FREE PROBABILITY FOR PROBABILISTS

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Abstract. This is an introduction to some of the most probabilistic aspects of free probability theory.

Introduction

Free probability is a non-commutative probability theory, in which the concept of independence of classical probability is replaced by that of freeness. This new concept incorporates both the probabilistic idea of non-correlation involved in the independence of random variables, and the algebraic notion of abscence of relations, like e.g. between the generators of a (non-abelian) free group. It was introduced abstractly by D. Voiculescu around 1983, in order to study some problems in the theory of von Neumann algebras, but some years later, around 1990, he realized that a concrete probabilistic model of free random variables is afforded by large independent random matrices. While this discovery has lead to an impressive series of advances in von Neumann algebras, some long standing open problems being solved with these new ideas, we shall not discuss these applications here, rather the purpose of these lectures is to try to explain to a probabilist audience why the theory of free random variables is an interesting and beautiful subject in itself, with deep connections with classical probability, and other related areas, such as harmonic analysis, and combinatorics. We hope that this brief introduction will help the reader find its way into the litterature on free probability. We shall take as a departure point a natural problem about hermitian matrices, whose solution involves free probability theory.

2. Large matrices

A frequent question, occuring both in mathematics and physics, is to determine the spectrum of the sum of two hermitian matrices. Knowing the respective spectra of two $N\times N$ hermitian matrices A and B, the set of possible spectra for $A+B$ is a subset of $\mathbb{R}^N$, which can be described explicitly and depends in a complicated way on the spectra of A and B. This problem is not an easy one, and indeed its solution was obtained only quite recently (see e.g. [F] for a discussion). When N becomes large, however, a remarkable phenomenon occurs, and it turns out that, roughly speaking, for almost all choices of the matrices A and B, with given spectra, the spectrum of $A+B$ is essentially the same, and can be computed explicitly, without knowing the detailed structure of the matrices A and B (i.e. their eigenvectors). In order to give a rigourous mathematical statement corresponding to the above assertions, we shall first introduce a probability measure on the set of matrices having the same spectrum as A. By the spectral theorem this set consists of all matrices UAU^*, where U describes the group of unitary $N \times N$ matrices. It is the orbit $\mathcal{O}_A$ of A under the adjoint action of $U(N)$, and as such it carries a unique invariant probability measure ρ_A, the image of the Haar measure on $U(N)$ by the map $U \mapsto UAU^*$.

Given an hermitian matrix A, with spectrum $\lambda_1, \dots, \lambda_N$ (counted with multiplicities), we denote by ν_A the empirical distribution on the set of its eigenvalues, i.e. the probability measure

$$\nu_A = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j}.$$

Knowledge of the spectrum (with multiplicities) and of ν_A are equivalent.

We can now state the result we had in mind.

Theorem 1. *Let for each N positive integer, A_N and B_N be two hermitian matrices, whose norm is bounded uniformly in N, and let ν_1 and ν_2 be two probability measures with compact supports on $\mathbb{R}$, such that $\nu_{A_N} \to \nu_1$ and $\nu_{B_N} \to \nu_2$ weakly as $N \to \infty$, then there exists a probability measure, depending only on ν_1 and ν_2, denoted $\nu_1 \boxplus \nu_2$, such that $\nu_{A'_N + B'_N} \to \nu_1 \boxplus \nu_2$ weakly, in probability, as $N \to \infty$, where A'_N and B'_N are random matrices chosen independently, with respective distributions ρ_{A_N} and ρ_{B_N}.*

Thus, if we are given two $N \times N$ hermitian matrices (with N large), and we only know their spectra, then we can bet, with a good chance to win, that the measure ν_{A+B} is well approximated by the measure $\nu_A \boxplus \nu_B$. In other words, loosely speaking, we can compute the spectrum of $A+B$, knowing only the spectra of A and B!

Since any probability measure with compact support can be approximated by measures like ν_A for arbitrarily large matrices, Theorem 1 introduces a binary operation $\boxplus$ on the set of compactly supported measures on the real line which, for reasons to be explained below, we shall call the free convolution of measures. This binary operation is clearly associative and commutative.

It is instructive to compare the preceding result with the case where all considered matrices are supposed to be diagonal. Let A be a self-adjoint diagonal matrix, then the set of all diagonal matrices having the same spectrum as A, the analogue of the set $\mathcal{O}_A$, is an orbit of the symmetric group S_N, acting by permutation of diagonal entries. Let us denote the normalized counting measure on this set by ξ_A, then one has, denoting by $\mu * \nu$ the usual convolution of two probability measures on $\mathbb{R}$,

Theorem 2. *Let for each N, A_N and B_N be two real diagonal matrices, and let ν_1 and ν_2 be two probability measures with compact supports, such that $\nu_{A_N} \to \nu_1$ and $\nu_{B_N} \to \nu_2$ weakly as $N \to \infty$, then $\nu_{A'_N + B'_N} \to \nu_1 * \nu_2$ weakly, in probability, as $N \to \infty$, where A'_N and B'_N are random matrices chosen independently, with respective distributions ξ_{A_N} and ξ_{B_N}.*

Although we do not know an adequate reference, this last result is probably well known. In fact, it is not difficult to deduce it from known concentration of measure results on the symmetric group, as e.g. in [M].

Coming back to free convolution, note that we have not said how to compute explicitly the measure $\nu_1 \boxplus \nu_2$ in terms of ν_1 and ν_2. This is where free probability comes in. As we shall see in the next section, Theorem 1 is the consequence of a more general result concerning large hermitian matrices, namely the fact that they give rise, asymptotically, to free random variables.

3. Free random variables

Before we describe the connection with large matrices, we introduce the necessary notions from free probability. We start with some purely algebraic definitions.

Let $\mathcal{A}$ be a complex algebra with a unit, and φ a $\mathbb{C}$-valued linear form on $\mathcal{A}$, such that $\varphi(1) = 1$. Although, in the examples that we shall consider, the algebra $\mathcal{A}$ will always be non-commutative, it will be convenient to think of the elements of this algebra as random variables, while the map φ should be considered as the expectation map of classical probability theory (here and in the sequel, we use the word classical for usual probability theory, as opposed to the non-commutative theory we develop).

We now introduce the basic notion of free probability theory.

Definition 1. *Let I be a set of indices, and $\mathcal{B}_i$, for $i \in I$, be subalgebras of $\mathcal{A}$, containing the unit, then the algebras $\mathcal{B}_i; i \in I$ are called free if one has $\varphi(a_1 \dots a_n) = 0$ each time $\varphi(a_j) = 0$ for all $j = 1, \dots, n$ and $a_j \in \mathcal{B}_{i_j}$ for some indices $i_1 \neq i_2 \neq \dots \neq i_n$.*

Pursuing our analogy with classical probability, if we think of the algebras $\mathcal{B}_i$ as algebras of random variables, measurable with respect to some sub-sigma field, then the above definition is a non-commutative analogue of the definition of independent subalgebras. In fact, although it might not be obvious at first sight, this definition captures the essence of both the notion of algebraic independence, and that of independence of sigma-algebras in classical probability. Let us stress however, that this definition is not a non-commutative extension of the notion of independence, indeed algebras generated by independent random variables, in the sense of classical probability theory, are not free in the sense of the above definition.

Before making some elementary comments, we shall give a convenient definition, whose analogy with classical probability should be obvious.

Definition 2. *Let $\mathcal{H}_i; i \in I$ be subsets of $\mathcal{A}$, they are called free if the subalgebras $\mathcal{B}_i(=$ unital algebra generated by $\mathcal{H}_i)$, for $i \in I$, are free.*

In order to get acquainted with freeness we shall make a few computations. So let $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ be two free subalgebras, and $b \in \mathcal{B}$, $c \in \mathcal{C}$, then we can write $b = b' + \bar{b}$ where $\bar{b} = \varphi(b)1$, so that $\varphi(b') = 0$. Similarly we have $c = c' + \bar{c}$. Then using the definition of freeness, we see that $\varphi(b'c') = 0$, thus

$$\begin{aligned}\varphi(bc) &= \varphi((b' + \bar{b})(c' + \bar{c})) \\ &= \varphi(b'c' + b'\bar{c} + \bar{b}c' + \bar{b}\bar{c}) \\ &= \varphi(b)\varphi(c)\end{aligned}$$

Here we have used $\varphi(\bar{b}c') = \varphi(\varphi(b)1.c') = \varphi(b)\varphi(c') = 0$, and similarly $\varphi(b'\bar{c}) = 0$.

This means that for two free elements b and c, their expectations factorize, exactly as in the case of independent variables. Let us now look at some more subtle example. Take $b_1, b_2 \in \mathcal{B}$, and $c_1, c_2 \in \mathcal{C}$, then we can expand

$$\begin{aligned}\varphi(b_1c_1b_2c_2) &= \varphi((b_1' + \bar{b}_1)(c_1' + \bar{c}_1)(b_2' + \bar{b}_2)(c_2' + \bar{c}_2)) \\ &= \varphi(b_1'c_1'b_2'c_2' + b_1'c_1'b_2'\bar{c}_2 + b_1'c_1'\bar{b}_2c_2' + b_1'c_1'\bar{b}_2\bar{c}_2 + \\ &\quad b_1'\bar{c}_1b_2'c_2' + b_1'\bar{c}_1b_2'\bar{c}_2 + b_1'\bar{c}_1\bar{b}_2c_2' + b_1'\bar{c}_1\bar{b}_2\bar{c}_2 + \\ &\quad \bar{b}_1c_1'b_2'c_2' + \bar{b}_1c_1'b_2'\bar{c}_2 + \bar{b}_1c_1'\bar{b}_2c_2' + \bar{b}_1c_1'\bar{b}_2\bar{c}_2 + \\ &\quad \bar{b}_1\bar{c}_1b_2'c_2' + \bar{b}_1\bar{c}_1b_2'\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2c_2' + \bar{b}_1\bar{c}_1\bar{b}_2\bar{c}_2)\end{aligned}$$

Using the fact that $\varphi(b_1'c_1'b_2'c_2') = 0$ by the freeness property, we are left with terms in which some expectation factorizes. For example we shall treat the term $\varphi(b_1'\bar{c}_1b_2'c_2') = \varphi(c_1)\varphi(b_1'b_2'c_2')$. By the factorization property already obtained one has $\varphi(b_1'b_2'c_2') = \varphi(b_1'b_2')\varphi(c_2') = 0$. The other terms can be treated by similar considerations, and after some straightforward manipulations we arrive at the formula

$$\varphi(b_1c_1b_2c_2) = \varphi(b_1b_2)\varphi(c_1)\varphi(c_2) + \varphi(b_1)\varphi(b_2)\varphi(c_1c_2) - \varphi(b_1)\varphi(b_2)\varphi(c_1)\varphi(c_2)$$

We observe that the computation of the expectation of the product $b_1c_1b_2c_2$ can be reduced to the computation of expectations in the subalgebras $\mathcal{B}$ and $\mathcal{C}$. This also shows that the result is different from the one we would have obtained with independent (commuting) random variables.

It is not difficult to see that the above computation can be generalized. More precisely, taking care of how the terms are successively reduced, the reader should check, and it is a good exercise do so, that the following is true

Proposition 1. *Let $\mathcal{B}_i; i \in I$ be free subalgebras in $\mathcal{A}$, and $a_1, \ldots, a_n \in \mathcal{A}$ such that for all $j = 1, \ldots, n$, one has $a_j \in \mathcal{B}_{i_j}$ for some $i_j \in I$. Let Π be the partition of $\{1, \ldots, n\}$ determined by $j \sim k$ if $i_j = i_k$. For each partition π of $\{1, \ldots, n\}$, let $\varphi_\pi = \prod_{\substack{\{j_1,\ldots,j_r\}\in\pi \\ j_1<\cdots<j_r}} \varphi(a_{j_1} \ldots a_{j_r})$, then there exists universal coefficients $c(\pi, \Pi)$, such that*

$$\varphi(a_1 \ldots a_n) = \sum_{\pi \leq \Pi} c(\pi, \Pi)\varphi_\pi$$

where the sum is over partitions π which are finer than Π.

In particular, this shows that $\varphi(a_1 \ldots a_n)$ can be computed explicitly in terms of the restriction of φ to the algebras $\mathcal{B}_i$. This is a reminiscence of the fact that the joint distribution of a family of independent random variables is completely determined if we know the distribution of each of the random variables.

As the example that we treated above suggests, the algorithm we have described for computing coefficients $c(\pi, \Pi)$ leads quickly to intractable calculations. Finding an explicit formula for these coefficients is a non trivial combinatorial problem, which fortunately has been solved by R. Speicher. We shall come back to describe his solution later. Let us just mention for the moment that it involves a certain class of partitions, called "non-crossing".

It is time now to state a result showing that the notion we have introduced is meaningful, i.e. that there exist non-trivial examples. In this respect the situation is optimum, and we have

Theorem 3. *Let $\mathcal{B}_i; i \in I$ be complex unital algebras, equipped with normalized (i.e. $\varphi_i(1) = 1$) linear forms φ_i, then there exists an algebra $\mathcal{A}$, with a normalized linear form φ, and unital injective morphisms $\iota_i : \mathcal{B}_i \to \mathcal{A}$, satisfying $\varphi_i = \varphi \circ \iota_i$, and the algebras $\iota_i(\mathcal{B}_i); i \in I$, are free in $\mathcal{A}$.*

The construction of the algebra $\mathcal{A}$ consists in enforcing the freeness condition in a rather straightforward way. One defines it as the "free amalgamated product" of the $\mathcal{B}_i$, more precisely, denote by $\mathcal{B}'_i$, for all $i \in I$, the subspace of elements in $\mathcal{B}_i$ with zero expectation, and then take for $\mathcal{A}$ the direct sum of $\mathbb{C}.1$ (1 is the unit in $\mathcal{A}$), and all spaces $\mathcal{B}'_{i_1} \otimes \ldots \otimes \mathcal{B}'_{i_n}$, where $i_1 \ldots i_n$ runs over all finite sequences in I satisfying $i_1 \neq i_2 \neq \ldots \neq i_n$. In order to turn $\mathcal{A}$ into an algebra, we need to specify the product of two elements $(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_m)$. If a_n and b_1 belong to distinct algebras, then the product is simply $a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m$, which belongs to $\mathcal{A}$. If a_n and b_1 belong to the <u>same</u> algebra $\mathcal{B}_i$, then $\varphi_i(a_n b_1)$ is not zero in general, so we write $a_n b_1 = (a_n b_1)' + \overline{a_n b_1}$, and put

$$\begin{aligned}(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_m) =& a_1 \otimes \ldots \otimes a_{n-1} \otimes (a_n b_1)' \otimes b_2 \otimes \ldots \otimes b_m \\ &+ \varphi_i(a_n b_1)(a_1 \otimes \ldots \otimes a_{n-1})(b_2 \otimes \ldots \otimes b_m)\end{aligned}$$

Since the "length" of $(a_1 \otimes \ldots \otimes a_{n-1})(b_2 \otimes \ldots \otimes b_m)$ is $m+n-2$, the computation can be repeated, and we are done after a finite number of steps. The injection ι_i is defined by $\iota_i(b) = \varphi_i(b).1 + b'$, and the linear form φ by $\varphi(a.1) = a$ for $a \in \mathbb{C}$, and $\varphi = 0$ on each space $\mathcal{B}'_{i_1} \otimes \ldots \otimes \mathcal{B}'_{i_n}$. It is not difficult to check that the pair $(\mathcal{A}, \varphi)$ fulfills the required conditions.

4. Free convolution

We are now going to be more specific about the kind of algebras that we are considering. Since we want to do some probability theory, it will be convenient to be able to say that a function of a random variable is again a random variable, and that a random variable X has a distribution, namely a probability measure μ_X, such that $\varphi(f(X)) = \int f(x)\mu_X(dx)$ for any bounded function f. The cleanest way to do that is to assume that our algebra $\mathcal{A}$ is a *von Neumann algebra*, which means that it is an algebra of bounded operators in some complex Hilbert space H, closed under taking adjoints of operators, and under taking limits for the weak operator topology (i.e. simple weak convergence on H). Since we want φ to be an analogue of the expectation map with respect to a probability measure, we need some positivity assumption. Positivity here will be taken in the sense of operator theory, namely an element X of $\mathcal{A}$ will be positive if it is a self-adjoint positive operator on H. The positivity requirement on φ is that it takes nonnegative values on positive operators in $\mathcal{A}$. We shall also assume that φ is continuous for the weak operator topology. You do not need to be familiar with von Neumann algebras in order to be able to read the following, in fact the only property of a von Neumann algebra that we will use is the stability with respect to functional calculus, namely if $X \in \mathcal{A}$ is a self-adjoint operator on H, then the operator $f(X)$, which can be defined by functional calculus for any bounded Borel function f on $\mathbb{R}$, still belongs to the algebra $\mathcal{A}$. Also, if X is self-adjoint then the map $f \mapsto \varphi(f(X))$, defined for Borel bounded functions, is given by $f \mapsto \int_{\mathbb{R}} f(x)\mu_X(dx)$, for a unique probability measure μ_X, with compact support on $\mathbb{R}$. The probability measure μ_X is called the distribution of X. Thus, self-adjoint operators in $\mathcal{A}$ behave somewhat like (real valued) random variables.

If $\mathcal{A}$ is a von Neumann algebra, and φ is a normalized, weakly continuous, positive linear form on $\mathcal{A}$, we shall call the couple $(\mathcal{A}, \varphi)$ a non-commutative probability space.

It turns out that the amalgamated free product of algebras exists in the category of non-commutative probability spaces, namely if the $(\mathcal{B}_i, \varphi_i)$ in Theorem 3 are non-commutative probability spaces, then we can choose for $(\mathcal{A}, \varphi)$ a non-commutative probability space, and the ι_i are weakly continuous. This can be seen by applying the GNS construction to the amalgamated free product constructed after Theorem 3. We refer to [VDN] for a detailed proof.

We are now in position to give the free-probabilistic definition of the free convolution of measures, which was introduced in Theorem 1. Let μ and ν be probability measures with compact support on $\mathbb{R}$, then there exists a non-commutative probability space $(\mathcal{A}, \varphi)$ and self-adjoint elements X, Y in $\mathcal{A}$ with respective distributions μ and ν, such that X an Y are free. In order to construct $(\mathcal{A}, \varphi)$ and X, Y, we can do the following: take the algebra $L^\infty(\mathbb{R}, \mu)$, which can be considered as a von Neumann algebra of operators on $L^2(\mu)$, elements of $L^\infty(\mathbb{R}, \mu)$ acting by multiplication on $L^2(\mu)$. The expectation map φ_μ (integration with respect to μ) turns the pair $(L^\infty(\mathbb{R}, \mu), \varphi_\mu)$ into a non-commutative probability space. The map $x \mapsto x$ on $\mathbb{R}$ defines a self-adjoint element of $L^\infty(\mathbb{R}, \mu)$ (since μ has compact support), which we call X. By the general construction described above, there exists a non-commutative probability space, containing $(L^\infty(\mathbb{R}, \mu), \varphi_\mu)$ and $(L^\infty(\mathbb{R}, \nu), \varphi_\nu)$ as free subalgebras, and thus we get a non-commutative probability space containing two free elements X and Y with respective distributions μ and ν. Since X and Y are bounded selfadjoint operators, the distribution of $X+Y$ is a probability measure with compact support on $\mathbb{R}$, and it is characterized by its moments. Expanding the expression $\varphi((X+Y)^n)$ and using Proposition 1, we see that the moments $\varphi((X+Y)^n)$ can be computed as polynomial functions of the moments of μ and ν, so that the distribution of $X+Y$ really depends only on the measures μ and ν and not on the particular realization of the operators X and Y, thus we can define $\mu \boxplus \nu$ as the distribution of $X+Y$, where X and Y is any pair of free random variables, with respective distributions μ and ν. Clearly this definition makes the operation $\boxplus$ the free analogue of the convolution of measures in classical probability.

We shall now recover Theorem 1 from the following result, which shows that large hermitian matrices give an asymptotic model for free random variables.

Theorem 4. *Let A_N, B_N, ν_1 and ν_2 be as in the statement of Theorem 1, and let X, Y be two self-adjoint elements of some non-commutative probability space $(\mathcal{A}, \varphi)$, with respective distributions ν_1 and ν_2, then for every non-commutative polynomial in two indeterminates, P, one has $\frac{1}{N} tr(P(A'_N, B'_N)) \to \varphi(P(X, Y))$ in probability, as $N \to \infty$, where A'_N and B'_N are random matrices chosen independently, with respective distributions ρ_{A_N} and ρ_{B_N}.*

Applying Theorem 4 to the non-commutative polynomials $P_n(X, Y) = (X+Y)^n$, we see that the moments of $A'_N + B'_N$ converge in probability towards that of $X+Y$. Since we are dealing with compactly supported measures, we conclude to the weak convergence of the associated distributions, and we get Theorem 1.

We shall not give a proof of Theorem 4 here, since this is a far from easy result. This is a consequence of a more general result from the paper [Vo2] (see also [VDN]), and another, more direct proof can also been found in [X].

It is now time we give a formula for computing the free convolution of two measures. For convolution of probability measures, one way to perform this computation is to use the Fourier transform, which converts convolution into multiplication. Observe that for probability measures with compact support, the logarithm of the Fourier transform can be expanded into a power series

$$\log(\int_{\mathbb{R}} e^{itx}\mu(dx)) = \sum_{n=1}^{\infty} \sigma_n(\mu)(it)^n.$$

This follows from a substitution of the power series

$$\int_{\mathbb{R}} e^{itx}\mu(dx) = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \int_{\mathbb{R}} x^n \mu(dx)$$

into the expansion of $\log(1+z)$. The coefficients $\sigma_n(\mu)$ are polynomial functions in the moments of μ, called the cumulants of the measure μ, and the multiplicativity property of the Fourier transform of convolution shows that these cumulants linearize the convolution, namely $\sigma_n(\mu * \nu) = \sigma_n(\mu) + \sigma_n(\nu)$. It turns out that the free convolution admits a similar description in terms of so-called "non-crossing cumulants", which are defined as follows. Let

$$G_\mu(\zeta) = \int_{\mathbb{R}} \frac{1}{\zeta - t} d\mu(t)$$

be the Cauchy transform of μ. This defines an analytic function on $\mathbb{C}\backslash\mathbb{R}$, such that $G_\mu(\bar{\zeta}) = \overline{G_\mu(\zeta)}$, and $G_\mu(\mathbb{C}^+) \subset \mathbb{C}^-$ where $\mathbb{C}^+$ and $\mathbb{C}^-$ denote respectively the sets of complex numbers with positive and negative imaginary part. We can expand this function in a Laurent series involving the moments of μ

$$G_\mu(\zeta) = \sum_{n=0}^{\infty} \zeta^{-n-1} \int_{\mathbb{R}} x^n \mu(dx)$$

This series can be inverted formally with respect to ζ, the formal inverse having the form

$$K_\mu(z) = \frac{1}{z} + \sum_{n=1}^{\infty} C_n z^{n-1}$$

where the coefficients C_n are polynomial functions of the moments $m_k = \int_{\mathbb{R}} x^k \mu(dx)$ of μ. The most convenient way to proceed in order to compute the C_n is to write the equation defining K_μ as

$$G_\mu(\zeta)K_\mu(G_\mu(\zeta)) = 1 + \sum_{n=1}^{\infty} C_n G_\mu(\zeta)^n = \zeta G_\mu(\zeta)$$

Equating the coefficients of ζ^{-n} in the second and third member of the above equality, we can evaluate the C_n in terms of the moments recursively. The first few values are

$$C_1 = m_1$$
$$C_2 = m_2 - m_1^2$$
$$C_3 = m_3 - 3m_1m_2 + 2m_1^3$$

These formulas can be inverted, and we get the moments in terms of the C_n as

$$\begin{aligned} m_1 &= C_1 \\ m_2 &= C_2 + C_1^2 \\ m_3 &= C_3 + 3C_1C_2 + C_1^3 \end{aligned}$$

The coefficients C_n, which we shall call the free cumulants of the measure μ play the role of cumulants for the free convolution, namely one has

Theorem 5. *For all compactly supported measures μ and ν, on $\mathbb{R}$, and all $n \geq 1$, one has*

$$C_n(\mu \boxplus \nu) = C_n(\mu) + C_n(\nu).$$

Since one can recover the moments of a measure from its free cumulants, this determines completely the measure $\mu \boxplus \nu$.

This result was first proved by Voiculescu in [Vo1], using free creation and annihilation operators (the formula has also been discovered independently around the same time, in the more restrictive context of random walks on free products of groups, see [W]). Later, Speicher used his combinatorial approach to freeness to give another proof of Theorem 5. We shall describe Speicher's proof in section 5.

Let us give an example of the computation of $\boxplus$. Let

$$\mu = \nu = \frac{1}{2}(\delta_0 + \delta_1)$$

one has

$$G_\mu(\zeta) = \frac{\zeta - 1/2}{\zeta(\zeta - 1)} \text{ and } K_\mu(z) = \frac{z + 1 + \sqrt{1 + z^2}}{2z}$$

This gives

$$K_{\mu\boxplus\mu}(z) = 1 + \frac{1}{z}\sqrt{1 + z^2}$$

and

$$G_{\mu\boxplus\mu}(\zeta) = \frac{1}{\sqrt{\zeta(\zeta - 1)}}$$

The Cauchy transform can be inverted to give

$$\mu \boxplus \mu(dx) = \frac{1}{\pi\sqrt{x(2 - x)}}dx \text{ on } [0, +2].$$

This is the famous arcsine distribution (here on the interval $[0, +2]$). If we recall Theorem 3, we have the following striking interpretation of this computation: take a large integer N, and two subspaces of $\mathbb{C}^N$, of dimension $[N/2]$, then for most choices of these subspaces, the sum of the corresponding orthogonal projections has an eigenvalue distribution which is well approximated by the arcsine distribution.

4. Theory of addition of free random variables

Once one has defined the free convolution of measures, one may try to develop the theory of addition of free random variables in parallel with the theory of addition of independent random variables. It turns out that most of the well known classical results, such as the law of large numbers, the central limit theorem, or the Lévy-Khintchine formula, have free analogues, and this develops into a beautiful new branch of mathematics, which sheds new lights on some well known results like the Wigner theorem on spectra of random gaussian matrices. We shall give a brief survey of these results below, but before that we need to extend the free convolution to probability measures with unbounded support.

4.1 Free convolution of measures with unbounded support. Let μ be an arbitrary probability measure on $\mathbb{R}$ and let

$$G_\mu(\zeta) = \int_{\mathbb{R}} \frac{1}{\zeta - t} d\mu(t)$$

be its Cauchy transform. Since μ may very well have no moment at all, we no longer have the expansion of G_μ into a power series in ζ^{-1}, however the following is nevertheless true, let

$$\Theta_{\alpha,\beta} = \{z = x + iy \mid y < 0; \alpha y < x < -\alpha y; |z| \leq \beta\}$$

For every $\alpha > 0$, there exists a real number $\beta > 0$ such that the function G_μ has a right inverse defined on the domain $\Theta_{\alpha,\beta}$, taking values in some domain of the form

$$\Gamma_{\gamma,\lambda} = \{z = x + iy \mid y > 0; -\gamma y < x < \gamma y; |z| \geq \lambda\}$$

with $\gamma, \lambda > 0$. Call K_μ this right inverse, and let $R_\mu(z) = K_\mu(z) - \frac{1}{z}$. The function R_μ is called the R-transform of the measure μ. Given another probability measure ν on $\mathbb{R}$, we shall define a new probability measure $\mu \boxplus \nu$ by the requirement that

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu$$

on some domain of the form $\Theta_{\alpha,\beta}$, where these three functions are defined. It turns out that this definition is meaningful, and it is clear that it coincides with the previous definition in the case where μ and ν have compact support. In fact, there is also an interpretation of $\mu \boxplus \nu$ as the distribution of the sum of two free (unbounded in general) self-adjoint elements affiliated to some non-commutative probability space, see [BV] for details.

4.2 Law of large numbers. The most general formulation of the law of large numbers for sums of independent equidistributed random variables asserts that the distribution of $\frac{1}{n}(X_1 + \ldots + X_n - M_n)$ converges weakly towards the Dirac measure at zero, for some constants M_n, if and only if the common distribution μ of the X_j satisfies $t\mu(\mathbb{R} \setminus [-t, t]) \to 0$ as $t \to \infty$. It turns out that exactly the same result holds if one replaces independent random variables by free ones. This is proved in [BP].

4.3 The Central Limit Theorem. Let μ be a probability measure having zero mean and finite variance σ, then one has $R_\mu(z) \sim \sigma z$ as $z \to 0$. Let now $X_1, \dots, X_n, \dots$ be a sequence of identically distributed free random variables, with common distribution μ, then it is easy to see that the distribution of $\frac{X_1 + \dots X_n}{\sqrt{n}}$ has R-transform $\sqrt{n} R_\mu(\frac{z}{\sqrt{n}})$, which converges, as $n \to \infty$, towards σz. Using continuity properties of the R-tranform, it is not difficult to see that this implies the following free central limit theorem

Theorem 6. *The distribution of* $\frac{X_1 + \dots + X_n}{\sqrt{n}}$ *converges weakly, as* $n \to \infty$, *towards the distribution with* R*-transform* σz.

The distribution appearing in the free central limit theorem can be computed, by inverting the R-transform, it is the famous Wigner semi-circular distribution given by the density $\frac{1}{2\pi\sigma}\sqrt{4\sigma - x^2}$ on the interval $[-2\sqrt{\sigma}, 2\sqrt{\sigma}]$. This central limit theorem, together with Theorem 3 on asymptotic freeness of random matrices, provides a conceptual framework for the well known result of Wigner, on the asymptotic behaviour of the spectra of large random matrices with gaussian entries.

4.4 Infinitely divisible distributions. There is a notion of infinitely divisible measures for the free additive convolution, which is the obvious one, namely a probability measure μ on the real line is said to be freely infinitely divisible if for every positive integer n, there exists a measure μ_n such that $\mu_n^{\boxplus n} = \mu$. The following characterization of freely infinitely divisible measures on $\mathbb{R}$ has been obtained by Bercovici and Voiculescu [BV].

Theorem 7. *A probability measure* μ, *on* $\mathbb{R}$ *is infinitely divisible if and only if the function* R_μ *has an analytic continuation to the whole of* $\mathbb{C}^+$, *with values in* $\mathbb{C}^- \cup \mathbb{R}$, *and one has*

$$\lim_{y \to 0, y \in \mathbb{R}} y R_\mu(iy) = 0$$

Furthermore any analytic function R *having the above properties is the* R*-transform of some probability measure.*

Functions such as those appearing in Theorem 7, have a Nevanlinna representation, which can be seen here as the free analogue of the Lévy-Khintchine formula. More precisely, let R be the R-transform of some infinitely divisible probability measure, then there exists a real number α, and a finite positive measure ν, on $\mathbb{R}$, such that

$$R(z) = \alpha + \int_{-\infty}^{+\infty} \frac{z + t}{1 - tz} d\nu(t)$$

The extreme points in this integral representation have an interpretation similar to the one of the classical Lévy-Khintchine formula, namely the function $R(z) = \alpha$ is the R-transform of a Dirac mass at the point α, the function $R(z) = \sigma z$, corresponding to a point mass at zero for ν, is the R transform of the semi-circular distribution with variance σ. Finally the probability measure with R-transform $R(z) = \lambda \frac{z+t}{1-tz}$ is the free analogue of the Poisson distribution, namely, it is the weak limit, as $n \to \infty$ of the measures $((1 - \frac{\lambda}{n})\delta_0 + \frac{\lambda}{n}\delta_t)^{\boxplus n}$ (the "free binomial distributions").

4.5 Stable distributions. One can define stable distributions exactly as for classical convolution, namely, a probability measure on $\mathbb{R}$ is called stable, if and only if the set of probability measures obtained from μ by applying affine transformations of $\mathbb{R}$, is stable under free convolution. The set of all free stable probability measures on the real line has been determined by Bercovici and Voiculescu [BV]. It turns out that there is a natural one-to-one correspondance between stable distributions and free stable distributions, and the domains of attraction are the same in the classical and the free cases. In fact, any free stable distribution is the image by an affine transformation of a distribution whose R-transform belongs to the following list

(1) $R(z) = e^{i\pi\theta} z^{\alpha-1}$ where $1 < \alpha \leq 2$, and $(\alpha - 2) \leq \theta \leq 0$.
(2) $R(z) = a + b \log z$ where $a \in \mathbb{C}^+ \cup \mathbb{R}$ and $b \geq -\Im a/\pi$.
(3) $R(z) = e^{i\pi\theta} z^{\alpha-1}$ where $0 < \alpha < 1$, and $1 \leq \theta \leq 1 + \alpha$.

In particular, the Cauchy distribution is a free stable distribution of stability index one.

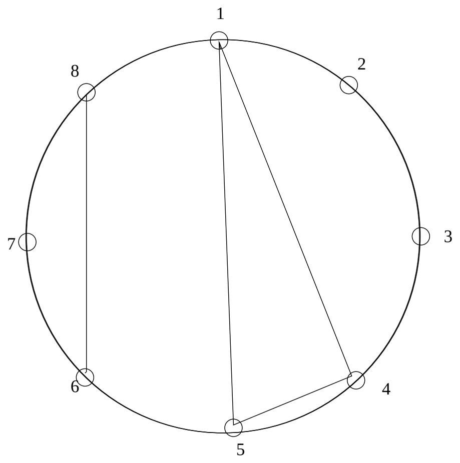
For these results, see [BVB]

5. Speicher's combinatorial approach to freenes

We shall now describe Speicher's combinatorial approach to the computation of the coefficients $c(\pi, \Pi)$ of Proposition 1, and some applications. A thorough discussion is given in [Sp2].

Let S be a totally ordered set. A partition of the set S is said to have a crossing if there exists a quadruple $(i, j, k, l) \in S^4$, with $i < j < k < l$, such that i and k belong to some class of the partition and j and l belong to another class. If a partition has no crossing, it is called non-crossing. The set of all non-crossing partitions of S is denoted by $NC(S)$, it is a lattice with respect to the dual refinement order (such that $\pi \leq \sigma$ if π is a finer partition than σ).

When $S = \{1, \ldots, n\}$, with its natural order, we will use the notation $NC(n)$. Here is an example with $n = 8$, $\pi = \{\{1, 4, 5\}, \{2\}, \{3\}, \{6, 8\}, \{7\}\}$.



$fig.1$

We draw a segment joining each point with the next point in the same class of the partition. The non-crossing condition means that segments should not intesect inside the circle.

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, then we shall define a family $R^{(n)}$ of n-multilinear forms on $\mathcal{A}$, for $n \geq 1$, by the following formula

$$\varphi(a_1 \ldots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \ldots, a_n)$$

Here, for $\pi \in NC(n)$, one has

$$R[\pi](a_1, \dots, a_n) = \prod_{V \in \pi} R^{(|V|)}(a_V)$$

where $a_V = (a_{j_1}, \dots, a_{j_k})$ if $V = \{j_1, \dots, j_k\}$ is a class of the partition π, with $j_1 < j_2 < \dots < j_k$. In particular $R[1_n] = R^{(n)}$ if 1_n is the partition with only one class. For example, one has, for all $a \in \mathcal{A}$,

$$\varphi(a) = R(a)$$

(we forget the superscript (n) when it is clear which n is considered), for $n = 2$

$$\varphi(a_1 a_2) = R(a_1, a_2) + R(a_1)R(a_2)$$

thus

$$R(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

is the covariance of a_1 and a_2. Finally,

$$\begin{aligned}\varphi(a_1 a_2 a_3) =& R(a_1, a_2, a_3) + R(a_1)R(a_2, a_3) + R(a_2)R(a_1, a_3)\\ &+ R(a_3)R(a_1, a_2) + R(a_1)R(a_2)R(a_3)\end{aligned}$$

which yields

$$\begin{aligned}R(a_1, a_2, a_3) =& \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3) - \varphi(a_2)\varphi(a_1 a_3)\\ &- \varphi(a_3)\varphi(a_1 a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)\end{aligned}$$

In general, for each n, one has

$$\varphi(a_1 \dots a_n) = R^{(n)}(a_1, \dots, a_n) + \text{terms involving } R^{(k)} \text{ for } k < n$$

so that the $R^{(n)}$ are well defined and can be computed by induction on n.

The non-crossing cumulants can be expressed explicitly in terms of the moments by the following formula

$$R^{(n)}(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} Moeb(\pi)\varphi[\pi](a_1, \dots, a_n)$$

Here $\varphi[\pi](a_1, \dots, a_n) = \prod_{V \in \pi} \varphi(a_{j_1} \dots a_{j_k})$ where $V = \{j_1, \dots, j_k\}$ are the classes of π, and $Moeb$ is the Möbius function on $NC(n)$ defined by

$$Moeb(\pi) = \prod_{V \in \pi} (-1)^{|V|} c_{|V|-1}$$

where $c_n = \frac{(2n)!}{n!(n+1)!}$ is the n^{th} Catalan number.

The connection between non-crossing cumulants and freeness is the following result of section 4 of [Sp1].

Proposition 2. *Let $(\mathcal{B}_i)_{i\in I}$ be free subalgebras of $\mathcal{A}$, and $a_1, \dots, a_n \in \mathcal{A}$ be such that a_j belongs to some $\mathcal{B}_{i_j}$ for each $j \in \{1, 2, \dots, n\}$. Then $R(a_1, \dots, a_n) = 0$ if there exists some j and k with $i_j \neq i_k$.*

Proposition 2 is the key to the computation of the coefficients $c(\pi, \Pi)$ of Proposition 1. Indeed, let $a_1, \dots, a_n$ be an arbitrary sequence in $\mathcal{A}$, such that each a_j belongs to one of the algebras $\mathcal{B}_i$, then we have

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \dots, a_n)$$

and in this sum the terms corresponding to partitions π having a class containing two elements j, k such that a_j and a_k belong to distinct algebras give a zero contribution. Thus we have to sum over partitions in which all j belonging to a certain class are such that a_j belongs to the same algebra, and the value of $R[\pi](a_1, \dots, a_n)$ can be expanded in terms of the restriction of φ to each of the subalgebras.

The multilinear forms $R^{(n)}$ allow us to recover the free cumulants, indeed one has

$$\int_{\mathbb{R}} x^n \, \mu(dx) = \varphi(X^n) = \sum_{\pi \in NC(n)} R[\pi](X, \dots, X)$$

Proposition 3. *Let $X \in \mathcal{A}$ be selfadjoint and have distribution $\mu(dx)$, then the free cumulants of the measure μ are given by the formula $C_n(\mu) = R^{(n)}(X, \dots, X)$, for $n = 1, 2 \dots$.*

Using Propositions 2 and 3, we can now give a proof of Theorem 5. Let X and Y be free random variables with respective distributions μ and ν, then the cumulants of $\mu \boxplus \nu$ are given, according to Proposition 3, by

$$C_n(\mu \boxplus \nu) = R^{(n)}(X + Y, \dots, X + Y)$$

Since $R^{(n)}$ is an n-linear form, we can expand $R^{(n)}(X+Y, \dots, X+Y)$, into a sum of terms $R^{(n)}(Z_1, Z_2, \dots, Z_n)$, where each Z_i is either X or Y. Applying Proposition 2, we see that all these terms are zero, except $R^{(n)}(X, \dots, X)$ and $R^{(n)}(Y, \dots, Y)$, thus we have

$$R^{(n)}(X + Y, \dots, X + Y) = R^{(n)}(X, \dots, X) + R^{(n)}(Y, \dots, Y)$$

and Theorem 5 follows from Proposition 3 again. The proofs of Propositions 2 and 3 can be found in [Sp1] or [Sp2].

6. Some further topics

6.1 Multiplicative free convolution. Given two unitary elements U, V, which are free in some non-commutative probability space $(\mathcal{A}, \varphi)$, we can form their product, which is again a unitary element. The distributions of U and V are this time probability measures say μ and ν, on the set T of complex numbers of modulus one, and the distribution of UV, which depends only on μ and ν, can be computed by the so-called Σ-transform. Let us introduce the ψ-function of a probability measure μ, on T, by

$$\psi_\mu(z) = \int_T \frac{z\xi}{1 - z\xi} d\mu(\xi)$$

This is a convergent power series in $D=\{z\in\mathbb{C}\,|\,|z|<1\}$, the open unit disk of $\mathbb{C}$, such that $\psi_\mu(0)=0$. Let $\mathcal{M}_*$ be the set of probability measures on T such that $\int_T \xi d\mu(\xi)\neq 0$. If $\mu\in\mathcal{M}_*$, the function $\frac{\psi_\mu}{1+\psi_\mu}$ has a right inverse, called $\tilde{\chi}_\mu$, defined in a neighbourhood of 0, such that $\tilde{\chi}_\mu(0)=0$, and we let $\Sigma_\mu(z)=\frac{1}{z}\tilde{\chi}_\mu(z)$ be the Σ-transform of μ. Then, for any measures $\mu,\nu\in\mathcal{M}_*$, one has $\mu\boxtimes\nu\in\mathcal{M}_*$ and

$$\Sigma_{\mu\boxtimes\nu}(z)=\Sigma_\mu(z)\Sigma_\nu(z)$$

in some neighbourhood of zero where these three functions are defined.

This formula was first found by Voiculescu in [V2]. His proof was quite complicated, and a simpler one has been given by U. Haagerup. A proof using non-crossing partitions, due to Nica and Speicher is in [NS1].

There is an analogue, for free multiplicative convolution on T, of the Lévy-Khinchine formula. This states that a probability measure on T is infinitely divisible, for the free multiplicative convolution, if and only if its Σ transform can be written as $\Sigma_\mu(z)=\exp(u(z))$ where u is an analytic function on D, taking values with nonnegative real parts. Such a function has a representation of the form

$$u(z)=i\alpha+\int_T\frac{1+\zeta z}{1-\zeta z}d\nu(\zeta)$$

for some finite positive measure ν on T, and real number α.

Finally one can also define multiplicative free convolution for measures on $\mathbb{R}_+$. We shall refer to [BV] for these topics.

We shall here make a remark on some features which distinguish free probability from classical probability. Let μ be a probability measure on $\mathbb{R}$, and let $\mathcal{G}$ be (a suitable branch of) the logarithm of its Fourier transform, then the set of positive real numbers t such that $t\mathcal{G}$ is again the logarithm of the Fourier transform of some probability measure is a closed additive subsemigroup of $\mathbb{R}_+$, containing the positive integers. It is equal to $\mathbb{R}_+$, exactly when μ is infinitely divisible, but it can also be reduced to the set of positive integers (e.g if μ is a Bernoulli distribution). In free probability, the analogue of the function $\mathcal{G}$ is the R-transform. Again, the set of t such that tR_μ is the R-transform of some probability measure is a closed additive subsemigroup of $\mathbb{R}_+$, but this time this subsemigroup always contains the interval $[1,+\infty[$. In fact the measure with R-transform tR_μ has a nice description in terms a free compression. Namely, let X be self-adjoint, with distribution μ in $(\mathcal{A},\varphi)$, and let π be a self-adjoint projection in $\mathcal{A}$, free with $\mathcal{A}$, and such that $\varphi(\pi)=\frac{1}{t}$, with $t\in[1,\infty[$. The distribution of μ is the Bernoulli distribution $(1-\frac{1}{t})\delta_0+\frac{1}{t}\delta_1$. The set $\pi\mathcal{A}\pi$ of elements of the form $\pi Z\pi$, for $Z\in\mathcal{A}$, is an algebra, with π as a unit, and $(\pi\mathcal{A}\pi,t\varphi)$ is a non-commutative probability space. The distribution of the element $t\pi X\pi$ in $(\pi\mathcal{A}\pi,t\varphi)$, has a distribution whose R-transform is given by tR_μ (try to show this using what you know about R-transforms and non-crossing partitions!). This shows that tR_μ is the R-transform of some probability measure, for all $t\geq 1$. As the example of Bernoulli distribution shows, there are probability measures μ for which tR_μ is the R-transform of some probability measure only for $t\in\{0\}\cup[1,\infty[$.

In order to close this section, let us note that the problem of finding the distribution of the commutator of two free self-adjoint random variables has been solved recently by Nica and Speicher [NS2].

6.2 More about random matrices. As we have seen in section 2, free probability provides us with a good understanding of the way that spectra of large matrices behave under addition. So far we have not said anything about eigenvectors. It turns out that free probability again has something to tell us about this. We shall again consider two large hermitian matrices A and B, whose spectra are known. Thinking of $A+B$ as a perturbation of the matrix A, we would like to know how the eigenvectors of $A+B$ are related to those of A. It is clear that if B is small compared to A, then the eigenvectors of $A+B$ should be close to those of A. Let us denote by $\gamma_1, \ldots, \gamma_N$ the normalized eigenvectors of A, associated with the eigenvalues $\lambda_1, \ldots, \lambda_N$, whereas we denote by $\xi_1, \ldots, \xi_N$ the eigenvectors of $A+B$, associated with the eigenvalue $\nu_1, \ldots, \nu_N$. The passage from the old basis to the new basis is given by the transition matrix $\langle \xi_k, \gamma_l \rangle$. In fact, since the eigenvectors are only defined up to some complex number of modulus one, we shall only consider the numbers $|\langle \xi_k, \gamma_l \rangle|^2$, which form a bistochastic matrix. These numbers are quite hard to tackle, and they may not have a definite asymptotic behaviour, as $N \to \infty$, so we shall evaluate them again some test functions. Let f and g be smooth functions on $\mathbb{R}$, then we shall look at the asymptotic behaviour of the expression

$$\sum_{1 \leq k,l \leq N} g(\nu_k) f(\lambda_l) |\langle \xi_k, \gamma_l \rangle|^2$$

We can rewrite this as

$$tr(g(A+B)f(A))$$

then using our result on asymptotic behaviour of large matrices, it is easy to see that, if the empirical distributions on the eigenvalues of A and B converge, as $N \to \infty$, towards μ and ν, and we choose matrices at random as in Theorem 3, then the expression

$$\frac{1}{N} tr(g(A'_N + B'_N) f(A'_N))$$

will converge, in probability, as $N \to \infty$, towards

$$\varphi(g(X+Y)f(X))$$

where X and Y are free self-adjoint elements, with respective distributions ν_1 and ν_2. In principle, we can compute the value of such an expression, for example if f and g are polynomials. In fact it is easy, by a positivity argument, to see that this value is given by $\int f(x)g(u)\rho(dx, du)$ where ρ is some probability measure on $\mathbb{R}^2$. It turns out that this probability measure can be desintegrated along the values of x as $\rho(dx, du) = k(x, du)\mu(dx)$, where $k(x, du)$ is a Markov transition kernel, which could be thought of as the limit of the bistochastic matrices $|\langle \xi_k, \gamma_l \rangle|^2$, and this Markov kernel can be explicitly computed, namely it is characterized by

$$\int_{\mathbb{R}} (\zeta - u)^{-1} k(x, du) = (F(\zeta) - x)^{-1} \qquad \text{for all } \zeta \in \mathbb{C} \setminus \mathbb{R}$$

for some function F on $\mathbb{C} \setminus \mathbb{R}$ such that $F(\bar{\zeta}) = \overline{F(\zeta)}$, $\quad F(\mathbb{C}^+) \subset \mathbb{C}^+$, $Im(F(\zeta)) \geq Im(\zeta)$ for $\zeta \in \mathbb{C}^+$, and $\frac{F(iy)}{iy} \to 1$ as $y \to +\infty$, $y \in \mathbb{R}$, and

$$G_\mu(F(\zeta)) = G_{\mu \boxplus \nu}(\zeta)$$

for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$. The map F is uniquely determined by these properties.

This result appears in [Bi], where it is applied to the theory of processes with free increments.

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