THE SYMBOLIC DYNAMICS OF TILING THE INTEGERS

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Abstract. A finite collection $\mathcal{P}$ of finite sets *tiles the integers* iff the integers can be expressed as a disjoint union of translates of members of $\mathcal{P}$. We associate with such a tiling a doubly infinite sequence with entries from $\mathcal{P}$. The set of all such sequences is a sofic system, called a *tiling system*. We show that, up to powers of the shift, every shift of finite type can be realized as a tiling system.

1. Introduction

For notation, terminology, and basic results of symbolic dynamics, see the book by D. Lind and B. Marcus [LM].

Let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a finite collection of finite subsets of the integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, called *prototiles*. We normalize the prototiles so that each has minimum 0. A *tile* is a translate of a prototile. A *tiling of the integers* by $\mathcal{P}$ is an expression of the integers as a disjoint union of tiles, $\mathbb{Z} = \bigcup(t_j + P_{k_j})$. Corresponding to this tiling is the point $x = (x_i) \in \prod_{i=-\infty}^{\infty}\{1, 2, \ldots, K\}$ defined by $x_i = k$ if and only if there exists j such that $i \in t_j + P_{k_j}$ and $k_j = k$. Thus we can think of a tiling as being given by a bi-infinite sequence of colors, where the colors are in one-to-one correspondence with the prototiles.

Let σ denote the *shift*, $(\sigma(x))_i = x_{i+1}$. The collection of points corresponding to tilings of the integers by $\mathcal{P}$, denoted $T(\mathcal{P})$, is closed and shift-invariant. We call $\sigma : T(\mathcal{P}) \to T(\mathcal{P})$ a *tiling system*. We first show that every tiling system is sofic. We then prove our main result: up to powers of the shift, every shift of finite type can be realized as a tiling system.

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Main Theorem. *Let $\sigma : \Sigma \to \Sigma$ be a shift of finite type. Then there is a positive integer m and a tiling system $\sigma : T \to T$ such that*

(1) *$T = T_0 \cup T_1 \cup \cdots \cup T_{m-1}$, where the T_i are closed and cyclically permuted by the shift σ.*
(2) *$\sigma^m : \Sigma \to \Sigma$ is topologically conjugate to every $\sigma^m : T_i \to T_i$.*

Corollary. *The set of topological entropies of tiling systems is the same as that of shifts of finite type, i.e., the set of logarithms of roots of Perron numbers.*

Remark. In the sequel we will sometimes, as is common in symbolic dynamics, call the space T a tiling system, the space Σ a shift of finite type, etc.

2. Tiling systems are sofic

Consider the following three examples.

(1) $\mathcal{P} = \{\{0\}, \{0,1\}\}$. It is more convenient to think of $\mathcal{P}$ as $\{R, BB\}$, (R = red, B = blue). Then $T(\mathcal{P})$ is the set of all bi-infinite indexed concatenations of R and B such that between any two consecutive occurrences of R there is an even number of B's, the well-known even system. In this case $T(\mathcal{P})$ is also a renewal system, although we do not use that fact here. Recall that a *renewal system* is the collection of indexed bi-infinite sequences which are concatenations of a finite set of finite words from some alphabet. In the sequel, we shall abuse notation and write $T(R, BB)$ in place of $T(\{\{0\}, \{0,1\}\})$.

(2) $\mathcal{P} = \{\{0\}, \{0,2\}\}$, which we replace by $\{R, B \_ B\}$. Then $T(\mathcal{P})$ is the renewal system generated by words R, BRB, and $BBBB$.

(3) $\mathcal{P} = \{R, BB \_ B, Y \_ \_ Y\}$, i.e., $\{\{0\}, \{0,1,3\}, \{0,3\}\}$. In this case $T(\mathcal{P})$ is not a renewal system.

To show that every tiling system is sofic, recall the proof that the even system $\sigma : T(R, BB) \to T(R, BB)$ is sofic — it is the image of the shift of finite type $\sigma : \widetilde{T}(R_1, B_1B_2) \to \widetilde{T}(R_1, B_1B_2)$ under the "drop the subscripts" map. Here $\widetilde{T}(R_1, B_1B_2)$ is the set of all bi-infinite indexed concatenations of R_1 and B_1B_2. We show that every "subscripted tiling system" is a shift of finite type. Clearly every tiling system can be obtained from a subscripted tiling system by dropping the subscripts.

Formally, let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a finite collection of prototiles. Write

$$P_k = \{0 = p_{k,1} < p_{k,2} < \cdots < p_{k,\ell_k}\}$$

and define $\widetilde{T} = \widetilde{T}(\mathcal{P})$ on alphabet $\{(k,\ell) : 1 \le k \le K, 1 \le \ell \le \ell_k\}$, by $x \in \widetilde{T}$ iff there is a tiling of the integers by members of $\mathcal{P}$, $\mathbb{Z} = \bigcup (t_j + P_{k_j})$, such that for every i, there exist $j = j(i)$ and $\ell = \ell(i)$ such that $i \in t_j + P_{k_j}$ and $x_i = (k_j, \ell)$. Equivalently, $x \in \prod \{(k,\ell) : 1 \le k \le K, 1 \le \ell \le \ell_k\}$ is in $\widetilde{T}$ if and only if for every i, $x_i = (k,\ell)$ and $1 \le \ell' \le \ell_k$ imply $x_{i+p_{k,\ell'} - p_{k,\ell}} = (k, \ell')$. Informally, if x_i is an element of a tile, then the other elements of that tile appear in the appropriate places of x.

The following result was proved in conversations with K. Schmidt in Warwick in 1994.

Theorem. *Every "subscripted tiling system" is a shift of finite type.*

Proof. Let L be the length of a longest prototile in $\mathcal{P}$. (For example, $B \_ B$ has length 3.) We show that $\widetilde{T} = \widetilde{T}(\mathcal{P})$ is a shift of finite type by showing that if $x \in \prod\{(k,\ell)\}$ and every solid L-word which appears in x appears in some point of $\widetilde{T}$, then $x \in \widetilde{T}$.

Suppose that every solid L-word which appears in x appears in some $y \in \widetilde{T}$. Let $x_i = (k,\ell)$. Since $p_{k,\ell_k} + 1 \leq L$, there exists $y \in \widetilde{T}$ such that

$$y_{i-p_{k,\ell}}, \dots, y_{i-p_{k,\ell}+p_{k,\ell_k}} = x_{i-p_{k,\ell}}, \dots, x_{i-p_{k,\ell}+p_{k,\ell_k}}.$$

But $y \in \widetilde{T}$ and $y_i = (k,\ell)$, so $y_{i-p_{k,\ell}+p_{k,\ell'}} = (k,\ell')$ for $1 \leq \ell' \leq \ell_k$. Hence $x \in \widetilde{T}$.

Informally, suppose that every solid L-word which appears in x appears in a subscripted tiling. Since no tile is longer than L, if x_i is an element of a tile, then the other elements of that tile appear in the appropriate places of x. Therefore $x \in \widetilde{T}$. $\square$

Corollary. *Every tiling system is sofic.*

Remark. We cannot use $L-1$ in the proof of the theorem. Again let $\mathcal{P} = \{R, BB\}$, so $\widetilde{T} = \widetilde{T}(R_1, B_1B_2)$ and $L = 2$. Every 1-word appearing in $x = \dots B_1B_1B_1 \dots$ appears in some point of $\widetilde{T}$, but $x \notin \widetilde{T}$.

Not every sofic system can be realized as a tiling system, as is shown by the following

Proposition. *A tiling system which has a point of period* 2 *must have at least two fixed points.*

Proof. The point of period 2 is $\dots abab \dots$, so there are two prototiles, each of which consists entirely of even integers or entirely of odd integers. Both tile the even integers and hence tile the integers. Then both $\dots aaa \dots$ and $\dots bbb \dots$ are in the tiling system. $\square$

Similarly, if a tiling system has a point of period 3 or one of period 4, then it must have at least one fixed point. The existence of a point of period greater than 4 does not imply the existence of a fixed point.

3. The main theorem

Main Theorem. *Let $\sigma : \Sigma \to \Sigma$ be a shift of finite type. Then there is a positive integer m and a tiling system $\sigma : T \to T$ such that*

(1) *$T = T_0 \cup T_1 \cup \dots \cup T_{m-1}$, where the T_i are closed and cyclically permuted by the shift σ.*
(2) *$\sigma^m : \Sigma \to \Sigma$ is topologically conjugate to every $\sigma^m : T_i \to T_i$.*

Proof. We may assume that $\Sigma = \Sigma_A$, the edge shift determined by a matrix A with nonnegative integer entries. A is the adjacency matrix of a directed graph G. The alphabet of Σ_A is the set of arcs (directed edges) of G and $x = (x_i) \in \Sigma_A$ if and only if for every i, the terminal vertex of x_i is the initial vertex of x_{i+1}.

For every positive integer m, $\sigma^m : \Sigma_A \to \Sigma_A$ is topologically conjugate to $\sigma : \Sigma_{A^m} \to \Sigma_{A^m}$. We find a tiling system $\sigma : T \to T$ and a positive integer m such that $T = T_0 \cup T_1 \cup \cdots \cup T_{m-1}$, the T_i are closed and cyclically permuted by the shift, and $\sigma : \Sigma_{A^m} \to \Sigma_{A^m}$ is topologically conjugate to $\sigma^m : T_0 \to T_0$. Hence $\sigma^m : \Sigma_A \to \Sigma_A$ is topologically conjugate to every $\sigma^m : T_i \to T_i$.

Suppose that A is $V \times V$. Choose $n > V$ so that

$$(V \max A_{ij})^{13n} < (n+1)!$$

Let $m = 13n$. Then every entry of $A^m = A^{13n}$ can be written (uniquely) as

$$c_1(1!) + c_2(2!) + \cdots + c_n(n!),$$

where $0 \leq c_k \leq k$ for $1 \leq k \leq n$.

We now construct the tiling system. The prototiles will be of two types: *barbells* and *racks* (to hold barbells). We will use the same terms for the corresponding tiles. In the sequel we will use colors to label prototiles. The symbols a, a' will stand for generic colors.

The barbells are the broken words of the form

$$a^2 \ \overset{\leftarrow\ 2r+1\ \rightarrow}{\_\ \cdots\ \_} \ a^2$$

for $0 \leq r \leq 2n-2$.

The racks are chosen from the broken words of the form HCT of length $13n+2J$, $1 \leq J \leq V$, where the *head* is

$$H = (a\ \_)^I a^{2n-2I}$$

for some I, $1 \leq I \leq V$; the *tail* is

$$T = (\_\ a)^J$$

for some J, $1 \leq J \leq V$; and the *center* is

$$C = a^{3n+i} \ \overset{\leftarrow 2k \rightarrow}{\_\ \cdots\ \_} \ a \ \overset{\leftarrow 2k \rightarrow}{\_\ \cdots\ \_} \ a^{8n-4k-1-i}$$

for some i and k, $0 \leq i \leq k-1$ and $1 \leq k \leq n$.

Given I, J with $1 \leq I, J \leq V$, write

$$(A^{13n})_{IJ} = c_1(1!) + c_2(2!) + \cdots + c_n(n!),$$

where $0 \le c_k \le k$ for $1 \le k \le n$. If $c_k \ne 0$, choose the racks to be the $c_k = c_k(I, J)$ broken words of the form

$$\left[(a\ \_)^I a^{2n-2I}\right] \left[a^{3n+i} \overset{\leftarrow 2k \rightarrow}{\_\cdots\_}\ a\ \overset{\leftarrow 2k \rightarrow}{\_\cdots\_}\ a^{8n-4k-1-i}\right] \left[(\_\ a)^J\right]$$

for $0 \le i \le c_k - 1$.

The barbells and racks have the following properties.

• The head $H = (a\ \_)^I a^{2n-2I}$ of a rack can be filled by the tail $T = (\_\ a')^J$ of a rack in a tiling if and only if $I = J$.

• Label the blanks in the center

$$C = a^{3n+i} \overset{\leftarrow 2k \rightarrow}{\_\cdots\_}\ a\ \overset{\leftarrow 2k \rightarrow}{\_\cdots\_}\ a^{8n-4k-1-i}$$

of a rack by $\{1, 2, \ldots, 4k\}$. Barbells can appear in a tiling only in the gaps in the centers of racks, starting only in odd places and straddling the a. Furthermore, the blanks in this center can be tiled by barbells in exactly $k!$ ways. To see this, define a permutation π of $\{1, 2, \ldots, k\}$ by $\pi(j) = \ell$ if and only if a barbell $a'a' \overset{\leftarrow 2r+1 \rightarrow}{\_\cdots\_}\ a'a'$ occupies places labelled $2j-1, 2j, 2(k+\ell)-1, 2(k+\ell)$.

• The heads of racks can appear in a tiling starting only at places which differ by multiples of $13n$.

Let T be the tiling system with prototiles the barbells and racks chosen above. Then $T = T_0 \cup T_1 \cup \cdots \cup T_{13n-1}$, where T_i is the set of indexed bi-infinite sequences in T in which the heads appear starting at places congruent to i modulo $13n$. Thus T_0 consists of all indexed bi-infinite concatenations of words of length $13n$, of the form $\bar{H}\bar{C}$, starting at multiples of $13n$, where H and C are the head and center of a rack, and $\bar{H}$ and $\bar{C}$ are the solid words resulting from filling them in a tiling. Recall that if $H = (a\ \_)^I a^{2n-2I}$, then it must be filled by a tail $T = (\_\ a')^I$. C can be filled only by barbells.

Define an edge shift as follows. Let G' be the directed graph with vertices $1, 2, \ldots, V$, and an arc from I to J for each rack with head $(a\ \_)^I a^{2n-2I}$, tail $(\_\ a)^J$, and center tiled by barbells. There are $(A^{13n})_{IJ}$ arcs from I to J, and an arc with head $(a\ \_)^I a^{2n-2I}$ can follow an arc with tail $(\_\ a')^J$ if and only if $I = J$. Therefore since $m = 13n$, the adjacency matrix of G' is A^m and $\sigma^m : T_0 \to T_0$ is topologically conjugate to $\sigma : \Sigma_{A^m} \to \Sigma_{A^m}$, which in turn is topologically conjugate to $\sigma^m : \Sigma_A \to \Sigma_A$. $\square$

Corollary. *The set of topological entropies of tiling systems is the same as that of shifts of finite type, i.e., the set of logarithms of roots of Perron numbers.*

References

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