ALGEBRAIC SHIFTING INCREASES RELATIVE HOMOLOGY

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ABSTRACT. We show that algebraically shifting a pair of simplicial complexes weakly increases their relative homology Betti numbers in every dimension.

More precisely, let $\Delta(K)$ denote the algebraically shifted complex of simplicial complex K, and let $\beta_j(K,L) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K,L;\boldsymbol{k})$ be the dimension of the jth reduced relative homology group over a field $\boldsymbol{k}$ of a pair of simplicial complexes $L \subseteq K$. Then $\beta_j(K,L) \leq \beta_j(\Delta(K),\Delta(L))$ for all j.

The theorem is motivated by somewhat similar results about Gröbner bases and generic initial ideals. Parts of the proof use Gröbner basis techniques.

1. INTRODUCTION

Algebraic shifting is a remarkable procedure that finds, for any simplicial complex K, a shifted (and hence combinatorially simpler) simplicial complex $\Delta(K)$ with many of the same properties as K. For instance, the f-vector and homology Betti numbers are preserved; Björner and Kalai [BK1] used this fact to characterize the f-vectors and Betti numbers of simplicial complexes.

However, the situation for pairs of complexes and relative homology is different. In a simple example on three vertices (Example 3.1), algebraically shifting a pair of complexes increases their relative homology in dimensions 0 and 1. Upon seeing this one example, Keith Pardue (private communication) conjectured that algebraic shifting always weakly increases relative homology in every dimension. Our main result (Theorem 5.2) is that this conjecture is true.

Pardue's conjecture was grounded in more than just this one simple example. Algebraic shifting, which takes place in exterior (anti-commutative) algebra, is similar to using Gröbner bases and generic initial ideals in commutative algebra (see Section 3). Quantities such as free resolution Betti numbers weakly increase upon taking generic initial ideals (see, *e.g.*, [Hu1, Hu2, Bi], and Section 3). Pardue's insight was that these results would carry over to algebraic shifting.

It would be ideal, then, to prove his conjecture by translating the algebraic shifting problem to a generic initial ideal problem, and then invoking the existing results. However, this approach has been unsuccessful, so far. The proof here, while motivated at points by Gröbner basis ideas (see Lemma 4.1), instead directly refines Björner and Kalai's correspondence between the homology of the original complexes and the combinatorics of the algebraically shifted complexes. The hope is that this result will serve as further evidence of the deeper connection between algebraic shifting and generic initial ideals.

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Background and notation on simplicial complexes, including homology, shifted complexes, and near-cones is in Section 2. Algebraic shifting is reviewed and compared to generic initial ideals in Section 3. In Section 4, we use Gröbner basis ideas to define a nice basis of a space associated with a pair of complexes (K, L), and then use this basis to compare key components of the homology groups of (K, L) and $(\Delta(K), \Delta(L))$. We prove our main result (Theorem 5.2) in Section 5.

2. Simplicial complexes

For basic definitions of simplicial complexes and their homology and relative homology, see, *e.g.*, [Mu, Chapter 1] or [St2, Section 0.3]. We allow the empty simplicial complex $\emptyset$ consisting of no faces; all other complexes must contain the empty set as a (-1)-dimensional face. We also allow the complex $\{\emptyset\}$ consisting of only the empty face, but we do distinguish between the two complexes $\emptyset$ and $\{\emptyset\}$. Let K_j denote the set of j-dimensional faces of a simplicial complex K. The f**-vector** of K is the sequence $(f_0, \ldots, f_{d-1})$, where $f_j = f_j(K) = |K_j|$ and $d - 1 = \dim K$. The same notion of f-vector will apply in this paper to every finite collection of sets.

Let $\boldsymbol{k}$ be a field, fixed throughout the paper. The jth **Betti number** of a simplicial complex K is $\beta_j = \beta_j(K) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K)$, where $\widetilde{H}_j(K)$ is the jth reduced homology group of K (with respect to $\boldsymbol{k}$). Similarly, the jth **relative Betti number** of a pair of simplicial complexes $L \subseteq K$ is $\beta_j = \beta_j(K, L) = \dim_{\boldsymbol{k}} \widetilde{H}_j(K, L)$, where $\widetilde{H}_j(K, L)$ is the jth reduced **relative homology group** of the pair (K, L) (with respect to $\boldsymbol{k}$).

"Reduced" homology means precisely to treat the empty set as a face of any non-empty complex, so β_0 is one less than the number of connected components of Δ, and hence one less than the "unreduced" β_0. Furthermore, $\beta_{-1} = 0$, unless $\Delta = \{\emptyset\}$, in which case $\beta_{-1} = 1$. Reduced relative homology, which also treats the empty set as a face of any non-empty complex, is the same as unreduced relative homology, except that $\beta_{-1}(\{\emptyset\}, \emptyset) = 1$; for any other pair of complexes, $\beta_{-1} = 0$.

Definition. If $S = \{s_1 < \cdots < s_j\}$ and $T = \{t_1 < \cdots < t_j\}$ are j-subsets of integers, then:

- $S \leq_P T$ under the standard **partial order** if $s_p \leq t_p$ for all p; and
- $S <_L T$ under the **lexicographic order** if there is a q such that $s_q < t_q$ and $s_p = t_p$ for $p < q$.

Lexicographic order is a total order which refines the partial order.

Definition. A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ is **shifted** if the set of j-dimensional faces of Δ is shifted for every j.

Björner and Kalai showed in [BK1] that shifted complexes are near-cones, which we now define.

Definition. A **near-cone** with **apex** v_0 is a simplicial complex Δ satisfying the following property: For all $F \in \Delta$, if $v_0 \notin F$ and $w \in F$, then

$$(F - \{w\}) \cup \{v_0\} \in \Delta.$$

For Δ a near-cone with apex v_0, let $B(\Delta) = \{F \in \Delta\colon F \cup \{v_0\} \notin \Delta\}$ and $\Delta' = \{F \in \Delta\colon v_0 \notin F, F \cup \{v_0\} \in \Delta\}$; then

$$\Delta = (v_0 * \Delta') \,\dot\cup\, B(\Delta),$$

where $*$ denotes topological join (so $v_0 * \Delta' = \Delta' \,\dot\cup\, \{\{v_0\} \,\dot\cup\, F\colon F \in \Delta'\}$). Both Δ' and $\Delta' \,\dot\cup\, B(\Delta)$ are subcomplexes of Δ. If $B(\Delta) = \emptyset$, then Δ is simply a **cone**.

Note, in particular, that $\emptyset$ and $\{\emptyset\}$ are near-cones (the condition in the definition is vacuous in this case) and that $\emptyset = v_0 * \emptyset$ and $\{\emptyset\} = (v_0 * \emptyset) \,\dot\cup\, \{\emptyset\}$. If Δ is a near-cone with apex v_0, then v_0 is one of the vertices of Δ, unless $\Delta = \emptyset$ or $\{\emptyset\}$.

It is not hard to see that shifted simplicial complexes are near-cones with apex 1.

Every $F \in B(\Delta)$ is maximal in Δ, so the collection of faces in $B(\Delta)$ forms an antichain. Further, $f(B(\Delta)) = \beta(\Delta)$, which follows by contracting $v_0 * \Delta'$ to v_0, leaving a sphere for every face in $B(\Delta)$ [BK1, Theorem 4.3]. In other words, if Δ is a near-cone with apex v_0, then

$$\beta_j(\Delta) = |\{F \in \Delta_j\colon v_0 \notin F,\ v_0 \,\dot\cup\, F \notin \Delta\}|. \tag{1}$$

This observation is generalized by [Du, Lemma 8]: If $\Gamma \subseteq \Delta$ is a pair of near-cones with common apex v_0, then

$$\begin{aligned}\beta_j(\Delta, \Gamma) = {} & |\{F \in (\Delta - \Gamma)_j\colon v_0 \notin F,\ v_0 \,\dot\cup\, F \notin \Delta\}| \\ & + |\{G \in (\Delta - \Gamma)_j\colon v_0 \in G,\ G - \{v_0\} \in \Gamma\}|.\end{aligned} \tag{2}$$

In light of the formulation of the homology of near-cones that equation (1) gives, equation (2) is approximately the near-cone equivalent of using the long exact sequence (e.g., [Mu, Theorem 23.3])

$$\cdots \to \widetilde{H}_j(L) \xrightarrow{i_*} \widetilde{H}_j(K) \xrightarrow{\pi_*} \widetilde{H}_j(K, L) \xrightarrow{\partial_*} \widetilde{H}_{j-1}(L) \to \cdots \tag{3}$$

to compute

$$\beta_j(K, L) = \dim(\mathrm{im}(\pi_*)_j) + \dim(\mathrm{im}(\partial_*)_j)$$

for an arbitrary pair (K, L).

3. Algebraic shifting

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with the same f-vector and Betti numbers. It also preserves many algebraic properties of the original complex. Algebraic shifting was introduced by Kalai in [Ka1]; our exposition is summarized from [BK1] and included for completeness (see also [BK2, Ka2]). We start with the exterior face ring.

Definition. Let K be a $(d-1)$-dimensional simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(\boldsymbol{k}V)$ denote the exterior algebra of the vector space $\boldsymbol{k}V$; it has a $\boldsymbol{k}$-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $S = \{e_{i_1} < \cdots < e_{i_j}\} \subseteq V$ (and $e_\emptyset = 1$). Note that $\Lambda(\boldsymbol{k}V) = \oplus_{j=0}^{n} \Lambda^j(\boldsymbol{k}V)$ is a graded $\boldsymbol{k}$-algebra, and that $\Lambda^j(\boldsymbol{k}V)$ has basis $\{e_S\colon |S| = j\}$. Let $(I_K)_j$ be the subspace of $\Lambda^{j+1}(\boldsymbol{k}V)$ generated by the basis $\{e_S\colon |S| = j+1,\ S \notin K\}$. Then $I_K := \oplus_{j=-1}^{d-1} (I_K)_j$ is the homogeneous graded ideal of $\Lambda(\boldsymbol{k}V)$ generated by

$\{e_S\colon S \notin K\}$. Let $\Lambda_j[K] := \Lambda^{j+1}(\boldsymbol{k}V)/(I_K)_j$. Then the graded quotient algebra $\Lambda[K] := \oplus_{j=-1}^{d-1}\Lambda_j[K] = \Lambda(\boldsymbol{k}V)/I_K$ is called the **exterior face ring** of K (over $\boldsymbol{k}$).

The exterior face ring is the exterior algebra analogue to the Stanley-Reisner face ring of a simplicial complex [St2]. For $x \in \boldsymbol{k}V$, let $\widetilde{x}$ denote the image of x in $\Lambda[K]$.

Definition (Kalai)**.** Let $\{f_1, \dots, f_n\}$ be a "generic" basis of $\boldsymbol{k}V$, *i.e.*,

$$f_i = \sum_{j=1}^{n} \alpha_{ij} e_j,$$

where the α_{ij}'s are n^2 transcendentals, algebraically independent over $\boldsymbol{k}$. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_j}$ for $S = \{i_1 < \cdots < i_j\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(K, \boldsymbol{k}) := \{S \subseteq [n]\colon \widetilde{f}_S \notin \operatorname{span}\{\widetilde{f}_R\colon R <_L S\}\}$$

be the **algebraically shifted complex** obtained from K; we will write $\Delta(K)$ instead of $\Delta(K, \boldsymbol{k})$ when the field is understood to be $\boldsymbol{k}$. In other words, the j-subsets of $\Delta(K)$ can be chosen by listing all the j-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_K and with respect to the f-basis.

The algebraically shifted complex $\Delta(K)$ is (as its name suggests) shifted, and is independent of the numbering of the vertices of K [BK1, Theorem 3.1].

It is easy to see that algebraic shifting preserves the f-vector, *i.e.*, $f_j(K) = f_j(\Delta(K))$. Björner and Kalai [BK1] showed that algebraic shifting also preserves Betti numbers, *i.e.*, $\beta_j(K) = \beta_j(\Delta(K))$. The reason lies in the relation between algebraic shifting and coboundaries. Define the **weighted coboundary operator** $\delta\colon \Lambda[K] \to \Lambda[K]$ by $\delta(x) = \widetilde{f}_1 \wedge x$, so

$$\delta(\widetilde{e}_S) = \widetilde{f}_1 \wedge \widetilde{e}_S = \sum_{j=1}^{n} \alpha_{1j}\widetilde{e}_j \wedge \widetilde{e}_S = \sum_{\substack{j \notin S\\ S\cup\{j\}\in K}} \pm\alpha_{1j}\widetilde{e}_{S\cup\{j\}}$$

(hence the name weighted coboundary operator). Betti numbers may be computed using this δ, *i.e.*, $\beta_j(K) = \dim_{\boldsymbol{k}}(\ker\delta)_j/(\operatorname{im}\delta)_j$ [BK1, pp. 289–290]. Furthermore, the action of δ on many members of the $\widetilde{f}$-basis is easy to describe: $\delta(\widetilde{f}_F)$ equals $\widetilde{f}_{1\dot\cup F}$ if $1 \notin F$ and $1 \dot\cup F \in \Delta(K)$, but is zero if $1 \in F$ (the third case, when $1 \notin F$, but $1 \dot\cup F \notin \Delta(K)$, is harder, and we shall not need it).

What about relative homology? First, we note a result of Kalai's [Ka2, Theorem 2.2] that if $L \subseteq K$ is a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$. For every pair (K, L), we may then consider the pair $(\Delta(K), \Delta(L))$. In contrast to the single complex case, however, the homology of (K, L) and $(\Delta(K), \Delta(L))$ need not coincide, as the following example shows.

Example 3.1. Let K be the simplicial complex on vertices $\{1, 2, 3\}$ whose maximal faces are $\{1\}$ and $\{2, 3\}$, and L be the subcomplex consisting of just the two vertices $\{1\}$ and $\{2\}$. The only shifted complex with three vertices and one edge has maximal faces $\{1, 2\}$ and $\{3\}$, so this must be $\Delta(K)$. Furthermore, L is the only simplicial

FIGURE 1. Example 3.1

complex with two vertices and no edges, so $\Delta(L) = L$. (See Figure 1.) But then it is easy to see that $(\Delta(K), \Delta(L))$ has non-trivial relative homology in dimensions 0 and 1, while (K, L) has no non-trivial relative homology.

Thus, the relative Betti numbers of $(\Delta(K), \Delta(L))$ are all at least as large as those of (K, L). Theorem 5.2 shows that this is true for any pair (K, L).

Algebraic shifting is the exterior algebra analogue of generic initial ideals and Gröbner bases in commutative algebra, in the following way. If I_K were instead a monomial ideal of a *polynomial* ring, then the algorithm used to create the list of non-faces of $\Delta(K)$ would instead create a list of monomials generating the generic initial ideal of I_K, denoted $\mathrm{Gin}(I_K)$. For further details of generic initial ideals, see, for instance [Ei, Section 15.9]. For more about the relationship between generic initial ideals and algebraic shifting, see [HT]. For a more general exterior algebra version of Gröbner bases and generic initial ideals, see [AHH, Section 1].

Theorem 5.2 bears some resemblance to results about generic initial ideals (Section 3). For instance, Hulett [Hu1, proof of Lemma 1.24], [Hu2, p. 2338] and Bigatti [Bi, proof of Theorem 3.7] have shown that for any homogeneous ideal I in a polynomial ring, the free resolution Betti numbers of its generic initial ideal $\mathrm{Gin}(I)$ are at least as large as those of I.

4. RELATIVE HOMOLOGY

In order to say anything about $(\Delta(K), \Delta(L))$, we must first consider (K, L). For $Q = K - L$ (the "Q" is for "quotient"), we define $\Delta(Q) = \Delta(K) - \Delta(L)$. This is primarily a combinatorial definition, with the algebra hidden in the computation of $\Delta(K)$ and $\Delta(L)$. We now examine how to interpret $\Delta(Q)$ algebraically. Let

$$\widetilde{Q} = \mathrm{span}\{\widetilde{e}_F \colon F \in Q\}.$$

It is not hard to see, then, that we may algebraically shift the subcomplex L using $\Lambda[K]$ instead of $\Lambda[L]$, by modding out by $\widetilde{Q}$ on $\Lambda[K]$ instead of by I_L on $\Lambda[L]$, since $\widetilde{Q} = \widetilde{I_L}$ (see [Du, Section 3]).

Lemmas 4.1 and 4.2 show how $\Delta(Q)$ is related to $\widetilde{Q}$, namely that $\Delta(Q)$ indexes a nice basis of $\widetilde{Q}$; the construction is motivated by Gröbner basis ideas. Then, guided by earlier results about $\widetilde{Q}$ (summarized here as Lemma 4.3), we use this basis of $\widetilde{Q}$ in Lemmas 4.4 and 4.5 to compare key subspaces of $\widetilde{H}(K, L)$ and $\widetilde{H}(\Delta(K), \Delta(L))$.

Lemma 4.1. *If $F \in \Delta(Q)$, then there is a unique linear combination $\sum_{\substack{G <_L F \\ G \in \Delta(L)}} a_G \widetilde{f}_G$, such that $\widetilde{f}_F - \sum_{\substack{G <_L F \\ G \in \Delta(L)}} a_G \widetilde{f}_G \in \widetilde{Q}$.*

Proof. Since $F \in \Delta(Q)$, and hence $F \notin \Delta(L)$, we have

$$\widetilde{f}_F - \sum_{G <_L F} a_G \widetilde{f}_G \in I_L = \widetilde{Q} \tag{4}$$

for some a_G. We may iterate this process on the $\widetilde{f}_G$'s for which $G \notin \Delta(L)$, replacing *them* by lexicographically earlier linear combinations that are equal modulo $I_L = \widetilde{Q}$ until every G in equation (4) is in $\Delta(L)$. This eventually terminates, since lexicographic order is a total order. (In Gröbner basis theory, this procedure is known as finding the normal form [AL, Definition 2.1.3].)

To show these a_G are unique, assume that also

$$\widetilde{f}_F - \sum_{\substack{G <_L F \\ G \in \Delta(L)}} b_G \widetilde{f}_G \in \widetilde{Q}. \tag{5}$$

Then by subtracting equation (5) from equation (4), we get

$$\sum_{\substack{G <_L F \\ G \in \Delta(L)}} (b_G - a_G) \widetilde{f}_G \in \widetilde{Q}. \tag{6}$$

If any $b_G - a_G$ in equation (6) is non-zero, let G_0 index the lexicographically last of these; then

$$(a_{G_0} - b_{G_0}) \widetilde{f}_{G_0} - \sum_{\substack{G <_L G_0 \\ G \in \Delta(L)}} (b_G - a_G) \widetilde{f}_G \in \widetilde{Q},$$

and

$$\widetilde{f}_{G_0} - \sum_{\substack{G <_L G_0 \\ G \in \Delta(L)}} \left(\frac{b_G - a_G}{a_{G_0} - b_{G_0}} \right) \widetilde{f}_G \in \widetilde{Q},$$

which contradicts $G_0 \in \Delta(L)$. □

Definition. By Lemma 4.1, we may define, for any $F \in \Delta(Q)$, $\widetilde{\gamma}_F$ supported on $\Delta(L)$ such that $\widetilde{f}_F - \widetilde{\gamma}_F \in \widetilde{Q}$.

Lemma 4.2. *$\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q)\}$ is a basis of $\widetilde{Q}$.*

Proof. We first show that $\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q)\}$ is linearly independent. Assume otherwise;

$$\sum_{F \in \Delta(Q)} b_F (\widetilde{f}_F - \widetilde{\gamma}_F) = 0, \tag{7}$$

where $b_{F_0} \neq 0$ for some $F_0 \in \Delta(Q)$. When expanding the sum on the left-hand side of equation (7) in the $\{\widetilde{f}_F \colon F \in \Delta(K)\}$ basis, the coefficient of $\widetilde{f}_{F_0}$ will be $b_{F_0} \neq 0$, since the $\widetilde{\gamma}_F$ are all supported on $\Delta(L)$, and so cannot cancel $\widetilde{f}_{F_0}$. So $\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q)\}$ is a set of $|\Delta(Q)|$ linearly independent vectors in $\widetilde{Q}$.

On the other hand, $\widetilde{Q}$ is a $|K - L| = |\Delta(Q)|$-dimensional vector space, so $\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q)\}$ must be a basis. □

Now we see how $\widetilde{Q}$ can help compute homology and relative homology. We adopt the shorthand

$$\delta^{-1}\widetilde{Q} = \{\widetilde{x} \in \Lambda[K] \colon \delta\widetilde{x} \in \widetilde{Q}\}.$$

Lemma 4.3. *For any pair of simplicial complexes $L \subseteq K$,*

(a) $\beta_j(L) = \dim((\delta^{-1}\widetilde{Q})/(\operatorname{im}\delta + \widetilde{Q}))_j$*; and*

(b) $\beta_j(K, L) = \dim((\ker\delta \cap \widetilde{Q})/(\delta\widetilde{Q}))_j$.

Proof. This is [Du, Lemmas 2 and 4], where the notation $\Lambda[\Sigma]$ was used in place of $\widetilde{Q}$. □

Lemma 4.3(b) suggests that in order to compute $\beta_j(K, L)$, we examine $\ker\delta \cap \widetilde{Q}$ and $\delta\widetilde{Q}$. However, $\operatorname{im}\delta \cap \widetilde{Q}$ turns out to be easier to handle than $\ker\delta \cap \widetilde{Q}$. The next two lemmas compare $\operatorname{im}\delta \cap \widetilde{Q}$ and $\delta\widetilde{Q}$ to subspaces of $\widetilde{Q}$ indexed by combinatorially defined sets of $\Delta(Q)$. These two comparisons will combine to prove the key inequality in the proof of Theorem 5.2.

Lemma 4.4. $\operatorname{im}\delta \cap \widetilde{Q} \subseteq \operatorname{span}\{\widetilde{f}_{1\dot{\cup}F} - \widetilde{\gamma}_{1\dot{\cup}F} \colon 1 \notin F,\ 1 \mathbin{\dot{\cup}} F \in \Delta(Q)\}$.

Proof. Let $\widetilde{x} \in \operatorname{im}\delta \cap \widetilde{Q}$. By Lemma 4.2, we can write

$$\widetilde{x} = \sum_{G \in \Delta(Q)} a_G(\widetilde{f}_G - \widetilde{\gamma}_G) \tag{8}$$

uniquely, since $\widetilde{x} \in \widetilde{Q}$. Similarly, by [BK1, equation (3.5)], we can also write

$$\widetilde{x} = \sum_{\substack{1 \notin F \\ 1\dot{\cup}F \in \Delta(K)}} b_F \widetilde{f}_{1\dot{\cup}F} \tag{9}$$

uniquely, since $\widetilde{x} \in \operatorname{im}\delta$. Now, by definition, the support of $\widetilde{\gamma}_G$ is entirely on $\Delta(L)$. Of course, the support of $\widetilde{x}$ in $\Delta(Q)$ must be the same in equations (8) and (9), so we must be able to write every $G \in \Delta(Q)$ such that $a_G \neq 0$ as $G = 1 \mathbin{\dot{\cup}} F$ for some F. Therefore, equation (8) can be rewritten as

$$\widetilde{x} = \sum_{\substack{1 \notin F \\ 1\dot{\cup}F \in \Delta(Q)}} a_{1\dot{\cup}F}(\widetilde{f}_{1\dot{\cup}F} - \widetilde{\gamma}_{1\dot{\cup}F}),$$

implying the lemma. □

Lemma 4.5. *There is a subspace $\widetilde{Q'}$ of $\widetilde{Q}$ such that*

$$\dim\delta\widetilde{Q'} = |\{F \in \Delta(Q) \colon 1 \notin F,\ 1 \mathbin{\dot{\cup}} F \in \Delta(Q)\}|.$$

Proof. Let $\widetilde{Q'} = \operatorname{span}\{\widetilde{f}_F - \widetilde{\gamma}_F \colon F \in \Delta(Q),\ 1 \notin F,\ 1 \mathbin{\dot{\cup}} F \in \Delta(Q)\}$. First note that by definition of $\widetilde{\gamma}_F$ (and $F \in \Delta(Q)$), each $\widetilde{f}_F - \widetilde{\gamma}_F \in \widetilde{Q}$, so $\widetilde{Q'}$ is a subspace of $\widetilde{Q}$. Clearly, we only need to show that $\{\delta(\widetilde{f}_F - \widetilde{\gamma}_F) \colon F \in \Delta(Q),\ 1 \notin F,\ 1 \mathbin{\dot{\cup}} F \in \Delta(Q)\}$ is linearly independent.

By definition of $\widetilde{\gamma}_F$ and Lemma 4.1, we may write each

$$\widetilde{f}_F - \widetilde{\gamma}_F = \widetilde{f}_F - \sum_{\substack{G <_L F \\ G \in \Delta(L)}} b_{F,G} \widetilde{f}_G$$

for some $b_{F,G}$'s. Furthermore, we are assuming $1 \dot\cup F \in \Delta(Q)$ for each F, so $1 \dot\cup F \in \Delta(K)$, and thus $\delta \widetilde{f}_F = \widetilde{f}_{1 \dot\cup F}$. For each $G <_L F$, if $1 \in G$, then $\delta \widetilde{f}_G = 0$; otherwise $1 \dot\cup G <_L 1 \dot\cup F \in \Delta(K)$, so $1 \dot\cup G \in \Delta(K)$, and $\delta \widetilde{f}_G = \widetilde{f}_{1 \dot\cup G}$. Therefore

$$\delta(\widetilde{f}_F - \widetilde{\gamma}_F) = \widetilde{f}_{1 \dot\cup F} - \sum_{\substack{G <_L F \\ 1 \notin G,\ G \in \Delta(L)}} b_{F,G} \widetilde{f}_{1 \dot\cup G}.$$

To show that $\{\delta(\widetilde{f}_F - \widetilde{\gamma}_F) \colon F \in \Delta(Q),\ 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)\}$ is linearly independent, assume

$$0 = \sum_{\substack{F \in \Delta(Q) \\ 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)}} c_F (\widetilde{f}_{1 \dot\cup F} - \sum_{\substack{G <_L F \\ 1 \notin G,\ G \in \Delta(L)}} b_{F,G} \widetilde{f}_{1 \dot\cup G}). \tag{10}$$

Now, $f_{1 \dot\cup G}$'s appearing in equation (10) all satisfy $G \in \Delta(L)$, while all the $\widetilde{f}_{1 \dot\cup F}$'s satisfy $F \in \Delta(Q)$, so there is no cancellation between the $\widetilde{f}_{1 \dot\cup G}$'s and the $\widetilde{f}_{1 \dot\cup F}$'s. But all the $\widetilde{f}_{1 \dot\cup F}$'s are distinct members of the $\{\widetilde{f}_H \colon H \in \Delta(K)\}$ basis, so there is no cancellation among the $\widetilde{f}_{1 \dot\cup F}$'s. Therefore all the c_F's must be zero, and the $\delta(\widetilde{f}_F - \widetilde{\gamma}_F)$'s are linearly independent. □

5. Proof of main theorem

We start with an easy lemma.

Lemma 5.1. *If I, J, and K are subspaces of a vector space and $I \subseteq K$, then*

$$\dim(K/I) = \dim((K \cap J)/(I \cap J)) + \dim((K + J)/(I + J)).$$

Proof. Simply expand the right-hand side as

$$\begin{aligned}(\dim(K \cap J) - \dim(I \cap J)) + ((\dim K + \dim J - \dim(K \cap J)) \\ - (\dim I + \dim J - \dim(I \cap J)))\end{aligned}$$

by the standard vector space argument $\dim(A + B) = \dim A + \dim B - \dim(A \cap B)$, applied twice. This expression then easily simplifies to $\dim K - \dim I = \dim(K/I)$. □

Theorem 5.2. *For any pair of simplicial complexes $L \subseteq K$,*

$$\beta_j(K, L) \leq \beta_j(\Delta(K), \Delta(L))$$

for all j.

Proof. Because $\Delta(K)$ and $\Delta(L)$ are shifted, and therefore near-cones with apex 1, we may use equations (1) and (2) to compute the homology of $\Delta(K)$, $\Delta(L)$, and $(\Delta(K), \Delta(L))$. The sets in these equations overlap in a nice way. In particular, if we let

$$\begin{aligned}
C_{KQ} &= \{F \in \Delta(Q)\colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\},\\
C'_{LQ} &= \{G \in \Delta(Q)\colon 1 \in G,\ G - 1 \in \Delta(L)\},\\
C_{LQ} &= \{F \in \Delta(L)\colon 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)\}, \text{ and}\\
C_{KL} &= \{F \in \Delta(L)\colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\},
\end{aligned}$$

then it is not hard to see, from equation (1), that

$$\begin{aligned}
\beta_j(\Delta(K)) &= |(C_{KQ})_j| + |(C_{KL})_j|,\\
\beta_j(\Delta(L)) &= |(C_{LQ})_j| + |(C_{KL})_j|,
\end{aligned}$$

and, from equation (2), that

$$\beta_j(\Delta(K), \Delta(L)) = |(C_{KQ})_j| + |(C'_{LQ})_j|.$$

(We name these sets "C" because they are combinatorial.) An easy bijection ($F \leftrightarrow 1 \dot\cup F = G$) shows that

$$|(C_{LQ})_j| = |(C'_{LQ})_{j+1}|.$$

Continuing the analogy begun at the end of Section 2 between formulas for homology of near-cones and the long exact sequence (3), C_{KQ} corresponds to $\operatorname{im}\pi_*$, C_{LQ} and C'_{LQ} correspond to $\operatorname{im}\partial_*$, and C_{KL} corresponds to $\operatorname{im} i_*$.

We can find "corresponding" subspaces in $\Lambda[K]$; define

$$\begin{aligned}
A_{KQ} &= (\ker\delta \cap \widetilde{Q})/(\operatorname{im}\delta \cap \widetilde{Q}),\\
A'_{LQ} &= (\operatorname{im}\delta \cap \widetilde{Q})/(\delta\widetilde{Q}),\\
A_{LQ} &= (\delta^{-1}\widetilde{Q})/(\ker\delta + \widetilde{Q}), \text{ and}\\
A_{KL} &= (\ker\delta + \widetilde{Q})/(\operatorname{im}\delta + \widetilde{Q}).
\end{aligned}$$

(We name these spaces "A" because they are algebraic.) Then by Lemma 5.1,

$$\beta_j(K) = \dim(A_{KQ})_j + \dim(A_{KL})_j;$$

by Lemma 4.3,

$$\begin{aligned}
\beta_j(L) &= \dim(A_{LQ})_j + \dim(A_{KL})_j, \text{ and}\\
\beta_j(K, L) &= \dim(A_{KQ})_j + \dim(A'_{LQ})_j;
\end{aligned}$$

and, by [Du, Lemma 5],

$$(A_{LQ})_j \cong (A'_{LQ})_{j+1}.$$

We will show how the dimension of each A subspace compares with the cardinality of the corresponding C set with the same subscript. Because algebraic shifting preserves homology,

$$\dim(A_{LQ})_j + \dim(A_{KL})_j = \beta_j(L) = \beta_j(\Delta(L)) = |(C_{LQ})_j| + |(C_{KL})_j| \tag{11}$$

and

$$\dim(A_{KQ})_j + \dim(A_{KL})_j = \beta_j(K) = \beta_j(\Delta(K)) = |(C_{KQ})_j| + |(C_{KL})_j|. \tag{12}$$

By Lemma 4.4,

$$\dim(\operatorname{im}\delta \cap \widetilde{Q})_{j+1} \leq |\{F \in \Delta(K)_j \colon 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)\}|,$$

and by Lemma 4.5,

$$\begin{aligned}\dim(\delta\widetilde{Q})_{j+1} &\geq \dim(\delta\widetilde{Q'})_{j+1}\\ &= |\{F \in \Delta(Q)_j \colon 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)\}|.\end{aligned}$$

(The index shift, of $j+1$ to j, in the above inequalities arises because the $(j+1)$-dimensional basis elements for $\operatorname{im}\delta \cap \widetilde{Q}$ and $\delta\widetilde{Q'}$ are the $(j+1)$-dimensional elements $\widetilde{f}_{1\dot\cup F} - \widetilde{\gamma}_{1\dot\cup F}$ and $\delta(\widetilde{f}_F - \widetilde{\gamma}_F)$, respectively, each of which is $(j+1)$-dimensional precisely when F is j-dimensional.) Since $\Delta(L)$ is the complement of $\Delta(Q)$ with respect to $\Delta(K)$, then,

$$\begin{aligned}\dim(A'_{LQ})_{j+1} &= \dim((\operatorname{im}\delta \cap \widetilde{Q})/(\delta\widetilde{Q}))_{j+1} = \dim(\operatorname{im}\delta \cap \widetilde{Q})_{j+1} - \dim(\delta\widetilde{Q})_{j+1}\\ &\leq |\{F \in \Delta(L)_j \colon 1 \notin F,\ 1 \dot\cup F \in \Delta(Q)\}|\\ &= |(C_{LQ})_j|,\end{aligned}$$

and so

$$\dim(A_{LQ})_j = \dim(A'_{LQ})_{j+1} \leq |(C_{LQ})_j| = |(C'_{LQ})_{j+1}|$$

Equation (11) then implies

$$\dim(A_{KL})_j \geq |(C_{KL})_j|,$$

so by equation (12),

$$\dim(A_{KQ})_j \leq |(C_{KQ})_j|,$$

and so finally

$$\beta_j(K, L) = \dim(A_{KQ})_j + \dim(A'_{LQ})_j \leq |(C_{KQ})_j| + |(C'_{LQ})_j| = \beta_j(\Delta(K), \Delta(L)).$$

□

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