Intrinsic Ultracontractivity, Conditional Lifetimes and Conditional Gauge for Symmetric Stable Processes on Rough Domains

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Abstract

For a symmetric α-stable process X on $\mathbf{R}^n$ with $0 < \alpha < 2$, $n \geq 2$ and a domain $D \subset \mathbf{R}^n$, let L^D be the infinitesimal generator of the subprocess of X killed upon leaving D. For a Kato class function q, it is shown that $L^D + q$ is intrinsic ultracontractive on a Hölder domain D of order 0. This is then used to establish the conditional gauge theorem for X on bounded Lipschitz domains in $\mathbf{R}^n$. It is also shown that the conditional lifetimes for symmetric stable process in a Hölder domain of order 0 are uniformly bounded.

Keywords and phrases: Symmetric stable processes, Feynman-Kac semigroup, conditional gauge theorem, logarithmic Sobolev inequality, intrinsic ultracontractivity.

Running Title: Conditional Gauge Theorem

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1 Introduction

A symmetric α-stable process X on $\mathbf{R}^n$ is a Lévy process whose transition density $p(t, x-y)$ relative to Lebesgue measure is uniquely determined by its Fourier transform

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t,x)dx = e^{-t|\xi|^\alpha}.$$

Here α must be in the interval (0, 2]. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. Brownian motion has been intensively studied due to its central role in modern probability theory and its numerous important applications in other scientific areas including many other branches of mathematics. In the sequel, symmetric stable processes are referred to the case when $0 < \alpha < 2$. During the last thirty years, there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable processes. Stable processes are now widely used in physics, operation research, queuing theory, mathematical finance and risk estimation. Recently some fine properties related to symmetric stable processes and Riesz potential theory, as counterparts to Brownian motion and Newtonian potential theory, have been studied, for example in [8]–[13].

Let X be a symmetric α-stable process in $\mathbf{R}^n$ with $n \geq 2$. It is well known that X is transient and has Green function $G(x,y) = A(n,\alpha)|x-y|^{\alpha-n}$ where

$$A(n,\alpha) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}. \tag{1.1}$$

Definition 1.1 *A Borel measurable function q on $\mathbf{R}^n$ is said to be in the* **Kato class $\mathbf{K}_{n,\alpha}$** *if*

$$\lim_{r\downarrow 0} \sup_{x\in\mathbf{R}^n} \int_{|x-y|\leq r} \frac{|q(y)|}{|x-y|^{n-\alpha}} dy = 0.$$

For $q \in \mathbf{K}_{n,\alpha}$ and $t > 0$, define

$$e_q(t) = \exp\left(\int_0^t q(X_s)ds\right).$$

For a domain $D \subset \mathbf{R}^n$, the gauge function g of (D,q) is defined by $g(x) = E^x[e_q(\tau_D)]$, $x \in D$ where $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Chung and Rao [16] showed the following *gauge theorem* for domain D with finite Lebesgue measure: the function g is either identically infinite or is bounded on D. In the latter case, (D,q) is said to be *gaugeable* (with respect to X). By Theorem 1 of Chung [14], $(D,0)$ is gaugeable for any domain D with finite Lebesgue measure. It is proved in [12] that (D,q) is gaugeable if and only if the first eigenvalue of $L^D + q$ is negative, where L^D is the non-positive definite infinitesimal generator of the part process X^D of X killed upon leaving domain D.

Now assume that D is a bounded Lipschitz domain. Recall that the Green function G_D and Poisson kernel K_D of X in D are determined by the following equations. For $x \in D$,

$$\begin{aligned} E^x\left[\int_0^{\tau_D} f(X_s)ds\right] &= \int_D G_D(x,y)f(y)dy \quad \text{for } f \geq 0 \text{ on } D, \\ E^x\left[\varphi(X_{\tau_D})\right] &= \int_{D^c} K_D(x,z)\varphi(z)dz \quad \text{for } \varphi \geq 0 \text{ on } D^c. \end{aligned}$$

Fix an $x_0 \in D$, it is shown in [13] (see also [9]) that for each $z \in \partial D$ and $x \in D$,

$$M_D(x,z) = \lim_{y \to z, y \in D} \frac{G_D(x,y)}{G_D(x_0,y)} \quad \text{exits and is finite.} \tag{1.2}$$

M_D is called the Martin kernel of X in D.

Definition 1.2 *Let D be a domain in $\mathbf{R}^n$. A locally integrable function f defined on D taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}\cap D} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be* **superharmonic respect to** *X^D if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,*

$$f(x) \geq E_x[f(X_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \tau_D].$$

A function h is said to be **harmonic with respect to** *X^D if both h and $-h$ are superharmonic respect to X^D.*

Suppose that $h \geq 0$ is superharmonic in D with respect to X^D. Then by Theorem 2.3 of [13], for any domain $D_1 \subset \overline{D_1} \subset D$, $E_x[h^-(X^D_{\tau_{D_1}})] < \infty$ and

$$h(x) \geq E_x[h(X^D_{\tau_{D_1}})] \quad \text{for every } x \in D_1.$$

This implies (cf. e.g. page 11 of Dynkin [21]) that

$$h(x) \geq E^x[h(X^D_t)].$$

Therefore one can define the h–conditioned stable process. Note that for each fixed $y \in D$ and $z \in \partial D$, $G_D(x,y)$ and $M_D(x,z)$ are harmonic functions in x with respect to $X^{D\setminus\{y\}}$ and X^D, respectively. For any $w \in \overline{D}^c$, $K_D(x,w)$ is superharmonic in x with respect to X^D. Therefore we can define the $G_D(\cdot,y)$–conditioned stable process, $K_D(\cdot,w)$–conditioned stable process and the $M_D(\cdot,z)$–conditioned stable process. The probability laws corresponding to these conditional stable processes will be denoted by P^x_y, P^x_w and P^x_z, respectively. The following conditional gauge theorem is established in Chen and Song [12].

Theorem 1.1 *Assume that D is a bounded $C^{1,1}$ smooth domain in $\mathbf{R}^n$, $q \in \mathbf{K}_{n,\alpha}$ and that (D,q) is gaugeable. Then exists a constant $c > 1$ such that*

$$c^{-1} \le \inf_{(x,z)\in D\times(\mathbf{R}^n\setminus\partial D)} E_z^x\left[e_q(\zeta)\right] \le \sup_{(x,z)\in D\times(\mathbf{R}^n\setminus\partial D)} E_z^x\left[e_q(\zeta)\right] \le c.$$

Recently it is shown in [13] that under the condition of Theorem 1.1, there is a constant $c > 1$ such that

$$c^{-1} \le \inf_{(x,z)\in D\times\partial D} E_z^x\left[e_q(\zeta)\right] \le \sup_{(x,z)\in D\times\partial D} E_z^x\left[e_q(\zeta)\right] \le c.$$

Gauge theorem for Brownian motion was first proved by Chung and Rao in [15] for bounded q and later was generalized to more general q by various authors. The conditional gauge theorem for Brownian motion was first proved by Falkner [22] for bounded q and a class of domains including bounded C^2 domains. Extensions of this result to q belonging to the Kato class and to bounded $C^{1,1}$ domains were given by Zhao in [32] and [33]. The conditional gauge theorem has also been generalized by Cranston, Fabes and Zhao [18] to diffusion processes whose infinitesimal generators are uniformly elliptic divergence form operators and to bounded Lipschitz domain in $\mathbf{R}^n$. For a more detailed story about gauge and conditional gauge theorem for Brownian motion, the interested reader is referred to the recent book of Chung and Zhao [17].

In this paper, we extend the above conditional gauge theorem for symmetric stable processes from bounded $C^{1,1}$ domains to bounded Lipschitz domains. We follow the idea from Chen and Song [12], proving conditional gauge theorem through intrinsic ultracontractivity, but with substantial improvements, motivated by Bañuelos [3]. In [12], sharp estimates on Green functions of bounded $C^{1,1}$ domains obtained in [11] were used to establish conditional gauge theorem. However these estimates are no longer available for bounded Lipschitz domains. We are able to circumvent it in this paper. We first show that, under the assumption that (D,q) is gaugeable, the Green function V_q of $L^D + q$ on D is bounded by a constant multiple of G_D. The latter is then used to prove the conditional gauge theorem on bounded Lipschitz domains. In order to establish $V_q \le c\,G_D$ on $D \times D$ for some $c > 0$, we show that operator $L^D + q$ is intrinsic ultracontractive on D. In fact we show that $L^D + q$ is intrinsic ultracontractive for any Hölder domain D of order 0. For definitions of intrinsic ultracontractivity and Hölder domain of order 0, see Definitions 3.1 and 3.2 below. We mention here that John domains, particularly bounded NTA domains and Lipschitz domains, are Hölder domains of order 0. Under the assumption that D has finite Lebesgue measure such that L^D is intrinsic ultracontractive, we show that there is a constant $c > 0$ such that for each nontrivial nonnegative superharmonic function h in D with respect to X^D, $\sup_{x\in D} E_h^x[\tau_D] \le c$. This especially implies that for a bounded Lipschitz domain D,

$$\sup_{x\in D,\, z\in\mathbf{R}^n} E_z^x[\tau_D] < \infty.$$

Previously the boundedness of conditional lifetimes was proved for bounded $C^{1,1}$ smooth domains for symmetric stable processes in [11] and [13]. Conditional lifetime estimates for Brownian motion in planar domains was first studied by M. Cranston and T. McConnell, in answering a question of K. L. Chung. The first extension to several dimensions for Brownian motion was done by M. Cranston, followed by many works on important extensions to more general domains and to elliptic diffusions (see [3], [4] and the references therein).

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2 Preliminaries

Throughout this paper, we assume that $n \geq 2$ and $0 < \alpha < 2$. Let X be a symmetric α-stable process in $\mathbf{R}^n$. It is well known that the Dirichlet form $(\mathcal{E}, \mathcal{F}^{\mathbf{R}^n})$ associated with X is given by

$$\begin{aligned}
\mathcal{E}(u,v) &= A(n,\alpha)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy \\
\mathcal{F}^{\mathbf{R}^n} &= \left\{u\in L^2(\mathbf{R}^n): \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dxdy<\infty\right\},
\end{aligned}$$

where $A(n,\alpha)$ is the constant in (1.1). As usual, we use $\{P_t\}_{t\geq 0}$ to denote the transition semigroup of X and G to denote the potential of $\{P_t\}_{t\geq 0}$; that is,

$$Gf(x) = \int_0^\infty P_t f(x)dt.$$

FROM now on, we assume that D is a domain in $\mathbf{R}^n$. Adjoin an extra point ∂ to D and set

$$X_t^D(\omega) = \begin{cases} X_t(\omega) & \text{if } t < \tau_D(\omega), \\ \partial & \text{if } t \geq \tau_D(\omega). \end{cases}$$

The process X^D is called the symmetric α-stable process killed upon leaving D, or simply the killed symmetric α-stable process on D. It is well known (cf. [23]) that the Dirichlet form corresponding to the killed symmetric α-stable process X^D on D is $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{F} = \{u \in \mathcal{F}^{\mathbf{R}^n} : \tilde{u} = 0 \text{ quasi everywhere on } D^c\},$$

where $\tilde{u}$ denotes a quasi continuous version of u.

We are going to use P_t^D and $p^D(t,x,y)$ to denote the transition semigroup and transition density of X^D respectively. L^D will used to denote the non–positive definite infinitesimal generator of X^D on $L^2(D,dx)$. G_D, K_D and M_D will be used to denote the Green function, Poisson kernel and Martin kernel of X on D respectively. From [11] we know that when D satisfies the uniform exterior cone condition, G_D and K_D are related by

$$K_D(x,z) = A(n,\alpha)\int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}}dy, \tag{2.1}$$

where $A(n,\alpha)$ is the constant in (1.1). Suppose that $h>0$ is a positive superharmonic function with respect to X^D. We define

$$p_h^D(t,x,y) = h(x)^{-1}p^D(t,x,y)h(y), \qquad t>0, x,y\in D.$$

It is easy to check that p_h^D is a transition density and it determines a Markov process. This process is called the h–conditioned symmetric stable process.

For any $y\in D$, the $G_D(\cdot,y)$–conditioned symmetric stable process is a Markov process with state space $(D\setminus\{y\})\cup\{\partial\}$, with lifetime $\zeta=\tau_{D\setminus\{y\}}$. We will use P_y^x and E_y^x to denote the probability and expectation with respect to this process.

For any $w\in\overline{D}^c$, the $K_D(\cdot,w)$–conditioned symmetric stable process is a Markov process with state space $D\cup\{\partial\}$, with lifetime $\zeta=\tau_D$. We will use P_w^x and E_w^x to denote the probability and expectation with respect to this process.

For any $z\in\partial D$, the $M_D(\cdot,z)$–conditioned symmetric stable process is a Markov process with state space $D\cup\{\partial\}$, with lifetime $\zeta=\tau_D$. We will use P_z^x and E_z^x to denote the probability and expectation with respect to this process.

In [11] we proved the following 3G Theorem.

Theorem 2.1 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n\geq 2$. Then there exists a constant $c=c(D,n,\alpha)>0$ such that*

$$\frac{G_D(x,y)G_D(y,w)}{G_D(x,w)} \leq \frac{c\,|x-w|^{n-\alpha}}{|x-y|^{n-\alpha}|y-w|^{n-\alpha}}, \quad x,y,w\in D, \tag{2.2}$$

$$\frac{G_D(x,y)K_D(y,z)}{K_D(x,z)} \leq \frac{c\,|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}, \quad x,y\in D, z\in D^c. \tag{2.3}$$

Using Theorem 2.1 above and the scaling property we easily get the following result.

Corollary 2.2 *There exists a constant $c=c(n,\alpha)>0$ such that for any ball B in $\mathbf{R}^n$ one has*

$$\frac{G_B(x,y)G_B(y,w)}{G_B(x,w)} \leq \frac{c\,|x-w|^{n-\alpha}}{|x-y|^{n-\alpha}|y-w|^{n-\alpha}}, \quad x,y,w\in B, \tag{2.4}$$

$$\frac{G_B(x,y)K_B(y,z)}{K_B(x,z)} \leq \frac{c\,|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}, \quad x,y\in B, z\in B^c. \tag{2.5}$$

3G Theorem actually holds for bounded Lipschitz domains.

Theorem 2.3 *Suppose that D is a bounded Lipschitz domain in $\mathbf{R}^n$. Then there exists a constant $c = c(D, n, \alpha) > 0$ such that*

$$\frac{G_D(x,y)G_D(y,w)}{G_D(x,w)} \le \frac{c\,|x-w|^{n-\alpha}}{|x-y|^{n-\alpha}|y-w|^{n-\alpha}}, \quad x, y, w \in D \tag{2.6}$$

$$\frac{G_D(x,y)M_D(y,z)}{M_D(x,z)} \le \frac{c\,|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}, \quad x, y \in D, z \in \partial D. \tag{2.7}$$

Proof. Note that the boundary Harnack inequality holds on D for positive harmonic functions in D with respect to the symmetric stable process X, due to Bogdan [8]. One can then prove (2.6) by repeating the argument in Section 6.2 of [17] so we omit the details here. Recalling (1.2), inequality (2.7) follows from (2.6) by passing $w \to z$. ∎

We need the following result on the decomposition of Kato class functions later on.

Lemma 2.4 *Let q have compact support. Then $q \in \mathbf{K}_{n,\alpha}$ if and only if, for any $\varepsilon > 0$, there is a function q_ε such that $q - q_\varepsilon$ is bounded and*

$$\sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|q_\varepsilon(y)|}{|x-y|^{n-\alpha}} dy \le \varepsilon$$

Proof. The proof is the same as that of Theorem 4.16 in [1]. ∎

In what follows, q is an arbitrary but fixed function in $\mathbf{K}_{n,\alpha}$. For a domain D in $\mathbf{R}^n$, define

$$T_t f(x) = E^x\left[e_q(t) f(X(t));\, t < \tau_D\right], \qquad x \in D.$$

The semigroup T_t admits an integral kernel $u_q(t,x,y)$ (cf. [12]). The infinitesimal generator of the semigroup T_t on $L^2(D, dx)$ is $L^D + q$. If D has finite Lebesgue measure, then it is known (see Theorem 3.3 of [12]) that $L^D + q$ has discrete spectrum. Let $\{\lambda_k,\, k = 0, 1, \cdots\}$ be all the eigenvalues of $L^D + q$ written in decreasing order, each repeated according to its multiplicity. Then $\lambda_k \downarrow -\infty$ and the corresponding eigenfunctions $\{\varphi_k,\, k = 0, 1, \cdots\}$ can be chosen that they form an orthonormal basis of $L^2(D, dx)$. We know that all the eigenfunctions φ_k are bounded and the first eigenfunction φ_0 can be chosen to be strictly positive in D (cf. [12]).

Definition 2.1 *A bounded Borel function f defined on $\mathbf{R}^n$ is said to a solution to the equation*

$$(L^D + q)f(x) = 0, \qquad x \in D \tag{2.8}$$

if it is continuous in D and for any open domain $D_0 \subset \overline{D_0} \subset D$,

$$f(x) = E^x[e_q(\tau_{D_0}) f(X_{\tau_{D_0}})], \qquad x \in D_0.$$

Clearly the first eigenfunction φ_0 of T_t is a positive solution of

$$(L^D + q + \lambda_0) f(x) = 0, \qquad x \in D.$$

For positive solutions of (2.8) we have the following uniform local Harnack inequality, which is applicable to φ_0 with $q + \lambda_0$ in place of q.

Theorem 2.5 *There exist two positive constants $r_0 = r_0(q)$ and $C = C(q) > 0$ such that for any solution f of (2.8) which is strictly positive on D and for any ball $B(x_0, r)$ with $0 < r \le r_0$ and $B(x_0, 2r) \subset D$, one has*

$$\sup_{x \in B(x_0,r)} f(x) \le C \inf_{x \in B(x_0,r)} f(x).$$

Proof. It follows from (2.5) and the assumption of $q \in \mathbf{K}_{n,\alpha}$ that there exists a positive number R_0 such that for any ball of radius $r \le R_0$ in $\mathbf{R}^n$,

$$\sup_{x \in B, z \in \overline{B}} \int_B \frac{G_B(x,y)|q(y)|K_B(y,z)}{K_B(x,z)} dy \le \frac{1}{2}.$$

By Jensen's inequality and Khas'minskii's lemma,

$$e^{-1/2} \le \inf_{x \in B, z \in \overline{B}^c} E_z^x[e_q(\tau_B)] \le \sup_{x \in B, z \in \overline{B}^c} E_z^x[e_q(\tau_B)] \le 2.$$

For any ball $B(x_0, r)$ with $0 < r \le R_0/2$ and $\overline{B(x_0, 2r)} \subset D$, we have for any $x \in B(x_0, 2r)$,

$$\begin{aligned} f(x) &= E^x[e_q(\tau_{B(x_0,2r)}) f(X_{\tau_{B(x_0,2r)}})] \\ &= \int_{B(x_0,2r)^c} E_z^x[e_q(\tau_{B(x_0,2r)})]\, f(z)\, K_{B(x_0,2r)}(x,z)\, dz \\ &\le 2E^x[f(X_{\tau_{B(x_0,2r)}})]. \end{aligned}$$

By Harnack inequality in [5], there exists a constant $c = c(n, \alpha) > 0$ such that

$$\sup_{x \in B(x_0,r)} E^x[f(X_{\tau_{B(x_0,2r)}})] \le c \inf_{x \in B(x_0,r)} E^x[f(X_{\tau_{B(x_0,2r)}})].$$

Therefore for any $x, y \in B(x_0, y)$,

$$\begin{aligned} f(x) &\le 2c\, E^y[f(X_{\tau_{B(x_0,2r)}})] \\ &\le 2c\, e^{1/2} \int_{B(x_0,2r)^c} E_z^y[e_q(\tau_{B(x_0,2r)})]\, f(z)\, K_{B(x_0,2r)}(y,z)\, dz \\ &= 2c\, e^{1/2} f(y), \end{aligned}$$

and the proof is now complete. ∎

From the Theorem above we immediately get the following Harnack inequality by a standard chain argument.

Theorem 2.6 *Suppose that K is a compact subset of D. There exists a constant $C = C(D, q, n, \alpha) > 0$ such that for any solution f of (2.8) which is strictly positive on D one has*

$$\sup_{x \in K} f(x) \leq C \inf_{x \in K} f(x).$$

3 Intrinsic Ultracontractivity

Let us first recall the definition of intrinsic ultracontractivity, due to Davies and Simon [20]. Suppose that H is a semibounded self-adjoint operator on $L^2(D)$ with D being a domain in $\mathbf{R}^n$ and that $\{e^{Ht}, t > 0\}$ is an irreducible positivity-preserving semigroup with integral kernel $a(t, x, y)$. Assume that the top of the spectrum μ_0 of H is an eigenvalue. In this case, μ_0 has multiplicity one and the corresponding eigenfunction φ_0, normalized by $\|\varphi_0\|_2 = 1$, can be chosen to be positive almost everywhere on D. φ_0 is called the ground state of H.

Let U be the unitary operator U from $L^2(D, \varphi_0^2(x)dx)$ to $L^2(D)$ given by $Uf = \varphi_0 f$ and define $\widetilde{H}$ on $L^2(D, \varphi_0^2(x)\, dx)$ by

$$\widetilde{H} = U^{-1}(H - \mu_0)\, U.$$

Then $e^{\widetilde{H}t}$ is an irreducible symmetric Markov semigroup on $L^2(D, \varphi_0^2(x)\, dx)$ whose integral kernel with respect to the measure $\varphi_0^2(x)dx$ is given by

$$\frac{e^{-\mu_0 t} a(t, x, y)}{\varphi_0(x)\varphi_0(y)}.$$

Definition 3.1 *H is said to be* **ultracontractive** *if e^{Ht} is a bounded operator from $L^2(D)$ to $L^\infty(D)$ for all $t > 0$. H is said to be* **intrinsically ultracontractive** *if $\widetilde{H}$ is ultracontractive; that is, $e^{\widetilde{H}t}$ is a bounded operator from $L^2(D, \varphi_0^2(x)\, dx)$ to $L^\infty(D, \varphi_0^2(x)\, dx)$ for all $t > 0$.*

Ultracontractivity is connected to logarithmic Sobolev inequalities. The connection between logarithmic Sobolev inequalities and L^p to L^q bounds of semigroups was first given by L. Gross [24] in 1975. E. Davies and B. Simons [1] adopted L. Gross's approach to allow $q = \infty$ and therefore established the connection between logarithmic Sobolev inequalities and ultracontractivity. (For an updated survey on the subject of logarithmic Sobolev inequalities and contractivity properties of semigroups, see [2], [25].) In [3], R. Bañuelos proved the intrinsic ultracontractivity for Shrödinger operators on uniformly Hölder domains of order $\alpha \in (0, 2)$ using logarithmic Sobolev inequality characterization. We will use the same strategy in this section; that is, establishing intrinsic ultracontractivity through logarithmic Sobolev inequalities.

In the rest of this section, unless otherwise specified, D is a domain in $\mathbf{R}^n$ with finite Lebesgue measure, q is a fixed function in the Kato class $\mathbf{K}_{n,\alpha}$. Recall that the semigroup $\{T_t,\, t > 0\})$ is defined as follows.

$$T_t f(x) = E^x \left[e_q(t) f(X(t));\, t < \tau_D \right], \qquad x \in D.$$

$\{\lambda_k : k = 0, 1, \cdots\}$ are all the eigenvalues of $L^D + q$ written in decreasing order, each repeated according to its multiplicity. $\{\varphi_k : k = 0, 1, \cdots\}$ are the corresponding eigenfunctions, normalized so that they form an orthonormal basis of $L^2(D, dx)$ and $\varphi_0 > 0$ on D.

The following result is proven in [12].

Theorem 3.1 *The logarithmic Sobolev inequality holds for functions in $(\mathcal{E}, \mathcal{F})$. That is, for any $\eta > 0$ and $f \in \mathcal{F} \cap L^\infty(D, dx)$, we have*

$$\int_D f^2 \log |f| dx \le \eta \mathcal{E}(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2,$$

with

$$\beta(\eta) = -\frac{n}{2\alpha} \log \eta + c$$

for some constant $c > 0$.

Recall that for any domain D in $\mathbf{R}^n$, the quasi–hyperbolic distance between any two points x_1 and x_2 in D is defined by

$$\rho_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta(x, \partial D)}$$

where the infimum is taken over all rectifiable curves γ joining x_1 to x_2 in D and $\delta(x, \partial D)$ is the Euclidean distance between x and ∂D. Fix a point $x_0 \in D$ which we call the center of D and we may assume without loss of generality that $\delta(x_0, \partial D) = 1$.

Definition 3.2 *A domain D in $\mathbf{R}^n$ is a* **Hölder domain of order 0** *if for a fixed $x_0 \in D$, there exist constants C_1 and C_2 such that for all $x \in D$,*

$$\rho_D(x_0, x) \le C_1 \log \left(\frac{1}{\delta(x, \partial D)} \right) + C_2.$$

It is shown in Smith and Stegenga [29] that a Hölder domain of order 0 is bounded. It is well known that John domains, in particular bounded NTA domains and Lipschitz domains, are Hölder domains of order 0 (cf. [3]).

Lemma 3.2 *If D is a Hölder domain of order 0, then for any $\beta > 0$, there exists a constant $C = C(D) > 0$ such that*

$$\int_D (\rho_D(x_0, x))^\beta u^2(x)dx \le C\mathcal{E}(u, u), \qquad u \in \mathcal{F}$$

Proof. From [29] we know that for any $\beta > 0$ we have

$$\int_D (\rho_D(x_0, x))^\beta dx < \infty.$$

It follows from the Sobolev inequality (see formula (1.5.20) of [23] or Theorem 1 on page 119 of [31]) that there is constant $C_1 > 0$ such that for any $u \in \mathcal{F}$,

$$\|u\|_{p_0} \le C_1\sqrt{\mathcal{E}(u, u)},$$

where p_0 is such that $1/p_0 = 1/2 - \alpha/(2n)$. Let $p = n/(n - \alpha)$ and $p' = n/\alpha$. Applying Hölder's inequality we get that for any $u \in \mathcal{F}$,

$$\begin{aligned}
\int_D (\rho_D(x_0, x))^\beta u^2(x)dx &\le \left(\int_D (\rho_D(x_0, x))^{\beta p'} dx\right)^{1/p'} \left(\int_D |u|^{p_0}(x)dx\right)^{(n-\alpha)/n} \\
&\le C_1 \left(\int_D (\rho_D(x_0, x))^{\beta p'} dx\right)^{1/p'} \mathcal{E}(u, u).
\end{aligned}$$

The proof is now complete. ∎

Theorem 3.3 *If D is a Hölder domain of order 0, then for any $\varepsilon > 0$ and any $\sigma > 0$ we have*

$$\int_D f^2 \log \frac{1}{\varphi_0} dx \le \varepsilon\mathcal{E}(f, f) + \beta(\varepsilon)\|f\|_2^2, \qquad f \in \mathcal{F}$$

with

$$\beta(\varepsilon) = c_1\varepsilon^{-\sigma} + c_2$$

for some positive constants c_1 and c_2. Here φ_0 is the ground state of $L^D + q$.

Proof. Let $W = \{Q_j\}$ be a Whitney decomposition of D. This is a decomposition of D into closed cubes Q with the following three properties (see [31] for details)

(1) for $j \ne k$, the interior of Q_j and the interior of Q_k are disjoint;

(2) if Q_j and Q_k intersects, then
$$\frac{1}{4} \le \frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q_k)} \le 4;$$

(3) for any j,

$$1 \leq \frac{\delta(Q_j, \partial D)}{\operatorname{diam}(Q_j)} \leq 4.$$

Let x_0 be a fixed point in D and $x_0 \in Q_0$. If $Q_k \in W$, we say that $Q_0 = Q(0) \to Q(1) \to \cdots \to Q(m) = Q_k$ is a Whitney chain connecting Q_0 and Q_k of length m if $Q(i) \in W$ for all i and $Q(i)$ and $Q(i+1)$ have touching edges for all i. We define the Whitney distance $d(Q_0, Q_k)$ to be the length of the shortest Whitney chain connecting Q_0 and Q_k. If $x \in Q_k$ we define $\tilde{\rho}(x_0, x) = d(Q_0, Q_k)$. It is well known and easy to prove that this distance is comparable with ρ_D, the quasi–hyperbolic distance.

By Theorem 2.5, property (3) of the Whitney decomposition, the boundedness of D and the equivalence of $\tilde{\rho}$ and ρ_D, there is a constant $C_1 = C_1(D) > 0$ such that for any $Q \in W$ we have

$$\sup_{x \in Q} \varphi_0(x) \leq C_1 \inf_{x \in Q} \varphi_0(x).$$

Therefore there exists a constant $C_2 = C_2(D) > 0$ such that

$$\varphi_0(x) \geq e^{-C_2 \rho_D(x_0, x)} \varphi_0(x_0), \qquad x \in D. \tag{3.1}$$

For any $p > 1$, let p' be its conjugate. By (3.1) and Lemma 3.2, we get that for any $\varepsilon > 0$ and $u \in \mathcal{F}$,

$$\begin{aligned}
\int_D u^2 \log \frac{1}{\varphi_0} dx &\leq C_2 \int_D \rho_D(x_0, x) u(x)^2 dx \\
&= C_2 \int_D \frac{\varepsilon^{1/p}}{\varepsilon^{1/p}} \rho_D(x_0, x) u(x)^2 dx \\
&\leq \varepsilon C_2 \int_D (\rho_D(x_0, x))^p u(x)^2 dx + C_2 \varepsilon^{-p'/p} \int_D u^2(x) dx \\
&\leq \varepsilon\, C_3\, \mathcal{E}(u, u) + C_2\, \varepsilon^{-p'/p} \int_D u^2(x) dx,
\end{aligned}$$

where C_3 is a positive constant depending on D only. The proof is now complete. ∎

Combining the two theorems above we get the following result.

Theorem 3.4 *If D is a Hölder domain of order 0, then for any $\varepsilon > 0$ and any $\sigma > 0$ we have*

$$\int_D f^2 \log \frac{|f|}{\varphi_0} dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \qquad f \in \mathcal{F} \cap L^\infty(D, dx)$$

with

$$\beta(\varepsilon) = -\frac{n}{2\alpha} \log \varepsilon + c_1 \varepsilon^{-\sigma} + c_2$$

for some positive constants c_1 and c_2.

With the result above, we can easily get our main result of this section.

Theorem 3.5 *Assume that D is a Hölder domain of order 0. Then $L^D + q$ is intrinsically ultracontractive. More precisely,*

$$\frac{e^{-\lambda_0 t} u_q(t,x,y)}{\varphi_0(x)\varphi_0(y)} \le e^{2M(t/2)} \quad \textit{for all } x, y \in D \textit{ and } t > 0,$$

where

$$M(t) = \frac{1}{t}\int_0^t A(\varepsilon)d\varepsilon$$

with

$$A(\varepsilon) = \begin{cases} -\frac{n}{2\alpha}\log\varepsilon + c_1\varepsilon^{-1/3} + c_2 & \textit{for } \varepsilon \le 1, \\ c_1 + c_2 & \textit{for } \varepsilon > 1. \end{cases}$$

for some positive constants c_1 and c_2.

Proof. By taking $\sigma = 1/3$ in Theorem 3.4, we get that for any $\varepsilon > 0$ and any $f \in \mathcal{F} \cap L^\infty(D, dx)$,

$$\int_D f^2 \log\frac{|f|}{\varphi_0} dx \le \varepsilon\mathcal{E}(f,f) + \beta_1(\varepsilon)\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2, \tag{3.2}$$

with

$$\beta_1(\varepsilon) = -\frac{n}{2\alpha}\log\varepsilon + c_1\varepsilon^{-1/3} + c_2$$

for some positive constants c_1 and c_2.

Suppose that $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ is the Dirichlet form on $L^2(m)$ with $m(dx) = \varphi_0^2 dx$ associated with the semigroup whose integral kernel with respect to the measure m is given by

$$\frac{e^{-\lambda_0 t} u_q(t,x,y)}{\varphi_0(x)\varphi_0(y)}.$$

Then

$$\widetilde{\mathcal{F}} = \{f : f\varphi_0 \in \mathcal{F}\}$$

and

$$\widetilde{\mathcal{T}}(f,h) = \mathcal{E}(f\varphi_0, h\varphi_0) - \int_D q f\varphi_0\, h\varphi_0\, dx + \lambda_0 \int_D fh\, dm$$

Since $q \in \mathbf{K}_{n,\alpha}$, by Theorem 3.2 of [30] there exists a constant $B > 0$ such that

$$\int_D |q| u^2 dx \le \frac{1}{2}\mathcal{E}(u,u) + B\int_D u^2 dx, \qquad u \in \mathcal{F}.$$

Thus

$$\widetilde{\mathcal{T}}(h,h) \ge \frac{1}{2}\mathcal{E}(h\varphi_0, h\varphi_0) - (B - \lambda_0)\int_D h^2 dm.$$

By putting $f = h\varphi_0$ in (3.2) we get that for $h \in \widetilde{\mathcal{F}} \cap L^\infty(D, dm)$,

$$\int_D h^2 \log|h|\, dm \le 2\varepsilon\, \widetilde{\mathcal{T}}(h,h) + (\beta_1(\varepsilon) + 2(B - \lambda_0)) \int_D h^2 dm + \|h\|^2_{L^2(m)} \log \|h\|_{L^2(m)}. \quad (3.3)$$

Therefore for $0 < \varepsilon \le 1$ and $h \in \widetilde{\mathcal{F}} \cap L^\infty(D, dm)$,

$$\int_D h^2 \log|h|\, dm \le \varepsilon\, \widetilde{\mathcal{T}}(h,h) + \beta_2(\varepsilon) \int_D h^2 dm + \|h\|^2_{L^2(m)} \log \|h\|_{L^2(m)}, \quad (3.4)$$

where

$$\beta_2(\varepsilon) = -\frac{n}{2\alpha} \log\varepsilon + c_3 \varepsilon^{-1/3} + c_4,$$

for some constants c_3, $c_4 > 0$. For $\varepsilon > 1$, since $\widetilde{\mathcal{T}}$ is nonnegative and (3.4) holds for $\varepsilon = 1$, we have for any $h \in \widetilde{\mathcal{F}} \cap L^\infty(D, dm)$,

$$\begin{aligned} \int_D h^2 \log|h| dm &\le \widetilde{\mathcal{T}}(h,h) + \beta_1(1) \int_D h^2 dm + \|h\|^2_{L^2(m)} \log \|h\|_{L^2(m)} \\ &\le \varepsilon \widetilde{\mathcal{T}}(h,h) + \beta_1(1) \int_D h^2 dm + \|h\|^2_{L^2(m)} \log \|h\|_{L^2(m)}. \quad (3.5) \end{aligned}$$

Combining (3.4) and (3.5) we get that for any $\varepsilon > 0$ and any $h \in \widetilde{\mathcal{F}} \cap L^\infty(D, dm)$,

$$\int_D h^2 \log|h| dm \le \varepsilon \widetilde{\mathcal{T}}(h,h) + A(\varepsilon) \|h\|^2_{L^2(m)} + \|h\|^2_{L^2(m)} \log \|h\|_{L^2(m)}, \quad (3.6)$$

with

$$A(\varepsilon) = \begin{cases} -\frac{n}{2\alpha} \log\varepsilon + c_5 \varepsilon^{-1/3} + c_6 & \text{for } \eta \le 1, \\ c_5 + c_6 & \text{for } \varepsilon > 1. \end{cases}$$

for some positive constants c_5 and c_6. By Corollary 2.2.8 of [19], we have

$$\frac{e^{-\lambda_0 t} u_q(t,x,y)}{\varphi_0(x)\varphi_0(y)} \le e^{2M(t/2)} \quad \text{for all } x, y \in D \text{ and } t > 0,$$

where

$$M(t) = \frac{1}{t} \int_0^t A(\varepsilon) d\varepsilon < \infty.$$

∎

Using the same argument as that for Theorem 6 in R. Smits [28], we have

Theorem 3.6 *Assume that D is a domain in $\mathbf{R}^n$ with finite Lebesgue measure such that $L^D + q$ is intrinsic ultracontractive. Then there exists $C > 0$ such that for any $t > 1$,*

$$e^{(\lambda_1 - \lambda_0)t} \le \sup_{x,y \in D} \left| \frac{e^{-\lambda_0 t} u_q(t,x,y)}{\varphi_0(x)\varphi_0(y)} - 1 \right| \le C e^{(\lambda_1 - \lambda_0)t}.$$

4 Conditional Lifetimes

Assume in this section that D is a domain in $\mathbf{R}^n$ with finite Lebesgue measure such that L^D is intrinsic ultracontractive, unless otherwise specified. In particular, we know from Theorem 3.5 that a Hölder domain of order 0 satisfies this assumption. Since $(D, 0)$ is gaugeable, the first eigenvalue μ_0 of L^D is negative. Let ϕ_0 be the ground state of L^D. Recall that P_D is the transition density function for the killed symmetric stable process X^D. Similar to Corollary 1 of Bañuelos [3], we have

Theorem 4.1 *Under the assumption given at the beginning of this section. There is a constant $c > 0$ such that*

(1) $e^{\mu_0 t}\phi_0(x)\phi_0(y) \le p^D(t,x,y) \le ce^{\mu_0 t}\phi_0(x)\phi_o(y)$ *for all* $x, y \in D$ *and* $t > 0$*;*

(2) Let SH^+ *denote all non-trivial nonnegative superharmonic functions in* D *with respect to* X^D*. Then*
$$\sup_{x\in D,\, h\in \mathrm{SH}^+} E_h^x[\tau_D] < \infty;$$

(3) For $h \in \mathrm{SH}^+$*,*
$$\lim_{t\to\infty} e^{-\mu_0 t} P_h^x(\tau_D > t) = \frac{\phi_0(x)}{h(x)} \int_D \phi_0(y)h(y).$$
In particular, $\lim_{t\to\infty} \frac{1}{t}\log P_h^x(\tau_D > t) = \mu_0$.

Proof. (1) It follows directly from Theorem 3.6.
(2) Note that for each $h \in \mathrm{SH}^+$, by (1)
$$h(x) \ge \int_D p^D(1,x,y)h(y)dy \ge e^{\mu_0}\phi_0(x)\int_D \phi_0(y)h(y)dy \quad \text{for } x \in D.$$
Therefore
$$\sup_{x\in D,\, h\in \mathrm{SH}^+} \frac{\phi_0(x)}{h(x)} \int_D \phi_0(y)h(y)\,dy \le e^{-\mu_0} < \infty.$$
Therefore by (1)
$$\begin{aligned}
&\sup_{x\in D,\, h\in \mathrm{SH}^+} E_h^x[\tau_D] \\
= \ &\sup_{x\in D,\, h\in \mathrm{SH}^+} \frac{1}{h(x)} \int_0^\infty \int_D p^D(t,x,y)h(y)\,dy \\
\le \ & c\int_0^\infty e^{\mu_0 t}\,dt \sup_{x\in D,\, h\in \mathrm{SH}^+} \frac{\phi_0(x)}{h(x)} \int_D \phi_0(y)h(y)\,dy < \infty.
\end{aligned}$$

(3) By Theorem 3.6

$$\begin{aligned}
& \lim_{t\to\infty} e^{-\mu_0 t} P_h^x(\tau_D > t) \\
= & \lim_{t\to\infty} e^{-\mu_0 t}\phi_0(x)^{-1}\int_D p^D(t,x,y)h(y)dy \\
= & \frac{\phi_0(x)}{h(x)}\int_D \phi_0(y)h(y).
\end{aligned}$$

■

When D is a bounded Lipschitz domain, $G_D(x,y)$, $M_D(x,z)$ and $K_D(x,w)$ are superharmonic functions in x with respect to X^D for each fixed $y \in D$, $z \in \partial D$ and $w \in \overline{D}^c$, respectively. The above theorem in particular implies that

Corollary 4.2 *(Conditional Lifetimes) Assume that D is a bounded Lipschitz domain. Then*

$$\sup_{x\in D,\, z\in \mathbf{R}^n} E_z^x[\tau_D] < \infty.$$

5 Conditional Gauge Theorem

Throughout this section, D is a bounded Lipschitz domain. Recall that L^D is the non-positive definite infinitesimal generator of the killed α-stable process on D. For $q \in \mathbf{K}_{n,\alpha}$, let $u_q(t,x,y)$ be the kernel of the following Feynman-Kac semigroup

$$T_t f(x) = E^x\left[e_q(t) f(X_t) 1_{\{t<\tau_D\}}\right].$$

Note that the semigroup T_t only depends on function q through $q1_D$ so we may assume that $q = 0$ off D. The following result is proven in [12].

Theorem 5.1 *Suppose that $q \in \mathbf{K}_{n,\alpha}$ is such that*

$$\sup_{x,y\in D}\int_D \frac{G_D(x,z)|q(z)|G_D(z,y)}{G_D(x,y)}dz \le \frac{1}{2}.$$

Then we have

$$e^{-1/2}G_D(x,y) \le V_q(x,y) \le 2G_D(x,y), \tag{5.1}$$

where

$$V_q(x,y) = \int_0^\infty u_q(t,x,y)dt.$$

Theorem 5.2 *Assume that $q \in \mathbf{K}_{n,\alpha}$ and (D,q) is gaugeable. Then there is a constant $c > 0$ such that*

$$V_q(x,y) \le c\, G_D(x,y) \quad \text{for all } x,y \in D.$$

Proof. By Lemma 2.4 and Theorem 2.3, the function q can be decomposed as $q = q_1 + q_2$ with q_1 bounded and $q_2 \in \mathbf{K}_{n,\alpha}$ satisfying

$$\sup_{x,y\in D} \int_D \frac{G_D(x,z)|q_2(z)|G_D(z,y)}{G_D(x,y)} dz \leq \frac{1}{2}. \tag{5.2}$$

Therefore by Theorem 5.1

$$e^{-1/2} G_D(x,y) \leq V_{q_2}(x,y) \leq 2G_D(x,y). \tag{5.3}$$

Let $\{\nu_k,\ k = 0, 1, \cdots\}$ be all the eigenvalues of $L^D + q_2$ written in decreasing order, each repeated according to its multiplicity. and let $\{\psi_k,\ k = 0, 1, \cdots\}$ be the corresponding eigenfunctions with $\psi > 0$ on D. We can assume that $\{\psi_k,\ k = 0, 1, \cdots\}$ form an orthonormal basis of $L^2(D, dx)$. Since by Khas'minskii's lemma $E_y^x[e_{q_2}(\zeta)] < 2$ for $x, y \in D$, we know from Theorem 3.11 of [12] that $\nu_0 < 0$.

By Theorem 3.6, there is a $t_1 > 1$ such that for all $t \geq t_1$ and all $x, y \in D$,

$$\frac{1}{2} \leq \frac{e^{-\nu_0 t} u_{q_2}(t,x,y)}{\psi_0(x)\psi_0(y)} \leq \frac{3}{2}.$$

Therefore for all $x, y \in D$,

$$\begin{aligned} V_{q_2}(x,y) &= \int_0^\infty u_{q_2}(t,x,y)dt \\ &\geq \int_{t_1}^\infty u_{q_2}(t,x,y)dt \\ &\geq \frac{1}{2}\psi_0(x)\psi_0(y) \int_{t_1}^\infty e^{\nu_0 t}\, dt \\ &\geq C_1\, \psi_0(x)\psi_0(y), \end{aligned} \tag{5.4}$$

for some positive constant $C_1 > 0$.

Since $L^D + q = (L^D + q_2) + q_1$ and q_1 is bounded, it follows from Theorem 3.4 of [20] that the first eigenfunction φ_0 of $L^D + q$ is comparable to the first eigenfunction ψ_0 of $L^D + q_2$, i.e., there exists a constant $C_2 > 1$ such that

$$C_2^{-1}\psi_0 \leq \varphi_0 \leq C_2\psi_0. \tag{5.5}$$

By Theorem 3.6 again, there is a $t_2 > 1$ such that for all $t \geq t_2$ and all $x, y \in D$,

$$\frac{1}{2} \leq \frac{e^{-\lambda_0 t} u_q(t,x,y)}{\varphi_0(x)\varphi_0(y)} \leq \frac{3}{2}.$$

Since the gauge function of (D,q) is assumed to be finite, the first eigenvalue λ_0 is negative by Theorem 3.11 of [12]. Therefore it follows from (5.5), (5.4) and (5.3) that for any $x,y \in D$,

$$\begin{aligned}\int_{t_2}^{\infty} u_q(t,x,y)dt &\leq \frac{3}{2}\varphi_0(x)\varphi_0(y)\int_{t_2}^{\infty} e^{\lambda_0 t}dt \\ &\leq C_3\varphi_0(x)\varphi_0(y) \leq C_4\psi_0(x)\psi_0(y) \\ &\leq C_5 V_{q_2}(x,y) \leq C_6 G_D(x,y),\end{aligned}$$

where C_3, C_4, C_5, C_6 are positive constants. Since q_1 is bounded,

$$\begin{aligned}\int_0^{t_2} u_q(t,x,y)dt &\leq e^{\|q_1\|_\infty t_2}\int_0^{t_2} u_{q_2}(t,x,y)dt \\ &\leq e^{\|q_1\|_\infty t_2} V_{q_2}(x,y) \leq 2e^{\|q_1\|_\infty t_2} G_D(x,y).\end{aligned}$$

Hence there exists a constant $C > 1$ such that

$$V_q(x,y) \leq C G_D(x,y), \qquad x,y \in D.$$

■

Applying the Theorem above and the 3G Theorem we can get the following results which correspond to Theorems 5.3–5.7 of [12]. The proofs are exactly the same as those of the corresponding results in [12] so the details are omitted.

Theorem 5.3 *Assume that (D,q) is gaugeable. Then for all $x,y \in D$ with $x \neq y$,*

$$V_q(x,y) = G_D(x,y) + \int_D V_q(x,u)q(u)G_D(u,y)du \tag{5.6}$$

$$V_q(x,y) = G_D(x,y) + \int_D G_D(x,u)q(u)V_q(u,y)du. \tag{5.7}$$

Theorem 5.4 *Assume that (D,q) is gaugeable. Then for all $(x,y) \in D \times D$ with $x \neq y$,*

$$E_y^x\left[e_q(\zeta)\right] = 1 + G_D(x,y)^{-1}\int_D V_q(x,w)q(w)G_D(w,y)dw.$$

Theorem 5.5 *Assume that (D,q) is gaugeable. Then for all $(x,y) \in D \times D$ with $x \neq y$,*

$$E_y^x\left[e_q(\zeta)\right] = \frac{V_q(x,y)}{G_D(x,y)}.$$

Theorem 5.6 *Assume that (D,q) is gaugeable. Then for There exist $c > 1$ such that*

$$c^{-1} \leq \inf_{x,y\in D} E_y^x\left[e_q(\zeta)\right] \leq \sup_{x,y\in D} E_y^x\left[e_q(\zeta)\right] \leq c.$$

Theorem 5.7 *Assume that (D,q) is gaugeable. There exists a constant $c>1$ such that*

$$c^{-1}\,G_D(x,y)\le V_q(x,y)\le c\,G_D(x,y) \quad \textit{for } x,y\in D.$$

That is, the Green function of D with respect to L^D+q is comparable to the Green function of D with respect to L^D.

Theorem 5.8 *Assume that (D,q) is gaugeable. For all $x\in D$ and $z\in \overline{D}^c$,*

$$E^x_z\left[e_q(\zeta)\right]=1+K_D(x,z)^{-1}\int_D V_q(x,w)q(w)K_D(w,z)dw.$$

Proof. For any Borel measurable subset $A\subset D$, by 2.1

$$\begin{aligned}
&\int_A \frac{G_D(x,y)|q(y)|K_D(y,z)}{K_D(x,z)}dy\\
&=\frac{A(n,\alpha)}{K_D(x,z)}\int_A G_D(x,y)|q(y)|\left(\int_D \frac{G_D(y,w)}{|w-z|^{n+\alpha}}dw\right)dy\\
&=\frac{A(n,\alpha)}{K_D(x,z)}\int_D \frac{G_D(x,w)}{|w-z|^{n+\alpha}}\left(\int_A \frac{G_D(x,y)|q(y)|G_D(y,w)}{G_D(x,w)}dy\right)dw.
\end{aligned}$$

The family of functions

$$\left\{\frac{G_D(x,\cdot)|q(\cdot)|G_D(\cdot,w)}{G_D(x,w)},\ x,w\in D\right\}$$

is uniformly integrable by (2.6) and therefore the family of functions

$$\left\{\frac{G_D(x,\cdot)|q(\cdot)|K_D(\cdot,z)}{K_D(x,z)},\ x\in D, z\in \overline{D}^c\right\}$$

is uniformly integrable. Using Fubini's theorem,

$$\begin{aligned}
&E^x_z\left[\int_0^\zeta e_q(t)|q(X_t)|dt\right]\\
&=\int_0^\infty E^x_z\left[e_q(t)|q(X_t)|;\ t<\zeta\right]dt\\
&=K_D(x,z)^{-1}\int_0^\infty E^x\left[e_q(t)|q(X_t)|K_D(X_t,z);\ t<\tau_D\right]dt\\
&=K_D(x,z)^{-1}\int_0^\infty\int_D u_q(t,x,y)|q(w)|K_D(w,z)dwdt\\
&=K_D(x,z)^{-1}\int_D V_q(x,w)|q(w)|K_D(w,z)dw<\infty.
\end{aligned}$$

Hence

$$\begin{aligned} E_z^x[e_q(\zeta)] &= 1 + E_z^x\left[\int_0^\zeta e_q(t)q(X_t)dt\right] \\ &= 1 + K_D(x,z)^{-1}\int_D V_q(x,w)q(w)K_D(w,z)dw. \end{aligned}$$

■

Theorem 5.9 *Assume that (D,q) is gaugeable. There exists a constant $c>1$ such that*

$$c^{-1} \le \inf_{(x,z)\in D\times\overline{D}^c} E_z^x\left[e_q(\tau_D)\right] \le \sup_{(x,z)\in D\times\overline{D}^c} E_z^x\left[e_q(\tau_D)\right] \le c.$$

Proof. By (2.1)

$$\begin{aligned} E_z^x\left[e_q(\tau_D)\right] &= 1 + \frac{1}{K_D(x,z)}\int_D V_q(x,w)q(w)\int_D \frac{G_D(w,v)}{|v-z|^{n+\alpha}}dvdw \\ &= \frac{A(n,\alpha)}{K_D(x,z)}\int_D \frac{G_D(x,v)}{|v-z|^{n+\alpha}}\left(1+\frac{1}{G_D(x,v)}\int_D V_q(x,w)q(w)G_D(w,v)dw\right)dv \\ &= \frac{A(n,\alpha)}{K_D(x,z)}\int_D \frac{G_D(x,v)}{|v-z|^{n+\alpha}}E_v^x\left[e_q(\zeta)\right]dv. \end{aligned}$$

The assertion of this theorem then follows from Theorem 5.6. ■

Theorem 5.10 *Assume that (D,q) is gaugeable. There exists a constant $c>1$ such that*

$$c^{-1} \le \inf_{(x,z)\in D\times\partial D} E_z^x\left[e_q(\tau_D)\right] \le \sup_{(x,z)\in D\times\partial D} E_z^x\left[e_q(\tau_D)\right] \le c.$$

Proof. It follows from the (2.7) that the family of functions

$$\left\{\frac{G_D(x,\cdot)|q(\cdot)|M_D(\cdot,z)}{M_D(x,z)}, x\in D, z\in\partial D\right\}$$

is uniformly integrable. A similar calculation as that given in the proof of Theorem 5.8 yields

$$E_z^x[e_q(\zeta)] = 1 + \frac{1}{M_D(x,z)}\int_D V_q(x,u)q(u)M_D(u,z)du.$$

Let x_0 be a fixed point in D. Then from [13] we know that

$$\begin{aligned} \lim_{y\to z}\frac{1}{G_D(x,y)}V_q(x,w)q(w)G_D(w,y) &= \lim_{y\to z}V_q(x,w)q(w)\frac{G_D(w,y)/G_D(x_0,y)}{G_D(x,y)/G_D(x_0,y)} \\ &= V_q(x,w)q(w)\frac{M_D(w,z)}{M_D(x,z)}. \end{aligned}$$

Now using the uniform integrability of the family of functions

$$\left\{ \frac{G_D(x,\cdot)|q(\cdot)|G_D(\cdot,y)}{G_D(x,y)},\ x,y \in D \right\}$$

and letting $y \to z \in \partial D$ in Theorem 5.4 yield

$$\lim_{y\to z} E_y^x[e_q(\zeta)] = 1 + \frac{1}{M_D(x,z)} \int_D V_q(x,u)q(u)M_D(u,z)du.$$

Therefore

$$\lim_{y\to z} E_x^y[e_q(\zeta)] = E_x^z[e_q(\zeta)] \tag{5.8}$$

and the theorem now follows from Theorem 5.6. ∎

Combining Theorems 5.6, 5.9 and 5.10 we get the following

Theorem 5.11 *(Conditional Gauge Theorem). Assume that (D,q) is gaugeable. There exists a constant $c>1$ such that*

$$c^{-1} \le \inf_{(x,z)\in D\times \mathbf{R}^n} E_z^x\left[e_q(\zeta)\right] \le \sup_{(x,z)\in D\times \mathbf{R}^n} E_z^x\left[e_q(\zeta)\right] \le c.$$

For $q \in \mathbf{K}_{n,\alpha}$, let $\psi_0 >$ be the ground state of $L^D + q$ and let $u_q(t,x,y)$ be the density kernel of $L^D + q$ with respect to the Lebesgue measure in D.

Theorem 5.12 *Suppose that (D,q) is gaugeable.*

(1) There is a constant $c>1$) such that

$$c^{-1}\phi_0(x) \le \psi_0(x) \le c\phi_0(x) \quad \textit{for all } x \in D;$$

(2) For each $t>0$ there is a constant $c_t > 1$ such that

$$c_t^{-1}p^D(t,x,y) \le u_q(t,x,y) \le c_t\, p^D(t,x,y) \quad \textit{for all } x,y \in D.$$

Proof. The proof is the same as that for Theorem 2 in Bañuelos [3] and is thus omitted. ∎

Remark. It is possible to extend conditional gauge theorem beyond bounded Lipschitz domains to domains having the following properties. Suppose that D is a domain having finite Lebesgue measure and $q \in \mathbf{K}_{n,\alpha}$ such that $L^D + q$ is intrinsic ultracontractive. Assume also that q admits a decomposition $q = q_1 + q_2$ with q_1 and q_2 satisfying (5.2). Then Theorem 5.2 remains true by exactly the same argument. Assume further that

$$\left\{ \frac{G_D(x,\cdot)|q(\cdot)|G_D(\cdot,y)}{G_D(x,y)} :\ x,y \in D \right\} \tag{5.9}$$

is uniformly integrable on D. Then Theorems 5.3–5.7 hold by the same proof as those for Theorems 5.3–5.7 in [12]. In this case, Theorem 5.12 holds as well. When D is a bounded Lipschitz domain, the above conditions are satisfied due to 3G Theorem 2.3.

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