ISOPERIMETRIC CONDITIONS AND DIFFUSIONS IN RIEMANNIAN MANIFOLDS

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May 28, 1998

Abstract. We study diffusions in Riemannian manifolds and properties of their exit time moments from smoothly bounded domains with compact closure. For any smoothly bounded domain with compact closure, Ω, and for each positive integer k, we characterize the kth exit time moment of Brownian motion, averaged over the domain Ω with respect to the metric density, using a variational quotient. We prove that for Riemannian manifolds satisfying an isoperimetric condition, the averaged k-th exit time moment of Brownian motion from domains of a fixed volume is bounded above. For the case of Euclidean space, we establish similar boundedness properties for a larger class of diffusions.

1. Introduction

In this note we study diffusions in a Riemannian manifold M and properties of their corresponding exit times from smoothly bounded, connected, open domains in M with compact closure. Our results relate boundedness properties of exit time moments and averaged exit time moments to isoperimetric properties of the underlying manifold. To formulate our results, we fix some notation.

Given a Riemannian manifold (M, g) we will denote by X_t a general diffusion on M with corresponding infinitesimal generator L. We let P_x be the probability associated to starting the process X_t at the point x and let E_x denote the corresponding expectation.

Our primary interest involves diffusions with associated infinitesimal generator arising as the Laplace operator with respect to some Riemannian metric on M. Given a Riemannian metric, g, we denote by Δ_g the Laplace operator with respect to g.

Let $\mathcal{D}$ be the space of smoothly bounded, connected, open domains in M with compact closure. Given $\Omega \in \mathcal{D}$, let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin \Omega\}$ be the first exit time of X_t from Ω.

Research at MSRI is supported in part by NSF grant DMS-9701755. The author was partially supported by a DSR grant from the University of South Florida

For k any positive integer, we will study properties of the maps $\mathcal{S}_k, \mathcal{A}_k : \mathcal{D} \to \mathbb{R}$ defined by

$$\mathcal{S}_k(\Omega) = \sup_{x\in\Omega} E_x[\tau^k] \tag{1.1}$$

$$\mathcal{A}_k(\Omega) = \int_\Omega E_x[\tau^k] dg \tag{1.2}$$

where dg is the volume form induced by g on M. Our first theorem provides a description of the domain functionals $\mathcal{A}_k$ in terms of variational quotients:

Theorem 1.1. *Let M be a Riemannian manifold with metric g. Let X_t be Brownian motion on M with respect to the metric g. Let Ω be a smoothly bounded open domain in M with the closure of Ω, denoted $\bar{\Omega}$, compact. Let $\mathcal{A}_k$ be defined as in (1.2). Let $\mathcal{F}_k$ be defined by*

$$\mathcal{F}_k = \left\{ f \in C^\infty(\bar{\Omega}); \int_\Omega f dx \neq 0,\ f = \Delta_g f = \cdots = \Delta_g^{k-1} f = 0 \text{ on } \partial\Omega \right\}. \tag{1.3}$$

Then, for k odd,

$$\mathcal{A}_k^{-1} = \inf_{f\in\mathcal{F}_k} \frac{\int_\Omega |\nabla \Delta_g^{[\![\frac{k}{2}]\!]} f|^2 dg}{\left(\int_\Omega f dg\right)^2} \tag{1.4}$$

and for k even,

$$\mathcal{A}_k^{-1} = \inf_{f\in\mathcal{F}_k} \frac{\int_\Omega |\Delta_g^{[\![\frac{k}{2}]\!]} f|^2 dg}{\left(\int_\Omega f dg\right)^2}. \tag{1.5}$$

Variational quotients such as those appearing in the statement of Theorem (1.1) have found a number of applications in both physics and geometry (a very nice survey can be found in [O1]). For example; when M is two dimensional Euclidean space and $k = 1$, $(\mathcal{A}_k(\Omega))^{-1}$ is perhaps better known as the *torsional rigidity* associated to a beam with cross-section Ω. In 1856, St-Venant conjectured that among domains of a given fixed area, torsional rigidity is maximized by a disk. The first complete proof of St-Venant's conjecture was given by Polya in 1948. Polya's approach to resolving the conjecture involved the study of the behavior of the variational quotient

$$\frac{\int_\Omega |\nabla f|^2 dx}{\left(\int_\Omega f dx\right)^2}$$

under (Schwarz) symmetrization. This approach had been used by Faber [F1] and Krahn [K1] to resolve the conjecture of Rayleigh concerning the behavior of the first Dirichlet eigenvalue: *Among smoothly bounded domains of fixed volume* *in Euclidean space, the principle Dirichlet eigenvalue is minimized by a ball.* To resolve this conjecture, one studies the behavior of the variational quotient

$$\frac{\int_\Omega |\nabla f|^2 dx}{\int_\Omega f^2 dx}$$

under (Schwarz) symmetrization.

The results of Faber and Krahn establish more than the conjecture of Rayleigh: they relate the size of the principle Dirichlet eigenvalue on domains in a Riemannian manifold to the size of the principle Dirichlet eigenvalue in a constant curvature comparison space, provided the Riemannian manifold satisfies an isoperimetric condition (cf Definition (2.1) below). Faber and Krahn's result can be concisely stated: *If (M, g) is a Riemannian manifold satisfying an isoperimetric condition with constant curvature space form M_κ, then Volume$_g(\Omega) = v_0$ implies that $\lambda_1(\Omega) \geq \lambda_1(B)$ where B is a geodesic ball in M_κ of volume v_0 and λ_1 is the first Dirichlet eigenvalue for the Laplace operator induced by the corresponding metric.* Our second theorem establishes a similar result for the averaged moments $\mathcal{A}_k(\Omega)$ and for the extremized moments $\mathcal{S}_k(\Omega)$.

Theorem 1.2. *Let (M, g) be a Riemannian manifold satisfying an isoperimetric condition with constant curvature comparison space form M_κ. Then for each positive integer k, Volume$_g(\Omega) = v_0$ implies that $\mathcal{A}_k(\Omega) \leq \mathcal{A}_k(B)$ where B is a geodesic ball in M_κ of volume v_0 and $\mathcal{A}_k$ is the kth averaged exit time moment for Brownian motion the corresponding metric. The same result holds for the functions $\mathcal{S}_k$ defined by* (1.1).

As an immediate corollary of Theorem (1.2), we get a result concerning the uniform boundedness of the maps $\mathcal{A}_k$: If $\mathcal{D}_v = \{\Omega \in \mathcal{D} : \mathrm{Vol}_g(\Omega) = v\}$, then for each positive integer k the map $\mathcal{A}_k : \mathcal{D}_v \to \mathbb{R}$ is bounded. This result also holds for the maps $\mathcal{S}_k$ (cf Corollary 4.1).

In the case where M is Euclidean space $\mathbb{R}^n$, we use the techniques employed to prove Theorem 1.2 to obtain a uniform boundedness result for more general diffusions (cf Theorem 5.1).

This paper is organized as follows. In section 2 we recall basic facts and definitions which will be used throughout. In particular, we provide a representation for the functions, $\mathcal{S}_k$ and $\mathcal{A}_k$ in terms of a heirarchy of boundary value problems on the underlying domain. We also provide necessary information on Schwarz symmetrization, including proofs of several elementary results which we were unable to locate in the literature. In section 3 we prove Theorem 1.1. In section 4 we prove Theorem 1.2. In section 5 we establish a sufficient condition for a diffusion on Euclidean space to admit uniformly bounded moments and uniformly bounded average moments of all orders, on the space of domains constrained to be of given volume.

Before proceeding a few words concerning background are in order. The literature concerning isoperimetric conditions and their various analytic and geometric implications is vast. Those looking for a nice introduction to the topic are directed to the survey article [O1], to the texts [B] and [C], and to the references therein. Much of the present work is inspired by investigations of the Euclidean case. In particular, the formulae for the variational quotients given in Theorem 1.1 where first discovered for $\mathbb{R}^n$ and will appear in [KM2]. Similarly, an early version of the rearrangement argument which appears in the proof of Theorem 1.2 first appears for the special case of $\mathbb{R}^n$ and $k = 2$ in [KM1]. The author would like to thank Kimberly Kinatader for a number of helpful conversations.

2. Basic results and definitions

Throughout this section, let (M, g) be a Riemannian manifold.

Let $(\mathcal{Y}, \mathcal{B})$ be a measurable space and $\{P_x\}_{x \in M}$ a family of probability measures on $(\mathcal{Y}, \mathcal{B})$. Let $\{X_t\}_{t \geq 0}$ denote a diffusion on M with generator L, a uniformly elliptic operator, and for which $P_x\{X_0 = x\} = 1$, for $x \in M$.

Let $\Omega \subset M$ be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by X_t from Ω by $\tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin \Omega\}$. For each $x \in M$, we will denote the expected value of a random variable Z under the probability measure P_x by $E_x(Z)$.

There is a useful relationship between the solution of a certain Poisson problem on the domain $\Omega \in \mathcal{D}$ and the expected value of the kth power of the first exit time of a particle driven by X_t from Ω starting at $x \in \Omega$. Suppose u_k solves the problem

$$\begin{gathered} L^k u_k + (-1)^{k-1} k! = 0 \text{ on } \Omega \\ u_k = L u_k = \cdots L^{k-1} u_k = 0 \text{ on } \partial\Omega. \end{gathered}$$

Note that u_k can be defined inductively by

$$\begin{aligned} L u_1 + 1 &= 0 \text{ on } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} L u_k + k u_{k-1} &= 0 \text{ on } \Omega \\ u_k &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.2}$$

Using the generalized Dynkin formula [H] (cf also [AK]), we have

$$\begin{aligned} E_x[u_k(X_0)] - E_x[u_k(X_\tau)] = &\sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\ &+ \frac{(-1)^k}{(k-1)!} E_x\left[\int_0^\tau s^{k-1} L^k u_k(X_s) ds\right]. \end{aligned}$$

Using the definition of u_k and τ, we obtain $u_k(x) = E_x\left[\tau^k\right]$.

Suppose that $L = \Delta_g$ is the Laplace operator, $Lf = \operatorname{div}(\nabla f)$. Let u_k be as defined in (2.2). Then, from the above, we have

$$\mathcal{S}_k(\Omega) = \sup_{x\in\Omega} u_k(x) \tag{2.3}$$

$$\mathcal{A}_k(\Omega) = \int_\Omega u_k(x)dg \tag{2.4}$$

where dg is the volume form induced by the metric g.

We now focus on those aspects of geometric measure theory that we will need in the sequel. Let $\Omega \subset M$ be a smoothly bounded, connected, open domain with compact closure. Suppose that $w \in C^0(\bar{\Omega}) \cap C^\infty(\Omega)$ satisfies $w|_{\partial\Omega} = 0$. Let $\mathbf{Reg}(w)$ denote the regular values of w ($\mathbf{Reg}(w)$ is an open set in $\mathbb{R}$).

For $\mu \geq 0$, define

$$\Omega(\mu) = \{x \in \Omega : |w(x)| > \mu\} \tag{2.5}$$

$$\Gamma(\mu) = \{x \in \Omega : |w(x)| = \mu\} = \partial\Omega(\mu). \tag{2.6}$$

Using the metric density, dg, define

$$V(\mu) = \mathrm{Vol}_g(\Omega(\mu)) \tag{2.8}$$

where Vol_g denotes the volume with respect to the density dg. It is well known (cf [F])that $V(\mu)$ is decreasing, has discontinuities only at those t for which $\Gamma(\mu)$ has positive g-volume, and is a smooth function on $\mathbf{Reg}(w)$. For $\mu \in \mathbf{Reg}(w)$, the derivative of $V(\mu)$ is given by the co-area formula:

$$V'(\mu) = -\int_{\Gamma(\mu)} |\nabla w|^{-1} dA_\mu \tag{2.9}$$

where ∇ denotes the gradient operator and dA_μ is the induced Riemannian density on $\Gamma(\mu)$. We will use the co-area formula to study the behavior of solutions to Poisson problems for smoothly bounded domains in a large class of Riemannian manifolds. We begin by recalling a number of standard definitions and constructions.

Let M_κ be a constant curvature space form of curvature κ and dimension n. Recall that for $\kappa > 0$, M_κ is given by the Euclidean sphere of radius κ centered at the origin, that for $\kappa = 0$, M_κ is n-dimensional Euclidean space, and that for $\kappa < 0$, M_κ is hyperbolic n space, which we model using the Poincaré disk. For each value of κ we fix a basepoint $p \in M_\kappa$: for $\kappa > 0$, let p be the south pole, for $\kappa \geq 0$, let p be the origin. For $r \in \mathbb{R}$, $r > 0$, we designate by $\mathbf{B}_\kappa(r)$ the open geodesic ball in M_κ of radius r, centered at $p \in M_\kappa$. Given $x \in M_\kappa$, we will denote by $|x|_\kappa$ the distance from x to $p \in M_\kappa$. If $\Omega \subset M_\kappa$, we will denote by $\mathrm{Vol}_\kappa(\Omega)$ the volume of Ω with respect to the natural density dg_κ.

We will be particularly interested in those manifolds which satisfy an isoperimetric condition:

Definition 2.1. *We say that a Riemannian manifold (M, g) satisfies an isoperimetric condition if there is a constant curvature space form, M_κ, such for all smoothly bounded domains with compact closure, $\Omega \subset M$, we have*

$$Vol_g(\Omega) = v_0 \text{ implies that } Area_g(\partial\Omega) \geq Area_\kappa(\partial B)$$

where B is a geodesic ball in M_κ of volume v_0 and Area is the surface area induced by the corresponding metric. If (M, g) satisfies an isoperimetric condition, we call the associated constant curvature space form, M_κ, the comparison space form.

We note that any constant curvature space form, M_κ, satisfies an isoperimetric condition with comparison space form M_κ.

Extensive work in the case of Euclidean space has provided direction as to how to use the existence of an isoperimetric condition to investigate properties of solutions to various elliptic boundary value problems for domains in $\mathbb{R}^n$. We will restrict our attention to an approach established by Schwarz in his study of the classical isoperimetric inequality (cf [O1], [B], and references therin). The approach associates to a domain $\Omega \subset \mathbb{R}^n$, and a nonnegative function $w : \Omega \to \mathbb{R}$, a rearranged domain, $\Omega^* \subset \mathbb{R}^n$ (a ball centered at the origin), and a symmetric decreasing function, $w^* : \Omega^* \to \mathbb{R}$, which is equimeasurable with w. Extensions of Schwarz symmetrization have been used by a number of authors to study the properties of Riemannian manifolds satisfying isoperimetric conditions (cf [O1], [B], [C]). We recall the definitions and basic properties of Schwarz symmetrization below and follow the methods of Faber and Krahn to obtain results which will be needed to establish Theorem 1.2.

Definition 2.2. *Let (M, g) be a Riemannian manifold satisfying an isoperimetric condition with comparison space form M_κ. Let $\Omega \subset M$ be a bounded, connected open domain. The symmetrization of Ω, denoted $\Omega^* \subset M_\kappa$, is the geodesic ball $\mathbf{B}_\kappa(r)$ satisfying $Vol_g(\Omega) = Vol_\kappa(\mathbf{B}_\kappa(r))$.*

Suppose that $w : \Omega \to \mathbb{R}$ is nonnegative. For $\mu > 0$, let

$$\Omega(\mu) = \{x \in \Omega : w(x) \geq \mu\}.$$

The symmetrization of w, denoted w^*, is the function $w^* : \Omega^* \to \mathbb{R}$ given by

$$w^*(x) = \sup\{\mu : x \in \Omega(\mu)^*\}. \tag{2.10}$$

The following properties of symmetric rearrangement follow easily from the definition:

(1) The rearrangement w^* depends only on the geodesic distance to $p \in M_\kappa$ and w^* is nonincreasing.
(2) If $w_1 \leq w_2$, then $w_1^* \leq w_2^*$.
(3) If w is continuous on $\bar{\Omega}$,

$$\sup_{x \in \Omega} w(x) = \sup_{x \in \Omega^*} w^*(x). \tag{2.11}$$

(4) The functions w and w^* are equimeasurable in the sense that $\mathrm{Vol}_g(\{x \in \Omega : w(x) \geq \mu\}) = \mathrm{Vol}_\kappa(\{x \in \Omega^* : w^*(x) \geq \mu\})$ for all nonnegative real μ. In particular, for $\mu > 0$,

$$\int_{\Omega(\mu)} w dg \leq \int_{\Omega(\mu)^*} w^* dg_\kappa \tag{2.12}$$

where dg is the volume form induced by g and dg_κ is the natural volume form on M_κ.

Suppose that $\Omega \subset M$ and that $w : \Omega \to \mathbb{R}$ is nonnegative. Let $T = \sup\{t : w(x) = t \text{ for some } x \in D\}$ and define a function $r : [0, T] \to \mathbb{R}$ by

$$r(\mu) = \sup\{s \in \mathbb{R}; \mathrm{Vol}_\kappa(\mathbf{B}_\kappa(s)) < V(\mu)\} \tag{2.13}$$

where $V(\mu)$ is as in (2.8). Then r is continuous on $[0, T]$, smooth on $\mathbf{Reg}(w) \cap (0, T)$, and is strictly decreasing. Let R be the inverse function for r. Then, if $t = r(\mu)$ where $V(\mu) = \mathrm{Vol}_\kappa(\mathbf{B}_\kappa(r(\mu)))$, we have

$$R(t) = w^*(x) \text{ for } x \text{ satisfying } |x|_\kappa = r(\mu). \tag{2.14}$$

We will need the following lemma, due to Faber-Krahn (cf [C]), whose proof we include for completeness:

Lemma 2.2. *Suppose that (M, g) satisfies an isoperimetric condition with comparison space M_κ. Let $w : \Omega \to \mathbb{R}$ be as above. Then, for $\mu \in$ $\mathbf{Reg}(w)$,*

$$\int_{\Gamma(\mu)} |\nabla w| dA_\mu \geq -Area_\kappa(\partial\mathbf{B}_\kappa(r(\mu))) \frac{1}{r'(\mu)}. \tag{2.15}$$

Proof. From the definition of r we have

$$\begin{aligned} V'(\mu) &= \frac{d}{d\mu} \mathrm{Vol}_\kappa(\mathbf{B}_\kappa(r(\mu))) \\ &= \mathrm{Area}_\kappa(\partial\mathbf{B}_\kappa(r(\mu))) r'(\mu) \end{aligned}$$

and hence

$$r'(\mu) = \frac{V'(\mu)}{\mathrm{Area}_\kappa(\partial\mathbf{B}_\kappa(r(\mu)))}.$$

Using (2.9), we obtain

$$r'(\mu) = -\frac{1}{\mathrm{Area}_\kappa(\partial\mathbf{B}_\kappa(r(\mu)))} \int_{\Gamma(\mu)} |\nabla w|^{-1} dA_\mu.$$

By Cauchy-Schwarz,

$$-r'(\mu) \int_{\Gamma(\mu)} |\nabla w| dA_\mu \leq \frac{A^2(\mu)}{\mathrm{Area}_\kappa(\partial\mathbf{B}_\kappa(r(\mu)))}. \tag{2.16}$$

The lemma follows from (2.16) and the isoperimetric condition.

The following theorem, due to Szegö for Euclidean space (cf [PS1] and also [B]), will be crucial in the proof of Theorem 1.2. The proof is very similar to the argument given in the Euclidean case in [B].

Theorem 2.1. *Let (M,g) be a Riemannian manifold satisfying an isoperimetric condition with comparison space form M_κ. Let $\Omega \subset M$ be a bounded, connected open domain and suppose that $f : \Omega \to I\!R$ is a positive Lipschitz function. Suppose that u solves*

$$\begin{aligned} \Delta_g u + f = 0 \ \ on\ \Omega \\ u = 0 \ \ on\ \partial\Omega. \end{aligned} \tag{2.17}$$

Let Ω^* be the symmetrization of Ω and let $u^*,\ f^* : \Omega^* \to I\!R$ be the symmetrizations of u and f, respectively. Let v be the solution of the symmetrized problem:

$$\begin{aligned} \Delta_\kappa v + f^* = 0 \ \ on\ \Omega^* \\ v = 0 \ \ on\ \partial\Omega^* \end{aligned} \tag{2.18}$$

where Δ_κ is the Laplace operator on M_κ. Then, $u^*(x) \leq v(x)$ for all $x \in \Omega^*$.

Proof. Let u be as above. Then, for all μ, u is constant on the boundary of $\Omega(\mu)$ and the divergence theorem gives

$$-\int_{\Omega(\mu)} \Delta_g u dg = \int_{\Gamma(\mu)} |\nabla u| dA_\mu. \tag{2.19}$$

Since u satisfies $\Delta_g u = -f$,

$$\begin{aligned} \int_{\Gamma(\mu)} |\nabla u| dA_\mu &= \int_{\Omega(\mu)} f dg \\ &\leq \int_{\Omega(\mu)^*} f^* dg_\kappa \end{aligned}$$

where we have used (2.12). Applying Lemma 2.2 to u, we conclude, using (2.15), that

$$-\frac{1}{r'(\mu)} \leq \frac{\int_{\Omega(\mu)^*} f^* dg_\kappa}{\text{Area}_\kappa(\partial \mathbf{B}_\kappa(r(\mu)))}.$$

We rewrite this as

$$-R'(\mu) \leq \frac{\int_{\Omega(\mu)^*} f^* dg_\kappa}{\text{Area}_\kappa(\partial \mathbf{B}_\kappa(r(\mu)))}. \tag{2.20}$$

Note that $\Omega(\mu)^*$ is a ball in the constant curvature comparison space M_κ and that f^* depends only on the geodesic distance from the center of $\Omega(\mu)^*$. Using (2.18), the Divergence Theorem, and the fact that v depends only on geodesic distance to the center of $\Omega(\mu)^*$, the right hand side of (2.20) can be integrated. If we integrate both sides of the inequality (2.20) with respect to μ from $\mu = T$ and use (2.14), we obtain

$$u^*(x) \leq v(x)$$

which concludes the proof of the proposition.

3. Variational Characterizations and the proof of Theorem (1.1)

Throughout this section let M be a Riemannian manifold with metric g, let Δ_g denote the Laplace operator, let Ω be as above and let $\mathcal{F}_k$ be given as in Theorem 1.1.

Let u_1 be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let ν be the outward pointing unit normal vector to ∂D. By the Divergence Theorem,

$$\int_\Omega (f\Delta_g u_1 - u_1 \Delta_g f) dg = \int_{\partial D} f(\langle \nabla u_1, \nu\rangle - u_1\langle \nabla f, \nu\rangle) dA_g = 0$$

where the pairing is given by the metric g and dA_g denotes the measure induced by g on ∂D. We conclude

$$\int_\Omega f dg = -\int_\Omega u_1 \Delta_g f dg. \tag{3.1}$$

If X is a vectorfield on Ω and $f \in \mathcal{F}_k$, then $\mathrm{div}(fX) = f\mathrm{div}(X) + \langle \nabla f, X\rangle$. By the Divergence Theorem,

$$\int_\Omega \mathrm{div}(fX) dg = \int_{\partial D} f\langle X, \nu\rangle dA_g = 0$$

and we conclude that

$$\int_\Omega f\mathrm{div}(X) dg = -\int_\Omega \langle \nabla f, X\rangle dg.$$

In particular, if u_1 is as defined in (2.1) and $X = \nabla u_1$, then

$$\int_\Omega f dg = \int_\Omega \langle \nabla u_1, \nabla f\rangle dg. \tag{3.2}$$

Lemma 3.1. *Let u_n be as defined by (2.2), let k be a positive integer, and let $f \in \mathcal{F}_k$. If $k = 2n$, then*

$$\int_\Omega f dg = \frac{(-1)^n}{n!}\int_\Omega u_n \Delta_g^n f dg. \tag{3.3}$$

If $k = 2n+1$, then

$$\int_\Omega f dg = \frac{(-1)^n}{(n+1)!}\int_\Omega \langle \nabla u_{n+1}, \nabla \Delta_g^n f\rangle dg \tag{3.4}$$

where the pairing is given by the metric g.

Proof. Suppose that $k = 2n$ and for $0 \le l \le n-1$, define

$$P_l = (\Delta_g^l u_n)(\Delta_g^{n-l} f) - (\Delta_g^{l+1} u_n)(\Delta_g^{n-(l+1)} f). \tag{3.5}$$

Then
$$\sum_{l=0}^{n-1} P_l = u_n \Delta_g^n f - f \Delta_g^n u_n.$$
Let ν be the outward pointing unit normal vector along ∂D. By the Divergence Theorem and the fact that $\Delta_g^l u_n = 0$ on $\partial\Omega$, for $l = 0, \ldots, n-1$,

(3.6)
$$\int_\Omega P_l dg = \int_{\partial D} (\Delta_g^l u_n)\langle \nabla \Delta_g^{n-l-1} f, \nu\rangle - (\Delta_g^{n-(l+1)} f)\langle \nabla \Delta_g^l u_n, \nu\rangle dA_g = 0.$$
Combining (3.5) and (3.6) and using that $\Delta_g^n u_n = (-1)^n n!$, we have established (3.3).

Suppose $k = 2n+1$ and for $0 \le l \le n$, define
$$R_l = (\Delta_g^l u_{n+1})(\Delta_g^{n+1-l} f) - (\Delta_g^{l+1} u_{n+1})(\Delta_g^{n+1-(l+1)} f).$$
Then
$$\sum_{l=0}^{n} R_l = u_{n+1} \Delta_g^{n+1} f - f \Delta_g^{n+1} u_{n+1}.$$
As above, we use the Divergence Theorem to see that
$$\int_\Omega R_l dg = \int_{\partial D} (\Delta_g^l u_{n+1})\langle \nabla \Delta_g^{n-l} f, \nu\rangle - (\Delta_g^{n-l} f)\langle \nabla \Delta_g^l u_{n+1}, \nu\rangle dA_g = 0.$$
Since $L^{n+1} u_{n+1} = (-1)^{n+1}(n+1)!$, we conclude
$$\int_\Omega f dg = \frac{(-1)^{n+1}}{(n+1)!} \int_D u_{n+1} \Delta_g (\Delta_g^n f).$$
By the Divergence Theorem,
$$\begin{aligned}\int_\Omega u_{n+1} \Delta_g(\Delta_g^n f) dg + \int_\Omega \langle \nabla u_{n+1}, \Delta_g^n f\rangle dg &= \int_\Omega \operatorname{div}(u_{n+1} \nabla \Delta_g^n f) dg \\ &= 0.\end{aligned}$$
We conclude that
$$\int_\Omega u_{n+1} \Delta_g(\Delta_g^n f) dg = -\int_\Omega \langle \nabla u_{n+1}, \nabla \Delta_g^n f\rangle dg$$
and we have established (3.4).

Proof of Theorem 1.1. The proof of Theorem 1.1 now procedes along the lines of the argument given in [KM2]. Suppose $k = 2n$ and, for $f \in \mathcal{F}_k$, consider the quotient
$$Q_k(f) = \frac{\left(\int_\Omega f dg\right)^2}{\int_\Omega |\Delta_g^n f|^2 dg}.$$

From (3.3)

$$Q_k(f) = \frac{\left(\frac{1}{n!}\right)^2 \left(\int_\Omega u_n \Delta_g^n f\, dg\right)^2}{\int_\Omega |\Delta_g^n f|^2 dg}.$$

Let $\mathcal{G}_k = \{h \in \mathcal{F}_n : h = \Delta_g^n f \text{ for some } f \in \mathcal{F}_k\}$. Let $\mathcal{H}_k$ be the completion of $\mathcal{G}_k$ in $W^{2,2}$, the Hilbert space of square integrable functions on Ω. If we denote the inner product of h and $w \in W^{2,2}$ by $\langle h, w\rangle$ and by $\|h\|$ the $W^{2,2}$ norm of h, then we can view Q_k as a map $Q_k : \mathcal{G}_k \subset \mathcal{H}_k \to \mathbb{R}$,

$$Q_k(g) = \left(\frac{1}{n!}\right)^2 \left(\frac{\langle u_n, h\rangle}{\|h\|}\right)^2.$$

Clearly, the domain of Q_k can be extended to nonzero elements of $\mathcal{H}_k$ and $Q_k(ch) = Q_k(h)$ for every nonzero scalar c. It follows that Q_k is maximized when $h \in \mathcal{H}_k$ is in the direction of $u_n \in \mathcal{H}_k$. If $h = cu_n$ we have $\Delta_g^n(c' u_{2n}) = h$, and computing $Q_k(cu_n)$ we see that

$$\begin{aligned}\sup_{h \in \mathcal{H}_k} Q_k(h) &= Q_k(cu_n) \\ &= \frac{\left(\int_\Omega u_{2n}\, dg\right)^2}{\int_\Omega \left(\Delta_g^n u_{2n}\, dg\right)^2}\end{aligned}$$

where we have applied (3.3) of Lemma 3.1 to the numerator. Note that $(\Delta_g^n u_{2n})^2 = \frac{(-1)^n}{n!}(2n)! u_n \Delta_g u_{2n}$. Applying Lemma 3.1 to the denominator we obtain

$$\begin{aligned}\sup_{h \in \mathcal{H}_k} Q_k(h) &= \frac{\left(\int_\Omega u_{2n}\, dg\right)^2}{(2n)! \int_\Omega u_{2n}\, dg} \\ &= \frac{1}{k!}\mathcal{A}_k(\Omega)\end{aligned}$$

which establishes (1.3) of Theorem 1.1.

The proof of (1.4) of Theorem 1.1 is similar. Suppose $k = 2n+1$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$\tilde{Q}_k(f) = \frac{\left(\int_\Omega f\, dg\right)^2}{\int_\Omega |\nabla \Delta_g^n f|^2 dg}.$$

From (3.4) of Lemma 3.1,

$$\tilde{Q}_k(f) = \frac{\left(\frac{1}{(n+1)!}\right)^2 \left(\int_\Omega \langle \nabla u_{n+1}, \nabla \Delta_g^n f\rangle dg\right)^2}{\int_\Omega |\nabla \Delta_g^n f|^2 dg}.$$

Let $C^\infty(\bar{\Omega}, \mathrm{IR}^n)$ be the space of smooth vectorfields on $\bar{\Omega}$. Let

$$\tilde{\mathcal{G}}_k = \{X \in C^\infty(\bar{\Omega}, \mathrm{IR}^n) : \text{there is } f \in \mathcal{F}_k, \text{ and } h \in \mathcal{F}_{n+1} \text{ with } h = \Delta_g^n f,\ X = \nabla h\}.$$

Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vectorfields square integrable with respect to the inner product determined by the metric g. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{\mathcal{G}}_k \subset \tilde{\mathcal{H}}_k \to \mathrm{IR}$,

$$\tilde{Q}_k(h) = \left(\frac{1}{(n+1)!}\right)^2 \left(\frac{\langle \nabla u_{n+1}, h\rangle}{\|h\|}\right)^2.$$

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars c, $\tilde{Q}_k(ch) = \tilde{Q}_k(h)$. It follows that $\tilde{Q}_k$ is maximized when when $h = c\nabla u_{n+1}$ where c is some nonzero constant. In particular, the maximizer is a smooth vectorfield h where $h = \nabla\Delta_g^n(cu_{2n+1})$ for some constant c, and we get that $h \in \mathcal{F}_n$. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$\begin{aligned}\sup_{h\in\tilde{\mathcal{H}}_k} \tilde{Q}_k(h) &= \tilde{Q}_k(c\nabla u_{n+1}) \\ &= \frac{\left(\int_\Omega u_{2n+1} dg\right)^2}{\int_\Omega \|\nabla\Delta_g^n u_{2n+1}\|^2 dg}\end{aligned}$$

where we have used (3.4) on the numerator.

Note that $\|\nabla\Delta_g^n u_{2n+1}\|^2 = \frac{(-1)^n}{(n+1)!}(2n+1)!\langle\nabla u_{n+1}, \nabla\Delta_g^n u_{2n+1}\rangle$. Applying (3.4) of Lemma 3.1 to the denominator we obtain

$$\begin{aligned}\sup_{h\in\tilde{\mathcal{H}}_k} \tilde{Q}_k(h) &= \frac{\left(\int_\Omega u_{2n+1} dg\right)^2}{(2n+1)!\int_\Omega u_{2n+1} dg} \\ &= \frac{1}{k!}\mathcal{A}_k(\Omega)\end{aligned}$$

which establishes (1.4) of Theorem 1.1.

4: Isoperimetric-type bounds and the proof of Theorem (1.2)

Proof of Theorem 1.2. For the case $k = 1$, we could proceed along the lines of the classical Faber-Krahn result using Theorem 1.1 (a nice presentation of the Faber-Krahn argument can be found in [C]). Unfortunately, the Faber-Krahn argument breaks down for $k > 1$. As an alternative, we use Theorem 2.1 and induction.

Let (M, g) be a Riemannian manifold satisfying an isoperimetric condition with comparison space form M_κ. Let $\Omega \subset M$ be a smoothly bounded domain with compact closure. Let u_1 be the solution of

$$\begin{aligned}\Delta_g u_1 + 1 &= 0 \text{ on } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega.\end{aligned}$$

Let Ω^* be the rearrangement of Ω and let u_1^* be the rearrangement of u_1^*. Let v_1 be the solution of the symmetrized problem:

$$\begin{aligned} \Delta_\kappa v_1 + 1 &= 0 \text{ on } \Omega^* \\ v_1 &= 0 \text{ on } \partial\Omega^*. \end{aligned} \tag{4.1}$$

By Theorem 2.1, $u_1^*(x) \leq v_1(x)$. Since u_1 and u_1^* are equimeasurable,

$$\int_\Omega u_1 dg \leq \int_{\Omega^*} v_1 dg_\kappa$$

which proves the theorem for the case $k = 1$.

Assume that the claim is true for all $k \leq m-1$. We need a lemma, which follows directly from the maximum principle:

Lemma 4.1. *Let Ω be a smoothly bounded domain in M and suppose f_1 and f_2 are nonnegative Hölder continuous functions on Ω satisfying $f_1 \geq f_2$. Suppose w_1 and w_2 satisfy*

$$\begin{aligned} \Delta w_i + f_i &= 0 \text{ on } \Omega \\ w_i &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then $w_1 \geq w_2$ on Ω.

Let u_k denote the solution of

$$\begin{aligned} \Delta_g u_k + k u_{k-1} &= 0 \text{ on } \Omega \\ u_k &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Let Ω^* be the rearrangement of Ω and let u_{k-1}^* be the rearrangement of u_{k-1}. Define v_j inductively as follows: for $j = 1$, let v_j be the solution of (4.1). Having defined v_{k-1} such that $u_{k-1}^* \leq v_{k-1}$, define v_k as the solution of

$$\begin{aligned} \Delta_\kappa v_k + k v_{k-1} &= 0 \text{ on } \Omega^* \\ v_k &= 0 \text{ on } \partial\Omega^*. \end{aligned}$$

By Lemma 4.1, $u_k^* \leq v_k$ and the inductive definition of the family v_j is complete. Since u_k and u_k^* are equimeasurable, we have

$$\int_\Omega u_k dg \leq \int_{\Omega^*} v_k dg_\kappa$$

which completes the proof of that portion of Theorem 1.2 which addresses the functional $\mathcal{A}_k$. The proof of the portion of Theorem 1.2 addressing the functional $\mathcal{S}_k$ follows from the above construction of $v_k(x)$ and (2.2). This concludes the proof of Theorem 1.2

Remark 4.1 There should be a convexity argument to establish Theorem 1.2. Such an argument was given in the case of Euclidean space by Aizenmann and Simon [AS]. For $k \geq 2$, it would be interesting to find an argument which used the variational formulae given in Theorem 1.1. I was unable to find such an argument that worked, in part due to the continuity properties of symmetric rearrangement.

As an immediate corollary to Theorem 1.2 we have the following:

Corollary 4.1. *Suppose that (M, g) is a Riemannian manifold which satisfies an isoperimetric condition with comparison space form M_κ. Suppose that $v \in I\!R$ is fixed and let $\mathcal{D}_v = \{\Omega \in \mathcal{D} : Vol_g(\Omega) = v\}$. Then, for each positive integer k, the maps*

$$\mathcal{S}_k : \mathcal{D}_v \to I\!R$$
$$\mathcal{A}_k : \mathcal{D}_v \to I\!R$$

are uniformly bounded.

5: Uniform bounds for exit time moments for a general class of diffusions

In this section we examine results concerning the existence of uniform bounds for general classes of diffusions on spaces which satisfy an isoperimetric conditions. We begin with some notation:

Definition 5.1. *Suppose that X_t is a diffusion on M and that $\mathcal{V} \subset \mathcal{D}$. We say that X_t has bounded kth moment on $\mathcal{V}$ if there is a real number N such that $\mathcal{S}_k(\Omega) < N$ for all $\Omega \in \mathcal{V}$. Similarly, we say that X_t has bounded average kth moment on $\mathcal{V}$ if there is a real number N such that $\mathcal{A}_k(\Omega) < N$ for all $\Omega \in \mathcal{V}$. When $\mathcal{V} = \mathcal{D}_v = \{\Omega \subset I\!R^n : Vol(\Omega) = v\}$, and X_t has bounded k-th moment on $\mathcal{V}$ for every choice of v, we say that X_t has bounded k-th moments.*

Let

$$C^\infty_+(M) = \{f \in C^\infty(M) : \inf_{x \in M} f(x) > 0\}$$

Our first result is that $C^\infty_+(M)$ acts on the space of diffusions admitting bounded exit time moments and on the space of diffusions admitting bounded average exit time moments.

Proposition 5.1. *Suppose that X_t is a diffusion on M with infintesimal generator L a uniformly elliptic operator. Suppose that X_t has bounded jth moments on $\mathcal{V} \subset \mathcal{D}$, $j \le k$. Suppose that $M = fL$ where $f \in C^\infty_+(M)$, and that Y_t is the diffusion corresponding to M. Then M has bounded jth moments on $\mathcal{V}$, $j \le k$. Similarly, Suppose if X_t has bounded jth average moments on $\mathcal{V} \subset \mathcal{D}$, $j \le k$, then M has bounded average jth moments on $\mathcal{V}$, $j \le k$.*

Proof. Suppose that $\inf f(x) = \mu,\ \mu > 0$. Let w_1 be the solution of

$$\begin{aligned} Mw_1 &= -1 \text{ on } \Omega \\ w_1 &= 0 \text{ on } \partial\Omega \end{aligned} \tag{5.1}$$

and let v_1 be the solution of

$$\begin{aligned} \mu L v_1 &= -1 \text{ on } \Omega \\ v_1 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.2}$$

Then w_1 is the solution of

$$\begin{aligned} Lw_1 &= -\frac{1}{f} \text{ on } \Omega \\ w_1 &= 0 \text{ on } \partial\Omega \end{aligned}$$

and hence,

$$\begin{aligned} L(w_1 - v_1) &= \frac{1}{\mu} - \frac{1}{f} \text{ on } \Omega \\ w_1 - v_1 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.3}$$

But $\frac{1}{\mu} \geq \frac{1}{f}$, and hence the right hand side of (5.3) is nonnegative. Since L is elliptic and $w_1 - v_1$ is zero on the boundary, we obtain $w_1 - v_1 \leq 0$. Since X_t has bounded first moment, we have proved the result for $k = 1$.

Having defined w_1 by (5.1) and v_1 by (5.2), define w_k and v_k inductively by

$$\begin{aligned} Mw_k + kw_{k-1} &= \text{ on } \Omega \\ w_k &= 0 \text{ on } \partial\Omega \end{aligned}$$

and

$$\begin{aligned} \mu L v_k + k v_{k-1} &= 0 \text{ on } \Omega \\ v_k &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Assume that $w_j - v_j \leq 0$ for all $j < k$. Then

$$\begin{aligned} L(w_k - v_k) &= \frac{kv_{k-1}}{\mu} - \frac{kw_{k-1}}{f} \text{ on } \Omega \\ w_k - v_k &= 0 \text{ on } \partial\Omega. \end{aligned}$$

It follows from the maximum principle that $w_k \leq v_k$, which proves the first part of the proposition. The second statement of the proposition follows from the first and the proof of the proposition is complete.

Proposition 5.1 and Theorem 1.2 have a number of corollaries. For example

Corollary 5.2. *Let (M, g) be a Riemannian manifold satisfying an isoperimetric condition and suppose that M is two dimensional. Let h be a metric conformally equivalent to g with conformal factor $\alpha \in C^\infty_+(M)$. Suppose that Y_t is Brownian motion on M with respect to h. Then Y_t has bounded exit time moments and bounded average exit time moments of all orders.*

We now restrict our attention to diffusions in Euclidean space $\mathbb{R}^n$ with the standard metric. We will be interested in classifying those diffusions which admit bounded kth moments.

Let X_t be a diffusion in Euclidean space with infinitesimal generator L, a second order differential operator with smooth coefficient. We introduce some notation:

Definition 5.2. *Let L be a second order differential operator with smooth coefficients on $\mathbb{R}^n$. We say that L is self-adjoint if, for every smoothly bounded domain $\Omega \subset \mathbb{R}^n$, with compact closure, L is self-adjoint with respect to Dirichlet conditions on $\partial\Omega$.*

It is easy to see that L is self-adjoint if and only if L is of divergence form.

Theorem 5.1. *Let Y_t be a diffusion on Euclidean space with infinitesimal generator L, a second order differential operator with smooth coefficients. Suppose that L is self-adjoint and uniformly elliptic. Then Y_t admits bounded moments of all orders.*

Proof. Suppose that $L = \operatorname{div}(a_{ij}\nabla)$ where a_{ij} is smooth and symmetric. Let $0 < \lambda$ be an ellipticity constant; for $0 < |\xi|$,

$$\lambda|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2.$$

Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain with compact closure. Let $G_L(x, y)$ be the Green's function for L on Ω. Let $u_1(x)$ be the solution of

$$\begin{aligned} Lu_1 + 1 &= 0 \text{ on } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then u_1 can be expressed in terms of G_L :

$$u_1(x) = \int_\Omega G_L(x, y)dy.$$

For $n > 2$, it is a theorem of [GW] that $G_L(x, y)$ can be estimated in terms of the Newtonian potential and the ellipticity constant. More precisely, there is a constant, K, depending only on dimension, such that

$$G_L(x, y) \leq K\lambda^{-1}|x - y|^{2-n}. \tag{5.4}$$

Let Ω^* be the rearrangement of Ω (Ω^* is the ball of volume $\mathrm{Vol}(\Omega)$ centered at the origin). Then, we have

$$\begin{aligned} \int_\Omega |x - y|^{2-n}dy &\leq \int_{\Omega^*} |y|^{2-n}dy \\ &\leq CR^2 \end{aligned} \tag{5.5}$$

where R is the radius of the ball of volume $\mathrm{Vol}(\Omega)$ centered at the origin and C is a constant depending only on dimension. Let τ be the first exit time of Y_t. Combining (5.4), (5.5), and (2.3), we conclude

$$\sup_{x\in\Omega} E_x[\tau] \leq K'\lambda^{-1}$$

with constant K' depending only on the volume of Ω and dimension. This proves that Y_t has bounded first moments. As in (2.2), let u_k be the solution of

$$\begin{aligned} Lu_k + ku_{k-1} &= 0 \text{ on } \Omega \\ u_k &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then u_k can be expressed in terms of G_L and u_{k-1} :

$$u_k(x) = \int_\Omega ku_{k-1}G_L(x,y)dy$$

and the theorem follows by induction.

It is clear from the proof that any process for which there is a kernel which can be bounded in terms of the Newtonian potential will have bounded first moments. A class of processes (of importance in modeling both physical and economic systems) for which this might be the case is the class of symmetric α-stable processes. To carry out a study of the boundedness properties for exit time moments for such a process on a domain $D \in \mathcal{D}$, one needs an estimate for the Green function, $G_D(x,y)$, associated to the process of the form

$$G_D(x,y) \leq C(\alpha,n)|x-y|^{\alpha-n}.$$

Such estimates will be developed elsewhere.

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