SOME FUNCTION SPACES RELATED TO THE BROWNIAN MOTION ON SIMPLE NESTED FRACTALS

KATARZYNA PIETRUSKA-PAŁUBA

ABSTRACT. In this paper we identify the domain of the Dirichlet form associated with the Brownian motion on simple nested fractals with an integral Lipschitz space. This result generalizes such an identification on the Sierpiński gasket, carried on by Jonsson in [9].

1. INTRODUCTION

Since the mid-eighties there has been an outburst of papers concerned with the Brownian motion on fractal spaces, dealing with the existence/uniqueness problem as well as investigating properties of the resulting process(es). A good source of references for those papers is [2].

The first set to be investigated has been the Sierpiński gasket, by far the simplest representant of the 'simple nested fractals' class (see [3],[7],[12]). The first constructions were purely probabilistic and the Brownian motion was obtained as a limit (in distribution) of appropriately scaled random walks on lattices, approximating the gasket (see [3], for example). Later it became clear that simpler approach to the problem is the one that uses Dirichlet forms rather than random walks (see [6]). Both constructions were then generalized to the class of 'simple nested fractals' (see Sec. 2.1 for definition) — probabilistically in [11], [13] and from the Dirichlet point of view in [5].

The Dirichlet form associated with the semigroup it generates is a local regular Dirichlet form on L^2 on the fractal with respect to the Hausdorff measure, which is invariant under those isometries. As Brownian motion on a simple nested fractal is unique (up to a trivial time-rescaling), both approaches are equivalent.

In the early paper of Barlow and Perkins [3] it was proven that all functions in the domain of the generator of the Brownian motion have good continuity properties (they are Hölder continuous) and that they cannot be extended beyond their fractal domain as differentiable functions.

Jonsson in [9] gave a precise description of the domain of the Dirichlet form associated with the Brownian motion on the Sierpiński gasket; it turns out that this domain coincides with certain integral Lipschitz space on the fractal in question. Those spaces as well as closely related Besov spaces on general sets are analyzed in depth in [10].

Jonsson conjectured that analogous result should hold for a much larger class of fractals. This is true indeed — by refinement of his methods we were able to extend

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the characterisation of the Dirichlet form domain to the class of all simple nested fractals. The advantage of simple nested fractals is that the Dirichlet form can be obtained as a limit of finite-dimensional forms (see Sec. 2.2), which is the main tool in our proof.

2. Preliminaries. Simple nested fractals and Dirichlet forms

Throughout the paper, $\#K$ denotes the cardinality of the set K and $C(K)$ — the space of continuous real-valued functions on K. Moreover, generic constants whose values are irrelevant to our purposes are denoted by c and they usually change from line to line.

2.1. **Simple nested fractals.** The class of simple nested fractals was first introduced by T. Lindstrøm in [13]. Our short exposition will be mostly based on that paper.

Let $N \geq 1$ be fixed. A transformation $\psi : \mathbf{R}^N \to \mathbf{R}^N$ is called a *similitude with scaling factor* L $(L > 1)$ if $\psi(x) = \frac{1}{L}U(x) + v$, where U is an isometry of $\mathbf{R}^N$ and $v \in \mathbf{R}^N$ is fixed.

Suppose that $\psi_1, ..., \psi_M, M \geq 2$, are given similitudes with common scaling factor L. In view of Hutchinson's result (see [4], [8], [13]) there exists a unique nonempty compact set $\mathcal{K} \subset \mathbf{R}^N$ such that

$$\mathcal{K} = \bigcup_{i=1}^{M} \psi_i(\mathcal{K}).$$

It is called the *self-similar fractal* generated by the family of similitudes $\psi_1, ...\psi_M$.

There are several explicit methods of constructing $\mathcal{K}$. Let us describe one of them, based on the set of fixed points of the similitudes ψ_i.

As all the similitudes ψ_i are contractions, there exist $x_1, ..., x_M$ such that $\psi_i(x_i) = x_i$ (these fixed points are not necessarily distinct). Denote by F the collection of those fixed points.

Definition 1. *$x \in F$ is called an essential fixed point of the system $\psi_1, ...\psi_M$ if there exists another fixed point $y \in F$ and two different transformations ψ_i, ψ_j such that $\psi_i(x) = \psi_j(y)$.*

Informally speaking, the essential fixed points are points through which parts of the fractal meet. Denote by V_0 the set of all essential fixed points.

Condition 1. $\#V_0 \geq 2$.

Example 1. *The Sierpiński gasket in* $\mathbf{R}^2$. Three transformations: $\psi_1(x) = \frac{x}{2}$, $\psi_2(x = \frac{x}{2} + (\frac{1}{2}, 0)$, $\psi_3(x) = \frac{x}{2} + (\frac{1}{4}, \frac{\sqrt{3}}{4})$. The system (ψ_1, ψ_2, ψ_3) has three fixed points: $(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and all of them are essential (see fig. 1).

Example 2. *The Lindstrøm snowflake* is an invariant set for the system of seven transformations $\psi_1, ...\psi_7$. Of the fixed points $x_1, ..., x_7$, all are essential fixed points, except for x_7 (see fig. 2)

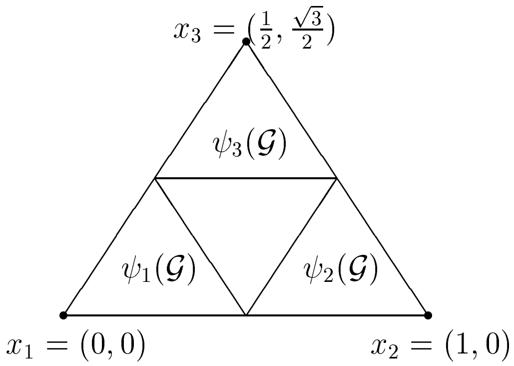

FIGURE 1. Transformations that define the Sierpiński gasket $\mathcal{G}$

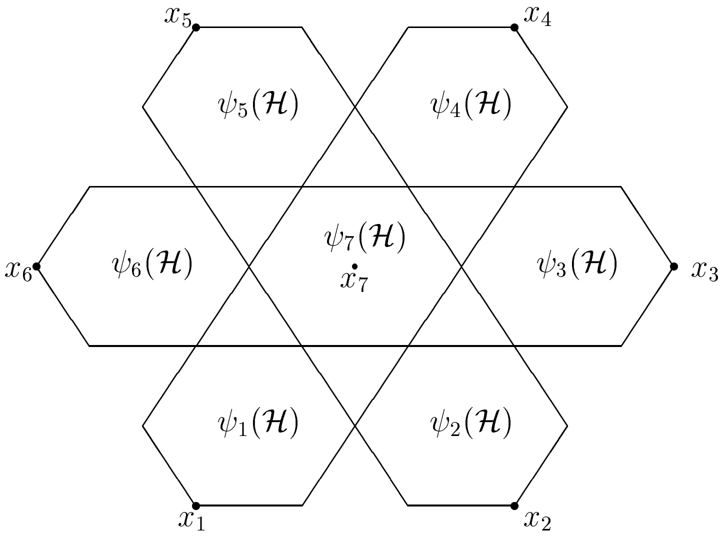

FIGURE 2. Transformations that define the Lindstrøm snowflake $\mathcal{H}$

For $A \subset \mathbf{R}^N$ let $\Psi(A) = \bigcup_{i=1}^M \psi_i(A)$ and $\Psi^n = \underbrace{\Psi \circ \cdots \circ \Psi}_{n \text{ times}}$. Then we define 'all the vertices at the n-th level' as

$$V_n = \Psi^n(V_0), \quad n = 1, 2...$$

$(V_n)_n$ is an increasing family of sets (it follows from $V_0 \subset \Psi(V_0)$). Let

$$V_\infty = \bigcup_{n=1}^{\infty} V_n.$$

The following holds true (see [4], p. 120):

Theorem 1. *$\mathcal{K} = \overline{V_\infty}$, the closure taken in the Euclidean topology of $\mathbf{R}^N$.*

Next, we impose the following

Condition 2 (Open Set Condition). The family $\psi_1, ...\psi_M$ satisfies the *open set condition* if there exists an open, bounded, nonempty set $U \subset \mathbf{R}^N$ such that $\psi_i(U) \cap \psi_j(U) = \emptyset$

for $i \neq j$ and

$$\Psi(U) = \bigcup_{i=1}^{M} \psi_i(U) \subset U.$$

If the open set condition is satisfied, then the Hausdorff dimension of $\mathcal{K}$ can be easily calculated (see Th. 8.6 of [4]) and is equal to

$$d_f(\mathcal{K}) = \frac{\log M}{\log L} \leq N \tag{2.1}$$

For instance, the Hausdorff dimension of the two-dimensional Sierpiński gasket equals to $\frac{\log 3}{\log 2}$ and of the Lindstrøm snowflake — to $\frac{\log 7}{\log 3}$. M is sometimes called the mass-scaling factor, and L – the length-scaling factor for the fractal.

Definition 2. *Let $m \geq 1$ be given. A set of the form $\psi_{i_1} \circ \cdots \circ \psi_{i_m}(\mathcal{K})$ is called an m-symplex (m-symplices are just scaled down copies of $\mathcal{K}$). Collection of all m-symplices will be denoted by $\mathcal{F}_m$. A set of the form $\psi_{i_1} \circ \cdots \circ \psi_{i_m}(V_0)$ is called an m-cell. For an m-symplex $S = \psi_{i_1} \circ \cdots \circ \psi_{i_m}(\mathcal{K})$, let $V(S) = \psi_{i_1} \circ \cdots \circ \psi_{i_m}(V_0)$ be the set of its vertices.*

$(i_1, ..., i_m)$ is called the *address* of the symplex $S = \psi_{i_1} \circ ... \circ \psi_{i_m}(\mathcal{K})$. Similarly, if $x = \psi_{i_1} \circ \cdots \circ \psi_{i_m}(x_0)$, $x_0 \in V_0$, then we call $(i_1, ... i_m)$ the *address* of the point x. Observe that addresses of symplices are uniquely determined, addresses of points from V_∞ in general are not.

Definition 3. *Let $m \geq 1$ be fixed. $x, y \in V_\infty$ are called m-neighbours if there exists an m-symplex S such that $x, y \in V(S)$ (in particular $x, y \in V_m$).*

After these definitions we can formulate the remaining three conditions.

Condition 3. Nesting. For each $m \geq 1$, and $S, T \in \mathcal{F}_m$, $S \cap T = V(S) \cap V(T)$. This condition is in fact equivalent to

$$\forall_{i,j \in \{1,...,M\}} \ \psi_i(\mathcal{K}) \cap \psi_j(\mathcal{K}) = \psi_i(V_0) \cap \psi_j(V_0).$$

Condition 4. Connectivity. Define the graph structure $E_{(1)}$ on V_1 as follows: we say that $(x, y) \in E_{(1)}$ if x and y are 1-neighbours. Then we require the graph $(V_1, E_{(1)})$ to be connected.

Condition 5. Symmetry. For $x, y \in V_0$ let $R_{x,y}$ be the reflection in the hyperplane bisecting the segment $[x, y]$. Then we stipulate that

$$\forall_{i \in \{1,...,M\}} \forall_{x,y \in V_0,\, x \neq y} \exists_{j \in \{1,...,M\}} \ R_{x,y}(\psi_i(V_0)) = \psi_j(V_0)$$

(natural reflections map 1-cells onto 1-cells).

Definition 4. *Fractals satisfying conditions 1-5 are called simple nested fractals.*

Remark 1. The most restrictive of those assumptions is nesting, as this condition requires the parts of the fractal to meet only through vertices. Sierpiński carpet, for example, does not fall into this category. However, this condition is crucial for the construction of the Dirichlet form.

Lemma 1. *1.*

$$\exists_{c_1,c_2>0} \text{ such that } \forall_{m\geq 1}\ c_2 M^m \leq \# V_m \leq c_1 M^m. \tag{2.2}$$

2.

$$\forall_{m\geq 1}\forall_{S\in\mathcal{F}_m}\forall_{n\geq m}\ \#[V_n\cap S] = \# V_{n-m}. \tag{2.3}$$

Proof. Denote by v_m the cardinality of V_m. Obviously $v_0 < v_1$, and also $v_1 < M\cdot v_0$ (as we deal with essential fixed points only).

Let $k_0 = Mv_0 - v_1 \geq 1$. Then for each m we have (nesting)

$$v_{m+1} = M\cdot v_m - k_0,$$

so that

$$v_m = M^m\left(v_0 - \frac{k_0}{M-1}\right) + \frac{k_0}{M-1}.$$

In view of definition of k_0 we have $(M-1)v_0 - k_0 = v_1 - v_0 > 0$ and (2.2) follows. (2.3) is obvious. □

Finally, for $m\geq 1$ let μ_m be the normalized counting measure on V_m, i.e.

$$\mu_m(\cdot) = \frac{1}{\# V_m}\sum_{x\in V_m}\delta_{\{x\}}(\cdot). \tag{2.4}$$

It is not hard to see that measures μ_m converge weakly towards the d_f-dimensional Hausdorff measure restricted to the fractal $\mathcal{K}$, μ.

2.2. **Dirichlet forms on simple nested fractals.** The material of this section is based on [1] and [5].

Suppose that the nested fractal $\mathcal{K}$, generated by the family of similitudes $\psi_1,...,\psi_M$ is given. Let $A = (a_{x,y})_{x,y\in V_0}$ be a <u>*conductivity matrix*</u> on V_0, i.e. a real-valued matrix that satisfies

1. $\forall_{x\neq y}\ a_{x,y}\geq 0$ and $a_{x,y} = a_{y,x}$,
2. $\forall_x\ \sum_y a_{x,y} = 0$ (so that $a_{x,x}\leq 0$).

We assume that A is <u>*irreducible*</u> i.e. the graph $(V_0, E_{(A)})$ is connected, where $E_{(A)}$ is the graph structure on V_0 related to the matrix A :

$$\text{for } x,y\in V_0\ \ (x,y)\in E_{(A)} \Leftrightarrow a_{x,y} > 0.$$

<u>*Dirichlet form on V_0 associated with A*</u> is defined as follows: for $f\in C(V_0)$ (i.e. for any $f: V_0\to\mathbf{R}$, as V_0 is finite)

$$\mathcal{E}_A^{(0)}(f,f) = \mathcal{E}_A(f,f) = \frac{1}{2}\sum_{x,y\in V_0} a_{x,y}(f(x)-f(y))^2 \geq 0. \tag{2.5}$$

We introduce two operators acting on Dirichlet forms, related to the geometrical structure of $\mathcal{K}$.

1. <u>*Reproduction.*</u> For $f\in C(V_1)$ let

$$\widetilde{\mathcal{E}}_A^{(1)}(f,f) = \sum_{i=1}^{M}\mathcal{E}_A^{(0)}(f\circ\psi_i, f\circ\psi_i).$$

The mapping $\mathcal{E}_A^{(0)} \mapsto \widetilde{\mathcal{E}_A^{(1)}}$ is called the reproduction map and will be denoted by $\mathcal{R}$.

2. *Decimation*. Given a symmetric form $\mathcal{E}$ on V_1, define its restriction to V_0, $\mathcal{E}|_{V_0}$, by:
for $f : V_0 \to \mathbf{R}$,

$$\mathcal{E}|_{V_0} = \inf\{\mathcal{E}(g, g) : g : V_1 \to \mathbf{R} \text{ and } g|_{V_0} = f\}. \tag{2.6}$$

This mapping is called the *decimation map* and is denoted by $\mathcal{D}e$.

Denote by $\mathbf{G}$ the symmetry group of V_0, i.e. the group of transformations generated by $\{R_{x,y} : x, y \in V_0\}$.

The following holds true (Lindstrøm [13] for the existence part, Sabot [14] for the uniqueness part):

Theorem 2. *There exist a unique number $\rho = \rho(\mathcal{K})$ and a unique (up to a multiplicative constant) irreducible conductivity matrix A on V_0, invariant under $\mathbf{G}$ and such that*

$$(\mathcal{D}e \circ \mathcal{R})(\mathcal{E}_A) = \frac{1}{\rho}(\mathcal{E}_A).$$

A is called the symmetric nondegenerate harmonic structure on V_0 (symmetric NDHS, or just NDHS). It follows then that $a_{x,y} > 0$ for $x \neq y$ and $\rho > 1$.

This nondegenerate harmonic structure is of particular interest for defining the Dirichlet form on the complete fractal.

Suppose from now that the conductivity matrix A is equal to the NDHS from theorem 2 which allows us to drop the subscript 'A'.

For $f \in C(V_m)$ define $\widetilde{\mathcal{E}}^{(m)}(f, f)$ in a manner similar to (2.6), i.e.

$$\widetilde{\mathcal{E}}^{(m)}(f, f) = \sum_{i_1,\ldots,i_m} \mathcal{E}^{(0)}(f \circ \psi_{i_1} \circ \ldots \circ \psi_{i_m}, f \circ \psi_{i_1} \circ \ldots \circ \psi_{i_m})$$

and further

$$\mathcal{E}^{(1)}(f, f) \stackrel{def}{=} \rho\widetilde{\mathcal{E}}^{(1)}(f, f), \;\; \mathcal{E}^{(m)}(f, f) \stackrel{def}{=} \rho^m\widetilde{\mathcal{E}}^{(m)}(f, f).$$

With this notation, as a consequence of nesting we have:

Theorem 3.

$$\forall_{f \in L^2(\mathcal{K},\mu)} \forall_{m=0,1,2,\ldots} \;\; \mathcal{E}^{(m+1)}(f, f) \geq \mathcal{E}^{(m)}(f, f).$$

Set

$$\mathcal{D} = \mathcal{D}(\mathcal{E}) = \{f \in L^2(\mathcal{K}, \mu) : \sup_m \mathcal{E}^{(m)}(f, f) < +\infty\}$$

and for $f \in \mathcal{D}$

$$\mathcal{E}(f, f) = \lim_{m\to\infty} \mathcal{E}^{(m)}(f, f). \tag{2.7}$$

Then $\mathcal{D} \subset C(\mathcal{K})$. Equation (2.7) defines a Dirichlet form on $L^2(\mathcal{K}, \mu)$ (a regular, symmetric, closed local bilinear form) that is invariant under $\mathbf{G}$.

In particular '$\mathcal{E}$ is closed' means that $\mathcal{D}$ is closed under the norm

$$\|f\|_{\mathcal{E}} \stackrel{def}{=} \sqrt{\mathcal{E}(f,f)+\|f\|_2^2}. \tag{2.8}$$

We can write the forms $\mathcal{E}^{(m)}$ in a more convenient way:

$$\mathcal{E}^{(m)}(f,f)=\rho^m \sum_{S\in\mathcal{F}_m} \widetilde{\mathcal{E}}_S^{(m)}(f,f), \tag{2.9}$$

where for $\mathcal{F}_m \ni S=\psi_{i_1}\circ\ldots\circ\psi_{i_m}(\mathcal{K})$

$$\begin{aligned}\widetilde{\mathcal{E}}_S^{(m)}(f,f) &= \mathcal{E}^{(0)}(f\circ\psi_{i_1}\circ\cdots\circ\psi_{i_m}, f\circ\psi_{i_1}\circ\cdots\circ\psi_{i_m}) = \\ &= \tfrac{1}{2}\sum_{x,y\in V(S)} a_{x,y}^{(m)}(f(x)-f(y))^2;\end{aligned}$$

$a_{x,y}^{(m)}$ being the 'm-th level conductivities' between $x,y\in V(S)$: if x,y are m-neighbours and $x=\psi_{i_1}\circ\cdots\circ\psi_{i_m}(\bar{x})$, $y=\psi_{i_1}\circ\cdots\circ\psi_{i_m}(\bar{y})$, then

$$a_{x,y}^{(m)}=a_{\bar{x},\bar{y}}$$

and if x,y are not m-neighbours we set $a_{x,y}^{(m)}=0$.

2.2.1. *Significance of the constant ρ.*. By analogy with the electrical circuit theory, ρ is called the resistance scaling factor. Denote

$$d_w=d_w(\mathcal{K}) \stackrel{def}{=} \frac{\log(M\rho)}{\log L}>1.$$

As the constant ρ is uniquely determined, this number can be considered to be one of characteristic numbers of the fractal $\mathcal{K}$ and is called 'the walk dimension of $\mathcal{K}$'. One has

$$\rho=\frac{L^{d_w}}{L^{d_f}}. \tag{2.10}$$

Next,

$$d_s=d_s(\mathcal{K}) \stackrel{def}{=} \frac{2d_f}{d_w}$$

is the spectral dimension of the set $\mathcal{K}$. In view of $\rho>1$ we have $d_s<2$, which will be decisive for our results in the sequel. For example, for the n-dimensional Sierpiński gasket $\mathcal{G}$ the walk dimension equals to $\frac{\log(n+3)}{\log 2}$ and its spectral dimension — to $\frac{2\log 3}{\log 5}<2$.

2.3. **Lipschitz spaces on d-sets.** (see Jonsson-Wallin [10], Jonsson [9])

Let F be an arbitrary closed subset of $\mathbf{R}^N$ ($N\geq 1$ is fixed) and d a real number, $0<d\leq N$. $B(x,r)$ denotes the closed ball with centre x and radius r. A positive Borel measure μ with support F is called a d-measure on F if there exist constants c_1 and c_2 such that

$$\forall_{x\in F}\forall_{0<r\leq 1}\;\; c_1 r^d\leq\mu(B(x,r))\leq c_2 r^d.$$

If F is a support or a d-measure, then F is called a <u>d-set</u>. Nested fractal $\mathcal{K}$ with Hausdorff dimension d_f is a d_f-set and the restriction of the d_f-dimensional Hausdorff measure to $\mathcal{K}$ is a d_f-measure (it follows from [4], pp. 120-121)

If F is a closed subset of $\mathbf{R}^N$ (not necessarily a d-set), and $\gamma > 0$ is a real number, then $Lip(\gamma, F)$ is the set of those bounded functions $f : F \to \mathbf{R}$ that satisfy the Hölder condition with exponent γ :

$$\exists_{M>0}\forall_{x,y\in F,|x-y|\leq 1}\ \ |f(x)| \leq M, \ \ |f(x)-f(y)| \leq M|x-y|^{\gamma}. \tag{2.11}$$

The norm of f in $Lip(\gamma, F)$ is the infimum of all the possible constants M in (2.11).

If, for some $d > 0$, $F \subset \mathbf{R}^N$ is a d-set and μ is a d-measure on F, then one defines the following 'integral' Lipschitz spaces on F :
for given $\alpha > 0$, $1 \leq p, q \leq +\infty$ the Lipschitz space $Lip(\alpha, p, q)(F)$ is the collection of all those functions $f \in L^p(F, \mu)$ for which, when $p, q < \infty$

$$\|(a_m^{(p)}(f))\|_{l_q} = \left(\sum_{m=1}^{\infty}(a_m^{(p)}(f))^q\right)^{\frac{1}{q}} < \infty,$$

where coefficients $a_m^{(p)}(f)$ are defined as

$$a_m^{(p)}(f) = 2^{m\alpha}\left(2^{md}\iint_{|x-y|<\frac{c_0}{2^m}}|f(x)-f(y)|^p d\mu(x)d\mu(y)\right)^{\frac{1}{p}} \tag{2.12}$$

for some $c_0 > 0$. For $p = \infty$ or $q = \infty$ take the L_∞ or l^∞ norms, as needed.

The space $Lip(\alpha, p, q)(F)$ is a Banach space with the norm

$$\|f\|_{Lip} = \|f\|_{L^p(\mathcal{K},\mu)} + \|(a_m^{(p)}(f))\|_{l_q}. \tag{2.13}$$

Note that if we replace the constant c_0 in (2.12) with another one we obtain the same function space and the norm changes to an equivalent one. The same remains true if we replace '2' by some other number greater than 1, see lemma 2 below.

Finally let us formulate the following property (Corollary 2, p. 498 of [9]):

Theorem 4. *For $0 < d \leq n$, $1 \leq p, q \leq \infty$ and α such that $\alpha - \frac{d}{p} = \gamma \in (0,1)$, $Lip(\alpha, p, q)(F)$ is continuously imbedded in $Lip(\alpha - \frac{d}{p}, F)$. In particular every $f \in Lip(\alpha, p, q)(F)$ is Hölder continuous with exponent γ.*

Jonsson ([9]) proves that on the N-dimensional Sierpiński gasket $\mathcal{G} \subset \mathbf{R}^N$ ($\mathcal{G}$ is a d_f-set with $d_f = \frac{\log(N+3)}{\log 2}$) the domain of the Dirichlet form defined by (2.7) coincides with the Lipschitz space

$$Lip(\alpha, 2, \infty)(\mathcal{G}), \ \ \text{with } \alpha = \frac{d_w}{2} = \frac{\log(N+3)}{2\log 2}$$

and the two norms: the Lipschitz norm and the Dirichlet norm are equivalent. He conjectured that a similar result should hold for a larger class of fractals. In this paper we prove that this conjecture holds for all simple nested fractals.

3. The main theorem

The natural scaling factor for the fractal $\mathcal{K}$ is $L = L(\mathcal{K})$, and in general $L \neq 2$. Therefore it is convenient to substitute L for 2 in the definition of Lipschitz coefficients, (2.12). This minor alteration changes the Lipschitz norm to an equivalent one. This is true in general: if $F \subset \mathbf{R}^N$ is a $d-$set, $\alpha > 0, 0 < p < \infty$ (α is considered fixed) let

$$b_m^{(p)}(f) = L^{m\alpha}\left(L^{md}\int\int_{|x-y|<\frac{c_0}{L^m}}|f(x)-f(y)|^p d\mu(x)d\mu(y)\right)^{\frac{1}{p}} \tag{3.1}$$

(similarly for $p = \infty$).
We have:

Lemma 2. *Let $F \subset \mathbf{R}^N$ be a d-set, $d \in (0, N]$, μ — a d-measure on F. Then there exists a constant $D > 0$ such that for each $f \in L^p(F, \mu)$ and for each $0 < q \leq \infty$*

$$\frac{1}{D}\|(a_m^{(p)}(f))\|_{l_q} \leq \|(b_m^{(p)}(f))\|_{l_q} \leq D\|(a_m^{(p)}(f))\|_{l_q}.$$

Proof. For $m \geq 1$, let $n(m)$ be the unique integer satisfying

$$2^{n(m)} \leq L^m < 2^{n(m)+1}$$

(for large m, $n(m)$ is the integer part of $m\frac{\log 2}{\log L}$) so that

$$\begin{aligned} b_m^{(p)}(f) &\leq (2^{n(m)+1})^{\alpha}\left((2^{n(m)+1})^d\int\int_{|x-y|<\frac{c_0}{2^{n(m)}}}|f(x)-f(y)|^p d\mu(x)d\mu(y)\right)^{\frac{1}{p}} \\ &= 2^{\alpha+\frac{d}{p}}a_{n(m)}^{(p)}(f) \end{aligned}$$

and

$$\begin{aligned} \|(b_m^{(p)}(f))\|_{l_q}^q = \textstyle\sum_m (b_m^{(p)}(f))^q &\leq (2^{\alpha+\frac{d}{p}})^q \textstyle\sum_m (a_{n(m)}^{(p)}(f))^q \\ &\leq (2^{\alpha+\frac{d}{p}})^q \|(a_m^{(p)}(f))\|_{l_q}^q \end{aligned}$$

(similarly for $q = \infty$). The opposite inequality follows by symmetry. □

From now on we restrict our attention to $F = \mathcal{K}$, $d = d_f$, $\alpha = \frac{d_w}{2}$, $p = 2, q = \infty$. We drop the superscript '2' in the definition of b_m's. The Lipschitz norm of f, with constant 'L' in place of '2' will be denoted by $\|f\|_{\mathcal{L}}$ (as explicitly written below).

After this technical lemma we can pass to our theorem.

Theorem 5. *Let $\alpha = \frac{d_w}{2}$ and let $\mathcal{L} = Lip(\alpha, 2, \infty)(\mathcal{K})$, endowed with the norm*

$$\|f\|_{\mathcal{L}} = \|f\|_2 + \|(b_m(f))\|_{\infty}.$$

Let $\mathcal{D}$ be the domain of the Dirichlet form on $\mathcal{K}$ with the norm (2.8). Then $\mathcal{L} = \mathcal{D}$ and the norms $\|f\|_{\mathcal{L}}$ and $\|f\|_{\mathcal{E}}$ are equivalent.

Proof. **Part 1.** First we prove that for $f \in \mathcal{D}$ we have $\|f\|_L \leq C\|f\|_{\mathcal{E}}$. To this end, it is enough to see that there exists a constant $c > 0$ such that for each $f \in \mathcal{D}$ and $m = 1, 2, ...$

$$\|b_m(f)\|_\infty^2 \leq c \cdot \mathcal{E}(f, f).$$

Take $f \in \mathcal{D}$. Then

$$\mathcal{E}(f, f) = \lim_{m\to\infty} c \cdot \left(\frac{L^{d_w}}{L^{d_f}}\right)^m \sum_{S\in\mathcal{F}_m} \sum_{x,y\in V(S)} a_{x,y}^{(m)} (f(x) - f(y))^2 < +\infty,$$

for some constant c. As all the conductivities $a_{x,y}^{(m)}$ are positive and chosen from a finite set, we have

$$\lim_{m\to\infty} \left(\frac{L^{d_w}}{L^{d_f}}\right)^m \sum_{S\in\mathcal{F}_m} \sum_{x,y\in V(S)} (f(x) - f(y))^2 \leq c \cdot \mathcal{E}(f, f).$$

Let $m > 0$ be fixed. We prove that for $c_0 = \inf\{|x - y| : x, y \in V_0\}$, for each $m \geq 1$

$$(b_m(f))^2 = L^{m(d_w+d_f)} \iint_{|x-y|<\frac{c_0}{L^m}} (f(x) - f(y))^2 d\mu(x)d\mu(y) \leq c \cdot \mathcal{E}(f, f). \tag{3.2}$$

Recall that the normalized counting measures on the sets V_n, μ_n's, converge weakly to the x^{d_f}–Hausdorff measure on the fractal, μ. For each $x \in \mathcal{K}$ and $r > 0$, $\mu(\partial(B(x, r))) = 0$. Moreover $\mathcal{D} \subset C(\mathcal{K})$, so that f in question is continuous (uniformly continuous in fact). As a consequence of the Portmanteau lemma it is enough to see that (3.2) holds for the approximations μ_n, i.e. that

$$\forall_{n>m} \;\; L^{m(d_w+d_f)} \iint_{|x-y|<\frac{c_0}{L^m}} (f(x) - f(y))^2 d\mu_n(x)d\mu_n(y) \leq c \cdot \mathcal{E}(f, f),$$

with constant not depending on n, m, f.

For $S \in \mathcal{F}_m$, let S_* be the union of S and all those symplices from $\mathcal{F}_m$ that have a point in common with S. Our choice of c_0 ensures that if $x \in S \in \mathcal{F}_m$ and $|x-y| < \frac{c_0}{L^m}$, then $y \in S_*$. Thus

$$\begin{aligned}\iint_{|x-y|<\frac{c_0}{L^m}} (f(x) - f(y))^2 d\mu_n(x)d\mu_n(y) \leq \\ \leq \frac{1}{2} \frac{1}{[\#V_n]^2} \sum_{S\in\mathcal{F}_m} \sum_{x\in S\cap V_n} \sum_{y\in S_*\cap V_n} (f(x) - f(y))^2.\end{aligned} \tag{3.3}$$

There exists $z_{xy} \in V(S)$ such that z_{xy} and x, as well as z_{xy} and y belong to the same m-symplex. As

$$(f(x) - f(y))^2 \leq 2[(f(x) - f(z_{xy}))^2 + (f(z_{xy}) - f(y))^2], \tag{3.4}$$

(3.3) is not greater than

$$\frac{c}{(\#V_n)^2}\sum_{S\in\mathcal{F}_m}\left(\sum_{x\in S\cap V_n}\sum_{y\in S_*\cap V_n}[(f(x)-f(z_{xy}))^2+(f(z_{xy})-f(y))^2]\right)$$
$$\leq \frac{c}{(\#V_n)^2}\sum_{S\in\mathcal{F}_m}\sum_{x\in S\cap V_n}\left(\sum_{z\in V(S)}(f(x)-f(z))^2\cdot\#[V_n\cap S]\right)$$
$$= \frac{c\cdot\#V_{n-m}}{(\#V_n)^2}\sum_{S\in\mathcal{F}_m}\left(\sum_{z\in V(S)}\sum_{x\in S\cap V_n}(f(x)-f(z))^2\right). \tag{3.5}$$

Let $m\geq 1$ and $S\in\mathcal{F}_m$ be fixed. S has unique address $(i_1,i_2,...,i_m)$, i.e.

$$S=\psi_{i_1}\circ\cdots\circ\psi_{i_m}(\mathcal{K}),\quad \text{for some }\ i_1,...,i_m\in\{1,...,M\}.$$

Its vertices $z\in V_m$ are of the form

$$z=\psi_{i_1}\circ\cdots\circ\psi_{i_m}(y)$$

and any $x\in S\cap V_n$ $(n>m)$ can be written as

$$x=\psi_{i_1}\circ\cdots\circ\psi_{i_m}\circ\psi_{i_{m+1}}\circ\ldots\psi_{i_n}(\bar{y}),$$

for some $y,\bar{y}\in V_0$. It follows that the double sum in round brackets in (3.5) is not greater than (there is no equality as there exist points with multiple addresses)

$$\sum_{y\in V_0}\sum_{i_{m+1},...,i_n}\sum_{\bar{y}\in V_0}((f(\psi_{i_1}\circ\cdots\circ\psi_{i_m}(y))-f(\psi_{i_1}\circ\cdots\circ\psi_{i_n}(\bar{y})))^2. \tag{3.6}$$

Next, we build a ‘path’ from z to x as follows:

$$\begin{aligned}
s_0 &= \psi_{i_1}\circ\cdots\circ\psi_{i_m}(y)=z\in V_m\cap S,\\
s_1 &= \psi_{i_1}\circ\cdots\circ\psi_{i_m}\circ\psi_{i_{m+1}}(y)\in V_{m+1}\cap S,\\
&\ \vdots\\
s_{n-m-1} &= \psi_{i_1}\circ\cdots\circ\psi_{i_{n-1}}(y)\in V_{n-1}\cap S,\\
s_{n-m} &= \psi_{i_1}\circ\cdots\circ\psi_{i_n}(\bar{y})=x\in V_n\cap S.
\end{aligned}$$

Using (3.4) repeatedly we get

$$\begin{aligned}
&(f(z)-f(x))^2)\leq\\
&\leq 2(f(s_0)-f(s_1))^2+4(f(s_1)-f(s_2))^2+...+2^{n-m}(f(s_{n-m-1})-f(s_{n-m}))^2.
\end{aligned}$$

Observe that for $l=0,1,...,n-m-1$ both points s_l and s_{l+1} belong to the $(m+l)$-symplex $S_l=\psi_{i_1}\circ\cdots\circ\psi_{i_{m+l}}(\mathcal{K})$, and that $s_l\in V(S_l)$. How many times can a given pair (s_l,s_{l+1}) of this form appear in such a path? It appears whenever the addresses of x and s_{l+1} coincide up to the $i_{m+l}-$th place, i.e. at most $M^{n-(l+1)}$ times. Summing up we arrive at:

$$(3.6)\leq c\cdot\sum_{r=m}^{n-1}\sum_{\mathcal{F}_r\ni R\subset S}\sum_{z\in V(R)}\sum_{w\in R\cap V_{r+1}}2^{r-m+1}M^{n-(r+1)}(f(z)-f(w))^2.$$

Every symplex $R \in \mathcal{F}_r$ is just a scaled-down copy of $\mathcal{K}$, therefore there exists a constant depending on $\mathcal{K}$ only such that

$$\sum_{z \in V(R), w \in R \cap V_{r+1}} (f(z) - f(w))^2 \leq c \cdot \widetilde{\mathcal{E}}_R^{(r+1)}(f,f),$$

with $\widetilde{\mathcal{E}}_R^{(r+1)}(f,f)$ defined by (2.9). To see this, connect x and z by a path $(p_0, p_1, ..., p_a)$ as follows: $p_0 = x, p_a = z$, $p_i \in V_{r+1} \cap R$ for $i = 0, 1, ..., a$ and for each $i = 0, ..., r-1$ (p_i, p_{i+1}) are $(r+1)$-neighbours. Next use the inequality $(a_1 + ... + a_p)^2 \leq p(a_1^2 + ... + a_p^2)$, valid for arbitrary real numbers a_i and positive integer p, then observe that a given pair (u, v) of $(r+1)$-neighbours can appear in such a path only a finite number of times and that this number does not change with r (R is just a scaled-down copy of $\mathcal{K}$ and $R \cap V_{r+1}$ — a scaled-down copy of V_1).

Summing over $\mathcal{F}_r \ni R \subset S$ and then over $S \in \mathcal{F}_m$ we end up with

$$\begin{aligned}
(3.5) \quad &\leq c \cdot \tfrac{M^{n-m}}{M^{2n}} \sum_{S \in \mathcal{F}_m} \sum_{r=m}^{n-1} \sum_{\mathcal{F}_r \ni R \subset S} 2^{r-m+1} M^{n-(r+1)} \widetilde{\mathcal{E}}_R^{(r+1)}(f,f) \\
&= c \cdot \tfrac{M^{n-m}}{M^{2n}} \sum_{r=m}^{n-1} 2^{r-m+1} M^{n-(r+1)} \sum_{R \in \mathcal{F}_r} \widetilde{\mathcal{E}}_R^{(r+1)}(f,f) \\
&= c \cdot \tfrac{M^{n-m}}{M^{2n}} \sum_{r=m}^{n-1} 2^{r-m+1} M^{n-(r+1)} \widetilde{\mathcal{E}}^{(r+1)}(f,f).
\end{aligned}$$

Recall that the sequence

$$\left(\frac{L^{d_w}}{L^{d_f}} \right)^r \widetilde{\mathcal{E}}^{(r)}(f,f)$$

increases towards $c \cdot \mathcal{E}(f,f)$ when $r \to \infty$, so that

$$(3.5) \leq c \cdot \mathcal{E}(f,f) \left[\frac{1}{M^m} \sum_{r=m}^{n-1} 2^{r-m+1} \frac{1}{M^{r+1}} \left(\frac{L^{d_f}}{L^{d_w}} \right)^{r+1} \right].$$

After summing up the resulting geometric series, using $L^{d_f} = M$, we obtain that the term in square brackets is not bigger than

$$\frac{1}{M^m} \cdot \frac{1}{L^{m d_w}} \cdot \frac{1}{1 - \frac{2}{L^{d_w}}} = \frac{c}{(L^{d_w + d_f})^m},$$

if $L^{d_w} > 2$. But as long as $d_s < 2$ (equivalently: $\rho > 1$) and $M \geq 2$ we have $L^{d_w} = \rho L^{d_f} = \rho M > 2$ (see the discussion at the end of section 2.2.1). Proof of the first part is completed.

Part 2. Now we show that $\mathcal{L} \subset \mathcal{D}$. Suppose that $g \in Lip(\alpha, 2, \infty)(\mathcal{K})$, i.e. $g \in L^2(\mathcal{K}, \mu)$, and that $\|(b_m(g))\|_\infty < +\infty$, with $b_m(g)$ defined by (3.1). We need to show that

$$\sup_{m \geq 1} \left(\frac{L^{d_w}}{L^{d_f}} \right)^m \widetilde{\mathcal{E}}^{(m)}(g,g) < c \cdot \|(b_m(g))\|_\infty^2.$$

Again,

$$\begin{aligned}\left(\tfrac{L^{d_w}}{L^{d_f}}\right)^m \widetilde{\mathcal{E}}^{(m)}(g,g) &= c\cdot \left(\tfrac{L^{d_w}}{L^{d_f}}\right)^m \sum_{S\in\mathcal{F}_m}\sum_{x,y\in V(S)} a^{(m)}_{x,y}(g(x)-g(y))^2 \\ &\leq c\cdot \left(\tfrac{L^{d_w}}{L^{d_f}}\right)^m \sum_{S\in\mathcal{F}_m}\sum_{x,y\in V(S)} (g(x)-g(y))^2.\end{aligned} \tag{3.7}$$

Fix $m\geq 1$ and $S\in\mathcal{F}_m$. Then for each $p\in S$

$$(g(x)-g(y))^2 \leq 2[(g(x)-g(p))^2+(g(p)-g(y))^2]. \tag{3.8}$$

Integrating both sides of (3.8) over $p\in S$ with respect to the Hausdorff measure μ we get

$$(g(x)-g(y))^2 \leq \frac{1}{\mu(S)}\left[\int_S (g(x)-g(p))^2 d\mu(p) + \int_S (g(y)-g(p))^2 d\mu(p)\right]$$

so that

$$\sum_{x,y\in V(S)} (g(x)-g(y))^2 \leq \frac{2\#[V(S)]}{\mu(S)} \sum_{x\in V(S)} \int_S (g(x)-g(p))^2 d\mu(p). \tag{3.9}$$

Let $(i_1,...i_m)$ be the address of the symplex S, i.e. $S=\psi_{i_1}\circ ...\circ\psi_{i_m}(\mathcal{K})$. Let a vertex $x\in V(S)$ be given. It is of the form $x=\psi_{i_1}\circ...\circ\psi_{i_m}(v)$ for some $v\in V_0$. Let $l\in\{1,...M\}$ be such an index that $v=\psi_l(v)$. Consider the following decreasing sequence of symplices:

$$\begin{aligned}\mathcal{F}_m &\ni S_0=\psi_{i_1}\circ...\circ\psi_{i_m}(\mathcal{K})=S,\\ \mathcal{F}_{m+1} &\ni S_1=\psi_{i_1}\circ...\circ\psi_{i_m}\circ\psi_l(\mathcal{K}),\\ \mathcal{F}_{m+2} &\ni S_2=\psi_{i_1}\circ...\circ\psi_{i_m}\circ\psi_l\circ\psi_l(\mathcal{K}),\\ &\vdots\\ \mathcal{F}_{m+r} &\ni S_r=\psi_{i_1}\circ...\circ\psi_{i_m}\circ(\psi_l)^r(\mathcal{K}),\\ &\vdots\end{aligned}$$

It is clear that for any $r=0,1,2...$, $x\in V(S_r)$ and that $\bigcap_{r=0}^\infty=\{x\}$.

Suppose $k\geq 1$ is given (to be chosen later on). Let

$$T_0=S_0,\ T_1=S_k,\ T_2=S_{2k},...\ \text{and so on.} \tag{3.10}$$

For any given $p_i\in T_i$, $i=1,2,..,\nu$ (ν — an arbitrary positive integer) we have

$$\begin{gathered}(g(x)-g(p_0))^2\leq 2((g(x)-g(p_\nu))^2+(g(p_\nu)-g(p_0))^2\leq\\ \leq 2(g(x)-g(p_\nu))^2+[4(g(p_0)-g(p_1))^2+8(g(p_1)-g(p_2))^2+\cdots+\\ +2^{\nu+1}(g(p_{\nu-1})-g(p_\nu))^2].\end{gathered}$$

Integrating this inequality $(\nu+1)$ times over $p_0\in T_0$, $p_1\in T_1$, $...p_\nu\in T_\nu$ and dividing both sides by $\mu(T_1)\cdot...\cdot\mu(T_\nu)$ we obtain:

$$\begin{aligned}&\int_{T_0}(g(x)-g(p_0))^2 d\mu(p_0)\leq \frac{2\mu(T_0)}{\mu(T_\nu)}\int_{T_\nu}(g(x)-g(p_\nu))^2 d\mu(p_\nu)\\ &+\sum_{r=0}^{\nu-1}2^{r+2}\frac{\mu(T_0)}{\mu(T_r)\mu(T_{r+1})}\int_{T_r}\int_{T_{r+1}}(g(p_r)-g(p_{r+1}))^2 d\mu(p_r)d\mu(p_{r+1}).\end{aligned}$$

As $(T_r)_{r\geq 0}$ is a decreasing sequence of symplices, for each $r\geq 0$ both p_r and p_{r+1} belong to T_r, so that $|p_r-p_{r+1}|\leq \operatorname{diam} T_r=\frac{\operatorname{diam}\mathcal{K}}{L^{m+rk}}$. Therefore we will not destroy the upper bound if we replace the integral over $\{p_r\in T_r, p_{r+1}\in T_{r+1}\}$ by an integral over $\{p_r\in S, p_{r+1}\in\mathcal{K}, |p_r-p_{r+1}|<\frac{\operatorname{diam}\mathcal{K}}{L^{m+rk}}\}$. Also, as $\mu(T_r)=\frac{1}{M^{m+rk}}$, we get

$$\begin{aligned}\int_{T_0}(g(x)-g(p))^2d\mu(p)&\leq 2M^{\nu k}\int_{T_\nu}(g(x)-g(p))^2d\mu(p)+\\ +\sum_{r=0}^{\nu-1}2^{r+2}M^{m+(2r+1)k}&\int_S\int_{|p-q|<\frac{\operatorname{diam}\mathcal{K}}{L^{m+rk}}}(g(p)-g(q))^2d\mu(p)d\mu(q).\end{aligned}\tag{3.11}$$

Combining (3.9) and (3.11) (recall that $T_0=S$ and $\mu(S)=\frac{1}{M^m}$) we obtain

$$\begin{aligned}\sum_{S\in\mathcal{F}_m}\sum_{x,y\in V(S)}(g(x)-g(y))^2\leq c\cdot\sum_{S\in\mathcal{F}_m}\sum_{x\in V(S)}\Bigg(2M^{\nu k}\int_{T_\nu}(g(x)-g(p))^2d\mu(p)\\ +\sum_{r=0}^{\nu-1}2^{r+2}M^{m+(2r+1)k}\int_S\int_{|p-q|<\frac{\operatorname{diam}\mathcal{K}}{L^{m+rk}}}(g(p)-g(q))^2d\mu(p)d\mu(q)\Bigg).\end{aligned}\tag{3.12}$$

As $x\in V(T_\nu)$, we have $|x-p_\nu|\leq\operatorname{diam}T_\nu=\frac{\operatorname{diam}\mathcal{K}}{L^{m+\nu k}}$. From Theorem 4 g is Hölder continuous with exponent $\frac{d_w-d_f}{2}$, as long as $\frac{d_w-d_f}{2}\in(0,1)$. This is true for all simple nested fractals: $d_w>d_f$ is equivalent to $d_s<2$. On the other hand we always have $d_w-d_f\leq 1$ (otherwise the resolvent density of the Brownian motion, being Hölder continuous with exponent d_w-d_f, would be a constant function). Therefore, in view of $L^{d_f}=M$,

$$\int_{T_\nu}(g(x)-g(p))^2d\mu(p)\leq c\cdot\mu(T_\nu)(\operatorname{diam}T_\nu)^{d_w-d_f}=c\cdot\left(\frac{1}{L^{m+\nu k}}\right)^{d_w}.$$

It follows for each $\nu,k\geq 1$

$$\begin{aligned}\sum_{S\in\mathcal{F}_m}\sum_{x,y\in V(S)}(g(x)-g(y))^2\leq cM^m\sum_{S\in\mathcal{F}_m}\sum_{x\in V(S)}2M^{\nu k}\int_{T_\nu}(g(x)-g(p))^2d\mu(p)+\\ +cM^m\sum_{S\in\mathcal{F}_m}\#[V(S)]\sum_{r=0}^{\nu-1}M^{m+(2r+1)k}2^{r+2}\int_{p\in S}\int_{|p-q|<\frac{\operatorname{diam}\mathcal{K}}{L^{m+rk}}}(g(p)-g(q))^2d\mu(q)d\mu(p)\end{aligned}\tag{3.13}$$

Finally, as $\#\mathcal{F}_m=M^m$, using (3.7), (3.13) and replacing $p\in S$ by $p\in\mathcal{K}$ in the last double integral above, we have

$$\begin{aligned}&\left(\tfrac{L^{d_w}}{L^{d_f}}\right)^m\widetilde{\mathcal{E}}^{(m)}(g,g)\leq\\ &\leq c\cdot\tfrac{L^{md_w}}{M^m}\left(\#\mathcal{F}_m\cdot M^m\left(\tfrac{1}{L^{m+\nu k}}\right)^{d_w}+M^m\sum_{r=0}^{\nu-1}M^{(r+1)k}M^{m+rk}2^{r+2}\frac{(b_{m+rk}(g))^2}{L^{(m+rk)(d_w+d_f)}}\right)\leq\\ &\leq c\cdot\left(\tfrac{M^m}{L^{\nu kd_w}}+\|(b_m(g))\|_\infty^2\sum_{r=0}^{\nu-1}\left(\frac{2M^k}{L^{kd_w}}\right)^r\right).\end{aligned}$$

As $\frac{M^k}{L^{kd_w}}=(L^{d_w-d_f})^k$ and $d_w>d_f$, we can choose k so big that $\left(\frac{2M^k}{L^{kd_w}}\right)<\frac{1}{2}$ and the resulting geometric series is convergent. For this choice of k we have, for all $m,\nu\geq 1$:

$$\left(\frac{L^{d_w}}{L^{d_f}}\right)^m\widetilde{\mathcal{E}}^{(m)}(g,g)\leq c\cdot\left(\frac{M^m}{L^{\nu kd_w}}+\|(b_m(g))\|_\infty^2\right).$$

The constant appearing in this estimate depends on the fractal only. Therefore, for fixed m, we can pass to infinity with ν, getting that

$$\left(\frac{L^{d_w}}{L^{d_f}}\right)^m \widetilde{\mathcal{E}}^{(m)}(g,g) \leq c \cdot \|(b_m(g))\|_\infty^2,$$

and finally

$$\mathcal{E}(g,g) \leq c \cdot \|(b_m(g))\|_\infty^2.$$

The proof is complete. □

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Katarzyna Pietruska-Pałuba, Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

E-mail address: kppmimuw.edu.pl