpha/2} + q$ with zero exterior condition on D^c is comparable to Martin kernel M_D.

Remark 3.1 Recently in [10] we were able to extend the 3G theorem and conditional gauge theorem established for bounded $C^{1,1}$ domains in Chen and Song [8], [9] to bounded Lipschitz domains. Thus Theorems 3.10 and 3.15 and (3.10) in Theorem 3.16 in fact hold on bounded Lipschitz domains as well. Thus under the condition that D is a bounded Lipschitz domain and (D,q) is gaugeable, the Martin kernel of the generalized Schrödinger operator $L = -(-\Delta)^{\alpha/2} + q$ with zero exterior condition on D^c is comparable to Martin kernel M_D and the Martin boundary for L coincides with the Euclidean boundary ∂D of D.

When D is a ball, we can actually get an explicit formula for the Martin kernel of D. This follows easily form the definition of the Martin kernel and the explicit formula for the Green functions of balls (see Corollary 4 of Blumenthal, Getoor and Ray [3]). We record this fact as follows.

Example. If $B = B(0,r)$, then

$$M_B(x,w) = \frac{(r^2 - |x|^2)^{\alpha/2}}{|x-w|^n}, \qquad x \in D, w \in \partial D.$$

From the formula above, we know that

$$h(x)=\int_{\partial B}\frac{(r^2-|x|^2)^{\alpha/2}}{|x-w|^n}dw, \qquad x\in D$$

is a positive harmonic function with respect to X^B. From Theorem 2.5 we know that h can not be a bounded function on B. In fact one can check directly in this case that for each $z\in\partial B$

$$\lim_{B\ni x\to z}h(x)=\infty.$$

In the Brownian motion case, the Martin boundary can be approached along Brownian paths. While for a symmetric stable process, we know from Lemma 6 of Bogdan [5] that, with probability 1, it will not hit ∂D upon first exiting from a bounded domain D satisfying the uniform exterior cone condition. Our next theorem gives the relationship between the Martin boundary and the (conditioned) stable paths.

Theorem 3.17 *Suppose D is a bounded Lipschitz domain. Then for every $x\in D$ and $z\in\partial D$,*

$$\begin{aligned}P^z_x\{\tau_D<\infty\} &= 1;\\ P^z_x\{\lim_{t\uparrow\tau_D}X(t)=z\} &= 1.\end{aligned}$$

Proof. Let $z\in\partial D$, $r_m\downarrow 0$, $B_m=(z,r_m)$, $D_m=D\setminus\overline{B_m}$ and set

$$T_m=\inf\{t>0: X_t\in B_m\}, \qquad R_m=\tau_{B_m\cap D}.$$

We may assume that $x\in D_m$ for $n\geq 1$. By Lemma 3.3, $M_D(\cdot,z)$ can be continuously extended onto $\overline{D_m}$ by setting $M_D(w,z)=0$ for $w\in\partial D\setminus B_m$. Since $M_D(\cdot,z)$ is harmonic in D_m with respect to X^D, we have by Theorem 2.4 and Lemma 3.12

$$\begin{aligned}M_D(x,z) &= E_x\left[M_D(X_{\tau_{D_m}},z)\right]\\ &= E_x\left[M_D(X_{T_m},z);T_m<\tau_D\right]\\ &= M_D(x,z)P^z_x\left(T_m<\tau_D\right).\end{aligned}$$

It follows that for all $n\geq 1$ we have

$$P^z_x\{T_m<\tau_D\}=1. \tag{3.13}$$

Note that for each fixed $z\in\partial D$, $M_D(x,z)$ is bounded in $x\in B^c_k\cap D$ by continuity. Let C_k denote its bound. Applying Lemmas 3.12 and 3.13 twice, we have for all $k<m$:

$$P^z_x\{T_m<\tau_D,\, R_k\circ\theta_{T_m}<\tau_D\} \;=\; E^z_x\left[P^{X_{T_m}}_z[R_k<\tau_D];\,T_m<\tau_D\right]$$

$$
\begin{aligned}
&= \frac{1}{M_D(x,z)} E_x\left[M(X_{T_m}, z) P_z^{X_{T_m}}[R_k < \tau_D];\, T_m < \tau_D\right] \\
&= \frac{1}{M_D(x,z)} E_x\left[E^{X_{T_m}}[M_D(X_{R_k}, z);\, R_k < \tau_D];\, T_m < \tau_D\right] \\
&\leq \frac{C_k}{M_D(x,z)} P_x\left(T_m < \tau_D\right). \qquad (3.14)
\end{aligned}
$$

By the definition of T_m and the quasi left continuity of the unconditioned process X, we have

$$
\begin{aligned}
\lim_{m\to\infty} P_x\{T_m < \tau_D\} &\leq P_x\{\lim_{m\to\infty} T_m \leq \tau_D\} \\
&\leq P_x\{T_{\{z\}} \leq \tau_D\} = 0
\end{aligned}
$$

because $z \in \partial D$ and by (2.1) with D in place of U that $P_x(X_{\tau_D} \in \partial D) = 0$. It follows from (3.14) that the left hand side there converges to zero as $m \to \infty$ for each k. Therefore there exists a subsequence $\{m_j\}$ such that

$$
\sum_{j=1}^{\infty} P_x^z\{T_{m_j} < \tau_D; R_k \circ \theta_{T_{m_j}} < \tau_D\} < \infty,
$$

and consequently by Borel–Cantelli lemma we have

$$
P_x^z\{[T_{m_j} < \tau_D; R_k \circ \theta_{T_{m_j}} < \tau_D] \text{ infinitely often }\} = 0.
$$

Together with (3.13) this implies that for $k \geq 1$ and P_x^z–a.e. ω, there exists an integer $N(\omega) < \infty$ such that

$$
X_t(\omega) \in B(z, r_k) \text{ for all } t \in [T_{N(\omega)}(\omega), \tau_D(\omega)).
$$

For each k let $N(k)$ be the smallest N for which the above is true. Then $T_{N(k)} \uparrow \tau_D$; otherwise, we would have $X_t = z$ for all $t \in [\lim_{k\to\infty} T_{N(k)}, \tau_D)$, which is impossible since $z \notin D_\partial$. This proves that $X_t \to z$ as $t \uparrow \tau_D$. ∎

Functions harmonic in D with respect to X^D do not come from solving Dirichlet exterior problems. Therefore the usual probabilistic interpretation of harmonic functions as solutions of Dirichlet problems is not true anymore. The following result, which follows easily from Theorem 3.17, provides some probabilistic interpretation to these kind of harmonic functions.

Theorem 3.18 *Suppose that D is a bounded Lipschitz domain and μ is a finite measure on ∂D. Define*

$$
h(x) = \int_{\partial D} M_D(x,z)\mu(dz).
$$

Then for any Borel measurable subset $A \subset \partial D$,

$$
P_x^h(\lim_{t\uparrow\tau_D} X(t) \in A) = \frac{1}{h(x)} \int_A M_D(x,z)\mu(dz).
$$

In particular, when $D = B(O,r)$ and $h(x) = \int_{\partial D} M_D(x,z)\sigma(dz)$, where σ is the surface measure, $\lim_{t\uparrow\tau_D} X(t)$ is distributed uniformly on ∂D under P_0^h.

4 Integral Representations of Positive Harmonic Functions

Functions which are (super)harmonic in D with respect to X are studied in Landkof [17] and Bogdan [5]. However, it seems that no one has studied the integral representations of this kind of (super)harmonic functions. We intend to establish such a representation. To prove the uniqueness of such a representation theorem we need the following result:

Lemma 4.1 *Suppose that D is a bounded Lipschitz domain. If a function f satisfies the following*

$$\int_{D^c} \frac{f(z)}{|y-z|^{n+\alpha}} dz = 0, \qquad \forall y \in D, \tag{4.1}$$

Then $f = 0$ almost everywhere on D^c.

Proof. Without loss of generality we can assume that the origin O is in D.

We claim that for all $0 \le m \le k$,

$$\int_{D^c} \prod_{j=1}^{m} (x_{i_j} - y_{i_j}) |x-y|^{-(n+\alpha+2k)} f(y) dy = 0, \qquad x \in D, \tag{4.2}$$

where for each j, $1 \le i_j \le n$. We are going to prove the claim by induction on k.

The case of $k = 0, m = 0$ follows by the assumption. Take the partial derivative of (4.1) with respect to x_j we get

$$0 = \frac{\partial}{\partial x_j} \int_{D^c} |x-y|^{-n-\alpha} f(y) dy = -(n+\alpha) \int_{D^c} (x_j - y_j) |x-y|^{-n-\alpha-2} f(y) dy,$$

Thus

$$\int_{D^c} (x_j - y_j) |x-y|^{-n-\alpha-2} f(y) dy = 0, \qquad x \in D, \tag{4.3}$$

that is, the claim is true for $k = m = 1$. Now take the partial derivative of (4.3) with respect to x_j we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_j} \int_{D^c} (x_j - y_j) |x-y|^{-n-\alpha-2} f(y) dy \\ &= \int_{D^c} |x-y|^{-n-\alpha-2} f(y) dy - (n+\alpha+2) \int_{D^c} (x_j - y_j)^2 |x-y|^{-n-\alpha-4} f(y) dy \end{aligned}$$

Summing the above from $j=1$ to $j=n$ we get

$$(\alpha+2)\int_{D^c}|x-y|^{-n-\alpha-2}f(y)dy=0,$$

which implies that the claim is true for the case of $k=1, m=0$. Therefore the claim is true for $k=1$.

Now we assume that the claim is true for all $0\leq m\leq k\leq N$. Take the partial derivative of (4.2) with respect to $x_{i_{m+1}}$ we get

$$\begin{aligned}
0 &= \frac{\partial}{\partial x_{i_{m+1}}}\int_{D^c}\prod_{j=1}^{m}(x_{i_j}-y_{i_j})|x-y|^{-(n+\alpha+2k)}f(y)dy\\
&= \int_{D^c}\frac{\partial}{\partial x_{i_{m+1}}}\left(\prod_{j=1}^{m}(x_{i_j}-y_{i_j})|x-y|^{-(n+\alpha+2k)}\right)f(y)dy\\
&= -(n+\alpha+2k)\int_{D^c}\prod_{j=1}^{m+1}(x_{i_j}-y_{i_j})|x-y|^{-(n+\alpha+2(k+1))}f(y)dy\\
&\quad +\int_{D^c}\frac{\partial}{\partial x_{i_{m+1}}}\left(\prod_{j=1}^{m}(x_{i_j}-y_{i_j})\right)|x-y|^{-(n+\alpha+2k)}f(y)dy\\
&= -(n+\alpha+2k)\int_{D^c}\prod_{j=1}^{m+1}(x_{i_j}-y_{i_j})|x-y|^{-(n+\alpha+2(k+1))}f(y)dy,
\end{aligned}$$

where in the last equality we used the induction assumption. Therefore the claim is true for all $0\leq m\leq k\leq N+1$, and hence the claim is always true.

Evaluate (4.2) at $x=O$ we get that for any non-negative integer k, and any multi–index $\beta=(\beta_1,\cdots,\beta_n)$ with $|\beta|=\beta_1+\cdots+\beta_n\leq k$,

$$\int_{D^c}y^\beta|y|^{-2k}\frac{f(y)}{|y|^{n+\alpha}}dy=0$$

where $y^\beta=y_1^{\beta_1}\cdots y_n^{\beta_n}$. Since the linear span of the set $\{y^\beta|y|^{-2k}:|\beta|\leq k\}$ is an algebra of real–valued continuous functions on D^c which separates points in D^c and vanishes at infinity, by the Stone–Weierstrass Theorem the linear span of $\{y^\beta|y|^{-2k}:|\beta|\leq k\}$ is dense in $C_\infty(D^c)$ with respect to the uniform topology. Here $C_\infty(D^c)$ is the space of continuous functions on D^c which vanishes at infinity. Thus for all $\phi\in C_\infty(D^c)$,

$$\int_{D^c}\phi(y)\frac{f(y)}{|y|^{n+\alpha}}dy=0$$

which implies that $f(y)|y|^{-(n+\alpha)}=0$ almost everywhere on D^c. Therefore $f=0$ almost everywhere on D^c. ∎

Theorem 4.2 *Suppose that D is a bounded domain in $\mathbf{R}^n$. If h and f are both harmonic in D with respect to X with $h = f$ in D, then $h = f$ in $\mathbf{R}^n$.*

Proof. Take $x_0 \in D$ and $B(x_0, r) \subset \overline{B(x_0, r)} \subset D$, then it follows from Theorem 2.2 that for any $x \in B(x_0, r)$,

$$E_x[h(X_{\tau_{B(x_0,r)}})] = h(x) = f(x) = E_x[f(X_{\tau_{B(x_0,r)}})].$$

Therefore we have

$$E_x[(h - f)(X_{\tau_{B(x_0,r)}})] = 0, \qquad x \in B(x_0, r).$$

By Theorem 1.4 of Chen and Song [8] we know that for all $x \in B(x_0, r)$,

$$\begin{aligned} E_x[(h-f)(X_{\tau_{B(x_0,r)}})] &= \int_{B(x_0,r)^c} K_{B(x_0,r)}(x,z)(h-f)(z)dz \\ &= A(n,\alpha)\int_{B(x_0,r)^c}\left(\int_{B(x_0,r)} G_{B(x_0,r)}(x,y)\frac{1}{|y-z|^{n+\alpha}}dy\right)(h-f)(z)dz \\ &= A(n,\alpha)\int_{B(x_0,r)} G_{B(x_0,r)}(x,y)\left(\int_{B(x_0,r)^c}\frac{(h-f)(z)}{|y-z|^{n+\alpha}}dz\right)dy. \end{aligned}$$

Therefore by general potential theory (see Section 5.2 of [11], for instance) we know that the function

$$y \mapsto \int_{B(x_0,r)^c} \frac{(h-f)(z)}{|y-z|^{n+\alpha}}dz$$

is zero almost everywhere on $B(x_0, r)$. Since the function above is continuous in $B(x_0, r)$, we have

$$\int_{B(x_0,r)^c} \frac{(h-f)(z)}{|x-z|^{n+\alpha}}dz = 0, \qquad \forall x \in B(x_0, r).$$

It follows from Lemma 4.1 we know that $u - f = 0$ almost everywhere on $B(x_0, r)^c$, and the proof is finished.. ∎

Remark 4.1 In fact the proof actually shows that for a function h harmonic in a domain D, the values of h in any ball $B(x_0, r) \subset \overline{B(x_0, r)} \subset D$ determine h uniquely.

Theorem 4.3 *Suppose D is a bounded Lipschitz domain in $\mathbf{R}^n$. If f is a non-negative harmonic function in D with respect to X, then there exists a unique finite measure μ on ∂D such that the restriction of f to D can be written as*

$$f(x) = \int_{D^c} K_D(x,z)f(z)dz + \int_{\partial D} M_D(x,z)\mu(dz), \qquad \forall x \in D. \tag{4.4}$$

Proof. First we are going to show that the difference

$$f(x) - \int_{D^c} K_D(x,z) f(z) dz$$

is non-negative. Take a sequence of domains D_m such that $D_m \subset \overline{D_m} \subset D_{m+1} \subset \overline{D_{m+1}} \subset D$ and set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$. For any $x \in D$, since $P_x(X_{\tau_D -} \neq X_{\tau_D}) = 1$, we know that $P_x(\tau_D = \tau_m \text{ for some } m \geq 1) = 1$. From Theorem 2.2 we know that

$$\begin{aligned} f(x) &= E_x f(X_{\tau_m}) \\ &= E_x[f(X_{\tau_D}); \tau_m = \tau_D] + E_x[f(X_{\tau_m}); \tau_m < \tau_D] \\ &\geq E_x[f(X_{\tau_D}); \tau_m = \tau_D]. \end{aligned}$$

Therefore

$$f(x) \geq E_x[f(X_{\tau_D})].$$

Therefore by Theorem 3.6 we know that there exists a unique finite measure μ on ∂D such that

$$f(x) - E_x[f(X_{\tau_D})] = \int_{\partial D} M_D(x,z)\mu(dz), \qquad \forall x \in D.$$

∎

From the above theorem we can easily get the following

Theorem 4.4 *If D is a bounded Lipschitz domain, then the restriction to D of any non-negative function f which is superharmonic in D with respect to X can be written as*

$$f(x) = \int_{D^c} K_D(x,z) f(z) dz + \int_D G_D(x,y)\nu(dy) + \int_{\partial D} M_D(x,z)\mu(dz),$$

where ν and μ are finite measures on D and ∂D respectively.

Proof. Similar to the first part of the proof of Theorem 4.3, we have that the function

$$f(x) - E_x[f(X_{\tau_D})]$$

is a non-negative function which vanishes outside D and is superharmonic in D with respect to X. Hence it is a non-negative function which is harmonic in D with respect to X^D. Now our claim follows from Theorem 3.8. ∎

However, the above decomposition is not unique anymore. This non-uniqueness is due to the following relation between the Green function G_D and Poisson kernel K_D established in Theorem 1.4 of Chen and Song [8]:

$$K_D(x,z) = A(n,\alpha) \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}} dy.$$

Using this relation, we can absorb the first term on the right hand side above, or part of it, into the second term.

5 Extensions

The results of this paper can be extended to dimension $n = 1$, by noting that Green functions and Poisson kernels on bounded open intervals for symmetric stable processes are explicitly known (see, for example, [3]). In particular (2.1)–(2.2) hold for bounded open interval U in $\mathbf{R}$. Note that the Kato class $\mathbf{K}_{1,\alpha}$ is defined by (3.8) for $0 < \alpha < 2$ (cf. Corollary 4 of [3] and [19]).

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