FUNCTIONS ON DISTRIBUTIVE LATTICES WITH THE CONGRUENCE SUBSTITUTION PROPERTY

Some Problems of Grätzer

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Abstract. Let L be a bounded distributive lattice. For $k \geqslant 1$, let $S_k(L)$ be the lattice of k-ary functions on L with the congruence substitution property (Boolean functions); let $S(L)$ be the lattice of all Boolean functions.

The lattices that can arise as $S_k(L)$ or $S(L)$ for some bounded distributive lattice L are characterized in terms of their Priestley spaces of prime ideals. For bounded distributive lattices L and M, it is shown that $S_1(L) \cong S_1(M)$ implies $S_k(L) \cong S_k(M)$. If L and M are finite, then $S_k(L) \cong S_k(M)$ implies $L \cong M$.

Some problems of Grätzer dating to 1964 are thus solved.

1. The Problem.

Let L be a bounded distributive lattice and let $k \geqslant 1$. A function $f : L^k \to L$ has the *congruence substitution property* if, for every congruence θ of L, and all $(a_1, b_1), \ldots, (a_k, b_k) \in \theta$, we have $f(a_1, \ldots, a_k)\theta f(b_1, \ldots, b_k)$. The set of all such functions forms a bounded distributive lattice, denoted $S_k(L)$ (also called the lattice of *Boolean* functions in [3]). Let $S(L)$ be the lattice of all Boolean functions of finite arity (on the variables x_1, x_2, …).

Grätzer has proposed the following problems [3]:

Problem 1 (Grätzer, 1964). *Let L and M be bounded distributive lattices such that $S_1(L) \cong S_1(M)$.*

Is $S_k(L)$ necessarily isomorphic to $S_k(M)$?

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Problem 2 (Grätzer, 1964). *Characterize those lattices isomorphic to $S_k(L)$ or $S(L)$ for some bounded distributive lattice L.*

(See also *General Lattice Theory* [4], Problem II.14.)

We solve both of these problems (Corollary 5.6, Theorem 6.7 and Theorem 6.9).

Grätzer has also proposed the following problem [3]: *Given a bounded distributive lattice L, find every bounded distributive lattice M such that $S_k(L) \cong S_k(M)$ (or such that $S(L) \cong S(M)$).* (In *General Lattice Theory* [4], Problem II.13, he asks: *To what extent do $S(L)$ and $S_k(L)$ determine the structure of L?*)

We prove that, for a finite distributive lattice L, $S_k(L)$ fully determines L; but there are infinitely many pairwise non-isomorphic finite distributive lattices L_1, L_2, ... such that $S(L) \cong S(L_n)$ (Theorem 7.1 and Note 7.2).

Along the way, we completely classify the Boolean functions on a bounded distributive lattice L (Theorem 4.7). Our central result is that $S_1\big(S_k(L)\big)$ is canonically isomorphic to $S_{k+1}(L)$ (Theorem 5.5).

Our proofs rely heavily on Priestley duality for distributive latttices.

2. Historical Background.

Functions on a general algebra with the congruence substitution property are the focus of the theory of *affine completeness*. (See, for instance, [6].)

It is obvious that every lattice polynomial on a bounded distributive lattice has the congruence substitution property, as does every Boolean algebra polynomial on a Boolean lattice. (For instance, $(x \wedge y) \vee z' \in S_3(L)$ if L is Boolean.) Grätzer proved the converse ([2], Theorem 1): Every function on a Boolean lattice with the congruence substitution property is a Boolean algebra polynomial (hence the term "Boolean function"). He also characterized those bounded distributive lattices such that every Boolean function is a lattice polynomial ([3], Corollary 3).

The key result for our purposes is the following

Theorem [3]. *Let L be a bounded distributive lattice with least element 0_L and greatest element 1_L. Let $k \geqslant 1$ and let $\mathbf{2} := \{0_L, 1_L\}$.*

For all $f : L^k \to L$, let $\phi_f : \mathbf{2}^k \to L$ be the restriction of f to $\mathbf{2}^k$.

(1) *For all f, $g \in S_k(L)$, $f = g$ if and only if $\phi_f = \phi_g$.*
(2) *Let $\phi : \mathbf{2}^k \to L$. There exists $f \in S_k(L)$ such that $\phi = \phi_f$ if and only if the interval $[\phi(\vec{b}), \phi(\vec{a}) \vee \phi(\vec{b})]$ is a Boolean lattice for all $\vec{a}$, $\vec{b} \in \mathbf{2}^k$ such that $\vec{a} < \vec{b}$.*

3. Mathematical Background, Terminology, and Notation (A Primer on Priestley Duality).

The central reference is [1].

Let L be a bounded distributive lattice; let $\mathbf{2} := \{0_L, 1_L\}$, where 0_L is the least element of L and 1_L is the greatest element. For a, $b \in L$, where $a \leqslant b$, let $[a, b]_L$ be the interval $\{\, c \in L \mid a \leqslant c \leqslant b \,\}$. Let $\operatorname{Con} L$ be the congruence lattice of L. For $\theta \in \operatorname{Con} L$ and a, $b \in L$, we write $a\theta b$ if $(a, b) \in \theta$.

For $k \geqslant 1$, a function $f : L^k \to L$ has the *congruence substitution property* if, for all $\theta \in \operatorname{Con} L$ and all a_1, b_1, ... , a_k, $b_k \in L$, $a_i \theta b_i$ $(i = 1, \ldots, k)$ implies $f(a_1, \ldots, a_k)\theta f(b_1, \ldots, b_k)$. The (bounded distributive) lattice of all such functions, also called the k-ary *Boolean* functions, is denoted $S_k(L)$.

If we view the members of $S_k(L)$ as functions depending on the variables x_1, ... , x_k, we can take the union

$$\bigcup_{k=1}^{\infty} S_k(L)$$

to get the (bounded distributive) lattice $S(L)$ of all (finitary) Boolean functions.

Let P be a poset. A *down-set* of P is a subset $U \subseteq P$ such that, for all $p \in P$ and $u \in U$, $p \leqslant u$ implies that $p \in U$. The poset of clopen down-sets of an ordered topological space P, partially ordered by inclusion, is a bounded distributive lattice, denoted $\mathcal{O}(P)$. (Meet is intersection, join is union, $0_{\mathcal{O}(P)}$ is $\emptyset$, and $1_{\mathcal{O}(P)}$ is P.)

A *Priestley space* P is a compact (partially) ordered topological space such that, for p, $q \in P$, $p \nleqslant q$ implies that $p \notin U$ and $q \in U$ for some $U \in \mathcal{O}(P)$. Given a bounded distributive lattice L, the poset $P(L)$ of prime ideals forms a Priestley space, with the subbasis

$$\bigl\{\{\, I \in P(L) \mid a \in I \,\}, \{\, I \in P(L) \mid a \notin I \,\}\bigr\}.$$

It is well known that L is isomorphic to $\mathcal{O}\bigl(P(L)\bigr)$ via the map

$$a \mapsto U_a := \{\, I \in P(L) \mid a \notin I \,\}.$$

It is also well known that every Priestley space P is order-homeomorphic (i.e., order-isomorphic and homeomorphic via the same function) to $P\bigl(\mathcal{O}(P)\bigr)$ by the map

$$p \mapsto I_p := \{\, U \in \mathcal{O}(P) \mid p \notin U \,\}.$$

Indeed, the category $\mathbf{D}$ of bounded distributive lattices with $\{0, 1\}$-preserving homomorphisms is dually equivalent to the category $\mathbf{P}$ of Priestley spaces with continuous order-preserving maps. [If L is a finite distributive lattice, and $\mathcal{J}(L)$

is its poset of join-irreducibles, then $L \cong \mathcal{O}\big(\mathcal{J}(L)\big)$. If P is a finite poset, then $P \cong \mathcal{J}\big(\mathcal{O}(P)\big)$.]

Under the dual equivalence functor, a map $f : L \to M$ in $\mathbf{D}$ corresponds to the map $\phi : P(M) \to P(L)$ in $\mathbf{P}$ given by $\phi(J) = f^{-1}(J)$ for all $J \in P(M)$. Similarly, a map $\phi : P \to Q$ in $\mathbf{P}$ corresponds to the map $f : \mathcal{O}(Q) \to \mathcal{O}(P)$ in $\mathbf{D}$ given by $f(V) = \phi^{-1}(V)$ for all $V \in \mathcal{O}(Q)$. (See [8]; [1], 10.25.)

If L, $M \in \mathbf{D}$, every prime ideal of $L \times M$ is of the form $I \times M$ or $L \times J$ where $I \in P(L)$ and $J \in P(M)$ ([1], Exercise 9.3). If M is a $\{0,1\}$-sublattice of $L \in \mathbf{D}$, then every $J \in P(M)$ is of the form $I \cap M$ for some $I \in P(L)$; moreover, the function $I \mapsto I \cap M$ is a continuous order-preserving map from $P(L)$ onto $P(M)$.

It is well known (Nachbin's Theorem, [4], Theorem II.1.22) that $L \in \mathbf{D}$ is Boolean if and only if $P(L)$ is an antichain (that is, distinct elements are incomparable).

In the sequel, let $P \in \mathbf{P}$ and let $L := \mathcal{O}(P)$.

Every clopen subset of P is a Priestley space; and for U, $V \in \mathcal{O}(P)$, $\mathcal{O}(U \setminus V)$ is isomorphic to $[U \cap V, U]$. Every clopen subset of $P \in \mathbf{P}$ is a finite union of sets of the form $U \setminus V$, where U,$V \in \mathcal{O}(P)$.

For all $Q \subseteq P$, let $\theta_Q := \{\, (U,V) \in L^2 \mid U \cap Q = V \cap Q \,\}$; if Q is a singleton $\{p\}$, we write θ_p. It is well known that $\operatorname{Con} L = \{\, \theta_Q \mid Q \subseteq P \text{ is closed} \,\}$ ([1], 10.27).

Given $U \subseteq P$, let $\overset{\circ}{\downarrow} U := \{\, p \in P \mid p < u \text{ for some } u \in U \,\}$; let $\operatorname{Max} U$ be the set of maximal elements of the poset U; let $U^0 := P \setminus U$ and let $U^1 := U$.

Let $\mathcal{S}_k(L)$ be the family of 2^k-tuples

$$\{\, (U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k} \in L^{2^k} \ \mid\ \text{for all } \vec{\delta}, \vec{\epsilon} \in \mathbf{2}^k,\ \vec{\delta} < \vec{\epsilon} \text{ implies } \overset{\circ}{\downarrow} U_{\vec{\delta}} \subseteq U_{\vec{\epsilon}} \,\}.$$

(Note that $\mathcal{S}_k(L)$ is a $\{0,1\}$-sublattice of L^{2^k}.)

For all $p \in P$, $\vec{\epsilon} \in \mathbf{2}^k$, let

$$I_{p,\vec{\epsilon}} := \{\, (U_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in \mathcal{S}_k(L) \mid p \notin U_{\vec{\epsilon}} \,\}.$$

We know that $P\big(\mathcal{S}_k(L)\big) = \{\, I_{p,\vec{\epsilon}} \mid p \in P, \vec{\epsilon} \in \mathbf{2}^k \,\}$.

An element $p \in P$ is *normal* if there exist U, $V \in L$ such that $p \in U$, $p \notin V$, and $[U \cap V, U]$ is a Boolean lattice; otherwise p is *special*. (Note that, if L is finite, every $p \in P$ is normal.)

For any ordered topological space R, let $P \ltimes R$ be the ordered topological space with underlying space $P \times R$ and partial ordering

$$\leqslant_{P \ltimes R} := \leqslant_{P \times R} \setminus \{\, \big((p,r),(p,r')\big) \in (P \times R)^2 \mid p \text{ is normal and } r \neq r' \,\}.$$

We denote the i-th component of $\vec{\epsilon} \in \mathbf{2}^k$ by ϵ_i $(1 \leqslant i \leqslant k)$; $\vec{\epsilon 0}$ denotes the element of $\mathbf{2}^{k+1}$ such that

$$(\vec{\epsilon 0})_i = \begin{cases} \epsilon_i & \text{if } 1 \leqslant i \leqslant k, \\ 0 & \text{if } i = k+1. \end{cases}$$

Similarly, we define $\vec{\epsilon 1} \in \mathbf{2}^{k+1}$; $\vec{\epsilon}\,'$ is the complement of $\vec{\epsilon}$ in $\mathbf{2}^k$.

4. The Lattice of k-ary Boolean Functions.

In this section, we completely characterize the k-ary Boolean functions on a bounded distributive lattice L (Theorem 4.7). In so doing, we obtain Grätzer's result that every $f \in S_k(L)$ is determined by its restriction to $\mathbf{2}^k$, where $\mathbf{2} := \{0_L, 1_L\}$; we also obtain a new description of the functions $\phi : \mathbf{2}^k \to L$ that are restrictions of Boolean functions to $\mathbf{2}^k$ [easily seen to be equivalent to Grätzer's ([3], Theorem)].

In the sequel, let P be a Priestley space and let L be the bounded distributive lattice $\mathcal{O}(P)$.

We begin with some trivial observations.

Note 4.1. *Let $U \in \mathcal{O}(P)$. Then $\overset{\circ}{\downarrow} U = U \setminus \operatorname{Max} U$.*

Proof. Every clopen subset of P is in $\mathbf{P}$, and so corresponds to the poset of prime ideals of some bounded distributive lattice. By Zorn's Lemma, every prime ideal of in such a lattice is contained in a maximal one. $\square$

Lemma 4.2. *Let U, V, $Q \subseteq P$. Then $U \cap Q = V \cap Q$ implies*

$$(P \setminus U) \cap Q = (P \setminus V) \cap Q. \qquad \square$$

Note 4.3. *Let U, $V \in \mathcal{O}(P)$. The following are equivalent:*

(1) $\overset{\circ}{\downarrow} U \subseteq V$;
(2) $U \setminus V$ *is an antichain;*
(3) $[U \cap V, U]_L$ *is a Boolean lattice;*
(4) $[V, U \cup V]_L$ *is a Boolean lattice.*

Proof. Clearly (1) is equivalent to (2), (2) is equivalent to (3), and (3) is equivalent to (4). $\square$

Lemma 4.4. *Let $f \in S_k(L)$. Then for all $U_1, \ldots, U_k \in L$,*

$$f(U_1, \ldots, U_k) = \bigcup_{\vec{\epsilon} \in \mathbf{2}^k} \bigcap_{i=1}^{k} f(\vec{\epsilon}) \cap U_i^{\epsilon_i}.$$

Proof. Let $p \in P$; let $U_1, \dots, U_k \in \mathcal{O}(P)$.

For $i = 1, \dots, k$, let

$$\epsilon_i = \begin{cases} 1 & \text{if } p \in U_i, \\ 0 & \text{if } p \notin U_i \end{cases}$$

(so that $p \in U_i^{\epsilon_i}$ and $U_i \theta_p \epsilon_i$). Hence $p \in f(U_1, \dots, U_k)$ if and only if $p \in f(\epsilon_1, \dots, \epsilon_k)$.

Now assume that $p \in \bigcap_{i=1}^k f(\vec{\epsilon}) \cap U_i^{\epsilon_i}$ for some $\vec{\epsilon} \in \mathbf{2}^k$. Then $U_i \theta_p \epsilon_i$ for $i = 1, \dots, k$, so that $f(U_1, \dots, U_k) \theta_p f(\vec{\epsilon})$ and hence $p \in f(U_1, \dots, U_k)$. $\square$

Lemma 4.5. *Let $f \in S_k(L)$. Then $\big(f(\vec{\epsilon})\big)_{\vec{\epsilon} \in \mathbf{2}^k}$ is in $\mathcal{S}_k(L)$.*

Proof. Let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ be such that $\vec{\delta} < \vec{\epsilon}$. Assume for a contradiction that $\overset{\circ}{\downarrow} f(\vec{\delta}) \nsubseteq f(\vec{\epsilon})$. Then there exist p, $q \in f(\vec{\delta})$ such that $p < q$ and $p \notin f(\vec{\epsilon})$.

Let $U \in \mathcal{O}(P)$ be such that $p \in U$ and $q \notin U$. Then $U \theta_p 1_L$ and $U \theta_q 0_L$.

For $i = 1, \dots, k$, let

$$U_i := \begin{cases} U & \text{if } \delta_i < \epsilon_i, \\ \delta_i & \text{otherwise,} \end{cases}$$

so that $U_i \theta_p \epsilon_i$ and $U_i \theta_q \delta_i$.

Hence $q \in f(U_1, \dots, U_k)$, so that $p \in f(U_1, \dots, U_k)$; but

$$p \notin f(U_1, \dots, U_k),$$

a contradiction. $\square$

Lemma 4.6. *Let $(U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$. Define $f : L^k \to L$ as follows: for U_1, $\dots$, $U_k \in L$, let*

$$f(U_1, \dots, U_k) := \bigcup_{\vec{\epsilon} \in \mathbf{2}^k} \bigcap_{i=1}^{k} U_{\vec{\epsilon}} \cap U_i^{\epsilon_i}.$$

Then $f \in S_k(L)$ and, for all $\vec{\epsilon} \in \mathbf{2}^k$, $f(\vec{\epsilon}) = U_{\vec{\epsilon}}$.

Proof. First we show that f is well defined. Let U_1, $\dots$, $U_k \in L$. Clearly $f(U_1, \dots, U_k)$ is a clopen subset of P. Let p, $q \in P$ be such that $p < q$ where $q \in f(U_1, \dots, U_k)$. We must show that $p \in f(U_1, \dots, U_k)$.

Assume not, for a contradiction. There exists $\vec{\delta} \in \mathbf{2}^k$ such that

$$q \in \bigcap_{i=1}^{k} U_{\vec{\delta}} \cap U_i^{\delta_i}.$$

For $i = 1, \dots, k$, let

$$\epsilon_i := \begin{cases} \delta_i & \text{if } p \in U_i^{\delta_i}, \\ 1 & \text{otherwise.} \end{cases}$$

For some $j \in \{1, \ldots, k\}$, $\delta_j = 0$ and $\epsilon_j = 1$ (or else

$$p \in \bigcap_{i=1}^{k} U_{\vec{\delta}} \cap U_i^{\delta_i},$$

a contradiction). Hence $\vec{\delta} < \vec{\epsilon}$. Thus $p \in U_{\vec{\epsilon}}$; and since

$$p \in \bigcap_{i=1}^{k} U_{\vec{\epsilon}} \cap U_i^{\epsilon_i},$$

we have $p \in f(U_1, \ldots, U_k)$, a contradiction. Hence $f : L^k \to L$ is well defined. Clearly $f \in S_k(L)$. (See Lemma 4.2.)

Finally, let $\vec{\epsilon} \in \mathbf{2}^k$. We will show that $f(\vec{\epsilon}) = U_{\vec{\epsilon}}$. Certainly $\epsilon_i^{\epsilon_i} = P$ for $i = 1, \ldots, k$, so

$$\bigcap_{i=1}^{k} U_{\vec{\epsilon}} \cap \epsilon_i^{\epsilon_i} = U_{\vec{\epsilon}}.$$

Now let $\vec{\delta} \in \mathbf{2}^k$ be distinct from $\vec{\epsilon}$. Then there exists $i \in \{\, 1, \ldots, k \,\}$ such that $\delta_i \neq \epsilon_i$. If $\delta_i = 0$ and $\epsilon_i = 1$, we have $\epsilon_i^{\delta_i} = \emptyset$. If $\delta_i = 1$ and $\epsilon_i = 0$, we have $\epsilon_i^{\delta_i} = \emptyset$. Hence

$$\bigcap_{i=1}^{k} U_{\vec{\delta}} \cap \epsilon_i^{\delta_i} = \emptyset.$$

Thus $f(\vec{\epsilon}) = U_{\vec{\epsilon}}$. $\square$

The main theorem of this section provides an alternate, unified proof of both [2], Theorem 1 and [3], Theorem. (Note the similarity with [5], Theorem 2.41, which the author came across after proving the main theorem: [5], Theorem 2.41 deals with normal forms for propositional formulas.) Our result extends these theorems by explicitly describing all possible k-ary Boolean functions.

Theorem 4.7. *The lattices $S_k(L)$ and $\mathcal{S}_k(L)$ are isomorphic.*

Define a map $\Phi : S_k(L) \to \mathcal{S}_k(L)$ as follows: for all $f \in S_k(L)$, let

$$\Phi(f) := \big(f(\vec{\epsilon})\big)_{\vec{\epsilon} \in \mathbf{2}^k}.$$

Define a map $\Psi : \mathcal{S}_k(L) \to S_k(L)$ as follows: for all $(U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$, let $\Psi\big((U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k}\big) : L^k \to L$ be the function defined for all $U_1, \ldots, U_k \in L$ by

$$\Psi\big((U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k}\big)(U_1, \ldots, U_k) := \bigcup_{\vec{\epsilon} \in \mathbf{2}^k} \bigcap_{i=1}^{k} U_{\vec{\epsilon}} \cap U_i^{\epsilon_i}.$$

Then Φ and Ψ are mutually-inverse order-isomorphisms.

Proof. The theorem follows from Lemmas 4.4–4.6. $\square$

The theorem implies that the generic unary Boolean function $f : L \to L$ is given by

$$f(U) = (U_0 \setminus U) \cup (U_1 \cap U)$$

where U_0, $U_1 \in L$ are such that $\overset{\circ}{\downarrow} U_0 \subseteq U_1$.

Example 4.8. Let P be the two-element chain $\{a, b\}$ where $a < b$; then $L = \mathcal{O}(P)$ is the three-element chain $\{\emptyset, a, ab\}$ (Figure 4.1).

Figure 4.1. The poset P and the lattice $L = \mathcal{O}(P)$.

Clearly $\overset{\circ}{\downarrow}\emptyset = \overset{\circ}{\downarrow}a = \emptyset$ and $\overset{\circ}{\downarrow}ab = a$ (Table 4.1).

$\mathbf{U}$	$\overset{\circ}{\downarrow}\mathbf{U}$
$\emptyset$	$\emptyset$
a	$\emptyset$
ab	a

Table 4.1. The members of $\mathcal{O}(P)$ *sans* their maximal elements.

Hence $\mathcal{S}_1(L)$ is the lattice

$$\{\, (\emptyset, \emptyset), (\emptyset, a), (\emptyset, ab), (a, \emptyset), (a, a), (a, ab), (ab, a), (ab, ab) \,\}$$

(Figure 4.2).

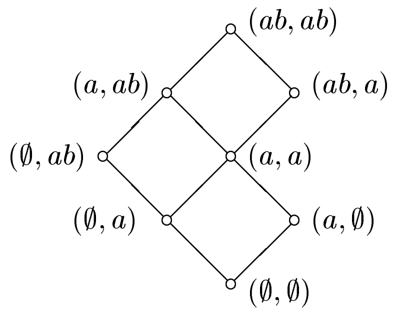

Figure 4.2. The lattice $\mathcal{S}_1(L)$.

The lattice $\mathcal{S}_2(L)$ has 52 elements, which we list in 2×2 matrix notation:

$$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & a \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & ab \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ a & \emptyset \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ a & a \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ a & ab \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ ab & a \end{pmatrix}\begin{pmatrix} \emptyset & \emptyset \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} \emptyset & a \\ \emptyset & \emptyset \end{pmatrix}\begin{pmatrix} \emptyset & a \\ \emptyset & a \end{pmatrix}\begin{pmatrix} \emptyset & a \\ \emptyset & ab \end{pmatrix}\begin{pmatrix} \emptyset & a \\ a & \emptyset \end{pmatrix}\begin{pmatrix} \emptyset & a \\ a & a \end{pmatrix}\begin{pmatrix} \emptyset & a \\ a & ab \end{pmatrix}\begin{pmatrix} \emptyset & a \\ ab & a \end{pmatrix}\begin{pmatrix} \emptyset & a \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} \emptyset & ab \\ \emptyset & a \end{pmatrix}\begin{pmatrix} \emptyset & ab \\ \emptyset & ab \end{pmatrix} \qquad \begin{pmatrix} \emptyset & ab \\ a & a \end{pmatrix}\begin{pmatrix} \emptyset & ab \\ a & ab \end{pmatrix}\begin{pmatrix} \emptyset & ab \\ ab & a \end{pmatrix}\begin{pmatrix} \emptyset & ab \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} a & \emptyset \\ \emptyset & \emptyset \end{pmatrix}\begin{pmatrix} a & \emptyset \\ \emptyset & a \end{pmatrix}\begin{pmatrix} a & \emptyset \\ \emptyset & ab \end{pmatrix}\begin{pmatrix} a & \emptyset \\ a & \emptyset \end{pmatrix}\begin{pmatrix} a & \emptyset \\ a & a \end{pmatrix}\begin{pmatrix} a & \emptyset \\ a & ab \end{pmatrix}\begin{pmatrix} a & \emptyset \\ ab & a \end{pmatrix}\begin{pmatrix} a & \emptyset \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} a & a \\ \emptyset & \emptyset \end{pmatrix}\begin{pmatrix} a & a \\ \emptyset & a \end{pmatrix}\begin{pmatrix} a & a \\ \emptyset & ab \end{pmatrix}\begin{pmatrix} a & a \\ a & \emptyset \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} a & a \\ a & ab \end{pmatrix}\begin{pmatrix} a & a \\ ab & a \end{pmatrix}\begin{pmatrix} a & a \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} a & ab \\ \emptyset & a \end{pmatrix}\begin{pmatrix} a & ab \\ \emptyset & ab \end{pmatrix} \qquad \begin{pmatrix} a & ab \\ a & a \end{pmatrix}\begin{pmatrix} a & ab \\ a & ab \end{pmatrix}\begin{pmatrix} a & ab \\ ab & a \end{pmatrix}\begin{pmatrix} a & ab \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} ab & a \\ a & a \end{pmatrix}\begin{pmatrix} ab & a \\ a & ab \end{pmatrix}\begin{pmatrix} ab & a \\ ab & a \end{pmatrix}\begin{pmatrix} ab & a \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} ab & ab \\ a & a \end{pmatrix}\begin{pmatrix} ab & ab \\ a & ab \end{pmatrix}\begin{pmatrix} ab & ab \\ ab & a \end{pmatrix}\begin{pmatrix} ab & ab \\ ab & ab \end{pmatrix}$$

Example 4.9. Let P be the three-element fence $\{a, b, c\}$ where $b > a < c$; then $L = \mathcal{O}(P)$ is the lattice $\{\,\emptyset, a, ab, ac, abc\,\}$ (Figure 4.3).

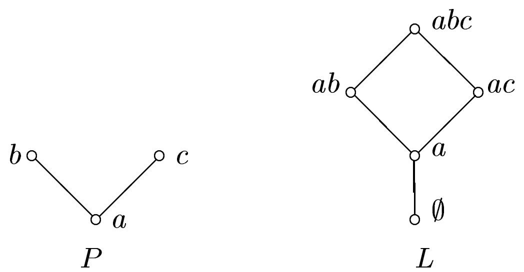

Figure 4.3. The poset P and the lattice $L = \mathcal{O}(P)$.

$\mathbf{U}$	$\overset{\circ}{\downarrow}\mathbf{U}$
$\emptyset$	$\emptyset$
a	$\emptyset$
ab	a
ac	a
abc	a

Table 4.2. The members of $\mathcal{O}(P)$ *sans* their maximal elements.

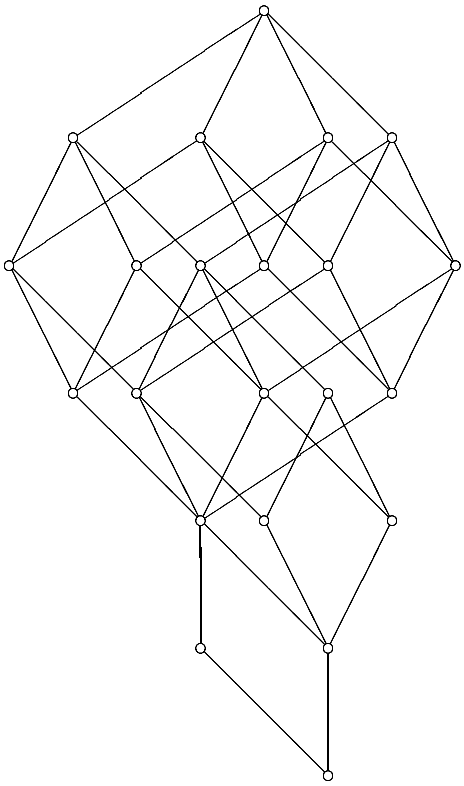

Figure 4.4. The lattice $\mathcal{S}_1(L)$.

Clearly $\overset{\circ}{\downarrow}\emptyset = \overset{\circ}{\downarrow}a = \emptyset$ and $\overset{\circ}{\downarrow}ab = \overset{\circ}{\downarrow}ac = \overset{\circ}{\downarrow}abc = a$ (Table 4.2).

Then $\mathcal{S}_1(L)$ is the lattice $\{\emptyset, a\} \times L \cup \{ab, ac, abc\} \times \{a, ab, ac, abc\}$ (Figure 4.4).

Example 4.10. Let Q be the four-element fence $\{w, x, y, z\}$ where $w < x > y < z$; then $M = \mathcal{O}(Q)$ is the lattice $\{\,\emptyset, w, y, wy, yz, wxy, wyz, wxyz\,\}$ (Figure 4.5).

Clearly $\overset{\circ}{\downarrow}\emptyset = \overset{\circ}{\downarrow}w = \overset{\circ}{\downarrow}y = \overset{\circ}{\downarrow}wy = \emptyset$, $\overset{\circ}{\downarrow}yz = \overset{\circ}{\downarrow}wyz = y$, and $\overset{\circ}{\downarrow}wxy = \overset{\circ}{\downarrow}wxyz = wy$ (Table 4.3).

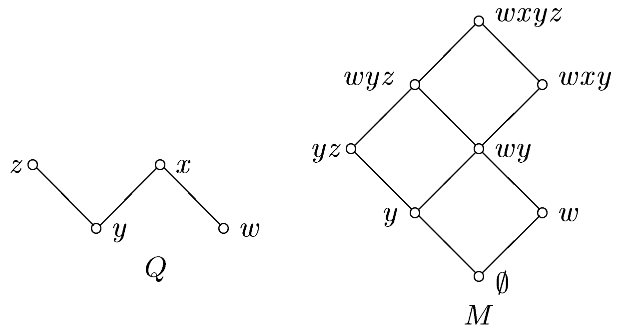

Figure 4.5. The poset Q and the lattice $M = \mathcal{O}(Q)$.

$\mathbf{U}$	$\overset{\circ}{\downarrow}\mathbf{U}$
$\emptyset$	$\emptyset$
y	$\emptyset$
yz	y
w	$\emptyset$
wy	$\emptyset$
wyz	y
wxy	wy
$wxyz$	wy

Table 4.3. The members of $\mathcal{O}(Q)$ *sans* their maximal elements.

Thus $\mathcal{S}_1(M)$ has 52 elements:

$(\emptyset,\emptyset)$ $(\emptyset,y)$ $(\emptyset,yz)$ $(\emptyset,w)$ $(\emptyset,wy)$ $(\emptyset,wyz)$ $(\emptyset,wxy)$ $(\emptyset,wxyz)$

$(y,\emptyset)$ (y,y) (y,yz) (y,w) (y,wy) (y,wyz) (y,wxy) $(y,wxyz)$

(yz,y) (yz,yz) (yz,wy) (yz,wyz) (yz,wxy) $(yz,wxyz)$

$(w,\emptyset)$ (w,y) (w,yz) (w,w) (w,wy) (w,wyz) (w,wxy) $(w,wxyz)$

$(wy,\emptyset)$ (wy,y) (wy,yz) (wy,w) (wy,wy) (wy,wyz) (wy,wxy) $(wy,wxyz)$

(wyz,y) (wyz,yz) (wyz,wy) (wyz,wyz) (wyz,wxy) $(wyz,wxyz)$

(wxy,wy) (wxy,wyz) (wxy,wxy) $(wxy,wxyz)$

$(wxyz,wy)$ $(wxyz,wyz)$ $(wxyz,wxy)$ $(wxyz,wxyz)$.

5. Boolean Functions on the Lattice of Boolean Functions: The Solution to Grätzer's First Problem.

In this section, we solve Problem 1 of §1, posed by Grätzer in 1964 (Corollary 5.6): The lattice $S_k(L)$ (for a bounded distributive lattice L) is determined up to isomorphism by the lattice $S_1(L)$. Indeed, we prove the surprising result that $S_{k+1}(L)$ is canonically isomorphic to $S_1\big(S_k(L)\big)$, the lattice of unary Boolean functions on the lattice of k-ary Boolean functions of L (Theorem 5.5).

Recall that $P \in \mathbf{P}$ and $L = \mathcal{O}(P)$.

Lemma 5.1. *Let $\vec{U} := (U_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k}$, $\vec{V} := (V_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that*

$$[U_{\vec{\delta}} \cap V_{\vec{\epsilon}}, U_{\vec{\delta}}]_L$$

is a Boolean lattice whenever $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ and $\vec{\delta} < \vec{\epsilon}$. Choose $W_{\vec{\epsilon}} \in [U_{\vec{\epsilon}} \cap V_{\vec{\epsilon}}, U_{\vec{\epsilon}}]_L$ for all $\vec{\epsilon} \in \mathbf{2}^k$.

Then $(W_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k}$ belongs to $\mathcal{S}_k(L)$.

Proof. Let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ be such that $\vec{\delta} < \vec{\epsilon}$. Then

$$\overset{\circ}{\downarrow} W_{\vec{\delta}} \subseteq \overset{\circ}{\downarrow} U_{\vec{\delta}} \subseteq U_{\vec{\epsilon}} \cap V_{\vec{\epsilon}} \subseteq W_{\vec{\epsilon}}$$

(using Note 4.3). □

Corollary 5.2. *Let $\vec{U} := (U_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k}$, $\vec{V} := (V_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that*

$$[U_{\vec{\delta}} \cap V_{\vec{\epsilon}}, U_{\vec{\delta}}]_L$$

is a Boolean lattice whenever $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ and $\vec{\delta} \leqslant \vec{\epsilon}$.

Then

$$[\vec{U} \wedge \vec{V}, \vec{U}]_{\mathcal{S}_k(L)}$$

is a Boolean lattice.

Proof. Let

$$\vec{W} := (W_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k} \in [\vec{U} \wedge \vec{V}, \vec{U}]_{\mathcal{S}_k(L)}.$$

Thus, for all $\vec{\epsilon} \in \mathbf{2}^k$, $W_{\vec{\epsilon}} \in [U_{\vec{\epsilon}} \cap V_{\vec{\epsilon}}, U_{\vec{\epsilon}}]_L$, so there exists $W'_{\vec{\epsilon}} \in [U_{\vec{\epsilon}} \cap V_{\vec{\epsilon}}, U_{\vec{\epsilon}}]_L$ such that $W_{\vec{\epsilon}} \cap W'_{\vec{\epsilon}} = U_{\vec{\epsilon}} \cap V_{\vec{\epsilon}}$ and $W_{\vec{\epsilon}} \cup W'_{\vec{\epsilon}} = U_{\vec{\epsilon}}$.

By Lemma 5.1, $\vec{W}' := (W'_{\vec{\epsilon}})_{\vec{\epsilon}\in\mathbf{2}^k}$ belongs to $\mathcal{S}_k(L)$; clearly $\vec{W} \wedge \vec{W}' = \vec{U} \wedge \vec{V}$ and $\vec{W} \vee \vec{W}' = \vec{U}$. □

Lemma 5.3. *Let $\vec{U}_0 := (U_{\vec{\epsilon}0})_{\vec{\epsilon}\in\mathbf{2}^k}$, $\vec{U}_1 := (U_{\vec{\epsilon}1})_{\vec{\epsilon}\in\mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that $(\vec{U}_0, \vec{U}_1)$ belongs to $\mathcal{S}_1\big(\mathcal{S}_k(L)\big)$.*

Then $\overset{\circ}{\downarrow} U_{\vec{\delta}0} \subseteq U_{\vec{\epsilon}1}$ for all $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ such that $\vec{\delta} < \vec{\epsilon}$.

Proof. Fix $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$ such that $\vec{\delta} < \vec{\epsilon}$. By Note 4.3,

$$[\vec{U}_0 \wedge \vec{U}_1, \vec{U}_0]_{\mathcal{S}_k(L)}$$

is a Boolean lattice.

For all $\vec{\eta} \in \mathbf{2}^k$, let

$$W_{\vec{\eta}} := \begin{cases} U_{\vec{\eta 0}} \cap U_{\vec{\eta 1}} & \text{if } \vec{\eta} < \vec{\epsilon}, \\ U_{\vec{\eta 0}} & \text{otherwise.} \end{cases}$$

Then $\vec{W} := (W_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$; indeed

$$\vec{W} \in [\vec{U}_0 \wedge \vec{U}_1, \vec{U}_0]_{\mathcal{S}_k(L)}.$$

Let $\vec{W}' := (W'_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that $\vec{W} \wedge \vec{W}' = \vec{U}_0 \wedge \vec{U}_1$ and $\vec{W} \vee \vec{W}' = \vec{U}_0$.

Clearly $W'_{\vec{\delta}} = U_{\vec{\delta 0}}$ and $W'_{\vec{\epsilon}} = U_{\vec{\epsilon 0}} \cap U_{\vec{\epsilon 1}}$. Hence $\overset{\circ}{\downarrow} U_{\vec{\delta 0}} \subseteq U_{\vec{\epsilon 1}}$. $\square$

Lemma 5.4. *Let $\vec{U}_0 := (U_{\vec{\epsilon 0}})_{\vec{\epsilon} \in \mathbf{2}^k}$, $\vec{U}_1 := (U_{\vec{\epsilon 1}})_{\vec{\epsilon} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that $(\vec{U}_0, \vec{U}_1)$ belongs to $\mathcal{S}_1(\mathcal{S}_k(L))$.*

Then for all $\vec{\epsilon} \in \mathbf{2}^k$, $\overset{\circ}{\downarrow} U_{\vec{\epsilon 0}} \subseteq U_{\vec{\epsilon 1}}$.

Proof. Fix $\vec{\epsilon} \in \mathbf{2}^k$. It suffices to prove that $[U_{\vec{\epsilon 0}} \cap U_{\vec{\epsilon 1}}, U_{\vec{\epsilon 0}}]_L$ is a Boolean lattice. Let $W \in [U_{\vec{\epsilon 0}} \cap U_{\vec{\epsilon 1}}, U_{\vec{\epsilon 0}}]_L$.

For all $\vec{\eta} \in \mathbf{2}^k$, let

$$(W_{\vec{\eta}}) := \begin{cases} U_{\vec{\eta 0}} \cap U_{\vec{\eta 1}} & \text{if } \vec{\eta} < \vec{\epsilon}, \\ W & \text{if } \vec{\eta} = \vec{\epsilon} \\ U_{\vec{\eta 0}} & \text{otherwise.} \end{cases}$$

Then $\vec{W} := (W_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$, and it lies in the Boolean interval

$$[\vec{U}_0 \wedge \vec{U}_1, \vec{U}_0]_{\mathcal{S}_k(L)}.$$

Let $\vec{W}' := (W'_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in \mathcal{S}_k(L)$ be such that $\vec{W} \wedge \vec{W}' = \vec{U}_0 \wedge \vec{U}_1$ and $\vec{W} \vee \vec{W}' = \vec{U}_0$. Clearly $W \cap W'_{\vec{\epsilon}} = U_{\vec{\epsilon 0}} \cap U_{\vec{\epsilon 1}}$ and $W \cup W'_{\vec{\epsilon}} = U_{\vec{\epsilon 0}}$. $\square$

Theorem 5.5. *The lattices $S_{k+1}(L)$ and $S_1(S_k(L))$ are isomorphic.*

Define a map

$$\Phi : \mathcal{S}_{k+1}(L) \to \mathcal{S}_1(\mathcal{S}_k(L))$$

as follows: for all $(U_{\vec{\zeta}})_{\vec{\zeta} \in \mathbf{2}^{k+1}} \in \mathcal{S}_{k+1}(L)$, let

$$\Phi\big((U_{\vec{\zeta}})_{\vec{\zeta} \in \mathbf{2}^{k+1}}\big) = \big((U_{\vec{\epsilon 0}})_{\vec{\epsilon} \in \mathbf{2}^k}, (U_{\vec{\epsilon 1}})_{\vec{\epsilon} \in \mathbf{2}^k}\big).$$

Define a map

$$\Psi : \mathcal{S}_1\big(\mathcal{S}_k(L)\big) \to \mathcal{S}_{k+1}(L)$$

as follows: for all $\big((U_{\vec{\epsilon}0})_{\vec{\epsilon}\in\mathbf{2}^k}, (U_{\vec{\epsilon}1})_{\vec{\epsilon}\in\mathbf{2}^k}\big) \in \mathcal{S}_1\big(\mathcal{S}_k(L)\big)$, *let*

$$\Psi\big((U_{\vec{\epsilon}0})_{\vec{\epsilon}\in\mathbf{2}^k}, (U_{\vec{\epsilon}1})_{\vec{\epsilon}\in\mathbf{2}^k}\big) = (U_{\vec{\zeta}})_{\vec{\zeta}\in\mathbf{2}^{k+1}}.$$

Then Φ *and* Ψ *are mutually-inverse order-isomorphisms.*

Proof. By Corollary 5.2 and Note 4.3, Φ is well defined. By Lemmas 5.3 and 5.4, Ψ is well defined. They are clearly order-preserving and inverses to each other. $\square$

As a corollary, we solve Grätzer's first problem ([3]; see §1):

Corollary 5.6. *Let* L, $M \in \mathbf{D}$ *be such that* $S_1(L) \cong S_1(M)$. *Then* $S_k(L) \cong S_k(M)$. $\square$

Example 5.7. Let L be the three-element chain. In Example 4.8, we computed $S_1(L)$ and $S_2(L)$. In Example 4.10, we computed $S_1(M)$ where $M \cong S_1(L)$. In both examples, we listed the elements of $\mathcal{S}_2(L)$ and $\mathcal{S}_1\big(\mathcal{S}_1(L)\big)$. The isomorphism of Theorem 5.5 can be easily seen by turning each 2×2 matrix of Example 4.8 into an ordered pair by grouping the rows together and using the isomorphism $S_1(L) \cong M$ given by

$$(\emptyset, \emptyset) \mapsto \emptyset$$
$$(\emptyset, a) \mapsto y$$
$$(\emptyset, ab) \mapsto yz$$
$$(a, \emptyset) \mapsto w$$
$$(a, a) \mapsto wy$$
$$(a, ab) \mapsto wyz$$
$$(ab, a) \mapsto wxy$$
$$(ab, ab) \mapsto wxyz.$$

6. The Priestley Dual of the Lattice of Boolean Functions: The Solution to Grätzer's Second Problem.

In this section, we solve Problem 2 of §1, posed by Grätzer in 1964 and restated in 1978 in his influential book (Theorems 6.7 and 6.9): We completely characterize the lattices that can arise as $S_k(L)$ or $S(L)$ for a bounded distributive lattice L. We do so in terms their Priestley spaces of prime ideals.

Recall that $P \in \mathbf{P}$ and $L = \mathcal{O}(P)$.

Note 6.1. *Let $p \in P$. The following are equivalent:*

(1) *p is normal;*
(2) *there exist U, $V \in \mathcal{O}(P)$ such that $U \setminus V$ is an antichain containing p;*
(3) *there exist $W \in \mathcal{O}(P)$ and a clopen subset C of P such that $p \in C \subseteq \operatorname{Max} W$.*

Proof. Note 4.3 gives the equivalence of (1) and (2) and the fact that (2) implies (3). To show that (3) implies (2), let U, $V \in \mathcal{O}(P)$ be such that $p \in U \setminus V \subseteq C$. Then $U \setminus V$ is an antichain. □

Lemma 6.2. *Let p, $q \in P$ and let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$. Assume that $p < q$ and $\vec{\delta} \geqslant \vec{\epsilon}$.*
Then $I_{p,\vec{\delta}} \subseteq I_{q,\vec{\epsilon}}$.

Proof. Let $(U_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in I_{p,\vec{\delta}}$. Then $p \notin U_{\vec{\delta}}$. Assume for a contradiction that $q \in U_{\vec{\epsilon}}$. Then $p \in \overset{\circ}{\downarrow} U_{\vec{\epsilon}}$ and hence $p \in U_{\vec{\delta}}$, a contradiction. □

Lemma 6.3. *Let $p \in P$ and let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$. Assume that p is special and that $\vec{\delta} \geqslant \vec{\epsilon}$.*
Then $I_{p,\vec{\delta}} \subseteq I_{p,\vec{\epsilon}}$.

Proof. Let $(U_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in I_{p,\vec{\delta}}$. Then $p \notin U_{\vec{\delta}}$ and $\overset{\circ}{\downarrow} U_{\vec{\epsilon}} \subseteq U_{\vec{\delta}}$, so, by Note 4.3, $U_{\vec{\epsilon}} \setminus U_{\vec{\delta}}$ is an antichain. Hence $p \notin U_{\vec{\epsilon}}$, by Note 6.1. □

Lemma 6.4. *Let p, $q \in P$ and let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$. Assume that $I_{p,\vec{\delta}} \subseteq I_{q,\vec{\epsilon}}$.*
Then $p \leqslant q$.

Proof. Assume for a contradiction that $p \nleqslant q$. Let $U \in \mathcal{O}(P)$ be such that $p \notin U$ and $q \in U$. Then $(U)_{\vec{\eta} \in \mathbf{2}^k} \in I_{p,\vec{\delta}} \setminus I_{q,\vec{\epsilon}}$, a contradiction. □

Lemma 6.5. *Let p, $q \in P$ and let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$. Assume that $I_{p,\vec{\delta}} \subseteq I_{q,\vec{\epsilon}}$.*
Then $\vec{\delta} \geqslant \vec{\epsilon}$.

Proof. Assume for a contradiction that $\vec{\delta} \ngeqslant \vec{\epsilon}$. For all $\vec{\eta} \in \mathbf{2}^k$, let

$$U_{\vec{\eta}} := \begin{cases} P & \text{if } \vec{\eta} \geqslant \vec{\epsilon}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $(U_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in I_{p,\vec{\delta}} \setminus I_{q,\vec{\epsilon}}$, a contradiction. □

Lemma 6.6. *Let $p \in P$ and let $\vec{\delta}$, $\vec{\epsilon} \in \mathbf{2}^k$. Assume that $I_{p,\vec{\delta}} \subseteq I_{p,\vec{\epsilon}}$ where $\vec{\delta} \neq \vec{\epsilon}$.*
Then p is special.

Proof. By Lemma 6.5, $\vec{\delta} > \vec{\epsilon}$.

Assume, for a contradiction, that p is normal. By Notes 4.3 and 6.1, there exist U, $V \in \mathcal{O}(P)$ such that $p \in U \setminus V$ and $\overset{\circ}{\downarrow} U \subseteq V$. For all $\vec{\eta} \in \mathbf{2}^k$, let

$$W_{\vec{\eta}} := \begin{cases} V & \text{if } \vec{\eta} \geqslant \vec{\delta}, \\ U & \text{if } \vec{\eta} \ngeqslant \vec{\delta}. \end{cases}$$

Then $(W_{\vec{\eta}})_{\vec{\eta} \in \mathbf{2}^k} \in I_{p,\vec{\delta}} \setminus I_{p,\vec{\epsilon}}$, a contradiction. □

Theorem 6.7. *The Priestley space of $S_k(L)$ is order-homeomorphic to the ordered space $P \ltimes \mathbf{2}^k$.*

Define the order-homeomorphism $\Phi : P\big(\mathcal{S}_k(L)\big) \to P \ltimes \mathbf{2}^k$ as follows: for all $p \in P$, $\vec{\epsilon} \in \mathbf{2}^k$, let $\Phi(I_{p,\vec{\epsilon}}) = (p, \vec{\epsilon}\,')$.

Proof. By Lemmas 6.4 and 6.5, Φ is well defined and order-preserving. By Lemmas 6.2 and 6.3, Φ is an order-embedding.

Obviously Φ is onto. Hence Φ is an order-isomorphism.

To prove that Φ is a homeomorphism, let

$$\Psi : P(L^{2^k}) \to P\big(\mathcal{S}_k(L)\big)$$

be the function sending $\{\, (U_{\vec{\epsilon}})_{\vec{\epsilon} \in \mathbf{2}^k} \in L^{2^k} \mid p \notin U_{\vec{\epsilon}} \,\}$ to $I_{p,\vec{\epsilon}}$ for all $p \in P$, $\vec{\epsilon} \in \mathbf{2}^k$. We know that Ψ is continuous. It is also a bijection. Since Priestley spaces are compact and Hausdorff, Ψ is a homeomorphism (see, for instance, [1], Lemma 10.7A). □

After seeing Theorem 6.7 for finite lattices, M. Maróti made the following observation:

Corollary 6.8. *If L is finite, then $\Big(\mathcal{J}\big(S_k(L)\big), <\Big)$ is isomorphic to*

$$\big(\mathcal{J}(L), <\big) \times (\mathbf{2}^k, \leqslant). \qquad \square$$

Theorem 6.9. *The Priestley space of $S(L)$ is order-homeomorphic to $P \ltimes \mathbf{2}^{\mathbb{N}}$.*

Proof. Clearly $P \ltimes \mathbf{2}^{\mathbb{N}}$ is a Priestley space. For all $k \in \mathbb{N}$, let

$$\pi_k : P \ltimes \mathbf{2}^{\mathbb{N}} \to P \ltimes \mathbf{2}^k$$

be the obvious projection; similarly, define $\pi_{kl} : P \ltimes \mathbf{2}^l \to P \ltimes \mathbf{2}^k$ for all k, $l \in \mathbb{N}$ such that $k \leqslant l$.

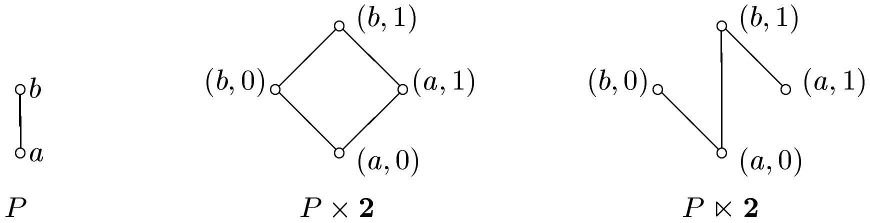

Figure 6.1. The posets P, $P \times \mathbf{2}$, and $P \ltimes \mathbf{2}$.

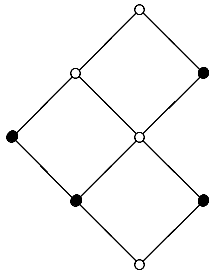

Figure 6.2. The lattice $S_1(L)$ and the poset $\mathcal{J}\big(S_1(L)\big)$.

We verify that $\big(P \ltimes \mathbf{2}^{\mathbb{N}}, (\pi_k : P \ltimes \mathbf{2}^{\mathbb{N}} \to P \ltimes \mathbf{2}^k)_{k \geqslant 1}\big)$ is the inverse limit of the directed system

$$\big((P \ltimes \mathbf{2}^k)_{k \geqslant 1}, (\pi_{kl} : P \ltimes \mathbf{2}^l \to P \ltimes \mathbf{2}^k)_{1 \leqslant k \leqslant l}\big)$$

in the category of Priestley spaces. $\square$

Example 6.10. Let P be the two-element chain $\{a, b\}$ of Example 4.8 and let $L = \mathcal{O}(P)$ (Figure 4.1). Figure 6.1 shows $P \times \mathbf{2}$ and $P \ltimes \mathbf{2}$.

Note that $P \ltimes \mathbf{2}$ is order-isomorphic to $\mathcal{J}\big(S_1(L)\big)$, so that $\mathcal{O}(P \ltimes \mathbf{2}) \cong S_1(L)$ (Figures 4.2 and 6.2).

Figure 6.3 shows P, $\mathbf{2}^2$, $P \times \mathbf{2}^2$, and $P \ltimes \mathbf{2}^2$.

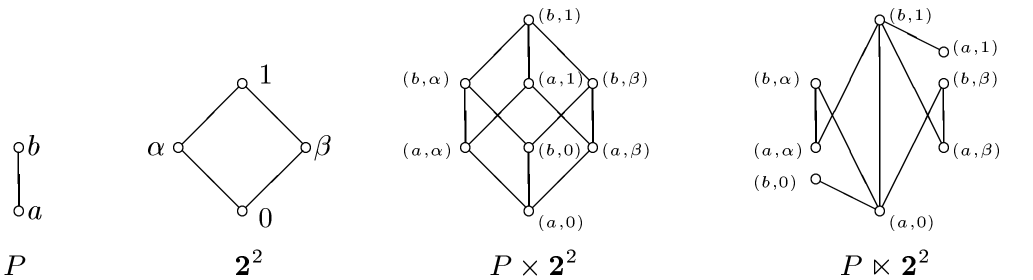

Figure 6.3. The posets P, $\mathbf{2}^2$, $P \times \mathbf{2}^2$, and $P \ltimes \mathbf{2}^2$.

Example 6.11. Let P be the three-element fence $\{a, b, c\}$ of Example 4.9 and let $L = \mathcal{O}(P)$ (Figure 4.3). Figure 6.4 shows P, $P \times \mathbf{2}$, and $P \ltimes \mathbf{2}$.

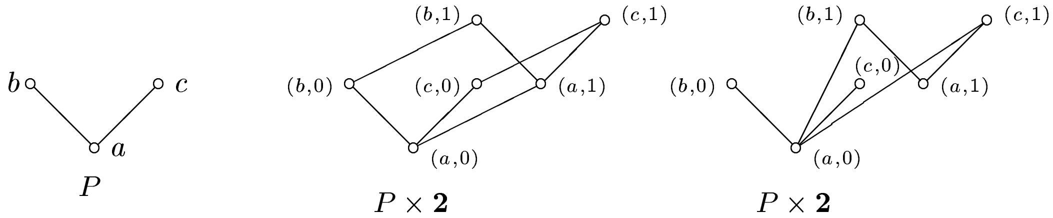

Figure 6.4. The posets P, $P \times \mathbf{2}$, and $P \ltimes \mathbf{2}$.

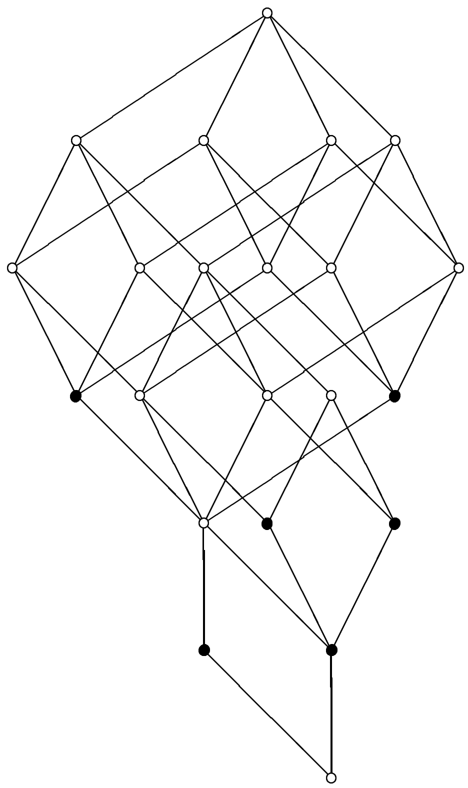

Figure 6.5. The lattice $\mathcal{S}_1(L)$ and the poset $\mathcal{J}\big(S_1(L)\big)$.

Note that $P \ltimes \mathbf{2}$ is order-isomorphic to $\mathcal{J}\big(S_1(L)\big)$, so that $\mathcal{O}(P \ltimes \mathbf{2}) \cong S_1(L)$ (Figures 4.4 and 6.5).

Indeed, $\mathcal{J}\big(S_1(L)\big) = \{\, (\emptyset, a), (\emptyset, ab), (\emptyset, ac), (a, \emptyset), (ab, \emptyset), (ac, \emptyset) \,\}$.

Example 6.12. Let Q be the four-element fence $\{w, x, y, z\}$ of Example 4.10 and let $M = \mathcal{O}(Q)$ (Figure 4.5). Figures 6.6 and 6.7 show Q, $Q \times \mathbf{2}$, and $Q \ltimes \mathbf{2}$.

Let P be the two-element chain of Example 6.10. Note that $Q \cong P \ltimes \mathbf{2}$ and that $Q \ltimes \mathbf{2} \cong (P \ltimes \mathbf{2}) \ltimes \mathbf{2}$ is order-isomorphic to $P \ltimes \mathbf{2}^2$ (Figure 6.3) under the isomorphism

$$(w, 0) \mapsto (a, \alpha)$$

$$(x, 0) \mapsto (b, \alpha)$$

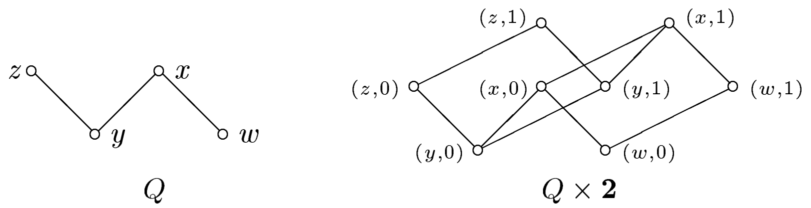

Figure 6.6. The posets Q and $Q \times \mathbf{2}$.

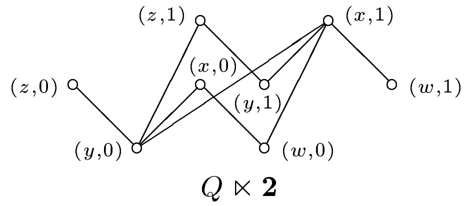

Figure 6.7. The poset $Q \ltimes \mathbf{2}$.

$$(y,0) \mapsto (a,0)$$

$$(z,0) \mapsto (b,0)$$

$$(w,1) \mapsto (a,1)$$

$$(x,1) \mapsto (b,1)$$

$$(y,1) \mapsto (a,\beta)$$

$$(z,1) \mapsto (b,\beta).$$

7. Recovering the Lattice from the Lattice of Boolean Functions.

In this section, we address Grätzer's remaining problem (see §1): We prove that a *finite* distributive lattice L is determined by its lattice of k-ary Boolean functions (Theorem 7.1), but *not* by the lattice of *all* Boolean functions (Note 7.2).

Theorem 7.1. *Let L, M be finite distributive lattices such that $S_k(L) \cong S_k(M)$.*

Then $L \cong M$.

Proof. Let $P := \mathcal{J}(L)$ and let $Q := \mathcal{J}(M)$. By Theorem 6.7 and Corollary 6.8, $P \ltimes \mathbf{2}^k \cong Q \ltimes \mathbf{2}^k$, so that $(P,<) \times (\mathbf{2}^k, \leqslant) \cong (Q,<) \times (\mathbf{2}^k, \leqslant)$. By [7],Theorem 3, $(P,<) \cong (Q,<)$, so that $P \cong Q$ and hence $L \cong M$. $\square$

Note 7.2. *Let L be a non-trivial finite distributive lattice. Let $\mathcal{M}$ be the family of finite lattices*

$$\{ S_k(L) \mid k \geqslant 1 \}.$$

Then $S(L) \cong S(M)$ for any $M \in \mathcal{M}$, but no two lattices in $\mathcal{M}$ are isomorphic.

Proof. The observation follows from Theorem 7.1 and the fact that, for any $N \in \mathbf{D}$, $S(N)$ is a limit of $\{ S_k(N) \mid k \geqslant 1 \}$ in the category $\mathbf{D}$. $\square$

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