A PIERI-TYPE FORMULA FOR ISOTROPIC FLAG MANIFOLDS

NANTEL BERGERON AND FRANK SOTTILE

Abstract. We give the formula for multiplying a Schubert class on an odd orthogonal or symplectic flag manifold by a special Schubert class pulled back from a Grassmannian of maximal isotropic subspaces. This is also the formula for multiplying a type B (respectively, type C) Schubert polynomial by the Schur P-polynomial p_m (respectively, the Schur Q-polynomial q_m). Geometric constructions and intermediate results allow us to ultimately deduce this from formulas for the classical flag manifold. These intermediate results are concerned with the Bruhat order of the Coxeter group $\mathcal{B}_\infty$, identities of the structure constants for the Schubert basis of cohomology, and intersections of Schubert varieties. We show these identities follow from the Pieri-type formula, except some 'hidden symmetries' of the structure constants. Our analysis leads to a new partial order on the Coxeter group $\mathcal{B}_\infty$ and formulas for many of these structure constants.

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Introduction

The cohomology of a flag manifold G/B has an integral basis of Schubert classes $\mathfrak{S}_w$ indexed by elements w of the Weyl group of G. The algebraic structure of these rings is known [9] with respect to a monomial basis, and there are methods (Schubert polynomials) for expressing the $\mathfrak{S}_w$ in terms of this basis [6, 7, 12, 17, 19, 25, 29]. Moreover, their multiplicative structure with respect to the Schubert basis is determined by Chevalley's formula [10]. Despite this, it remains an open problem to give a closed or bijective formula for the integral structure constants $c^w_{u\,v}$ defined by the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \;=\; \sum_w c^w_{u\,v}\, \mathfrak{S}_w .$$

These $c^w_{u\,v}$ are non-negative as they count the flags in a suitable triple intersection of Schubert varieties. They are expected to be related to the enumeration of chains in the Bruhat order of the Weyl group (see [3] and the references therein).

Of particular interest are Pieri-type formulas which describe the constants $c^w_{u\,v}$ when $\mathfrak{S}_v$ is a special Schubert class pulled back from a Grassmannian projection (G/P, P maximal parabolic), as these determine the ring structure for the cohomology of G/P when P is any parabolic subgroup. When G is $SL_n\mathbb{C}$, a Pieri-type formula for multiplication by a special Schubert class was described [25] in terms of the Weyl group element wu^{-1}. A formula in terms of chains in the Bruhat order was conjectured [1] and given a geometric proof [31]. Our main results are the analogous formulas when G is $Sp_{2n}\mathbb{C}$ or $So_{2n+1}\mathbb{C}$ and $\mathfrak{S}_v$ is a special Schubert class pulled back from a Grassmannian of maximal isotropic subspaces. These are common generalizations of the Pieri-type formulas for $SL_n\mathbb{C}$, Chevalley's formula, and Pieri-type formulas for Grassmannians of maximal isotropic subspaces [8].

Our proof uses results on the Bruhat order, identities of these structure constants, a decomposition of intersections of Schubert varieties, and formulas in the cohomology of the $SL_n\mathbb{C}$-flag manifold to explicitly determine a triple intersection of Schubert varieties. This shows the coefficients in the Pieri-type formula are the intersection number of a linear space with a collection of quadrics. Some intermediate results, including a fundamental identity and some additional 'hidden symmetries' of the structure constants, are deduced from constructions on $SL_n\mathbb{C}$-flag manifolds [3]. This analysis leads to other results, including a new partial order on the infinite Coxeter group $\mathcal{B}_\infty$ and a monoid for chains in this order as in [4]. We show how the Pieri-type formula implies our fundamental identity, use the identities to express many structure constants in terms of the Littlewood-Richardson coefficients for the multiplication of

Schur P– (or Q–) functions [33], and apply the hidden symmetries to the enumeration of chains in the Bruhat order.

1. Statement of results

Schubert classes in the cohomology of the flag manifolds $So_{2n+1}\mathbb{C}/B$ and $Sp_{2n}\mathbb{C}/B$ form integral bases indexed by elements of the Weyl group $\mathcal{B}_n$. We represent $\mathcal{B}_n$ as the group of permutations w of $\{-n, \ldots, -2, -1, 1, \ldots, n\}$ satisfying $w(-a) = -w(a)$ for $1 \leq a \leq n$. Let $\mathfrak{B}_w$ denote the Schubert class indexed by $w \in \mathcal{B}_n$ in $H^*So_{2n+1}\mathbb{C}/B$ and $\mathfrak{C}_w$ that in $H^*Sp_{2n}\mathbb{C}/B$. The degree of these classes is $2 \cdot \ell(w)$, where the *length* $\ell(w)$ of w is

$$\#\{0 < i < j \leq n \mid w(i) > w(j)\} \quad + \sum_{i>0>w(i)} |w(i)|.$$

For an integer i, let $\overline{\imath}$ denote $-i$. For each $1 \leq m \leq n$, define $v_m \in \mathcal{B}_n$ by

$$\overline{m} = v_m(1) < 0 < v_m(2) < \cdots < v_m(n).$$

This indexes a (maximal isotropic) special Schubert class in either cohomology ring, written as $p_m := \mathfrak{B}_{v_m}$ and $q_m := \mathfrak{C}_{v_m}$. We state the Pieri-type formula for the products $\mathfrak{B}_w \cdot p_m$ and $\mathfrak{C}_w \cdot q_m$ in terms of chains in the Bruhat order on $\mathcal{B}_n$. For this, we need a definition.

Definition 1.1. *The* 0-Bruhat order $\leq_0$ *on* $\mathcal{B}_n$ *is defined recursively as follows:* $u \lessdot_0 w$ *is a cover in the* 0*-Bruhat order if and only if*

(1) $\ell(u) + 1 = \ell(w)$, *and*
(2) $u^{-1}w$ *is a reflection of the form* $(\overline{\imath},\, i)$ *or* $(\overline{\imath},\, j)(\overline{\jmath},\, i)$ *for some* $0 < i < j \leq n$.

Chevalley's formula [10] may be stated as follows:

$$\begin{aligned} \mathfrak{B}_u \cdot p_1 &= \sum_{u \lessdot_0 w} \mathfrak{B}_w \\ \mathfrak{C}_u \cdot q_1 &= \sum_{u \lessdot_0 w} \chi(u^{-1}w)\mathfrak{C}_w, \end{aligned} \tag{1}$$

where $\chi(u^{-1}w)$ is the number of transpositions in the reflection $u^{-1}w$.

We enrich the 0-Bruhat order in two complementary ways. Write the two types of covers in the 0-Bruhat order as $u \lessdot_0 (\overline{\beta},\, \beta)u$ and $u \lessdot_0 (\overline{\beta},\, \overline{\alpha})(\alpha,\, \beta)u$ where $0 < \alpha < \beta \leq n$. The *labeled* 0*-Bruhat réseau* is a labeled directed multigraph with vertex set $\mathcal{B}_n$ and labeled edges between covers in the 0-Bruhat order given by the following rule: If $u \lessdot_0 (\overline{\beta},\, \beta)u$, then a single edge is drawn with label β. If $u \lessdot_0 (\overline{\beta},\, \overline{\alpha})(\alpha,\, \beta)u$, then two edges are drawn with respective labels $\overline{\alpha}$ and β. Thus if $u \lessdot_0 w$, then $\chi(u^{-1}w)$ counts the edges from u to w in this 0-Bruhat réseau. The *labeled* 0*-Bruhat order* is obtained from this réseau by removing edges with negative integer labels.

Given a (saturated) chain γ in either of these structures, let $\operatorname{end}(\gamma)$ denote the endpoint of γ. A *peak* in a chain γ is an index $i \in \{2, \ldots, m{-}1\}$ with $a_{i-1} < a_i > a_{i+1}$, where $a_1, a_2, \ldots, a_m$ is the sequence of edge labels in γ. A *descent* is an index $i < m$ with $a_i > a_{i+1}$ and an *ascent* is an index $i < m$ with $a_i < a_{i+1}$.

Theorem A. (Pieri-type formula) *Let $u \in \mathcal{B}_n$ and $m > 0$. Then*

I. (Odd-orthogonal Pieri-type formula)

$$\mathfrak{B}_u \cdot p_m \;=\; \sum \mathfrak{B}_{\mathrm{end}(\gamma)},$$

the sum over all chains γ in the labeled 0*-Bruhat order of $\mathcal{B}_n$ which begin at u, have length m, and no peaks.*

II. (Symplectic Pieri-type formula)

$$\mathfrak{C}_u \cdot q_m \;=\; \sum \mathfrak{C}_{\mathrm{end}(\gamma)},$$

(a) *the sum over all chains γ in the labeled* 0*-Bruhat réseau of $\mathcal{B}_n$ which begin at u, have length m, and no descents.*

(b) *the same sum, except with no ascents.*

This generalizes Chevalley's formula and the Pieri-type formulas for $SL_n\mathbb{C}/B$, which are expressed in [31] as a sum of certain labeled chains in the Bruhat order on the symmetric group $\mathcal{S}_n$ with no ascents/no descents. The duality of these two formulas, one in terms of peaks for an order, and the other in terms of descents/ascents for an enriched structure on that order has connections with other dualities in combinatorics. These include Fomin's duality of graded graphs [15, 16] and Stembridge's theory of enriched P-partitions [34], where peak and descent sets play a complementary role. These relations are explored in [2], which extends the theory developed in [5] to the ordered structures of this manuscript.

Example 1.2. *Represent permutations $w \in \mathcal{B}_3$ by their values $w(1)\,w(2)\,w(3)$. Consider the products $\mathfrak{B}_{3\bar{1}2} \cdot p_2$ and $\mathfrak{C}_{3\bar{1}2} \cdot q_2$. Figure 1 shows the part of the 0-Bruhat réseau of height 2 above $3\bar{1}2$ in $\mathcal{B}_3$. (Erase edges with negative labels to obtain its*

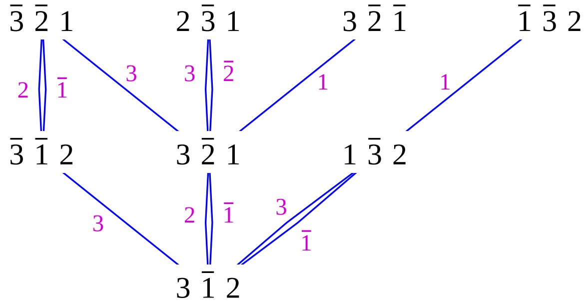


FIGURE 1. Chains above $3\bar{1}2$

analog in the 0-Bruhat order.) Chains of length 2 are peakless, so by Theorem A I, we have

$$\mathfrak{B}_{3\bar{1}2} \cdot p_2 \;=\; 2\,\mathfrak{B}_{\bar{3}\bar{2}1} + \mathfrak{B}_{2\bar{3}1} + \mathfrak{B}_{3\bar{2}\bar{1}} + \mathfrak{B}_{\bar{1}\bar{3}2}.$$

Every chain in Figure 1 with increasing labels may be paired with a chain with decreasing labels having the same underlying permutations, and this pairing exhausts all chains. Thus, by Theorem A II, we have

$$\mathfrak{C}_{3\bar{1}2} \cdot q_2 \;=\; 2\,\mathfrak{C}_{\bar{3}\bar{2}1} + 2\,\mathfrak{C}_{2\bar{3}1} + \mathfrak{C}_{3\bar{2}\bar{1}} + \mathfrak{C}_{\bar{1}\bar{3}2}.$$

If λ is a *strict partition* (decreasing integral sequence $n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$), then λ determines a unique *Grassmannian permutation* $v(\lambda) \in \mathcal{B}_n$ where $v(i) = \overline{\lambda_i}$ for $i \leq k$ and $0 < v(k+1) < \cdots < v(n)$. If $k = 1$ and $\lambda_1 = m$, then $v_m = v(\lambda)$. The Schubert classes $P_\lambda := \mathfrak{B}_{v(\lambda)}$ and $Q_\lambda := \mathfrak{C}_{v(\lambda)}$ are pullbacks of Schubert classes from the Grassmannians of maximal isotropic subspaces $So_{2n+1}\mathbb{C}/P_0$ and $Sp_{2n}\mathbb{C}/P_0$, where P_0 is the maximal parabolic associated to the simple root of exceptional length.

Formulas for products of these P- and Q-classes are known [33] as these classes are specializations of Schur P- and Q-functions [21, 28]. Our proof of Theorem A uses identities among the structure constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$ defined by the following formulas.

$$\mathfrak{B}_u \cdot P_\lambda \;=\; \sum_w b^w_{u\,\lambda}\, \mathfrak{B}_w \qquad \text{and} \qquad \mathfrak{C}_u \cdot Q_\lambda \;=\; \sum_w c^w_{u\,\lambda}\, \mathfrak{C}_w$$

If $u, w, v(\lambda) \in \mathcal{B}_n$, then these constants do not depend upon n.

Iterating Chevalley's formula (1) shows that if either of $b^w_{u\,\lambda}$ or $c^w_{u\,\lambda}$ is non-zero, then $u <_0 w$ and $\ell(w) - \ell(u)$ equals $|\lambda|$, the sum of the parts of λ. In fact the constant $b^w_{u\,\lambda}$ determines and is determined by the constant $c^w_{u\,\lambda}$: Let $s(w)$ count the sign changes ($\{i \mid i > 0 > w(i)\}$) in w. Then the map $\mathfrak{C}_w \mapsto 2^{s(w)}\mathfrak{B}_w$ embeds $H^*Sp_{2n}\mathbb{C}/B$ into $H^*So_{2n+1}\mathbb{C}/B$ and induces an isomorphism of their rational cohomology rings. Thus it suffices to work in $H^*Sp_{2n}\mathbb{C}/B$. This is fortunate, as a key geometric result, Theorem 3.4(2), holds only for $Sp_{2n}\mathbb{C}/B$.

Let f^w_u count the saturated chains in the interval $[u, w]_0$ and g^w_u count the saturated chains in the réseau $[u, w]_0$. Iterating Chevalley's formula (1) with $u = e$, the identity permutation, we obtain the following expressions.

$$p_1^m \;=\; \sum_{|\lambda|=m} f_e^{v(\lambda)}\, P_\lambda \qquad \text{and} \qquad q_1^m \;=\; \sum_{|\lambda|=m} g_e^{v(\lambda)}\, Q_\lambda$$

Multiplying the first expression by $\mathfrak{B}_u$ and collecting the coefficients of $\mathfrak{B}_w$ in the resulting expansion (likewise for the second expression) gives the following proposition.

Proposition 1.3. *Let $u, w \in \mathcal{B}_n$. Then*

$$f^w_u \;=\; \sum_{|\lambda|=\ell(w)-\ell(u)} f_e^{v(\lambda)}\, b^w_{u\,\lambda} \qquad \textit{and} \qquad g^w_u \;=\; \sum_{|\lambda|=\ell(w)-\ell(u)} g_e^{v(\lambda)}\, c^w_{u\,\lambda}.$$

Theorem A and Proposition 1.3 show a close connection between chains in the 0-Bruhat order/réseau and the structure constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$. This justifies an elucidation of the basic properties of the 0-Bruhat order and réseau, which we do in Sections 2 and 6. These structures have a remarkable property and there are related fundamental identities among the structure constants.

Theorem B. *Suppose $u <_0 w$ and $x <_0 z$ in $\mathcal{B}_n$ with $wu^{-1} = zx^{-1}$. Then*

(1) *The map $v \mapsto vu^{-1}x$ induces an isomorphism of labeled intervals in the 0-Bruhat order and 0-Bruhat réseau $[u, w]_0 \xrightarrow{\sim} [x, z]_0$.*

(2) *For any strict partition* λ,

$$b^w_{u\,\lambda} \;=\; b^z_{x\,\lambda} \qquad \text{and} \qquad c^w_{u\,\lambda} \;=\; c^z_{x\,\lambda}.$$

We prove Theorem B(1) in Section 2.1 using combinatorial methods. Theorem B(2) is a consequence of a geometric result (Theorem 3.3) proven in Section 4. Both parts of Theorem B are key to our proof of the Pieri-type formula. Interestingly, the Pieri-type formula and Theorem B(1) together imply Theorem B(2):

For any composition $\alpha = (\alpha_1, \ldots, \alpha_s)$ with each $\alpha_i \geq 0$, let $p_\alpha := p_{\alpha_1} \cdots p_{\alpha_s}$, $q_\alpha := q_{\alpha_1} \cdots q_{\alpha_s}$, and $I(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{s-1}\}$. The *peak set* of a (maximal) chain in a labeled order is the set of indices of peaks in the chain. Given a chain in a labeled réseau, its *descent set* (respectively *ascent set*) is the set of indices of descents (respectively ascents) in the chain.

Corollary 1.4. *Let* $u, w, x, z \in \mathcal{B}_n$.

(1) *Let* α *be any composition. Then the coefficient of* $\mathfrak{B}_w$ *in the product* $\mathfrak{B}_u \cdot p_\alpha$ *is the number of chains in the interval* $[u, w]_0$ *in the labeled* 0*-Bruhat order with peak set contained in* $I(\alpha)$.

(2) *Let* α *be any composition. Then the coefficient of* $\mathfrak{C}_w$ *in the product* $\mathfrak{C}_u \cdot q_\alpha$ *is the number of chains in the interval* $[u, w]_0$ *in the labeled* 0*-Bruhat réseau with descent set contained in* $I(\alpha)$. *This is also the number with ascent set contained in* $I(\alpha)$.

(3) *Suppose the Pieri-type formula (Theorem A) holds. Then the intervals* $[u, w]_0$ *and* $[x, z]_0$ *have the some number of chains with peak set* $I(\alpha)$ *for every composition* α *if and only if for every strict partition* λ, $b^w_{u\,\lambda} = b^z_{x\,\lambda}$. *The same statement holds for ascent/descent sets for chain in the réseaux and the coefficients* $c^w_{u\,\lambda}, c^z_{x\,\lambda}$. *In particular, Theorem B(1) implies Theorem B(2).*

Moreover, the numbers in 1 and 2 depend only upon the multiset $\{\alpha_1, \ldots, \alpha_s\}$.

Parts 1 and 2 follow from Theorem A. For 3, note that the Schur P-polynomials (respectively Q-polynomials) are linear combinations of the p_α (respectively the q_α) [27, III.8.6]. This linear combination gives a formula for $b^w_{u\,\lambda}$ (respectively $c^w_{u\,\lambda}$) in terms of chains with given peak sets (respectively, given ascent/descent sets).

Let $\zeta \in \mathcal{B}_n$. By Theorem B(1), we may define $\eta \preceq \zeta$ if there is a $u \in \mathcal{B}_n$ with $u \leq_0 \eta u \leq_0 \zeta u$ and $\mathcal{L}(\zeta) := \ell(\zeta u) - \ell(u)$ whenever $u \leq_0 \zeta u$. Then $(\mathcal{B}_n, \prec)$ is a graded partial order with rank function $\mathcal{L}(\cdot)$. By the identity of Theorem B(2), we may define $b^\zeta_\lambda := b^{\zeta u}_{u\,\lambda}$ and $c^\zeta_\lambda := c^{\zeta u}_{u\,\lambda}$ for any $u \in \mathcal{B}_n$ with $u \leq_0 \zeta u$ and $|\lambda| = \mathcal{L}(\zeta)$.

These coefficients satisfy one obvious identity, $c^\zeta_\lambda = c^{\zeta^{-1}}_\lambda$, as $c^w_{u\,v} = c^{\omega_0 w}_{\omega_0 u\,v}$ where $\omega_0 \in \mathcal{B}_n$ is the longest element. They also satisfy two others, which we call hidden symmetries. Let $\rho \in \mathcal{B}_n$ be the permutation defined by $\rho(i) = i - 1 - n$ for $1 \leq i \leq n$. Then ρ is the element with largest rank in $(\mathcal{B}_n, \prec)$. Let $\gamma \in \mathcal{B}_n$ be defined by $\gamma(1) = 2, \gamma(2) = 3, \ldots, \gamma(n) = 1$, so that $\gamma = (\overline{1}, \overline{2}, \ldots, \overline{n})(1, 2, \ldots, n)$.

Theorem C. *For any* $\zeta \in \mathcal{B}_n$,

(1) $\mathcal{L}(\zeta) = \mathcal{L}(\rho\zeta\rho)$ *and for any strict partition* λ, *we have* $b^\zeta_\lambda = b^{\rho\zeta\rho}_\lambda$ *and* $c^\zeta_\lambda = c^{\rho\zeta\rho}_\lambda$.

(2) *If* $a\cdot\zeta(a)>0$ *for all* a, *then* $\mathcal{L}(\zeta)=\mathcal{L}(\gamma\zeta\gamma^{-1})$ *and for any strict partition* λ, *we have* $b^{\zeta}_{\lambda} = b^{\gamma\zeta\gamma^{-1}}_{\lambda}$ *and* $c^{\zeta}_{\lambda} = c^{\gamma\zeta\gamma^{-1}}_{\lambda}$.

We prove a strengthening of Theorem B, relaxing the condition of equality of wu^{-1} and zx^{-1} to that of *shape equivalence.* Permutations $\eta,\zeta\in\mathcal{B}_n$ are shape equivalent if there exist sets $I: 0<i_1<\cdots<i_s\leq n$ and $J: 0<j_1<\cdots<j_s\leq n$ such that η acts as the identity on $\{1,\ldots,n\}\setminus I$, ζ acts as the identity on $\{1,\ldots,n\}\setminus J$, and $\eta(i_k)=i_l$ if and only if $\zeta(j_k)=j_l$.

Theorems B (the stronger version) and C allow us to determine many of the constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$, showing they equal certain Littlewood-Richardson coefficients $b^{\kappa}_{\mu\,\lambda}$ and $c^{\kappa}_{\mu\,\lambda}$for Schur P- and Q-functions. These are defined by the identities

$$P_\mu\cdot P_\lambda \;=\; \sum_\kappa b^{\kappa}_{\mu\,\lambda}\,P_\kappa \qquad\text{and}\qquad Q_\mu\cdot Q_\lambda \;=\; \sum_\kappa c^{\kappa}_{\mu\,\lambda}\,Q_\kappa.$$

A combinatorial formula for these coefficients was given by Stembridge [33].

Definition 1.5. *Let* μ,κ *be strict partitions with* $\mu\subset\kappa$. *We say that a permutation* $\zeta\in\mathcal{B}_n$ *has skew shape* κ/μ *if*

(1) *Either* ζ *or* ζ^ρ *is shape equivalent to* $v(\kappa)v(\mu)^{-1}$, *or*
(2) *If* $a\cdot\zeta(a)>0$ *for all* a, *and one of* $\zeta,\zeta^\gamma,\zeta^{\gamma^2},\ldots,\zeta^{\gamma^{n-1}}$ *is shape equivalent to* $v(\kappa)v(\mu)^{-1}$.

Corollary 1.6. *If* $u\leq_0 w$ *are permutations in* $\mathcal{B}_n$ *and* wu^{-1} *has a skew shape* κ/μ, *then for any strict partition* λ *we have*

$$b^w_{u\,\lambda} \;=\; b^{\kappa}_{\mu\,\lambda} \qquad\textit{and}\qquad c^w_{u\,\lambda} \;=\; c^{\kappa}_{\mu\,\lambda}.$$

We call the partial order $\prec$ the *Lagrangian order* and transfer the labeling from the 0-Bruhat order to obtain the labeled Lagrangian order. In the same fashion, we may transfer the labeling and multiple edges of the 0-Bruhat réseau to $(\mathcal{B}_n,\prec)$, obtaining the (labeled) *Lagrangian réseau.* By Corollary 1.4(3), Theorem C has a purely enumerative corollary.

Corollary 1.7. *For any* $\zeta\in\mathcal{B}_n$,

(1) *For any subset* S *of* $\{2,\ldots,\mathcal{L}(\zeta)-1\}$, *the intervals* $[e,\zeta]_\prec$ *and* $[e,\rho\zeta\rho]_\prec$ *in the Lagrangian order have the same number of chains with peak set* S.
(2) *For any subset* S *of* $\{1,\ldots,\mathcal{L}(\zeta)-1\}$, *the intervals* $[e,\zeta]_\prec$ *and* $[e,\rho\zeta\rho]_\prec$ *in the Lagrangian réseau order have the same number of chains with descent set* S *and the same number of chains with ascent set* S, *and these two numbers are equal.*
(3) *If* $a\cdot\zeta(a)>0$ *for all* a, *then the same is true for* $[e,\zeta]_\prec$ *and* $[e,\gamma\zeta\gamma^{-1}]_\prec$.

In general, $[e,\zeta]_\prec\not\simeq[e,\rho\zeta\rho]_\prec$, (See Figure 6 in Example 5.2) and if $a\cdot\zeta(a)>0$ for all a, then in general, $[e,\zeta]_\prec\not\simeq[e,\gamma\zeta\gamma^{-1}]_\prec$. (See Figure 7 in Example 5.12.)

Let $\mathrm{supp}(\zeta):=\{a>0\mid\zeta(a)\neq a\}$, the *support* of ζ. A permutation $\zeta\in\mathcal{B}_n$ is *reducible* if it has a non-trivial factorization $\zeta=\eta\cdot\xi$ with $\mathcal{L}(\zeta)=\mathcal{L}(\eta)+\mathcal{L}(\xi)$ where η and ξ have disjoint supports ($\eta\cdot\xi=\xi\cdot\eta$). If $\eta\cdot\xi=\xi\cdot\eta$ with $\mathcal{L}(\xi\cdot\eta)=\mathcal{L}(\eta)+\mathcal{L}(\xi)$, then $\eta\cdot\xi$ is a *disjoint product.* Permutations $w\in\mathcal{B}_n$ have unique factorizations into irreducibles.

We represented $\mathcal{B}_n \subset \mathcal{S}_{\pm[n]}$, the group of permutations of $\{-n, \ldots, -1, 1, \ldots, n\}$. We also have $\mathcal{S}_n \hookrightarrow \mathcal{B}_n$, and the image consists of those ζ with $a \cdot \zeta(a) > 0$ for every a: For $\eta \in \mathcal{S}_n$, let $\overline{\eta} \in \mathcal{S}_{-[n]}$ be the permutation such that $\overline{\eta}(-i) = -\eta(i)$. Then $\eta\overline{\eta} \in \mathcal{B}_n$. For $\zeta \in \mathcal{B}_n$, define $\delta(\zeta) = 1$ if ζ is in the image of $\mathcal{S}_n$, and $\delta(\zeta) = 0$ otherwise.

In Section 6.1 we establish the following result.

Lemma 1.8. *Let $\zeta \in \mathcal{B}_n$ and suppose ζ is irreducible. Then $\mathcal{L}(\zeta) \geq \#\mathrm{supp}(\zeta) - \delta(\zeta)$. If $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$, then either*

(1) *we have $\delta(\zeta) = 1$ and there exists a cycle $\eta \in \mathcal{S}_n$ with $\zeta = \eta\overline{\eta}$, or*
(2) *we have $\delta(\zeta) = 0$ and ζ is a single cycle in $\mathcal{S}_{\pm[n]}$.*

Definition 1.9. *If every irreducible factor η of ζ satisfies $\mathcal{L}(\eta) = \#\mathrm{supp}(\eta) - \delta(\eta)$, then we say that ζ is minimal.*

If $\zeta \in \mathcal{B}_n$ is minimal, then set

$$\begin{aligned}\theta(\zeta) &:= 2^{\#\{\textit{irreducible factors of } \zeta\}-1},\\ \chi(\zeta) &:= 2^{\#\{\textit{irreducible factors } \eta \textit{ of } \zeta \textit{ with } \delta(\eta)=1\}}.\end{aligned}$$

If ζ is not minimal, then set $\theta(\zeta) = \chi(\zeta) = 0$.

We state the Pieri-type formula in terms of the permutation wu^{-1}.

Theorem D. *Let $u, w \in \mathcal{B}_n$ and $m \leq n$. Then*

$$\mathfrak{B}_u \cdot p_m = \sum \theta(wu^{-1})\mathfrak{B}_w \qquad \textit{and} \qquad \mathfrak{C}_u \cdot q_m = \sum \chi(wu^{-1})\mathfrak{C}_w,$$

the sum over all $w \in \mathcal{B}_n$ with $u \leq_0 w$ and $\ell(w) - \ell(u) = m$.

This is similar to the form of the Pieri-type formula for $SL_n\mathbb{C}/B$ in [25], which is in terms of the cycle structure of the permutation wu^{-1}. This also generalizes the form of the the Pieri formula for Grassmannians of maximal isotropic subspaces [8]. Our proof for $Sp_{2n}\mathbb{C}/B$ shows these multiplicities to arise from the intersection of a linear subspace of $\mathbb{P}^{2n-1}$ with a collection of quadrics, one for each irreducible factor η of wu^{-1} with $\delta(\eta) = 1$, similar to the proof of the Pieri-type formula for maximal isotropic Grassmannians in [32]. We relate the two formulations (Theorems A and D) of the Pieri-type formula in sections 6.3 and 6.4.

Example 1.10. *We return to Example 1.2. For $u = 3\overline{1}2$ and w equal to each of $\overline{3}\,\overline{2}1$, $2\overline{3}1$, $32\overline{1}$, and $\overline{1}\,\overline{3}2$ in turn, wu^{-1} is the permutation in $\mathcal{S}_{\pm[3]}$:*

$$(12)(\overline{1}\,\overline{2})\,(3\overline{3}), \quad (132)(\overline{1}\,\overline{3}\,\overline{2}), \quad (12\overline{1}\,\overline{2}), \quad \textit{and} \quad (13\overline{1}\,\overline{3}).$$

As permutations in $\mathcal{B}_3$, the first has 2 irreducible factors, for these we have $\delta((12)(\overline{1}\,\overline{2})) = 1$ and $\delta((3\overline{3})) = 0$. The other permutations are irreducible with $\delta((132)(\overline{1}\,\overline{3}\,\overline{2})) = 1$ and $\delta((12\overline{1}\,\overline{2})) = \delta((13\overline{1}\,\overline{3})) = 0$. Thus the values of θ are $2, 1, 1, 1$ and of χ are $2, 2, 1, 1$, which shows the two forms of the Pieri-type formula agree on this example.

We remark that the last two permutations, $(12\overline{1}\,\overline{2})$ and $(13\overline{1}\,\overline{3})$, are shape equivalent.

While we use the cohomology rings of complex varieties, our results and methods are valid for the Chow rings [18] and l-adic (étale) cohomology [11] of these same varieties over any field not of characteristic 2.

This paper is organized as follows: Section 2 contains basic combinatorial definitions and properties of the Bruhat order on $\mathcal{B}_\infty$ analogous to those of the symmetric group established in [3, 4]. Section 3 contains the basic geometric definitions. In Section 4, we use geometry to establish the main identity, Theorem B(2). In Section 5, we establish additional geometric results and prove Theorem C. In Section 6, we establish further combinatorial properties of the Lagrangian order and réseau needed for the proof of the Pieri-type formula, which is given in Section 7.

2. Orders on $\mathcal{B}_\infty$

We derive the basic properties of the 0-Bruhat order on $\mathcal{B}_\infty$ analogous to properties of the k-Bruhat order on $\mathcal{S}_\infty$. Further properties are developed in Section 6.

Let $\#S$ be the cardinality of a finite set S. For an integer j, its absolute value is $|j|$ and let $\overline{\jmath} := -j$. Likewise, for a set P of integers, define $\overline{P} := \{\overline{\jmath} \mid j \in P\}$ and $\pm P := P \cup \overline{P}$. Set $[n] := \{1, \ldots, n\}$ and let $\mathcal{S}_{\pm[n]}$ be the group of permutations of $\pm[n]$. Let e be the identity permutation in $\mathcal{S}_{\pm[n]}$ and ω_0 the longest element in $\mathcal{S}_{\pm[n]}$: $\omega_0(i) = \overline{\imath}$. Then $\mathcal{B}_n$ is the subgroup of $\mathcal{S}_{\pm[n]}$ for which $\omega_0 w \omega_0 = w$ and $\omega_0 \in \mathcal{B}_n$. We also have $\mathcal{B}_n \subset \mathcal{S}_{[\overline{n},n]}$, the symmetric group on $[\overline{n}, n] := \pm[n] \cup \{0\}$. We refer to elements of these groups as permutations. Permutations $w \in \mathcal{B}_n$ are often represented by their values $w(1)\, w(2)\, \ldots\, w(n)$. For example, $2\,4\,\overline{3}\,\overline{1} \in \mathcal{B}_4$. The *length* $\ell(w)$ of $w \in \mathcal{B}_n$ is

$$\ell(w) \quad = \quad \#\{0 < i < j \mid w(i) > w(j)\} \;+ \sum_{i>0>w(i)} |w(i)|.$$

Thus $\ell(2\,4\,\overline{3}\,\overline{1}) = 4 + 4 = 8$. Note that ω_0 is the longest element in $\mathcal{B}_n$.

An important class of permutations are the *Grassmannian permutations*, those $v \in \mathcal{B}_n$ for which $v(1) < v(2) < \cdots < v(n)$. Such a permutation is determined by its initial negative values. If $v(k) < 0 < v(k+1)$, define $\lambda(v)$ to be the decreasing sequence $\overline{v(1)} > \overline{v(2)} > \cdots > \overline{v(k)}$. Note that $\ell(v) = \overline{v(1)} + \cdots + \overline{v(k)} =: |\lambda(v)|$. Likewise, given a decreasing sequence μ of positive integers (a *strict partition*) with $n \geq \mu_1$, let $v(\mu)$ be the Grassmannian permutation with $\lambda(v(\mu)) = \mu$. We write $\mu \subset \lambda$ for strict partitions μ, λ if $\mu_i \leq \lambda_i$ for all i, equivalently, if $v(\mu) \leq_0 v(\lambda)$.

The inclusion $\pm[n] \hookrightarrow \pm[n+1]$ induces inclusions $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ and $\mathcal{S}_{\pm[n]} \hookrightarrow \mathcal{S}_{\pm[n+1]}$. Define $\mathcal{B}_\infty := \bigcup_n \mathcal{B}_n$ and $\mathcal{S}_{\pm\infty} := \bigcup_n \mathcal{S}_{\pm[n]}$.

In this representation, $\mathcal{B}_\infty$ has three types of reflections, which are, as elements of $\mathcal{S}_{\pm\infty}$:

$$\begin{array}{lcll} t_{i\,j} & := & (\overline{\jmath},\, \overline{\imath})(i,\, j) & \\ t_j & := & (\overline{\jmath},\, j) & \qquad \text{for } 0 < i < j. \\ t_{\overline{\imath}\,j} & := & (\overline{\jmath},\, i)(\overline{\imath},\, j) & \end{array}$$

These reflections act on positions on the right and on values on the left. The *Bruhat order* on $\mathcal{B}_\infty$ is defined by its covers: $u \lessdot w$ if $\ell(u) + 1 = \ell(w)$ and $u^{-1}w$ is a reflection.

For each $k = 0, 1, \ldots,$ define the *k-Bruhat order* (on $\mathcal{B}_n$ or $\mathcal{B}_\infty$) by its covers: Set $u \lessdot_k w$ if $u \lessdot w$ and

$$u^{-1}w \quad \text{is one of} \quad \begin{cases} t_{ij} & \text{with } i \le k < j, \quad \text{or} \\ t_j & \text{with } k < j, \quad \text{or} \\ t_{\bar{\imath}j} & \text{with } k < j. \end{cases}$$

For example, Figure 2 shows all covers $w \in \mathcal{B}_4$ of $u = 2\,4\,\overline{3}\,\overline{1}$, the reflection $u^{-1}w$, and for which k this is a cover in the k-Bruhat order.

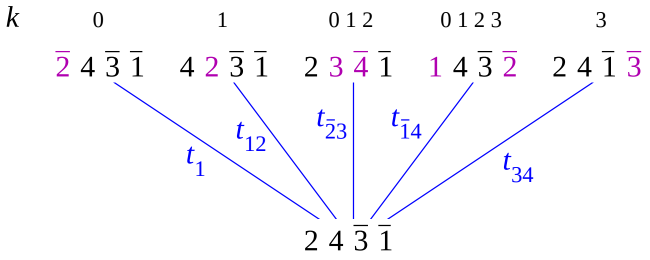


FIGURE 2. Covers of $2\,4\,\overline{3}\,\overline{1}$

2.1. **The 0-Bruhat order.** While these orders are analogous to the k-Bruhat orders on $\mathcal{S}_\infty$ [3, 4, 26, 31], only the 0-Bruhat order on B_∞ enjoys most properties of the k-Bruhat orders on $\mathcal{S}_\infty$. This is because the 0-Bruhat order is an induced suborder of the 0-Bruhat order on $\mathcal{S}_{\pm\infty}$.

The *length* $l(w)$ of a permutation $w \in \mathcal{S}_{\pm\infty}$ counts the inversions of w:

$$l(w) \;:=\; \#\{i < j \mid w(i) > w(j)\}.$$

The Bruhat order $(\lessdot)$ on $\mathcal{S}_{\pm\infty}$ is defined by its covers: $u \lessdot w$ if and only if wu^{-1} is a transposition and $l(w) = l(u) + 1$. If $k \in \mathbb{Z}$, this is a cover (written $\lessdot_k$) in the k-Bruhat order $(\lessdot_k)$ on $\mathcal{S}_{\pm\infty}$ (or $\mathcal{S}_{\pm[n]}$) if $wu^{-1} = (a, b)$ with $a < k < b$. The k-Bruhat order has a non-recursive characterization, needed below:

Proposition 2.1 ([3], Theorem A)**.** *Let $u, w \in \mathcal{S}_{\pm\infty}$ and $k \in \mathbb{Z}$. Then $u \lessdot_k w$ if and only if*

(1) *$a < k < b$ implies $u(a) \le w(a)$ and $u(b) \ge w(b)$.*
(2) *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a < k < b$.*

For the remainder of this section, we will be concerned with the case $k = 0$.

Theorem 2.2. *The 0-Bruhat order on $\mathcal{B}_\infty$ is the order induced from the 0-Bruhat order on $\mathcal{S}_{\pm\infty}$ by the inclusion $\mathcal{B}_\infty \hookrightarrow \mathcal{S}_{\pm\infty}$.*

Proof. For $u, w \in \mathcal{B}_\infty$, it is straightforward to verify

$$u \lessdot_0 ut_j \iff u \lessdot_0 u(\bar{\jmath}, j)$$

and

$$u \lessdot_0 ut_{\bar{\imath}j} \iff u \lessdot_0 u(\bar{\jmath}, i) \lessdot_0 u(\bar{\jmath}, i)(\bar{\imath}, j).$$

Thus $u <_0 w \Rightarrow u \lessdot_0 w$, and so $<_0$ is a suborder of $\lessdot_0$.

To show this suborder is induced, suppose that $u \lhd_0 w$ with $u, w \in \mathcal{B}_\infty$, and argue by induction on $l(w) - l(u)$. Suppose $u \lessdot_0 v \lhd_0 w$. If $v = u(\bar{\jmath}, j) = ut_j$, then $v \in \mathcal{B}_n$ and we are done by induction.

Suppose now that $v = u(\bar{\jmath}, i) \notin \mathcal{B}_n$. By the involution $x \mapsto \omega_0 x \omega_0$ of $(\mathcal{S}_{\pm\infty}, \lhd_0)$, $\omega_0 v \omega_0 = u(\bar{\imath}, j)$ also satisfies $u \lessdot_0 \omega_0 v \omega_0 \lhd_0 w$. The criteria of Proposition 2.1 show that either $0 < u(i)$ and $0 < u(j)$ so that $u \lessdot_0 u(\bar{\jmath}, j) \lhd_0 w$, or else $u(j) \cdot u(i) < 0$ and so $v \lessdot_0 v(\bar{\imath}, j) \lhd_0 w$. In the first case, $u(\bar{\jmath}, j) = ut_j \in \mathcal{B}_n$, and in the second, $v(\bar{\imath}, j) = ut_{\bar{\imath},j} \in \mathcal{B}_n$, which completes the proof. ∎

Remark 2.3. If $u(\bar{\jmath}, i) \lessdot_0 u$, then either $ut_{ij} \lessdot_0 u$ or else both $ut_i \lessdot_0 u$ and $ut_j \lessdot_0 u$.

Example 2.4. *We illustrate Theorem 2.2 in Figure 3. There, the elements of $\mathcal{B}_3$ are boxed.*

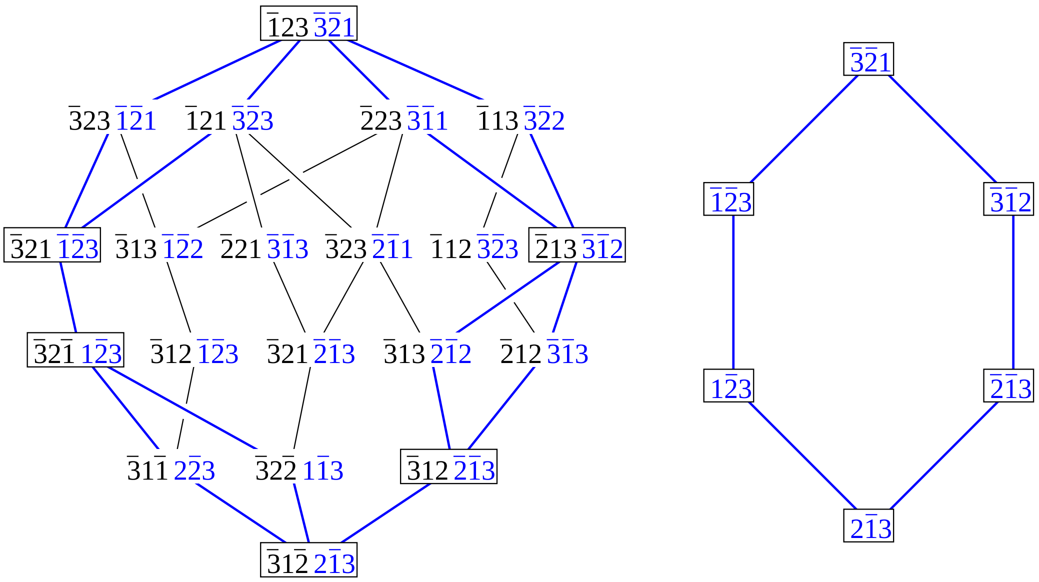


FIGURE 3. The intervals $[\bar{3}\,1\,\bar{2}\,2\,\bar{1}\,3,\ \bar{1}\,2\,3\,\bar{3}\,\bar{2}\,1]_{\lhd_0}$ and $[2\,\bar{1}\,3,\ \bar{3}\,\bar{2}\,1]_0$.

This relation between the two partial orders $(\mathcal{B}_\infty, <_0)$ and $(\mathcal{S}_{\pm\infty}, \lhd_0)$ makes many properties of $(\mathcal{B}_\infty, <_0)$ easy corollaries of the analogous results for $(\mathcal{S}_{\pm\infty}, \lhd_0)$ (established in [3, 4]). We discuss these properties in the remainder of this section, omitting proofs.

The 0-Bruhat order has a non-recursive characterization:

Proposition 2.5. *Let $u, w \in \mathcal{B}_\infty$. Then $u \leq_0 w$ if and only if*

(1) $0 < i \Longrightarrow u(i) \geq w(i)$, *and*
(2) $0 < i < j$ *and* $u(i) < u(j) \Longrightarrow w(i) < w(j)$.

For $P \subset \{1,2,\ldots\} = \mathbb{N}$, let $\#P \in \mathbb{N} \cup \{\infty\}$ be the cardinality of P. Then the inclusion $P \hookrightarrow \mathbb{N}$ induces compatible inclusions

$$\begin{tikzcd} \mathcal{B}_{\#P} \arrow[r, hook, "\varepsilon_P"] \arrow[d, hook] & \mathcal{B}_\infty \arrow[d, hook] \\ \mathcal{S}_{\pm\#P} \arrow[r, hook, "\varepsilon_P"] & \mathcal{S}_{\pm\infty} \end{tikzcd}$$

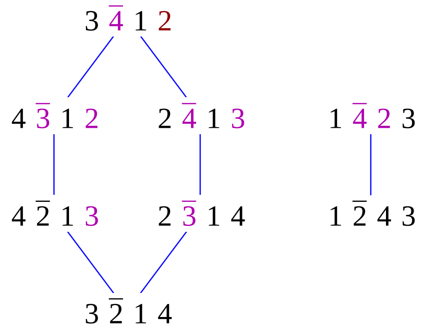
Shape equivalence is the equivalence relation on $\mathcal{B}_\infty$ induced by $u \sim \varepsilon_P(u)$ for $P \subset \mathbb{N}$ and $u \in \mathcal{B}_{\#P}$. Let $[u,w]_0 := \{v \mid u \leq_0 v \leq_0 w\}$ denote the interval in the 0-Bruhat order between u and w, a finite graded poset. A corollary of Theorems 2.2 and Theorem E(i) of [3] is the following fundamental result about the 0-Bruhat order on $\mathcal{B}_\infty$.

Theorem B(1) *Suppose $u,w,x,z \in \mathcal{B}_\infty$ with wu^{-1} shape equivalent to zx^{-1}. Then $[u,w]_0 \simeq [x,z]_0$. If $\varepsilon_P(wu^{-1}) = zx^{-1}$, then this isomorphism is given by*

$$[u,w]_0 \ni v \longmapsto \varepsilon_P(vu^{-1})x \in [x,z]_0.$$

This property is not shared by the k-Bruhat order on $\mathcal{B}_\infty$, for any $k > 0$.

Example 2.6. *Consider the following two intervals in the 1-Bruhat order on $\mathcal{B}_4$:*



Note that $(3\overline{2}14)^{-1} \cdot (3\overline{4}12) = (1\overline{2}43)^{-1} \cdot (1\overline{4}23) = (\overline{2},\,\overline{4})(2\,,4)$ and the two intervals are not isomorphic.

Remark 2.7. For any $\zeta \in \mathcal{B}_\infty$, there is a $u \in \mathcal{B}_\infty$ with $u \leq_0 \zeta u$: Suppose $\{a \in \pm\mathbb{N} \mid a > \zeta(a)\} = \{a_1,\ldots,a_m\}$ with $\zeta(a_1) < \zeta(a_2) < \cdots < \zeta(a_m)$. Further let $\{a_{m+1} < a_{m+2} < \cdots\} = \mathbb{N} \setminus \{|a_1|,|a_2|,\ldots,|a_m|\}$. If we define $u(j) = a_j$, then Proposition 2.5 implies that $u \leq_0 \zeta u$. Note that if $[m] = \{a > 0 \mid \zeta(a) \neq a\}$, then ζu is a Grassmannian permutation.

By Theorem B(1), we may define a new partial order on $\mathcal{B}_\infty$, which we call the *Lagrangian order:* For $\eta,\zeta \in \mathcal{B}_\infty$, set $\eta \preceq \zeta$ if there is a $u \in \mathcal{B}_\infty$ with $u \leq_0 \eta u \leq_0 \zeta u$. By Remark 2.7, it has a unique minimal element e. This order is graded by the rank, $\mathcal{L}(\zeta)$, where $\mathcal{L}(\zeta) := \ell(\zeta u) - \ell(u)$ whenever $u \leq_0 \zeta u$. These notions have definitions independent of $\leq_0$:

Definition-Theorem 2.8 (*cf.* Definition 3.2.2 [3])**.** *Let $\eta,\zeta \in \mathcal{B}_\infty$.*

(1) *Then* $\eta \preceq \zeta$ *if and only if*
  (i) $a \in \pm\mathbb{N}$ *with* $a > \eta(a) \Longrightarrow \eta(a) \geq \zeta(a)$, *and*
  (ii) $a, b \in \pm\mathbb{N}$ *with* $a < b$, $a > \zeta(a), b > \zeta(b)$, *and* $\zeta(a) < \zeta(b) \Longrightarrow \eta(a) < \eta(b)$.

(2) $$\mathcal{L}(\zeta) \;=\; \sum_{a, 0>\zeta(a)} |\zeta(a)| \;-\; \#\{(a,b) \mid 0 < a < b, a = \zeta(a), a > \zeta(b)\}$$
$$-\; \#\{(a,b) \mid a < b, a > \zeta(a), b > \zeta(b), \zeta(a) > \zeta(b)\} \;-\; \sum_{0>a>\zeta(a)} |a|.$$

Proof. Let u be the permutation with $u \leq_0 \zeta u$ constructed from ζ in Remark 2.7. If $u \leq_0 \eta u \leq_0 \zeta u$, then η satisfies the conditions in (i), and conversely.

For (ii), consider the difference $\ell(\zeta u) - \ell(u)$. The length of ζu is the first sum, plus the number of inversions of the form $0 < i \leq n < j$ with $\zeta u(i) > \zeta u(j) = u(j)$. In the construction of u, each of these is also an inversion in u involving positions $0 < i \leq n < j$, and so are canceled in the difference. The second term counts the remaining inversions of this type in u, the third term counts the inversions with $0 < i < j \leq n$ in u, and the fourth term is $\sum_{i>0>u(i)} |u(i)|$. ∎

The Lagrangian order is the $\mathcal{B}_\infty$-counterpart of the Grassmann-Bruhat order $\multimap$ on $\mathcal{S}_\infty$ of [3, 4]. This is defined as follows: Let $\eta, \zeta \in \mathcal{S}_\infty$. Then $\eta \multimap \zeta$ if and only if there is a $u \in \mathcal{S}_\infty$ with $u \lhd_0 \eta u \lhd_0 \zeta u$. The Grassmann-Bruhat order is ranked with $||\eta|| := l(\eta u) - l(u)$ whenever $u \lhd_0 \eta u$. Let $s(\zeta)$ count the sign changes $\{a > 0 \mid 0 > \zeta(a)\}$ in ζ. We have the following relation between these two orders.

Corollary 2.9.
(1) $(\mathcal{B}_\infty, \prec)$ *is an induced suborder of* $(\mathcal{S}_{\pm\infty}, \multimap)$.
(2) *For* $\zeta \in \mathcal{B}_\infty (\subset \mathcal{S}_{\pm\infty})$, *we have* $\mathcal{L}(\zeta) = (||\zeta|| + s(\zeta))/2$.

Proof. The first statement is a consequence of Theorem 2.2. For the second statement, consider any maximal chain in $[e, \zeta]_\prec$ (in $\mathcal{B}_\infty$). By Theorem 2.2, this gives a maximal chain in $[e, \zeta]_\multimap$ (in $\mathcal{S}_{\pm\infty}$), where covers of the form $\eta \lessdot t_{ab}\,\eta$ are replaced by $\eta \multimap (a,b)\eta \multimap (a,b)(\overline{a},\overline{b})\eta$. Thus $||\zeta|| = \mathcal{L}(\zeta) + \tau$, where τ counts the covers in that chain if the form $\eta \lessdot t_{ab}\,\eta$. Since only covers of the form $\eta \lessdot t_b\eta$ contribute to $s(\zeta)$, we have $||\zeta|| = 2\mathcal{L}(\zeta) - s(\zeta)$. ∎

Let $\eta, \zeta \in \mathcal{B}_\infty$. If $\zeta \cdot \eta = \eta \cdot \zeta$ with $\mathcal{L}(\eta \cdot \zeta) = \mathcal{L}(\eta) + \mathcal{L}(\zeta)$, and neither of ζ or η is the identity, then $\eta \cdot \zeta$ is the *disjoint product* of η and ζ. (In general $\mathcal{L}(\eta \cdot \zeta) \leq \mathcal{L}(\eta) + \mathcal{L}(\zeta)$.) If a permutation cannot be factored in this way, it is *irreducible*. Permutations $\zeta \in \mathcal{B}_\infty$ factor uniquely into irreducibles. This is described in terms of non-crossing partitions [23]: (A non-crossing partition of $\pm\mathbb{N}$ is a set partition such that if $a < c < b < d$ with a, b in a part π and c, d in a part π', then $\pi = \pi'$, as otherwise π, π' are crossing.)

First, consider ζ as an element of $\mathcal{S}_{\pm\infty}$. Let Π be the finest non-crossing partition of $\pm\mathbb{N}$ which is refined by the partition given by the cycles of ζ. For each non-singleton part π of Π, let ζ_π be the product of the cycles of ζ which partition π. (These ζ_π are the irreducible factors of ζ, as an element of $\mathcal{S}_{\pm\infty}$.) Since $\zeta \in \mathcal{B}_\infty$, for each such part

π of Π, either $\pi = \overline{\pi}$ or else one of $\pi, \overline{\pi}$ consists solely of positive integers. In the first case, ζ_π is an irreducible factor of ζ (as an element of $\mathcal{B}_\infty$), and in the second, $\zeta_\pi\zeta_{\overline{\pi}}$ is an irreducible factor of ζ.

The main result concerning this disjointness is the following:

Theorem 2.10 (*cf.* Theorem G (i) [3])**.** *Suppose $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of $\zeta \in \mathcal{B}_\infty$ into irreducibles. Then the map $(\eta_1, \ldots, \eta_s) \mapsto \eta_1 \cdots \eta_s$ induces an isomorphism*

$$[e, \zeta_1]_\prec \times \cdots \times [e, \zeta_s]_\prec \xrightarrow{\ \sim\ } [e, \zeta]_\prec .$$

This factorization into irreducibles suggests defining a type B non-crossing partition to be a non-crossing partition Π whose blocks π are either stable under ω_0 ($\pi = \overline{\pi}$), or else $\pi, \overline{\pi}$ are distinct, with one consisting solely of positive integers. These differ from the non-crossing partitions of type B introduced by Reiner [30], which form a graded lattice. Figure 4 shows the partitions of $\pm[2]$ defined here.

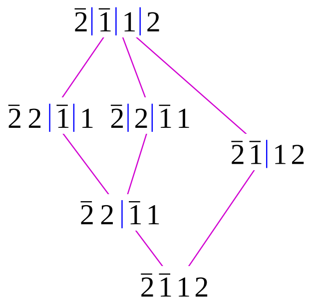


FIGURE 4. Non-crossing partitions in $\pm[2]$

We summarize some properties of $(\mathcal{B}_\infty, \prec)$.

Theorem 2.11 (*cf.* Theorem 3.2.3 of [3])**.**

1. *$(\mathcal{B}_\infty, \prec)$ is a graded poset with minimal element e and rank function $\mathcal{L}(\cdot)$.*
2. *The map $\lambda \mapsto v(\lambda)$ exhibits the lattice of strict partitions as an induced suborder of $(\mathcal{B}_\infty, \prec)$.*
3. *If $u \leq_0 \zeta u$, then $\eta \mapsto \eta u$ induces an isomorphism $[e, \zeta]_\prec \xrightarrow{\sim} [u, \zeta u]_0$.*
4. *If $\eta \preceq \zeta$, then $\xi \mapsto \xi\eta^{-1}$ induces an isomorphism $[\eta, \zeta]_\prec \xrightarrow{\sim} [e, \zeta\eta^{-1}]_\prec$.*
5. *For every infinite set $P \subset \mathbb{N}$, the map $\varepsilon_P : \mathcal{B}_\infty \to \mathcal{B}_\infty$ is an injection of graded posets. Thus if $\eta, \zeta \in \mathcal{B}_\infty$ are shape equivalent, then $[e, \zeta]_\prec \simeq [e, \eta]_\prec$.*
6. *The map $\eta \mapsto \eta\zeta^{-1}$ induces an order-reversing isomorphism between $[e, \zeta]_\prec$ and $[e, \zeta^{-1}]_\prec$.*

If $\zeta \in \mathcal{B}_\infty$ is factored into into disjoint cycles in $\mathcal{S}_{\pm\infty}$, the resulting cycles have one of two forms:

$$(a,\, b,\, \ldots,\, c) \qquad \text{or} \qquad (a,\, b,\, \ldots,\, c,\, \overline{a},\, \overline{b},\, \ldots,\, \overline{c})$$

with $|a|, |b|, \ldots, |c|$ distinct. Furthermore, every cycle $\eta = (a,\, b,\, \ldots,\, c)$ of the first type is paired with another, $\overline{\eta} := (\overline{a},\, \overline{b},\, \ldots,\, \overline{c})$, also of the first type. This motivates a 'cycle notation' for permutations $\zeta \in \mathcal{B}_\infty$. Write $\langle a, b, \ldots, c\rangle$ for the product $(a, b, \ldots, c) \cdot (\overline{a}, \overline{b}, \ldots, \overline{c})$ and $\langle a, b, \ldots, c]$ for cycles $(a, b, \ldots, c, \overline{a}, \overline{b}, \ldots, \overline{c})$ of the second type. Call either of these *cycles* in $\mathcal{B}_\infty$. In some examples and figures, the commas may be omitted. With this notation, Figure 5 shows the Lagrangian order on $\mathcal{B}_3$. The thickened lines are between skew Grassmannian permutations $v(\lambda)v(\mu)^{-1}$ for $\mu \subset \lambda$.

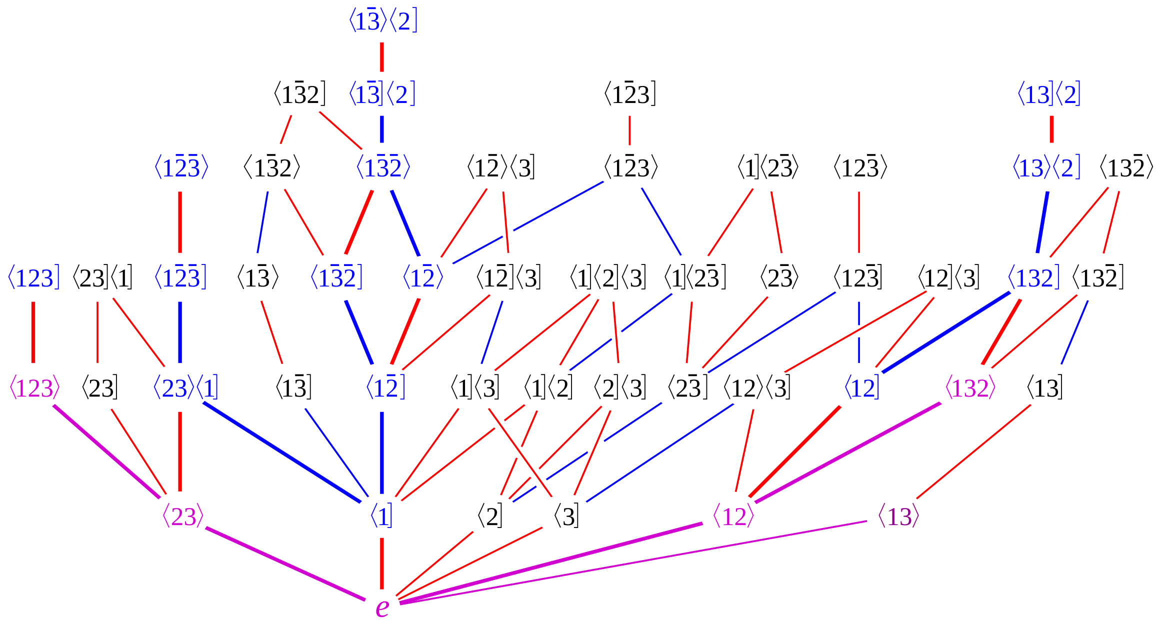


FIGURE 5. The Lagrangian order on $\mathcal{B}_3$

2.2. **A monoid for the Lagrangian order.** The 'Schubert vs. Schur' structure constants $b^w_{u\,\lambda}$ and $c^w_{u\,\lambda}$ are related to the enumeration of (saturated) chains in the 0-Bruhat order on $\mathcal{B}_\infty$, and hence to the enumeration of chains in the Lagrangian order on $\mathcal{B}_\infty$. We develop the elementary theory of chains in these orders along the lines of [4].

A chain in either $[u, \zeta u]_0$ or $[e, \zeta]_\prec$ is a particular factorization of ζ into transpositions t_b and $t_{a\,b}$. We give an algorithm for finding a chain in $[e, \zeta]_\prec$. For this, set $t_{\overline{b}\,b} = t_{b\,\overline{b}} = t_b$.

Algorithm 2.12 (*cf.* Remark 3.1.2 [3])**.**
`input:` *A permutation* $\zeta \in \mathcal{B}_\infty$.
`output:` *Permutations* $\zeta, \zeta_1, \ldots, \zeta_m = e$ *such that*

$$e \prec\!\!\cdot\; \zeta_{m-1} \prec\!\!\cdot\; \cdots \prec\!\!\cdot\; \zeta_1 \prec\!\!\cdot\; \zeta$$

is a saturated chain in the Lagrangian order.

Output ζ. *While* $\zeta \neq e$, *do*

1 *Choose* $b \in \mathbb{N}$ *maximal subject to* $b > \zeta(b)$.

2 *Choose a minimal subject to* $a \leq \zeta(b) < \zeta(a)$.
3 $\zeta := \zeta t_{a\,b}$, *output* ζ.

Before every execution of 3, $\zeta t_{a\,b} \lessdot \zeta$. *Moreover, this algorithm terminates in* $\mathcal{L}(\zeta)$ *iterations and the reverse of the sequence produced is a chain in* $[e,\zeta]_{\prec}$.

The Lagrangian order has a theory of reduced decompositions, analogous to the usual theory for the Coxeter group B_∞ with respect to the weak order. (Also analogous to that for the Grassmann-Bruhat order on $\mathcal{S}_\infty$ [4].) We express this in the context of monoids.

Define a monoid $\mathcal{M}$ with 0 and generators $\mathbf{t}_{a\,b}, \mathbf{t}_b$ for integers $0 < a < b$, one for each reflection of the Weyl group $\mathcal{B}_\infty$. To simplify the list of relations these satisfy, set $\mathbf{t}_{b\,b} := \mathbf{t}_b$. Also, if $\mathbf{w} \equiv \mathbf{u}$ is a relation between words $\mathbf{w}$ and $\mathbf{u}$ in these generators, then $\mathbf{w}^{\mathrm{op}} \equiv \mathbf{u}^{\mathrm{op}}$, where $\mathbf{w}^{\mathrm{op}}$ is the word $\mathbf{w}$ read backwards. That said, these generators are subject to the following relations:

$$
(2)\qquad
\begin{array}{llcll}
\text{(i)} & \mathbf{t}_{a\,c}\mathbf{t}_a\mathbf{t}_{a\,b} & \equiv & \mathbf{t}_{a\,b}\mathbf{t}_{b\,c}\mathbf{t}_b & \text{if } a<b<c,\\
\text{(ii)} & \mathbf{t}_{b\,c}\mathbf{t}_{c\,d}\mathbf{t}_{a\,c} & \equiv & \mathbf{t}_{b\,d}\mathbf{t}_{a\,b}\mathbf{t}_{b\,c} & \text{if } a<b<c<d\\
\text{(iii)} & \mathbf{t}_{a\,b}\mathbf{t}_{c\,d} & \equiv & \mathbf{t}_{c\,d}\mathbf{t}_{a\,b} & \text{if } a\leq b<c\leq d \text{ or } a<c<d<b,\\
\text{(iv)} & \mathbf{t}_{a\,c}\mathbf{t}_b & \equiv & \mathbf{t}_b\mathbf{t}_{a\,c} \ \equiv\ 0 & \text{if } a<b\leq c \text{ or } a=b=c,\\
\text{(v)} & \mathbf{t}_{a\,c}\mathbf{t}_{b\,d} & \equiv & \mathbf{t}_{b\,d}\mathbf{t}_{a\,c} \ \equiv\ 0 & \text{if } a\leq b<c\leq d,\\
\text{(vi)} & \mathbf{t}_{b\,c}\mathbf{t}_{a\,b}\mathbf{t}_{b\,c} & \equiv & \mathbf{t}_{a\,b}\mathbf{t}_{b\,c}\mathbf{t}_{a\,b} \ \equiv\ 0, & \text{if } a<b<c.
\end{array}
$$

These hold because $\mathcal{M}$ is a sub monoid of the monoid for the Grassmann-Bruhat order [4] in the same way the Lagrangian order is an induced suborder of the Grassmann-Bruhat order on $\mathcal{S}_\infty$.

The relation between $\mathcal{M}$ and the Lagrangian order on $\mathcal{B}_\infty$ is obtained via a faithful representation of $\mathcal{M}$ as linear operators on $\mathbb{Q}\mathcal{B}_\infty$: Define linear operators $\widehat{\mathbf{t}}_{a\,b}$ and $\widehat{\mathbf{t}}_b$ on $\mathbb{Q}\mathcal{B}_\infty$ by

$$
\begin{aligned}
\widehat{\mathbf{t}}_{a\,b}.\zeta &:= \begin{cases} t_{a\,b}\zeta & \text{if } \mathcal{L}(\zeta)+1 = |t_{a\,b}\zeta| \\ 0 & \text{otherwise.} \end{cases}\\
\widehat{\mathbf{t}}_{b}.\zeta &:= \begin{cases} t_{a}\zeta & \text{if } \mathcal{L}(\zeta)+1 = |t_{b}\zeta| \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

To simplify the following statement, let the index α represent either of b or $a\,b$.

Theorem 2.13 (*cf.* Theorem 1.1 [4]).

(1) *The operators* $\widehat{\mathbf{t}}_\alpha$ *satisfy the relations (1.1), and a composition of operators is characterized by its value at the identity. That is,* $\widehat{\mathbf{t}}_{\alpha'_m}\cdots\widehat{\mathbf{t}}_{\alpha'_1} = \widehat{\mathbf{t}}_{\alpha_n}\cdots\widehat{\mathbf{t}}_{\alpha_1}$ *if and only if* $\widehat{\mathbf{t}}_{\alpha'_m}\cdots\widehat{\mathbf{t}}_{\alpha'_1}e = \widehat{\mathbf{t}}_{\alpha_n}\cdots\widehat{\mathbf{t}}_{\alpha_1}e$.

(2) *For* $\mathbf{x} = \mathbf{t}_{\alpha_n}\cdots\mathbf{t}_{\alpha_2}\mathbf{t}_{\alpha_1} \in \mathcal{M}$, *the map* $\mathbf{x} \mapsto \widehat{x} := \widehat{\mathbf{t}}_{\alpha_n}\cdots\widehat{\mathbf{t}}_{\alpha_2}\widehat{\mathbf{t}}_{\alpha_1}$ *is a faithful representation of* $\mathcal{M}$.

(3) *The following map is a well defined bijection:*

$$
\begin{aligned}
\mathcal{M} &\longrightarrow \mathcal{B}_\infty \cup \{0\},\\
\mathbf{x} &\longmapsto \widehat{\mathbf{x}}e.
\end{aligned}
$$

(4) *For* $\zeta, \eta \in \mathcal{B}_\infty$, $\eta \preceq \zeta$ *if and only if there exists* $\mathbf{x} \in \mathcal{M}$ *such that* $\zeta = \widehat{\mathbf{x}}\eta$.
(5) *The set* $\mathbf{R}(\zeta) = \{\widehat{\mathbf{x}} : \widehat{\mathbf{x}}e = \zeta\}$ *corresponds to the set of all maximal chains in* $[e, \zeta]_\prec$.

We call the elements of $\mathbf{R}(\zeta)$ the $\prec$*-reduced decompositions* of ζ.

3. Isotropic flag manifolds and maximal Grassmannians

Let V denote either $\mathbb{C}^{2n+1}$ equipped with non-degenerate symmetric bilinear form or $\mathbb{C}^{2n}$ equipped with a non-degenerate alternating bilinear form. In the first case, V an *odd orthogonal (vector) space*, and in the second, a *symplectic (vector) space*. A linear subspace K of V is *isotropic* if the restriction of the form to K is identically zero. Isotropic subspaces have dimension at most n. An *isotropic flag* in V is a sequence $E_\bullet$ of isotropic subspaces:

$$E_\bullet \ : \ E_{\overline{n}} \ \subset \ E_{\overline{n-1}} \ \subset \ \cdots \ \subset \ E_{\overline{1}},$$

where $\dim E_{\overline{\imath}} = n + 1 - i$. Let $K^\perp$ be the annihilator of a subset K of V. Given an isotropic flag $E_\bullet$ in V, we obtain a canonical complete flag in V (also written $E_\bullet$) by defining $E_i := E^\perp_{\overline{i+1}}$ for $i = 1, \ldots, n$, and in the odd orthogonal case, $E_0 := E^\perp_{\overline{1}}$. Henceforth, flags will always be complete, although we may only specify $E_{\overline{n}}, \ldots, E_{\overline{1}}$.

The group G of linear transformations of V which preserve the given form acts transitively on the set of isotropic flags in V. Since the stabilizer of an isotropic flag is a Borel subgroup B of G, this exhibits the set of isotropic flags as the homogeneous space G/B. Here, G is either $So_{2n+1}\mathbb{C}$ (odd orthogonal) or $Sp_{2n}\mathbb{C}$ (symplectic). Similarly, G acts transitively on the set of maximal isotropic subspaces of V, exhibiting it as the homogeneous space G/P_0. Here P_0 is the stabilizer of a maximal isotropic subspace, a maximal parabolic subgroup associated to the simple root of exceptional length. Let $\pi : G/B \twoheadrightarrow G/P_0$ be the projection map.

The rational cohomology rings [9] of G/B for both the symplectic and odd-orthogonal flag manifolds are isomorphic to

$$\mathbb{Q}[x_1, \ldots, x_n]/\langle e_i(x_1^2, \ldots, x_n^2), i = 1, \ldots, n\rangle,$$

where $e_i(a_1, \ldots, a_n)$ is the ith elementary symmetric polynomial in $a_1, \ldots, a_n$. However, their integral cohomology rings differ [14]:

$$\begin{aligned}
H^*Sp_{2n}\mathbb{C}/B &\simeq \mathbb{Z}[x_1, \ldots, x_n]/\left\langle e_i(x_1^2, \ldots, x_n^2)\right\rangle. \\
H^*So_{2n+1}\mathbb{C}/B &\simeq \mathbb{Z}[x_1, \ldots, x_n, c_1, \ldots, c_n]/I, \\
I &= \left\langle e_i(x_1^2, \ldots, x_n^2),\, 2c_i - e_i(x_1, \ldots, x_n),\, c_{2i} = (-1)^i c_i^2 \right\rangle.
\end{aligned}$$

These rings have another description in terms of Schubert classes.

3.1. **Schubert varieties.** Since an isotropic flag $E_\bullet \in G/B$ is also a complete flag in V, we have a canonical embedding $G/B \hookrightarrow \mathbb{F}\ell(V)$, the manifold of complete flags in V. Similarly, there is an embedding $G/P_0 \hookrightarrow \mathbf{G}_n(V)$, the Grassmannian of n-dimensional subspaces of V. We use these maps to understand some structures of G/B.

Given $w \in \mathcal{B}_n$ and an isotropic (complete) flag $E_\bullet \in G/B$, the *Schubert variety* $Y_w E_\bullet$ of G/B is the collection of all flags $F_\bullet \in G/B$ satisfying

$$\dim E_i \cap F_{\overline{j}} \quad \geq \quad \#\{n \geq l \geq j \mid w(l) \leq i\} \tag{3}$$

for each $j \in [n]$ and $-n \leq i \leq n$ ($i \neq 0$ in the symplectic case). This has codimension $\ell(w)$ in G/B. Also, $Y_w E_\bullet \subset Y_u E_\bullet$ if and only if $u \leq w$ in the Bruhat order. The Schubert cell $Y_u^\circ E_\bullet$ is the set of flags $F_\bullet$ for which equality holds in (3). These are the flags in $Y_u E_\bullet$ which are not in any sub-Schubert variety ($Y_w E_\bullet$ with $u \leq w$).

If now $E_\bullet \in \mathbb{F}\ell(V)$ and $w \in \mathcal{S}_{[\overline{n},n]}$ ($\mathcal{S}_{\pm[n]}$ in the symplectic case), then the Schubert variety $X_w E_\bullet$ of $\mathbb{F}\ell(V)$ is the collection of flags $F_\bullet \in \mathbb{F}\ell(V)$ satisfying 3) for all $\overline{n} \leq i, j \leq n$ ($i, j \neq 0$ in the symplectic case). Furthermore, if $w \in \mathcal{B}_n$ and $E_\bullet \in G/B$, then

$$Y_w E_\bullet \;=\; G/B \bigcap X_w E_\bullet$$

and this is a scheme-theoretic equality [24].

The Schubert cells constitute a cellular decomposition of G/B. Thus *Schubert classes*, the cohomology classes Poincaré dual to the fundamental cycles of Schubert varieties, form $\mathbb{Z}$-bases for these cohomology rings. Write $\mathfrak{B}_w$ for the class $[Y_w E_\bullet]$ in $H^* So_{2n+1}\mathbb{C}/B$ Poincaré dual to the fundamental cycle of $Y_w E_\bullet$ of $So_{2n+1}\mathbb{C}/B$ and $\mathfrak{C}_w$ for the corresponding class in $H^* Sp_{2n}\mathbb{C}/B$. Since these are bases, there are integral structure constants $b^w_{u\,v}$ and $c^w_{u\,v}$ for $u, w, v \in \mathcal{B}_n$ defined by the identities:

$$\mathfrak{B}_u \cdot \mathfrak{B}_v \;=\; \sum_w b^w_{u\,v} \mathfrak{B}_w \qquad \text{and} \qquad \mathfrak{C}_u \cdot \mathfrak{C}_v \;=\; \sum_w c^w_{u\,v} \mathfrak{C}_w.$$

Let $s(w)$ count the number of sign changes in the permutation w. Then the isomorphism of rational rings is induced by the map [7]:

$$\mathfrak{C}_w \;\longmapsto\; 2^{s(w)} \mathfrak{B}_w.$$

Thus

$$2^{s(u)+s(v)} b^w_{u\,v} \;=\; 2^{s(w)} c^w_{u\,v}. \tag{4}$$

It suffices to establish identities and formulas for $Sp_{2n}\mathbb{C}/B$. We do this, because a crucial geometric result (Theorem 3.4(2)) does not hold for $So_{2n+1}\mathbb{C}/B$.

Two flags $E_\bullet, E'_\bullet$ are *opposite* if $\dim(E_i \bigcap E'_{\overline{i}}) = 1$ for all i. In what follows, Y_u and Y'_v, will always denote Schubert varieties defined by fixed, but arbitrary, opposite isotropic flags. A consequence of Kleiman's theorem on the transversality of a general translate [22], results in [13], and some combinatorics, is the following proposition.

Proposition 3.1. *Let $u, w \in \mathcal{B}_n$. Then $Y_u \bigcap Y'_{\omega_0 w} \neq 0$ if and only if $u \leq w$ in the Bruhat order. If $u \leq w$, then $Y_u, Y'_{\omega_0 w}$ meet generically transversally, and the intersection cycle is irreducible of dimension $\ell(w) - \ell(u)$.*

The top-dimensional component of H^*G/B is generated by the class of a point $[\mathrm{pt}] = \mathfrak{B}_{\omega_0}$ or $\mathfrak{C}_{\omega_0}$. The map $\deg : H^*G/B \to \mathbb{Z}$ selects the coefficient of $[\mathrm{pt}]$ in a cohomology class. The *intersection pairing* on H^*G/B is the composition

$$\beta, \gamma \in H^*G/B \;\longmapsto\; \deg(\beta \cdot \gamma).$$

By Proposition 3.1, the product $[Y_u]\cdot[Y_v]$ is the cohomology class $[Y_u\bigcap Y'_v]$. In particular, when $v=\omega_0 u$, these intersections are single, reduced points, so that $[Y_u]$ and $[Y_{\omega_0 u}]$ are dual under the intersection pairing. Thus

$$c^w_{u\,v} \;=\; \deg(\mathfrak{C}_u\cdot\mathfrak{C}_{\omega_0 w}\cdot\mathfrak{C}_v),$$

which is also the number of points in the intersection

$$Y_u\bigcap Y'_{\omega_0 w}\bigcap Y''_v,$$

where Y''_v is defined by a flag $E''_\bullet$ opposite to both $E_\bullet$ and $E'_\bullet$ (which define Y_u and $Y'_{\omega_0 w}$).

We derive a useful description of flags in $Y^\circ_u\bigcap {Y'}^\circ_{\omega_0 w}$ when $u\leq_0 w$. For $S\subset\mathbb{C}^m$, let $\langle S\rangle$ be the linear span of S.

Lemma 3.2. *Suppose that $u\leq_0 w$ and $E_\bullet, E'_\bullet$ are opposite isotropic flags in V. Then there are algebraic functions $g_{\overline{\jmath}}: Y^\circ_u E_\bullet\bigcap Y^\circ_{\omega_0 w}E'_\bullet\to V$ for $1\leq j\leq n$ such that for each flag $F_\bullet\in Y^\circ_u E_\bullet\bigcap Y^\circ_{\omega_0 w}E'_\bullet$ and each $1\leq j\leq n$,*

(1) $F_{\overline{\jmath}}=\langle g_{\overline{n}}(F_\bullet),\ldots,g_{\overline{\jmath}}(F_\bullet)\rangle$, *and*

(2) $g_{\overline{\jmath}}(F_\bullet)\in E_{u(j)}\bigcap E'_{\overline{w(j)}}$.

Proof. The representation of Schubert cells via parameterized matrices give V-valued functions $f_{\overline{\jmath}}$ defined on the Schubert cell $Y^\circ_u E_\bullet$ such that if $F_\bullet$ is a flag in that cell, then $F_{\overline{\jmath}}=\langle f_{\overline{n}}(F_\bullet),\ldots,f_{\overline{\jmath}}(F_\bullet)\rangle$, and $f_{\overline{\jmath}}\in E_{u(j)}$.

Construct the functions $g_{\overline{\jmath}}$ inductively. First, set $g_{\overline{n}}(F_\bullet):=f_{\overline{n}}(F_\bullet)$ for $F_\bullet\in Y^\circ_u E_\bullet\bigcap Y^\circ_{\omega_0 w}E'_\bullet$. Since $F_{\overline{n}}\subset E_{u(n)}\bigcap E'_{\overline{w(n)}}$, conditions 1 and 2 are satisfied for $g_{\overline{n}}$. Suppose we have constructed $g_{\overline{\imath}}$ for $n\geq i>j$. Let $g_{\overline{\jmath}}(F_\bullet)$ be the intersection of $E'_{\overline{w(j)}}$ with the affine space

$$W_j\;:=\;f_{\overline{\jmath}}(F_\bullet)\;+\;\langle g_{\overline{\imath}}(F_\bullet)\mid i>j \text{ and } w(i)<w(j)\rangle.$$

There is a unique point of intersection: Since $F_\bullet\in Y^\circ_{\omega_0 w}E'_\bullet$,

$$\dim E'_{\overline{w(j)}}\bigcap F_{\overline{\jmath}}\;=\;\#\{i\mid i\geq j \text{ and } w(i)\geq w(j)\}.$$

Since $u\leq_0 w$, if $i>j$ and $w(i)<w(j)$, then necessarily $u(i)<u(j)$, by condition 2 of Proposition 2.5. Hence $W_j\subset E_{u(j)}$ and so $g_{\overline{\jmath}}(F_\bullet)\in E_{u(j)}\bigcap E'_{\overline{w(j)}}$. ∎

Schubert varieties Υ_λ of G/P_0 are indexed by strict partitions λ: decreasing sequences $n\geq\lambda_1>\cdots>\lambda_k$ of positive integers. The projection map $\pi: G/B\twoheadrightarrow G/P_0$ maps Schubert varieties to Schubert varieties, with $\pi Y_u=\Upsilon_\lambda$, where λ consists of the positive numbers among $\{\overline{u(1)},\overline{u(2)},\ldots,\overline{u(n)}\}$ arranged in decreasing order. For a strict partition λ, let $v(\lambda)$ be the Grassmannian permutation whose (initial) negative values are $\overline{\lambda_1}<\overline{\lambda_2}<\cdots<\overline{\lambda_k}$. Let λ^c be the decreasing sequence obtained from the integers in $[n]$ which do not appear in λ. Then $Y_{v(\lambda)}=\pi^{-1}\Upsilon_\lambda$ and

$$\pi\;:\;Y_{\omega_0 v(\lambda)}\;\longrightarrow\;\Upsilon_{\lambda^c}$$

is birational. One may see this by considering typical elements of their Schubert cells.

Set $P_\lambda:=[\Upsilon_\lambda]$ in $H^*So_{2n+1}\mathbb{C}/P_0$ (equivalently $P_\lambda:=[Y_{v(\lambda)}]$ in $H^*So_{2n+1}\mathbb{C}/B$) and let Q_λ be the corresponding class in the symplectic case. For $1\leq m\leq n$, the *special*

Schubert variety $\Upsilon_{(m)}$ is the collection of all maximal isotropic subspaces which meet a fixed $(n+1-m)$-dimensional isotropic subspace. Let p_m (respectively q_m) denote the class $P_{(m)}$ in either $H^*So_{2n+1}\mathbb{C}/B$ or $H^*So_{2n+1}\mathbb{C}/P_0$ (respectively the class $Q_{(m)}$ in either $H^*Sp_{2n}\mathbb{C}/B$ or $H^*Sp_{2n}\mathbb{C}/P_0$).

We are particularly interested in the constants $b^w_{u\,\lambda} := b^w_{u\,v(\lambda)}$ and $c^w_{u\,\lambda} := c^w_{u\,v(\lambda)}$ which give the structure of the cohomology of G/B as a module over the cohomology of G/P_0. Using the intersection pairing and the projection formula (*cf.* [18, 8.1.7]), we have

$$\begin{aligned} c^w_{u\,\lambda} &= \deg(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w} \cdot \pi^*(Q_\lambda)) \\ &= \deg(\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w}) \cdot Q_\lambda), \end{aligned}$$

and a similar formula for $b^w_{u\,\lambda}$. Our main technique will be to find formulas for $\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0 w})$ by studying the effect of the map π on the cycle $Y_u \cap Y'_{\omega_0 w}$.

To that end, define $\mathcal{Y}^w_u := \pi(Y_u \cap Y'_{\omega_0 w})$. These cycles $\mathcal{Y}^w_u$ are, like Schubert varieties, defined up to translation by G. In the theorems below, write $\mathcal{Y}^w_u = \mathcal{Y}^z_x$ to mean that the cycles may be carried onto each other by an element of G. (The proofs are more explicit.)

The main result of Section 4 concerns these cycles.

Theorem 3.3. *Let $u, w \in \mathcal{B}_n$ with $u \leq_0 w$. Then*
1. *The map $\pi : Y_u \cap Y'_{\omega_0 w} \to \mathcal{Y}^w_u$ has degree 1.*
2. *If we have $x, z \in \mathcal{B}_n$ with $z \leq_0 z$ and wu^{-1} shape equivalent to zx^{-1}, then $\mathcal{Y}^w_u = \mathcal{Y}^z_x$.*

This implies the identity of Theorem B(2). As a consequence of Theorem 3.3(2), define $\mathcal{Y}_\zeta := \mathcal{Y}^{\zeta u}_u$ for any $\zeta, u \in \mathcal{B}_n$ with $u \leq_0 \zeta u$.

These cycles satisfy more identities. Let W be a $2m$-dimensional symplectic space. We study the manifolds of maximal isotropic subspaces of V, W, and $V \oplus W$, together with a map

$$\Xi \ : \ Sp_{2n}\mathbb{C}/P_0 \times Sp_{2m}\mathbb{C}/P_0 \ \longrightarrow \ Sp_{2n+2m}\mathbb{C}/P_0,$$

defined by $(H, K) \mapsto H \oplus K$. Also define $\rho \in \mathcal{B}_n$ by $\rho(i) = i-1-n$ for $1 \leq i \leq n$ and $\gamma \in \mathcal{B}_n$ by $\gamma(i) = i+1$ for $1 \leq i < n$ and $\gamma(n) = 1$.

Theorem 3.4.
1. *For any $\zeta \in \mathcal{B}_n$, $\mathcal{Y}_\zeta \ = \ \mathcal{Y}_{\zeta^{-1}}$.*
2. *If $\eta \cdot \zeta$ is a disjoint product in $\mathcal{B}_{n+m}$ with $\eta' \in \mathcal{B}_n$ shape equivalent to η and $\zeta' \in \mathcal{B}_m$ shape equivalent to ζ, then*
$$\Xi(\mathcal{Y}_{\eta'} \times \mathcal{Y}_{\zeta'}) \ = \ \mathcal{Y}_{\eta\cdot\zeta}.$$
3. *For any $\zeta \in \mathcal{B}_n$,*
$$\mathcal{Y}_\zeta \ = \ \mathcal{Y}_{\rho\zeta\rho}.$$
4. *For any $\zeta \in \mathcal{B}_n$ with $a \cdot \zeta(a) > 0$ for every a,*
$$\mathcal{Y}_\zeta \ = \ \mathcal{Y}_{\gamma\zeta\gamma^{-1}}.$$

The first statement follows from the observation that $Y_u \bigcap Y_{\omega_0\zeta u} = Y_{\omega_0\zeta u} \bigcap Y_{\omega_0\zeta^{-1}(\omega_0\zeta u)}$. Since $\mathcal{L}(\zeta) = \dim \mathcal{Y}_\zeta$, Theorem C(1) follows from part (3), by the projection formula. Similarly, Theorem C(2) follows from part (4). We remark that part (2) is true only for the symplectic case, while (1), (3), and (4) hold also for the odd orthogonal flag manifold. Statements (2), (3), and (4) are proven in Section 5.

4. Identities of structure constants

We establish Theorem 3.3 which implies Theorem B(2). As in Section 2, many results and methods are similar to those of [3] for analogous results about $SL_n\mathbb{C}/B$. Our discussions are therefore brief. The results here hold for both $So_{2n+1}\mathbb{C}/B$ and $Sp_{2n}\mathbb{C}/B$, with nearly identical proofs. We only provide justification for $Sp_{2n}\mathbb{C}/B$.

Let $H_2 = \langle h, \overline{h}\rangle \simeq \mathbb{C}^2$ be a symplectic space of dimension 2. Then the orthogonal direct sum $V \oplus H_2$ is a symplectic space of dimension $2n+2$. For each $1 \leq p \leq n+1$, define embeddings $\psi_p, \psi_{\overline{p}} \,:\, Sp_{2n}\mathbb{C}/B \hookrightarrow Sp_{2n+2}\mathbb{C}/B$, the space of isotropic flags in $V \oplus H_2$, by

$$(\psi_p E_\bullet)_j \;=\; \begin{cases} E_{j+1} & j \leq \overline{p}\,(<0) \\ \langle E_j, h\rangle & \overline{p} < j < 0 \end{cases}.$$

Define $\psi_{\overline{p}}$ by replacing h with $\overline{h}$ in the definition above. We find the effect of these maps on cohomology by determining the image of a Schubert variety under ψ_p.

First, define two maps between $\mathcal{B}_n$ and $\mathcal{B}_{n+1}$. For every $1 \leq p \leq n+1$ and $q \in \pm[n+1]$, define the injection $\varepsilon_{p,q} : \mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ by:

$$\varepsilon_{p,q}(w)(j) \;=\; \begin{cases} w(j) & j < p \text{ and } |w(j)| < |q| \\ w(j)-1 & j < p \text{ and } w(j) \leq -|q| \\ w(j)+1 & j < p \text{ and } w(j) \geq |q| \\ q & j = p \\ w(j-1) & j > p \text{ and } |w(j)| < |q| \\ w(j-1)-1 & j > p \text{ and } w(j) \leq -|q| \\ w(j-1)+1 & j > p \text{ and } w(j) \geq |q| \end{cases}.$$

Let $/_p : \mathcal{B}_{n+1} \to \mathcal{B}_n$ be the left inverse of $\varepsilon_{p,q}$, defined by $\varepsilon_{p,w(p)}(w/_p) = w$. If we represent permutations as permutation matrices, then the effect of $/_p$ on $w \in \mathcal{B}_{n+1}$ is to erase the pth and $\overline{p}$th columns and the $w(p)$th and $\overline{w(p)}$th rows. The effect of $\varepsilon_{p,q}$ on w is to expand its permutation matrix with new pth, $\overline{p}$th columns and qth, $\overline{q}$th rows filled with zeroes, except for 1s at positions (q,p) and $(\overline{q},\overline{p})$. For example:

$$\begin{aligned} \varepsilon_{3,\overline{2}}(\overline{2}\,3\,4\,1) &= \overline{3}\,4\,\overline{2}\,5\,1 \\ \varepsilon_{3,2}(\overline{2}\,3\,4\,1) &= \overline{3}\,4\,2\,5\,1 \end{aligned} \qquad \text{and} \qquad 4\,\overline{1}\,5\,\overline{2}\,3/_4 \;=\; 3\,\overline{1}\,4\,2.$$

These definitions imply the following proposition (*cf.* [31], Lemma 12).

Proposition 4.1. *Let $w \in \mathcal{B}_n$, $1 \leq p, |q| \leq n+1$, and $E_\bullet$ any isotropic flag. Then*

$$\psi_p Y_w E_\bullet \;\subset\; Y_{\varepsilon_{p,q}(w)} \psi_{\overline{q}} E_\bullet.$$

Recall that e is the identity permutation and Y_e is the flag manifold G/B.

Corollary 4.2. *Let $E_\bullet, E'_\bullet$ be opposite isotropic flags. Then for any $q \in \pm[n+1]$, $\psi_q E_\bullet, \psi_{\overline{q}} E'_\bullet$ are opposite flags, and for any $1 \leq p \leq n+1$, $\psi_p Y_w E_\bullet$ equals either of the following cycles*

$$Y_{\varepsilon_{p,\overline{n+1}}(w)} \psi_{n+1} E_\bullet \bigcap Y_{\varepsilon_{p,n+1}(e)} \psi_{\overline{n+1}} E'_\bullet \qquad Y_{\varepsilon_{p,n+1}(w)} \psi_{\overline{n+1}} E_\bullet \bigcap Y_{\varepsilon_{p,\overline{n+1}}(e)} \psi_{n+1} E'_\bullet.$$

Proof. It is straightforward to check that the flags are opposite. Moreover, by Proposition 4.1, $\psi_p Y_w E_\bullet$ is a subset of either intersection, as $Y_e = Sp_{2n}\mathbb{C}/B$. Since

$$\ell(\varepsilon_{p,\overline{n+1}}(w)) \;=\; \ell(w) + n + p, \qquad \ell(\varepsilon_{p,n+1}(w)) \;=\; \ell(w) + n + 1 - p,$$

and $\dim Sp_{2n}\mathbb{C}/B = n^2$, Proposition 3.1 implies that all three cycles are irreducible with the same dimension, proving their equality. ∎

Corollary 4.3. *For any $w \in \mathcal{B}_n$ and $1 \leq p \leq n$, we have*

$$(\psi_p)_* \mathfrak{C}_w \;=\; \mathfrak{C}_{\varepsilon_{p,\overline{n+1}}(w)} \cdot \mathfrak{C}_{\varepsilon_{p,n+1}(e)} \;=\; \mathfrak{C}_{\varepsilon_{p,n+1}(w)} \cdot \mathfrak{C}_{\varepsilon_{p,\overline{n+1}}(e)}.$$

Remark 4.4. As in Section 4.2 of [3], the Schubert classes $\mathfrak{C}_{\varepsilon_{p,n+1}(e)}$ and $\mathfrak{C}_{\varepsilon_{p,\overline{n+1}}(e)}$ are certain special Schubert classes from Grassmannian projections. Were the corresponding Pieri-type formulas known, we could deduce formulas for $(\psi_p)_*\mathfrak{C}_w$. Only one of these classes is a special Schubert class from G/P_0:

$$\mathfrak{B}_{\varepsilon_{1,\overline{n+1}}(e)} \;=\; p_{n+1} \qquad \text{and} \qquad \mathfrak{C}_{\varepsilon_{1,\overline{n+1}}(e)} \;=\; q_{n+1}.$$

We deduce formulas for $(\psi_1)_*\mathfrak{B}_w$ and $(\psi_1)_*\mathfrak{C}_w$ from this.

Recall that $H^*Sp_{2n}\mathbb{C}/B$ is generated by $x_1, \ldots, x_n$. These classes are Chern classes of certain line bundles on $Sp_{2n}\mathbb{C}/B$: Let $\mathcal{E}_\bullet \to Sp_{2n}\mathbb{C}/B$ be the flag of bundles whose fibre at $E_\bullet$ is $E_\bullet$. Then $x_i = -c_1(\mathcal{E}_{\overline{i+1}}/\mathcal{E}_{\overline{i}})$. Thus

$$\psi_p^*(x_i) \;=\; \begin{cases} x_i & i < p \\ 0 & i = p \\ x_{i-1} & i > p \end{cases}.$$

Theorem 4.5. *Let $v \in \mathcal{B}_{n+1}$. Then*

1. *In $H^*So_{2n+1}\mathbb{C}/B$, $\psi_1^*\mathfrak{B}_v = \sum \theta(\varepsilon_{1,\overline{n+1}}(y)v^{-1})\ \mathfrak{B}_y$, the sum over all $y \in \mathcal{B}_n$ with $\ell(v) = \ell(y)$.*
2. *In $H^*Sp_{2n}\mathbb{C}/B$, $\psi_1^*\mathfrak{C}_v = \sum \chi(\varepsilon_{1,\overline{n+1}}(y)v^{-1})\ \mathfrak{C}_y$, the sum over all $y \in \mathcal{B}_n$ with $\ell(v) = \ell(y)$.*

Proof. These are consequences of the projection formula and Corollary 4.3. For the first,

$$\psi_1^*\mathfrak{B}_v \;=\; \sum_{y \in \mathcal{B}_n} \deg(\psi_1^*\mathfrak{B}_v \cdot \mathfrak{B}_{\omega_0 y})\mathfrak{B}_y.$$

But

$$\deg(\psi_1^*\mathfrak{B}_v \cdot \mathfrak{B}_{\omega_0 y}) \;=\; \deg(\mathfrak{B}_v \cdot (\psi_1)_*(\mathfrak{B}_{\omega_0 y})) \;=\; \deg(\mathfrak{B}_v \cdot \mathfrak{B}_{\varepsilon_{1,n+1}(\omega_0 y)} \cdot q_{n+1})$$

Since $\varepsilon_{1,n+1}(\omega_0 y) = \omega_0(\varepsilon_{1,\overline{n+1}}(y))$ and $\ell(\varepsilon_{1,\overline{n+1}}(y)) = \ell(y) + n + 1$, the result follows by the Pieri-type formula (Theorem D). ∎

Lemma 4.6. *Suppose $u <_0 w$ in $\mathcal{B}_{n+1}$ and $u(p) = w(p) = q$ for some $1 \leq p \leq n+1$. Then*

(1) *$u/_p <_0 w/_p$ and $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$.*
(2) *For any opposite isotropic flags $E_\bullet, E'_\bullet$ in V,*

$$\psi_p\left(Y_{u/_p}E_\bullet \bigcap Y_{\omega_0 w/_p}E'_\bullet\right) \;=\; Y_u\psi_{\overline{q}}E_\bullet \bigcap Y_{\omega_0 w}\psi_q E'_\bullet.$$

Proof. Since $u <_0 w$ and $u(p) = w(p)$, Proposition 2.5 implies that $u/_p <_0 w/_p$. Moreover, wu^{-1} is shape equivalent to $w/_p(u/_p)^{-1}$, so the first statement follows from Theorem B(1).

For (2), Proposition 4.1 gives the inclusion $\subset$. By Corollary 4.2, $\psi_{\overline{q}}E_\bullet$ and $\psi_q E'_\bullet$ are opposite flags. Thus, by Proposition 3.1 and (1), both sides are irreducible and have the same dimension, proving their equality. ∎

Theorem 4.7. *Suppose $u <_0 w$ in $\mathcal{B}_{n+1}$ and $u(p) = w(p) = q$ for some $1 \leq p \leq n+1$. Then for any strict partition λ, we have*

$$b^{w}_{u\,\lambda} \;=\; b^{w/_p}_{u/_p\,\lambda} \qquad \text{and} \qquad c^{w}_{u\,\lambda} \;=\; c^{w/_p}_{u/_p\,\lambda}.$$

Proof. We first study the map $\Psi : Sp_{2n}\mathbb{C}/P_0 \hookrightarrow Sp_{2n+2}\mathbb{C}/P_0$, defined by $K \mapsto \langle K, h\rangle$. Here, $Sp_{2n+2}\mathbb{C}/P_0$ is the Grassmannian of maximal isotropic subspaces of $V \oplus H_2$. If $E_\bullet, E'_\bullet$ are opposite isotropic flags in V, the analog of Corollary 4.2 is

$$\Psi(\Upsilon_\lambda E_\bullet) \;=\; \Upsilon_\lambda \psi_{n+1}E_\bullet \bigcap \Upsilon_{(n+1)}\psi_{\overline{n+1}}E'_\bullet,$$

where $(n+1)$ is a decreasing sequence of length 1. We leave this to the reader. As this intersection is generically transverse, $\Psi_* Q_\lambda = Q_\lambda \cdot q_{n+1}$. Thus

$$\Psi^* Q_\lambda \;=\; \begin{cases} Q_\lambda & \lambda_1 < n+1 \\ 0 & \lambda_1 = n+1 \end{cases}.$$

To see this, note that $\Psi^* Q_\lambda = \sum_\mu d^\mu_\lambda Q_\mu$, where

$$\begin{aligned} d^\mu_\lambda &:= \deg((\Psi^* Q_\lambda)\cdot Q_{\mu^c}) \\ &= \deg(Q_\lambda \cdot \Psi_*(Q_{\mu^c})) \\ &= \deg(Q_\lambda \cdot q_{n+1} \cdot Q_{\mu^c}) \;=\; \delta^\mu_\lambda, \end{aligned}$$

the Kronecker delta, by the Pieri formula for isotropic Grassmannians [8].

Consider the commutative diagram:

$$\begin{array}{ccccc} & & Y_{u/_p}\bigcap Y'_{\omega_0 w/_p} & \xrightarrow{\;\psi_p\;} & Y_u \bigcap Y'_{\omega_0 w} & & \\ & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} & & \\ \mathcal{Y}^{w/_p}_{u/_p} & = & \pi(Y_{u/_p}\bigcap Y'_{\omega_0 w/_p}) & \xrightarrow{\;\Psi\;} & \pi(Y_u \bigcap Y'_{\omega_0 w}) & = & \mathcal{Y}^w_u \end{array}$$

Thus $[\mathcal{Y}^w_u] = \Psi_*[\mathcal{Y}^{w/_p}_{u/_p}]$ and the maps π have the same degree δ as the horizontal maps are isomorphisms.

Let λ be a strict partition. Then $c^w_{u\,\lambda} = \delta\cdot\deg(Q_\lambda\cdot[\mathcal{Y}^w_u])$ which is

$$\delta\cdot\deg(Q_\lambda\cdot\Psi_*[\mathcal{Y}^{w/_p}_{u/_p}]) = \delta\cdot\deg(\Psi^*(Q_\lambda)\cdot[\mathcal{Y}^{w/_p}_{u/_p}]) = \delta\cdot\deg(Q_\lambda\cdot[\mathcal{Y}^{w/_p}_{u/_p}]) = c^{w/_p}_{u/_p\,\lambda}.$$

Lemma 4.8. *Suppose $u <_0 w$ and $x <_0 z$ in $\mathcal{B}_n$ with $wu^{-1} = zx^{-1}$ and $u(i) \neq w(i)$ for all $i \in [n]$. Then, if Y_u, Y_x, (respectively $Y'_{\omega_0 w}, Y'_{\omega_0 z}$) are defined with the same flags, there is commutative diagram*

$$\begin{tikzcd}
Y_u \bigcap Y'_{\omega_0 w} \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y_x \bigcap Y'_{\omega_0 z} \arrow[dl, "\pi"] \\
& \pi(Y_u \bigcap Y'_{\omega_0 w}) \;=\; \pi(Y_x \bigcap Y'_{\omega_0 z}) &
\end{tikzcd}$$

where f is an isomorpism between Zariski open subsets of $Y_u \bigcap Y'_{\omega_0 w}$ and $Y_x \bigcap Y'_{\omega_0 z}$, and the maps π have degree 1.

Proof of Theorem B(2). Let $u, w, x, z \in \mathcal{B}_n$ with wu^{-1} shape equivalent to zx^{-1} and λ a strict partition. If $u(p) = w(p)$, then $u/_p <_0 w/_p$ with $w/_p(u/_p)^{-1}$ shape equivalent to wu^{-1}. Thus we may assume that the hypotheses of Lemma 4.8 hold, by Theorem 4.7. But then $\pi_*[Y_u \bigcap Y'_{\omega_0 w}] = \pi_*[Y_x \bigcap Y'_{\omega_0 z}]$, hence $\pi_*(\mathfrak{C}_u\cdot\mathfrak{C}_{\omega_0 w}) = \pi_*(\mathfrak{C}_x\cdot\mathfrak{C}_{\omega_0 z})$, showing $c^w_{u\,\lambda} = c^z_{x\,\lambda}$.

Proof of Lemma 4.8. Let $g_{\overline{\jmath}}$ for $1 \leq j \leq n$ be the functions of Lemma 3.2, defined for all $F_\bullet \in Y^\circ_u \bigcap Y'^\circ_{\omega_0 w}$. For such $F_\bullet$, let $f(F_\bullet)$ be the flag whose $\overline{\jmath}$-th subspace is

$$\langle g_{u^{-1}x(\overline{n})}(F_\bullet), \ldots, g_{u^{-1}x(\overline{\jmath})}(F_\bullet)\rangle.$$

Then f is an isomorphism between the intersections of Schubert cells, which are Zariski dense in the intersections of Schubert varieties, and the diagram is commutative. Since we may assume w is a Grassmannian permutation, and in this case, the map $\pi : Y_{\omega_0 w} \to \pi(Y_{\omega_0 w})$ has degree 1, it follows that the maps π (which have the same degree) have degree 1.

5. More identities

5.1. **Product decomposition.** Suppose that W is a $2m$-dimensional complex symplectic vector space and consider the map

$$\Xi \;:\; Sp_{2m}\mathbb{C}/P_0 \times Sp_{2n}\mathbb{C}/P_0 \;\longrightarrow\; Sp_{2m+2n}\mathbb{C}/P_0$$

defined by

$$\Xi \;:\; (H, K) \;\longmapsto\; H \oplus K,$$

where $H \subset W$ and $K \subset V$ are maximal isotropic (Lagrangian) subspaces.

Theorem 5.1. *Let $\eta, \zeta \in \mathcal{B}_{n+m}$ with $\eta\cdot\zeta$ a disjoint product and $\#\mathrm{supp}(\eta) \leq m$, $\#\mathrm{supp}(\zeta) \leq n$. Then for any $\eta' \in \mathcal{B}_m$ and $\zeta' \in \mathcal{B}_n$ with $\eta \sim \eta'$ and $\zeta \sim \zeta'$ ($\sim$ is shape equivalence), there is an element g of $Sp_{2m+2n}\mathbb{C}$ such that*

$$\Xi(\mathcal{Y}_{\eta'} \times \mathcal{Y}_{\zeta'}) \;=\; g(\mathcal{Y}_{\eta\cdot\zeta}).$$

Proof. This is a consequence of Lemma 5.2.1 of [3], the analogous fact for the classical flag manifold and Grassmannian. Restricting that result to the symplectic flag manifold and Grassmannian of maximal isotropic subspaces proves the theorem. ∎

Remark. This does not hold for the odd orthogonal case. In fact, even the map Ξ cannot be defined: If V, W are odd-orthogonal spaces, then $V \oplus W$ is an even-dimensional space.

5.2. **Some hidden symmetries.** Define the permutation $\rho \in \mathcal{B}_n$ by $\rho(i) = i-1-n$. For example, in $\mathcal{B}_4$, we have $\rho = \langle 1, \overline{4}\rangle\langle 2, \overline{3}\rangle$, in the cycle notation of Section 2. In general, $\rho = \langle 1, \overline{n}\rangle\langle 2, \overline{n-1}\rangle \cdots \langle \lfloor\frac{n+1}{2}\rfloor, \overline{\lfloor\frac{n+2}{2}\rfloor}\rangle$. For a permutation $\zeta \in \mathcal{B}_n$, let $\zeta^\rho := \rho\zeta\rho$ denote the conjugation of ζ by ρ.

A (saturated) chain in either the Lagrangian order or Lagrangian réseau gives a sequence $a_1, \ldots, a_m$ of labels. The *peak set* of that chain consists of those $j \in \{2, 3, \ldots, m-1\}$ with $a_{j-1} < a_j > a_{j+1}$, its *descent set* of those $i \in \{1, 2, \ldots, m-1\}$ with $a_i > a_{i+1}$, and its *ascent set* those i with $a_i < a_{i+1}$.

Theorem C(1). *Let $\zeta \in \mathcal{B}_n$. Then $\mathcal{L}(\zeta) = \mathcal{L}(\zeta^\rho)$ and if λ is a strict partition with $|\lambda| = \mathcal{L}(\zeta)$, then*

$$b^\zeta_\lambda = b^{\zeta^\rho}_\lambda \qquad \text{and} \qquad c^\zeta_\lambda = c^{\zeta^\rho}_\lambda.$$

Proof. The first statement follows from Corollary 2.9 as $||\zeta|| = ||\zeta^\rho||$ in $(\mathcal{S}_{\pm[n]}, \prec\!\!\!\cdot\,)$ and both ζ and ζ^ρ have the same number of sign changes. The last statement is a consequence of Lemma 5.4 below, which shows $\pi_*(\mathfrak{C}_u \cdot \mathfrak{C}_{\omega_0\zeta u}) = \pi_*(\mathfrak{C}_x \cdot \mathfrak{C}_{\omega_0\zeta^\rho x})$ whenever $u <_0 \zeta u$ and $x <_0 \zeta^\rho x$. ∎

Example 5.2. *In $\mathcal{B}_4$, let $\zeta = \langle 134\rangle\langle 2]$. Then $\zeta^\rho = \langle 142\rangle\langle 3]$. Consider the intervals $[e, \zeta]_\prec$ and $[e, \zeta^\rho]_\prec$ in the labeled Lagrangian réseau displayed in Figure 6. While they are not isomorphic, they have the same rank, the same number of maximal chains, 80, and the underlying orders each have 5 chains. Moreover, they each have 2 chains with peak set $\{3\}$, and one each with peak sets $\{2\}, \{4\}$, and $\{2, 4\}$. The réseaux have the same number of chains with fixed descent sets. The jth component of the following vector records the number of chains with descent set equal to the position of the 1's in the binary representation of $j-1$:*

$$(0, 2, 6, 4, 6, 12, 8, 2, 2, 8, 12, 6, 4, 6, 2, 0)$$

Definition 5.3. *Let $E_\bullet, E'_\bullet$ be opposite isotropic flags in V. Define a flag $\widetilde{E}_\bullet$ by:*

$$\widetilde{E}_\bullet \;:\; E_1 \bigcap E'_{\overline{1}} \;\subset\; \cdots \;\subset\; E_n \bigcap E'_{\overline{1}} \;\subset\; (E_{\overline{n}} + E'_{\overline{1}}) \;\subset\; \cdots \;\subset\; (E_{\overline{2}} + E'_{\overline{1}}) \;\subset\; V.$$

Define $\widetilde{E}'_\bullet$ the same way, but with the roles of $E_\bullet$ and $E'_\bullet$ reversed. This gives opposite flags $\widetilde{E}_\bullet, \widetilde{E}'_\bullet$, and since $(A \bigcap B)^\perp = (A^\perp + B^\perp)$, they are isotropic.

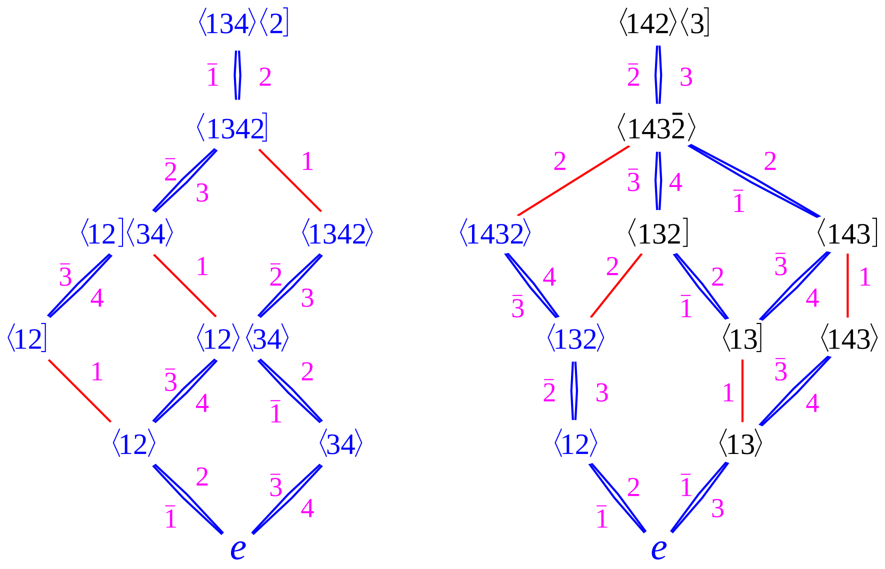


FIGURE 6. Conjugation by ρ on labeled intervals

Lemma 5.4. *Suppose $u, w, x, z \in \mathcal{B}_n$ with $\rho u^{-1} w \rho = x^{-1} z$, and $u(j) \neq w(j)$ for $1 \leq j \leq n$. Then, for any opposite isotropic flags $E_\bullet, E'_\bullet$ in V, there is a commutative diagram*

$$\begin{tikzcd}
Y_u E_\bullet \bigcap Y_{\omega_0 w} E'_\bullet \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y_x \widetilde{E}_\bullet \bigcap Y_{\omega_0 z} \widetilde{E}'_\bullet \arrow[dl, "\pi"] \\
& \pi(Y_u E_\bullet \bigcap Y_{\omega_0 w} E'_\bullet) = \pi(Y_x \widetilde{E}_\bullet \bigcap Y_{\omega_0 z} \widetilde{E}'_\bullet) &
\end{tikzcd}$$

with f an isomorphism of Zariski open subsets of $Y_u E_\bullet \bigcap Y_{\omega_0 w} E'_\bullet$ and $Y_x \widetilde{E}_\bullet \bigcap Y_{\omega_0 z} \widetilde{E}'_\bullet$.

Proof. Let $G_\bullet, G'_\bullet$ be oppsite (not necessarily isotropic) flags in V. Define $G_\bullet^+$ to be

$$G_\bullet^+ \;:\; G_{\overline{n-1}} \bigcap G'_{n-1} \;\subset\; G_{\overline{n-2}} \bigcap G'_{n-1} \;\subset\; \cdots \;\subset\; G'_{n-1} \;\subset\; V.$$

Define $G'^+_\bullet$ to be

$$G'^+_\bullet \;:\; G_{\overline{n}} \;\subset\; (G_{\overline{n}} + G'_{\overline{n}}) \;\subset\; \cdots \;\subset\; (G_{\overline{n}} + G'_{n-1}) \;\subset\; V.$$

For $\zeta \in \mathcal{S}_{\pm[n]}$, let ζ^+ be the conjugation of ζ by the cycle $(\overline{n}, \ldots, \overline{1}, 1, \ldots, n)$. In Section 5.3 of [3], the following proposition is proven:

Proposition 5.5. *Let $u, w, x, z \in \mathcal{S}_{\pm[n]}$ with $u \lhd_0 w$, $x \lhd_0 z$, $(u^{-1}w)^+ = x^{-1}z$, and w is a Grassmannian permutation with descent 0, $(w(\overline{n}) < \cdots < w(\overline{1})$ and $w(1) < \cdots < w(n))$. If $\pi : \mathbb{F}\ell V \twoheadrightarrow \mathbf{G}_n(V)$ is the projection, then there is a commutative* *diagram:*

$$\begin{tikzcd} X_uG_\bullet \bigcap X_{\omega_0 w}G'_\bullet \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & X_xG^+_\bullet \bigcap X_{\omega_0 z}G'^+_\bullet \arrow[dl, "\pi"] \\ \pi(X_uG_\bullet \bigcap X_{\omega_0 w}G'_\bullet) & = & \pi(X_xG^+_\bullet \bigcap X_{\omega_0 z}G'^+_\bullet) \end{tikzcd}$$

with f an isomorpism of Zariski open subsets of $X_uG_\bullet \bigcap X_{\omega_0 w}G'_\bullet$ and $X_xG^+_\bullet \bigcap X_{\omega_0 z}G'^+_\bullet$.

It suffices to prove Lemma 5.4 with w a Grassmannian permutation, by Lemma 4.8. Observe that $(\widetilde{E}_\bullet, \widetilde{E}'_\bullet)$ is the result of n applications of the map $(E_\bullet, E'_\bullet) \mapsto (E_\bullet{}^+, E'_\bullet{}^+)$. Similarly, $\rho = (\overline{n}, \ldots, \overline{2}, \overline{1}, 1, \ldots, n)^n$. Thus, iterating Proposition 5.5 n times gives the commutative diagram in $\mathbb{F}\ell V$ and $\mathbf{G}_n V$:

$$\begin{tikzcd} X_uE_\bullet \bigcap X_{\omega_0 w}E'_\bullet \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & X_x\widetilde{E}_\bullet \bigcap X_{\omega_0 z}\widetilde{E}'_\bullet \arrow[dl, "\pi"] \\ \pi(X_uE_\bullet \bigcap X_{\omega_0 w}E'_\bullet) & = & \pi(X_x\widetilde{E}_\bullet \bigcap X_{\omega_0 z}\widetilde{E}'_\bullet) \end{tikzcd}$$

Restricting this to the subset of isotropic flags gives the diagram of the lemma. ∎

These same arguments prove the analog of Lemma 5.4 for $So_{2n+1}\mathbb{C}/C$.

5.3. **More hidden symmetries.** Until now, we have deduced identities in $H^*Sp_{2n}\mathbb{C}/B$ by rectricting constructions involving Schubert subvarieties of $\mathbb{F}\ell(V)$ to those in $Sp_{2n}\mathbb{C}/B$ via the embedding $Sp_{2n}\mathbb{C}/B \hookrightarrow \mathbb{F}\ell(V)$. This is the geometric counterpart of the embedding $\mathcal{B}_n \hookrightarrow \mathcal{S}_{\pm[n]}$ studied in Section 2. Here, we explore the geometry of the map $\iota : \mathcal{S}_n \hookrightarrow \mathcal{B}_n$, where a permutation $w \in \mathcal{S}_n$ is extended to act on $-[n]$ by $w(-i) = -w(i)$. An immediate consequence of Definition-Theorem 2.8 and its analog for $\prec\!\!\lhd$ is the following lemma.

Lemma 5.6. *The map ι is an embedding of Bruhat orders $(\mathcal{S}_n, \lhd) \hookrightarrow (\mathcal{B}_n, \leq)$ and it respects the length functions in each order. Furthermore, $\iota(\mathcal{S}_n)$ consists of those permutations $\zeta \in \mathcal{B}_n$ with $a \cdot \zeta(a) > 0$ for all a.*

Let $L, L^\perp$ be complementary Lagrangian subspaces in V. The pairing $(x, y) \in L \oplus L^\perp \mapsto \beta(x, y)$, where β is the alternating form, identifies them as linear duals. Given a subspace H of L, let $H^\perp \subset L^\perp$ denote its annihilator in $L^\perp$. Then $H + H^\perp$ is a Lagrangian subspace of V.

Let $\mathbb{F}\ell(L)$ be the space of complete flags $F_\bullet := F_1 \subset F_2 \subset \cdots \subset F_n = L$ in L. Note that here $\dim F_i = i$. For each $k = 0, 1, \ldots, n$, define an injective map

$$\varphi_k \ : \ \mathbb{F}\ell(L) \longrightarrow Sp_{2n}\mathbb{C}/B$$

by

$$(5) \qquad (\varphi_k F_\bullet)_{\overline{\jmath}} = \begin{cases} F_{n+1-j} & j \geq n-k+1 \\ F_k + F^\perp_{k+j-1} & j \leq n-k+1 \end{cases} .$$

Then $(\varphi_k F_\bullet)_{\overline{1}} = (F_k + F_k^\perp)$ is Lagrangian, showing that $\varphi_k F_\bullet$ is an isotropic flag.

For $w\in\mathcal{S}_n$ the Schubert variety $X_wE_\bullet$ of $\mathbb{F}\ell(L)$ consists of those flags $F_\bullet\in\mathbb{F}\ell(L)$ satisfying

$$\dim E_a\bigcap F_b = \#\{b\geq l \mid w(l)+a\geq n+1\}. \tag{6}$$

We determine the image of Schubert varieties of $\mathbb{F}\ell(L)$ under these maps φ_k.

Define $\epsilon_k:\mathcal{S}_n\to\mathcal{B}_n$ by

$$(\epsilon_k w)(j) \;=\; \begin{cases} w(j+k) & 1\leq j\leq n-k\\ \overline{w(n{+}1{-}j)} & n-k<j\leq n\end{cases} \tag{7}$$

Note that $\epsilon_0 w=\iota(w)$.

Lemma 5.7. *Let $u,w\in\mathcal{S}_n$ with $u\lhd_k w$. Then ϵ_k induces an isomorphism of graded posets $[u,w]_{\lhd_k}\xrightarrow{\sim}[u,w]_0$ and $\iota(wu^{-1})=\epsilon_k w(\epsilon_k u)^{-1}$.*

Proof. A consequence of the definitions is that, for $u,w\in\mathcal{S}_n$,

$$u\lhd_k w \iff \epsilon_k u<_0\epsilon_k w,$$

and $\iota(wu^{-1})=\epsilon_k w(\epsilon_k u)^{-1}$. The Lemma follows immediately from these observations. ∎

Corollary 5.8. *The map ι is an embedding of ranked orders $\iota:(\mathcal{S}_\infty,\lessdot)\hookrightarrow(\mathcal{B}_\infty,\prec)$.*

Let $w^\vee$ be defined by $w^\vee(j)=n+1-w(j)$. Then $X_wE_\bullet$ and $X_{w^\vee}E'_\bullet$ are dual under the intersection pairing, where $E_\bullet, E'_\bullet$ are opposite flags.

Lemma 5.9. *With these definitions, $\varphi_kX_wE_\bullet$ is a subset of either*

$$Y_{\epsilon_k w}\varphi_nE_\bullet \quad\text{or}\quad Y_{\omega_0\epsilon_k w^\vee}\varphi_0E_\bullet.$$

Proof. Let $F_\bullet\in X_wE_\bullet$. We show $\varphi_kF_\bullet\in Y_{\epsilon_k w}\varphi_nE_\bullet$, that is, for each $-n\leq i\leq n$ and $1\leq j\leq n$ $(i\neq 0)$,

$$\dim(\varphi_nE_\bullet)_i\bigcap(\varphi_kF_\bullet)_{\overline{\jmath}} \;\geq\; \#\{n\geq l\geq j\mid i\geq\epsilon_k w(l)\}. \tag{8}$$

Suppose that $j>n-k+1$. Then $(\varphi_kF_\bullet)_{\overline{\jmath}}=F_{n+1-j}\subset L=(\varphi_nE_\bullet)_{\overline{1}}$. If $n\geq l\geq j$, then $(\epsilon_k w)(l)=\overline{w(n{+}1{-}l)}<0$. Thus if $i>0$, (8) holds as both sides equal $n+1-j$. Suppose $i<0$. Then $(\varphi_nE_\bullet)_i=E_{n+1-\overline{\imath}}$ and so the left side of 8) is

$$\begin{aligned}\dim E_{n+1-\overline{\imath}}\bigcap F_{n+1-j} \;&\geq\; \#\{m\leq n+1-j\mid w(m)+n+1-\overline{\imath}\geq n+1\}\\ &=\; \#\{n\geq l\geq j\mid i\geq\overline{w(n{+}1{-}l)}=\epsilon_k w(l)\}.\end{aligned}$$

Now suppose that $j\leq n-k+1$. Then $(\varphi_kF_\bullet)_{\overline{\jmath}}=F_k+F^\perp_{k+j-1}$. Thus the left side of (8) is

$$\dim(\varphi_nE_\bullet)_i\bigcap F_k \;+\; \dim(\varphi_nE_\bullet)_i\bigcap F^\perp_{k+j-1}. \tag{9}$$

If $i<0$, then $(\varphi_nE_\bullet)_i\subset L$ and only the first term of (9) contributes. By the previous paragraph, this is

$$\dim(\varphi_nE_\bullet)_i\bigcap F_k \;\geq\; \#\{n\geq l\geq k\mid i\geq\epsilon_k w(l)\}.$$

If $k \geq l$, then $\epsilon_k w(l) > 0 > i$, showing this equals the right side of (8).

If now $i > 0$, then $(\varphi_n E_\bullet)_i = L + E_{n-i}^\perp$. Thus (9) is

$$k + \dim E_{n-i}^\perp \bigcap F_{k+j-1}^\perp \;=\; k + n - \dim(E_{n-1} + F_{k+j-1}).$$

But this is $k + n - \dim E_{n-1} - \dim F_{k+j-1} + \dim E_{n-1} \bigcap F_{k+j-1}$, which is at least

$$\begin{aligned} i - j + 1 + \#\{k+j-1 \geq m \geq 1 \mid w(m) + n - 1 \geq n+1\} \\ &= n - j + 1 - \#\{n \geq m \geq k+j \mid w(m) \geq i+1\} \\ &= k + \#\{n-k \geq l \geq j \mid w(l+k) \leq i\} \\ &= k + \#\{n-k \geq l \geq j \mid \epsilon_k w(l) \leq i\}. \end{aligned}$$

This equals the right side of (8) since $l > n-k$ implies $\epsilon_k w(l) < 0 < i$.

Similar arguments show $\varphi_k F_\bullet' \in Y_{\omega_0 \epsilon_k w^\vee} \varphi_0 E_\bullet$. ∎

Corollary 5.10. *Let $u, w \in \mathcal{S}_n$ with $u \lhd_k w$ and $E_\bullet, E_\bullet' \in \mathbb{F}\ell(L)$ be opposite flags. Then $\varphi_n E_\bullet, \varphi_0 E_\bullet'$ are opposite isotropic flags, and*

$$\varphi_k\left(X_u E_\bullet \bigcap X_{w^\vee} E_\bullet'\right) \;=\; Y_{\epsilon_k u} \varphi_n E_\bullet \bigcap Y_{\omega_0 \epsilon_k w} \varphi_0 E_\bullet'.$$

Proof. Lemma 5.9 gives the inclusion $\subset$ and it is easy to see that $\varphi_n E_\bullet, \varphi_0 E_\bullet'$ are opposite. By Lemma 5.7, both sides have the same dimension, proving equality. ∎

For each $k = 1, 2, \ldots, n$, let $\pi_k : \mathbb{F}\ell(L) \to \mathbf{G}_k(L)$ be the projection induced by $E_\bullet \mapsto E_k$. As in Lemma 4.8, if $u \lhd_k w$ in $\mathcal{S}_n$ and $E_\bullet, E_\bullet'$ are opposite flags in $\mathbb{F}\ell(L)$, then the intersection $X_u E_\bullet \bigcap X_{w^\vee} E_\bullet'$ is mapped birationally onto its image $\pi_k(X_u E_\bullet \bigcap X_{w^\vee} E_\bullet')$ in $\mathbf{G}_k(L)$. Furthermore, the image cycle depends only upon $\eta := wu^{-1}$. Denote it by $\mathcal{X}_\eta$.

Define $\Phi_k : \mathbf{G}_k(L) \to Sp_{2n}\mathbb{C}/B$ by $H \mapsto (H + H^\perp)$. Then $\Phi_k \circ \pi_k = \pi \circ \varphi_k$ and we have the following corollary.

Corollary 5.11. *For any $\eta \in \mathcal{S}_n$, $\mathcal{Y}_{\iota(\eta)} = \Phi_k(\mathcal{X}_\eta)$, where $k = \#\{a \mid a < \eta(a)\}$.*

Recall that $\gamma := \iota(1, 2, \ldots, n) \in \mathcal{B}_n$.

Theorem C(2). *Let $\zeta \in \iota(\mathcal{S}_n)$. Then $\mathcal{L}(\zeta) = \mathcal{L}(\zeta^\gamma)$ and if λ is a strict partition with $|\lambda| = \mathcal{L}(\zeta)$, then*

$$b_\lambda^\zeta = b_\lambda^{\zeta^\gamma} \qquad \textit{and} \qquad c_\lambda^\zeta = c_\lambda^{\zeta^\gamma}.$$

Proof. The first statement follows from Lemma 5.6 as $||\eta|| = ||\eta^\gamma||$ for $\eta \in \mathcal{S}_n$. The last statement is a consequence of the identity $\mathcal{X}_\eta = \mathcal{X}_{\eta^\gamma}$ (Proposition 5.5) and Corollary 5.11. ∎

Example 5.12. *Let $\eta = (1, 2, 4, 3)$. Then $\zeta = \iota(\eta) = \langle 1, 2, 4, 3\rangle$ and $\zeta^\gamma = \langle 1, 4, 2, 3\rangle$. The labeled intervals $[e, \eta]_{\lessdot}$ and $[e, \zeta]_{\prec}$ are isomorphic. Consider the intervals $[e, \zeta]_{\prec}$ and $[e, \zeta^\gamma]_{\prec}$ in the labeled Lagrangian réseau displayed in Figure 7. While they are not isomorphic, they have the same rank, the same number of maximal chains, 16, and the underlying orders each have 2 maximal chains. Moreover, they each have a* *peakless chain and one with peak set $\{2\}$. The réseaux each have 2 increasing chains, 2 decreasing chains, 6 with descent set $\{1\}$, and 6 with descent set $\{2\}$.*

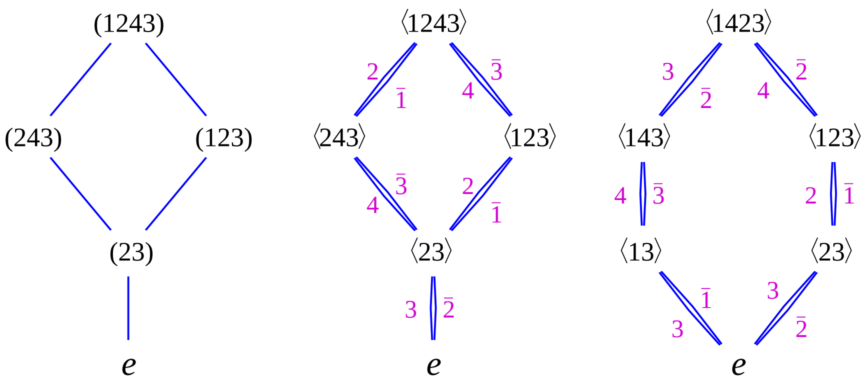


FIGURE 7. Conjugation by γ on labeled intervals

6. MINIMAL PERMUTATIONS AND LABELED RÉSEAUX

6.1. **Minimal permutations.** For a cycle $\zeta \in \mathcal{B}_\infty$, let $\delta(\zeta) = 1$ if ζ has the form $\iota(\eta)$ for $\eta \in \mathcal{S}_\infty$ and $\delta(\zeta) = 0$ otherwise. Note that $\delta(\zeta) = 1$ if and only if $a > 0$ implies $\zeta(a) > 0$.

Lemma 6.1. *Let $\zeta \in \mathcal{B}_\infty$ be a cycle. Then $\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta)$.*

Recall that $s(\zeta)$ counts the number of sign changes in ζ.

Proof. A saturated chain in $[e, \zeta]_{\triangleleft}$ (in $\mathcal{S}_{\pm\infty}$) gives a factorization of ζ into transpositions. If ζ consists of two cycles in $\mathcal{S}_{\pm\infty}$, then $||\zeta|| \;\geq\; 2(\#\mathrm{supp}(\zeta) - 1)$ and by Corollary 2.9(2),

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - 1 + s(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta),$$

with equality only if $s(\zeta) = 0$, that is, only if $\delta(\zeta) = 1$.

Similarly, if ζ is a single cycle in $\mathcal{S}_{\pm\infty}$, then $||\zeta|| \;\geq\; 2\#\mathrm{supp}(\zeta) - 1$. Since $\delta(\zeta) = 0$ and $s(\zeta) \geq 1$, Corollary 2.9(2) gives

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta).$$

Corollary 6.2. *If $\zeta \in \mathcal{B}_\infty$ is irreducible and $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$, then ζ is a single cycle.*

Proof. Recall that if $\eta, \xi \in \mathcal{B}_\infty$ have disjoint supports, then $\mathcal{L}(\eta \cdot \xi) \geq \mathcal{L}(\eta) + \mathcal{L}(\xi)$, with equality only when $\eta \cdot \xi$ is a disjoint product. Thus, by Lemma 6.1,

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \delta(\zeta),$$

with equality only when ζ is a single cycle.

Definition 6.3. *A minimal cycle is a cycle $\zeta \in \mathcal{B}_\infty$ for which $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - \delta(\zeta)$. For minimal cycles, $s(\zeta) + \delta(\zeta) = 1$. A permutation $\zeta \in \mathcal{B}_\infty$ is minimal if each of its irreducible factors are minimal cycles.*

Corollary 6.4. *If $\eta, \zeta \in \mathcal{B}_\infty$ with $\eta \prec \zeta$ and ζ is minimal, then so is η*

Proof. By Theorem 2.10, we may assume ζ is irreducible. Then ζ is a single cycle and the result follows by induction on $\mathcal{L}(\zeta)$, similar to the proof of Lemma 6.1. ∎

Corollary 6.5. *Let $\zeta \in \mathcal{B}_\infty$. If $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of ζ into irreducibles, then*

$$\mathcal{L}(\zeta) \;\geq\; \#\mathrm{supp}(\zeta) - \sum_i \delta(\zeta_i),$$

with equality only if ζ is minimal.

Lemma 6.6. *If $\zeta \in \mathcal{B}_\infty$ is a minimal cycle with $\delta(\zeta) = 0$, then there is a unique $a \in \mathrm{supp}(\zeta)$ with $a > 0 > \zeta(a) =: \overline{\alpha}$. Furthermore,*

$$\begin{array}{lll} 1 \lessdot t_\alpha \preceq \zeta & & \text{if } a \leq \alpha \\ 1 \preceq t_\alpha\zeta \lessdot \zeta & & \text{if } a \geq \alpha. \end{array}$$

Proof. Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle with $\delta(\zeta) = 0$. Then $s(\zeta) = 1$, so there is a unique $a > 0$ with $\overline{\alpha} := \zeta(a)$. We prove the lemma when $a > \alpha$: If $a = \alpha$, then $\zeta = t_a$, as ζ is irreducible and if $a < \alpha$, then replacing ζ by ζ^{-1}, reduces to the case $a > \alpha$.

Suppose $a > \alpha$. We claim that if $b > a$, then $\zeta(b) > \alpha$. The lemma follows from this claim. Indeed, then condition (ii) of Definition 2.8(1) is satisfied, and hence $t_\alpha\zeta \prec \zeta$. Since $a < \alpha$, we have $\mathrm{supp}(t_\alpha\zeta) = \mathrm{supp}(\zeta)$. As $\delta(t_\alpha\zeta) = 1$, it follows that $\mathcal{L}(t_\alpha\zeta) \geq \#\mathrm{supp}(\zeta) - 1 = \mathcal{L}(\zeta) - 1$ and thus $t_\alpha\zeta \lessdot \zeta$.

Let ζ be a permutation for which the claim does not hold with $\#\mathrm{supp}(\zeta)$ minimal. By Algorithm 2.12, $\eta := \zeta t_{y\,x} \lessdot \zeta$, where x is maximal in $\mathrm{supp}(\zeta)$ and y is minimal subject to $y \leq \zeta(x) < \zeta(y) \leq x$. Note that $y \neq a$. Since $a < x$, $\delta(\eta) = 0$. As ζ is minimal and $\eta \lessdot \zeta$, either η is irreducible and $\mathrm{supp}(\eta) \subsetneq \mathrm{supp}(\zeta)$ or else η is the disjoint product of two minimal cycles with x in the support of one and $|y|$ in the support of the other.

If $y < 0$, then $y = \overline{a}$ as $\zeta(y) > 0$. Then $\eta(a) = \overline{\zeta(x)}$ and $\eta(x) = \alpha$, so $\mathrm{supp}(\eta) \subset \mathrm{supp}(\zeta)$ and so η is irreducible with x in the support of one component and a in the other. But $x > a > \alpha = \eta(x) > \overline{\zeta(x)} = \eta(a)$, contradicting disjointness.

Suppose now that $y > 0$. Then $\zeta(x) > \alpha$, for otherwise $y = \overline{a} < 0$. Thus $x > b$. Since $b > a > \alpha > \eta(b)$ with $\eta(a) = \overline{\alpha}$, the component η' of η whose support contains a also contains b and $\mathrm{supp}(\eta') \subsetneq \mathrm{supp}(\zeta)$ with $\delta(\eta') = 0$, contradicting the minimality of $\#\mathrm{supp}(\zeta)$. ∎

6.2. **The Grassmann-Bruhat order on $\mathcal{S}_\infty$.** We develop some additional combinatorics for the symmetric group $\mathcal{S}_\infty$. For $k \in \mathbb{N}$, the k-Bruhat order $(\lhd_k)$ on $\mathcal{S}_\infty$ is the analog of the k-Bruhat order on $\mathcal{B}_{\pm\infty}$. The interval $[u, w]_k$ in the k-Bruhat order depends only upon wu^{-1}, so we define the *Grassmann-Bruhat order* on $\mathcal{S}_\infty$

(the Lagrangian order on $\mathcal{B}_\infty$ is its analog) by $\eta \lessdot \zeta$ if there is a $k \in \mathbb{N}$ and $u \in \mathcal{S}_\infty$ with $u \lessdot_k \eta u \lessdot_k \zeta u$. This order is ranked by $||\zeta|| = l(\zeta u) - l(u)$ for $u \lessdot_k \zeta u$, and it has an independent description:

Definition 3.2.2 of [3] *Let $\eta, \zeta \in \mathcal{S}_\infty$. Then $\eta \lessdot \zeta$ if and only if*

(1) $a < \eta(a) \Rightarrow \eta(a) \leq \zeta(a)$
(2) $a > \eta(a) \Rightarrow \eta(a) \geq \zeta(a)$
(3) *If $a < \zeta(a)$ and $b < \zeta(b)$ (respectively $a > \zeta(a)$ and $b > \zeta(b)$) with $a < b$ and $\zeta(a) < \zeta(b)$, then $\eta(a) < \eta(b)$.*

Covers $\eta \lessdot \zeta$ correspond to transpositions $(\alpha,\, \beta) = \zeta\eta^{-1}$ and we construct a labeled Hasse diagram for $(\mathcal{S}_\infty, \lessdot)$, labeling such a cover with the greater of α, β. By Theorem 3.2.3 of [3], the map $\eta \mapsto \zeta\eta^{-1}$ induces an order-reversing isomorphism between $[e, \zeta]_{\lessdot}$ and $[e, \zeta^{-1}]_{\lessdot}$, preserving the edge labels. Also, if $P = \{p_1, p_2, \dots\} \subset \mathbb{N}$ and $\varepsilon_P : \mathcal{S}_P \hookrightarrow \mathcal{S}_\infty$ is the map induced by the inclusion $P \hookrightarrow \mathbb{N}$ (these maps induce shape equivalence), then ε_P induces an isomorphism $[e, \zeta]_{\lessdot} \xrightarrow{\sim} [e, \varepsilon_P(\zeta)]_{\lessdot}$, preserving the relative order of the edge labels. Specifically, an edge label i of $[e, \zeta]_{\lessdot}$ is mapped to the label p_i of $[e, \varepsilon_P(\zeta)]_{\lessdot}$. Lastly, we remark that Algorithm 2.12 restricted to $\mathcal{S}_\infty$, and with $t_{a\,b}$ replaced by the transposition $(a,\, b)$, gives a chain in the $\lessdot$-order on $\mathcal{S}_\infty$ from e to ζ.

Lemma 6.7. *Let $\zeta \in \mathcal{S}_\infty$ and suppose that x is maximal subject to $x \neq \zeta(x)$. Then, for any α*

(1) $(\alpha,\, x)\zeta \lessdot \zeta \implies \zeta^{-1}(x) \leq \zeta(x) = \alpha.$
(2) $(\alpha,\, x) \lessdot \zeta \implies \zeta^{-1}(x) \geq \zeta(x) = \alpha.$

Proof. For 1, let $\eta := (\alpha,\, x)\zeta \lessdot \zeta$ and set $a = \zeta^{-1}(x)$ and $b = \zeta^{-1}(\alpha)$. Note that $a \neq b$ and $\eta(b) = x$. We claim that $b = x$ and $a < \alpha$, which will establish 1.

Suppose $b \neq x = \eta(b)$. Then, by the maximality of x, $b < \eta(b)$ and so the definition of $\lessdot$ implies $\eta(b) \leq \zeta(b) = \alpha$. Since $\alpha < x$, this implies $x < x$, a contradiction. Suppose now that $a > \alpha = \eta(a)$. By the definition of $\lessdot$, this implies that $\eta(a) \geq \zeta(a) = x$, and so $a > x$, contradicting the maximality of x.

The second assertion follows from the first by applying the anti-isomorphism $\eta \mapsto \eta\zeta^{-1}$ between $[e, \zeta]_{\lessdot}$ and $[e, \zeta^{-1}]_{\lessdot}$,

$$e \lessdot (\alpha,\, x) \lessdot \zeta \iff (\alpha,\, x)\zeta^{-1} \lessdot \zeta^{-1}. \qquad \blacksquare$$

A cycle $\zeta \in \mathcal{S}_\infty$ is *minimal* if $||\zeta|| + 1 = \#\mathrm{supp}(\zeta)$. A permutation is *minimal* if it is the disjoint product of minimal cycles. A maximal chain in an interval $[e, \zeta]_{\lessdot}$ is *peakless* if we do not have $a_{i-1} < a_i > a_{i+1}$ for any $i = 2, \dots, ||\zeta||-1$, where $a_1, \dots, a_{||\zeta||}$ is the sequence of labels in that chain.

Lemma 6.8. *Suppose $\zeta \in \mathcal{S}_\infty$ is a minimal cycle. Then there is a unique peakless chain in the labeled interval $[e,\, \zeta]_{\lessdot}$. If β is the smallest label in such a chain, then the transposition of that cover is (α, β) where $\alpha < \beta$ are the two smallest elements of* $\mathrm{supp}(\zeta)$.

Proof. We argue by induction on $||\zeta||$, which we assume is at least 2, as the case $||\zeta|| = 1$ is immediate. Replacing ζ by a shape equivalent permutation if necessary, we may assume that $\mathrm{supp}(\zeta) = [n]$, so that $||\zeta|| = m = n-1$.

Replacing ζ by ζ^{-1} would only reverse such a chain, so we may assume that $a := \zeta^{-1}(n) < b := \zeta(n)$. We claim that $(b,\, n)\zeta = \zeta(a,\, n) \lessdot \zeta$. Given this, the conclusion of the lemma follows. Indeed, let $\eta := (b,\, n)\zeta$. Since $\eta(n) = n$, this is an irreducible minimal permutation in $\mathcal{S}_{n-1}$. By the inductive hypothesis, $[e,\eta]_{\lessdot}$ has a unique chain with labels $\beta_1 > \cdots > \beta_k < \cdots < \beta_{n-2}$, and each $\beta_i < n$. The unique extension of this to a chain in $[e,\zeta]_{\lessdot}$ has $\beta_{n-2} < \beta_{n-1} = n$. This is the unique such chain in $[e,\zeta]_{\lessdot}$ as $\eta \lessdot \zeta$ is the unique terminal cover in $[e,\zeta]_{\lessdot}$ with edge label n, by Lemma 6.7. Also note that unless $n = 2$, $1 \leq a < b$, which proves the second part of 1.

By Algorithm 2.12, if y is chosen minimal so that $y \leq \zeta(n) = b < \zeta(y)$, then $\zeta(y,\, n) \lessdot \zeta$. We show that $y = a$, which will establish the claim and complete the proof.

Suppose $y \neq a$. Since $a < b < n = \zeta(a)$, the minimality of y implies that $y < a$. But then $\zeta(y,\, n)$ consists of two cycles η and η' and we have $\eta(a) = n$ and $\eta'(y) = b$. Since $y < a < b < n$, these cycles are not disjoint, so we have

$$
\begin{aligned}
n - 2 \;=\; ||\eta \cdot \eta'|| \;&>\; ||\eta|| + ||\eta'|| \\
&\geq\; \#\mathrm{supp}(\eta) - 1 + \#\mathrm{supp}(\eta') - 1 \;=\; n - 2,
\end{aligned}
$$

a contradiction. ∎

6.3. **The labeled Lagrangian order.** The labeled Lagrangian and 0-Bruhat orders on $\mathcal{B}_\infty$ are obtained from the Hasse diagrams of the underlying orders by labeling each cover with the integer β, where that cover is either $\zeta \prec\!\!\cdot\, t_\beta\zeta$ ($u \prec\!\!\cdot\, t_\beta u$) or $\zeta \prec\!\!\cdot\, t_{\alpha\,\beta}\zeta$ ($u \prec\!\!\cdot\, t_{\alpha\,\beta}u$). Recall that the map $\iota : \mathcal{S}_\infty \to \mathcal{B}_\infty$ maps the labeled Grassmann-Bruhat order on $\mathcal{S}_\infty$ isomorphically onto its image in the labeled Lagrangian order, preserving edge labels. The Pieri-type formula for $So_{2n+1}\mathbb{C}/B$ has two formulations (Theorems A and D), which we relate here. We say that a chain in $[e,\zeta]_{\prec}$ is *peakless* if in its sequence $\beta_1, \ldots, \beta_m$ of labels, we do not have $\beta_{i-1} < \beta_i > \beta_{i+1}$, for any $i = 2, \ldots, m-1$.

Lemma 6.9. *Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle. Then there is a unique peakless chain in the labeled interval $[e,\zeta]_{\prec}$. If $\delta(\zeta) = 0$, then the minimal label corresponds to the cover whose reflection is of the form t_a.*

Proof. If $\delta(\zeta) = 1$ this is an immediate consequence of Lemma 6.8. Suppose that $\delta(\zeta) = 0$. Replacing ζ by a shape equivalent permutation if necessary, we may assume that $\mathrm{supp}(\zeta) = [n]$ and $n > 1$. Replacing ζ by ζ^{-1} if necessary, we may assume that $a := \zeta^{-1}(n) < b := \zeta(n)$. As in the proof of Lemma 6.8, $(b,n)\zeta \lessdot \zeta$ in the Grassmann-Bruhat order on $\mathcal{S}_{\pm\infty}$. By Remark 2.3, either $t_{b\,n}\zeta \prec\!\!\cdot\, \zeta$ or else we have both $t_b\zeta \prec\!\!\cdot\, \zeta$ and $t_n\zeta \prec\!\!\cdot\, \zeta$. The second case implies $s(\zeta) > 1$, contradicting the minimality of ζ. Thus $\eta := t_{b\,n}\zeta \prec\!\!\cdot\, \zeta$.

Then η is a minimal cycle with $\delta(\eta) = 0$ and $\mathrm{supp}(\eta) = [n-1]$. Appending the cover $\eta \xrightarrow{n} \zeta$ to the unique peakless chain in $[e,\eta]_{\prec}$ gives a peakless chain in $[e,\zeta]_{\prec}$.

Moreover, η is the unique permutation with $\eta \lessdot \zeta$ and $\eta \xrightarrow{n} \zeta$, showing the uniqueness of this chain. ∎

For any $\zeta \in \mathcal{B}_\infty$, let $\Pi(\zeta)$ be the number of peakless chains in $[e,\zeta]_\prec$.

Lemma 6.10. *If $\eta, \zeta \in \mathcal{B}_\infty$ are disjoint, then*

$$\Pi(\eta\cdot\zeta) \;=\; 2\,\Pi(\eta)\cdot\Pi(\zeta).$$

Proof. For $\xi \in \mathcal{B}_\infty$, let $W(\xi)$ be the multiset of words formed from labels of maximal chains in $[e,\xi]_\prec$. The alphabet of these words is a subset of $\mathrm{supp}(\xi)$. Thus $W(\eta)$ and $W(\zeta)$ have disjoint alphabets. Note that $W(\eta\cdot\zeta)$ consists of all pairs of words in $W(\eta)\times W(\zeta)$. The lemma follows from Lemma 6.11, a combinatorial result concerning peakless words and shuffles.

For a set A of words in an ordered alphabet $\mathcal{A}$, let $\mathrm{peak}(A)$ be the subset of peakless words from A. Suppose that A' is another set of words with a different alphabet $\mathcal{A}'$ and fix some total order on the disjoint union $\mathcal{A}\coprod\mathcal{A}'$ which extends the given orders on each of $\mathcal{A},\mathcal{A}'$. Let $\mathrm{sh}(A,A')$ be all shuffles of pairs of words in $A\times A'$.

Lemma 6.11. *The natural restriction map $sh(A,A') \twoheadrightarrow A\times A'$ induces a 2 to 1 map*

$$\mathrm{peak}(\mathrm{sh}(A,A')) \;\twoheadrightarrow\; \mathrm{peak}(A)\times\mathrm{peak}(A').$$

Proof. It is clear that the restriction map takes a peakless word in $\mathrm{sh}(A,A')$ to a pair of peakless words in $A\times A'$. Given a pair of peakless words $(\omega,\omega')\in A\times A'$, there are exactly two shuffles of ω,ω' which are peakless: Suppose the minimal letter a in ω is greater than the minimal letter in ω'. Then these two shuffles differ only in their subwords consisting of a and u', where u' is that subword of ω' consisting of all letters less than a. Then u' is a segment of ω', as ω' is peakless. The two subwords of peakless shuffles are $a.u'$ and $u'.a$. ∎

Lemma 6.12. *Let $\zeta\in\mathcal{B}_\infty$ and suppose there is a peakless chain in $[e,\zeta]_\prec$. Then ζ is minimal.*

Proof. Suppose by way of contradiction that $\zeta\in\mathcal{B}_\infty$ is irreducible and not minimal, but $\Pi(\zeta)\neq 0$. We may further assume that among all such permutations, ζ has minimal rank, and that $\mathrm{supp}(\zeta)=[n]$. Let $\beta_1>\cdots>\beta_k<\cdots<\beta_m$ be the labels in a peakless chain in $[e,\zeta]_\prec$. Replacing ζ by ζ^{-1} if necessary (which merely reverses the chain), we may assume that $\beta_m=n$ and so $\beta_1\neq n$, by Lemma 6.7 and Theorem 2.2. Let η be the penultimate member of this chain. Then $\Pi(\eta)\neq 0$, as the initial segment of this chain gives a peakless chain in $[e,\eta]_\prec$. Thus η is a minimal permutation, by our assumption on ζ, and so

$$|\eta| \;\leq\; \#\mathrm{supp}(\eta)-\delta(\eta) \;\leq\; n-\delta(\zeta) \;<\; \mathcal{L}(\zeta) \;=\; |\eta|+1,$$

as ζ is not minimal and $\eta\lessdot\zeta$ so $\delta(\eta)\geq\delta(\zeta)$. Therefore the weak inequalities must be equalities, so that $\mathrm{supp}(\eta)=[n]$. Since $\beta_1>\cdots>\beta_k<\cdots<\beta_{m-1}$ are the labels of a chain in $[e,\eta]_\prec$ and $\beta_{m-1}<n$, we must have $\beta_1=n$, as $\mathrm{supp}(\eta)=[n]$. But this contradicts our earlier observation about β_1. ∎

We relate the two formulations of the Pieri formula in the odd-orthogonal case. For $\zeta \in \mathcal{B}_\infty$, define

$$\theta(\zeta) \;=\; \begin{cases} 2^{\#\{\text{irreducible factors of } \zeta\}-1} & \text{if } \zeta \text{ is minimal} \\ 0 & \text{otherwise} \end{cases}$$

Corollary 6.13. *For $\zeta \in \mathcal{B}_\infty$, $\Pi(\zeta) = \theta(\zeta)$.*

Proof. This is clear if ζ is minimal as both $\Pi(\zeta)$ and $\theta(\zeta)$ satisfy the same recursion, by Lemmas 6.9 and 6.10. ∎

6.4. **The Lagrangian réseau.** The enumerative significance of the structure constants $c^w_{u\,v}$ is best expressed in terms of maximal chains in certain directed multigraphs associated to intervals in the Bruhat order, which we call labeled réseaux. A cover $\eta \lessdot \zeta$ in the Lagrangian order corresponds to a reflection $\eta\zeta^{-1}$, which is either of the form $t_{a\,b}$ or of the form t_a. The *Lagrangian réseau* on $\mathcal{B}_\infty$ is the labeled directed multigraph where a cover $\eta \lessdot \zeta$ in the Lagrangian order with $\eta\zeta^{-1} = t_a$ is given a single edge $\eta \xrightarrow{a} \zeta$ and a cover with $\eta\zeta^{-1} = t_{a\,b}$ is given two edges $\eta \xrightarrow{\overline{a}} \zeta$ and $\eta \xrightarrow{b} \zeta$. We obtain the labeled Lagrangian order from this réseau by erasing those edges whose negative labels.

In the Grassmann-Bruhat order on $\mathcal{S}_\infty$, there are two conventions for labeling a cover $\eta \lessdot \zeta$: This cover gives a transposition $(\alpha, \beta) := \eta\zeta^{-1}$ with $\alpha < \beta$, and we may choose either α or β. For want of a better term, we call the consistent choice of α the lower convention, and the consistent choice of β the upper convention. We make use of the following fact.

Proposition 6.14. *Let $\eta \in \mathcal{S}_\infty$. If there is a chain in $[e, \eta]_{\lessdot}$ with decreasing labels in the lower convention, then there is a chain in $[e, \eta]_{\lessdot}$ with decreasing labels in the upper convention, and these chains are unique. The same is true for chains with increasing labels, and in either case η is minimal.*

A chain with increasing labels is an *increasing chain* and one with decreasing labels is a *decreasing chain.*

Lemma 6.15. *Let $\zeta \in \mathcal{B}_\infty$ be a minimal cycle. Then the réseau $[e, \zeta]_{\prec}$ has an increasing chain. If $\delta(\zeta) = 1$, then there are at least 2 increasing chains.*

Proof. Consider the peakless chain in the labeled order $[e, \zeta]_{\prec}$:

$$e \xrightarrow{\alpha_1} \zeta_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} \zeta_m \;=\; \zeta \tag{10}$$

Let α_k be the minimal label in this chain. Then $\zeta_{k-1} = \iota(\eta_{k-1})$ for some $\eta_{k-1} \in \mathcal{S}_\infty$. To see this, if $\delta(\zeta) = 0$, then by Lemma 6.9, the label α_k corresponds to the only cover whose reflection is not in $\iota(\mathcal{S}_\infty)$, and so $\delta(\zeta_{k-1}) = 1$.

The pullback of the initial segment of this chain to $[e, \eta_{k-1}]_{\lessdot}$ gives a decreasing chain (with labels $\alpha_1, \ldots, \alpha_{k-1}$) in the upper convention. Consider the unique decreasing chain

$$e \xrightarrow{\beta_1} \eta_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{k-1}} \eta_{k-1}$$

in the lower convention. Then

$$e \xrightarrow{\overline{\beta_1}} \iota(\eta_1) \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) \;=\; \zeta_{k-1}$$

is an increasing chain in the réseau $[e,\zeta_{k-1}]_\prec$. Concatenating the end of the peakless chain (10) onto this gives an increasing chain

$$e \xrightarrow{\overline{\beta_1}} \iota(\eta_1) \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) \xrightarrow{\alpha_k} \cdots \xrightarrow{\alpha_m} \zeta$$

in the réseau $e,\zeta]_\prec$.

Suppose $\delta(\zeta)=1$ and consider the middle portion of this increasing chain:

$$\iota(\eta_{k-2}) \xrightarrow{\overline{\beta_{k-1}}} \iota(\eta_{k-1}) = \zeta_{k-1} \xrightarrow{\alpha_k} \zeta_k.$$

Let $\overline{b}$ be the label of the other edge between ζ_{k-1} and ζ_k. Then we claim that $\overline{\beta_{k-1}} < \overline{b}$ so that replacing $\zeta_{k-1} \xrightarrow{\alpha_k} \zeta_k$ by $\zeta_{k-1} \xrightarrow{\overline{b}} \zeta_k$ gives a second increasing chain in the réseau.

To see this, first note that $\beta_{k-1} = b$ is impossible as these are consecutive covers in the Lagrangian order (see relation (iv) of Equation (2)). Define η by $\iota(\eta)=\zeta$ and pull this chain back to $[e,\eta]_\lhd$. It is the unique peakless chain in $[e,\eta]_\lhd$ and α_k is the minimal label. By Lemma 6.8(1), $b = \min(\mathrm{supp}(\eta))$ and so $\beta_{k-1} \geq b$. ∎

Lemma 6.16. *Let $\zeta \in \mathcal{B}_\infty$ and suppose there is an increasing chain in the réseau $[e,\zeta]_\prec$. Then ζ is minimal. If ζ is a minimal cycle, then there are precisely $2^{\delta(\zeta)}$ such chains.*

Proof. Let

$$e \xrightarrow{\beta_1} \zeta_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} \zeta_m \;=\; \zeta \tag{11}$$

be an increasing chain in $[e,\zeta]_\prec$. Suppose that $\beta_{k-1} < 0 < \beta_k$. Then for $i<k$, $\delta(\zeta_i)=1$. Define $\eta_i \in \mathcal{S}_\infty$ by $\iota(\eta_i)=\zeta_i$ for $i<k$. Then

$$e \xrightarrow{\overline{\beta_1}} \eta_1 \xrightarrow{\overline{\beta_2}} \cdots \xrightarrow{\overline{\beta_{k-1}}} \eta_{k-1}$$

is a decreasing chain in $[e,\eta]_\lhd$, with the lower labeling convention. Let

$$e \xrightarrow{\alpha_1} \xi_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} \xi_{k-1} \;=\; \eta_{k-1}$$

be the unique decreasing chain in the upper labeling convention. Concatenating the image of this chain in $[e,\zeta]_\prec$ with the end of the chain (11) gives a peakless chain

$$e \xrightarrow{\alpha_1} \iota(\xi_1) \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} \iota(\xi_{k-1}) = \zeta_{k-1} \xrightarrow{\beta_k} \cdots \xrightarrow{\beta_m} \zeta \tag{12}$$

in the interval $[e,\zeta]_\prec$ in the Lagrangian order. By Lemma 6.12, ζ is necessarily minimal.

Suppose now that ζ is a minimal cycle, then the réseau $[e,\zeta]_\prec$ has an increasing chain and the peakless chain (12) is unique. Consider another increasing chain

$$e \xrightarrow{\beta'_1} \zeta'_1 \xrightarrow{\beta'_2} \cdots \xrightarrow{\beta'_m} \zeta'_m \;=\; \zeta \tag{13}$$

and form η_i', ξ_i', α_i',and k' as for the original chain 11). If $k = k'$, then the chains (11), (13) coincide: The final segments agree, by the uniqueness of (12), as do their initial segments, by Proposition 6.14.

If $\delta(\zeta) = 0$, then the minimal label in the peakless chain (12) (either α_{k-1} or β_k) corresponds to the cover whose reflection has the form t_a. As $\delta(\zeta_{k-1}) = 1$, this must be β_k and so $k = k'$ and the chain (11) is the unique increasing chain in the réseau $[e, \zeta]_{\prec}$.

Suppose now that $\delta(\zeta) = 1$ and $k < k'$. Since $\alpha_i = \alpha_i'$ for $i < k$, $\beta_i = \beta_i'$ for $i \geq k'$, and $\alpha_1 > \cdots > \alpha_{k-1}$ and $\beta_k < \cdots < \beta_m$, we must have $k' = k + 1$. But then $\xi_i = \xi_i'$ for $i < k$ and also $\zeta_i = \zeta_i'$ for $i \geq k$, and so the two chains (11) and (13) agree except for the label of the cover $\zeta_{k-1} \lessdot \zeta_k$. Thus there are at most 2 increasing chains in the réseau $[e, \zeta]_{\prec}$ and their underlying permutations coincide. ∎

Example 6.17. *Suppose $\eta = (1, 2, 5, 3, 4)$ is a permutation in $\mathcal{S}_5$. Consider $\zeta = \langle 1, 2, 5, 3, 4\rangle = \eta\overline{\eta}$, a permutation in $\mathcal{B}_5$. Figure 8 shows the réseau $[e, \zeta]_{\prec}$. In this*

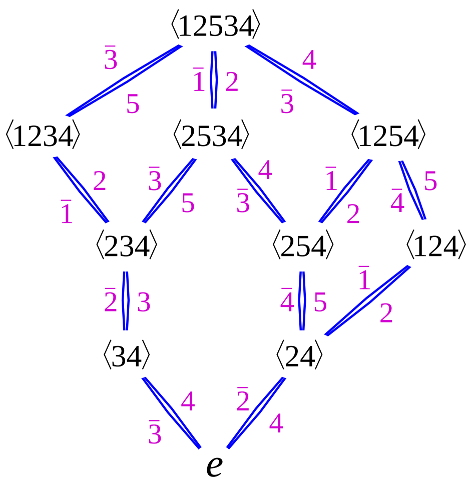


FIGURE 8. The interval $[e, \langle 1, 2, 5, 3, 4\rangle]_{\prec}$.

réseau, there are two increasing chains

$$e \xrightarrow{\overline{3}} \langle 3, 4\rangle \xrightarrow{\overline{2}} \langle 2, 3, 4\rangle \xrightarrow{\overline{1}} \langle 1, 2, 3, 4\rangle \xrightarrow{5} \langle 1, 2, 5, 3, 4\rangle$$
$$e \xrightarrow{\overline{3}} \langle 3, 4\rangle \xrightarrow{\overline{2}} \langle 2, 3, 4\rangle \xrightarrow{2} \langle 1, 2, 3, 4\rangle \xrightarrow{5} \langle 1, 2, 5, 3, 4\rangle$$

which correspond to the unique peakless chain in $[e, \eta]_{\lessdot}$ with $\beta_1 > \beta_2 > \beta_3 < \beta_4$

$$e \xrightarrow{4} (3, 4) \xrightarrow{3} (2, 3, 4) \xrightarrow{2} (1, 2, 3, 4) \xrightarrow{5} (1, 2, 5, 3, 4)\,.$$

Let $I(\zeta)$ count the increasing chains in the réseau $[e, \zeta]_{\prec}$.

Lemma 6.18. *If $\eta, \zeta \in \mathcal{B}_\infty$ are disjoint, then $I(\eta \cdot \zeta) = I(\eta) \cdot I(\zeta)$.*

Proof. As with Lemma 6.10, this is a consequence of the analogous bijection concerning increasing words among shuffles of words with disjoint alphabets. ∎

For $\zeta \in \mathcal{B}_\infty$, define

$$\chi(\zeta) \;=\; \begin{cases} 2^{\#\{\text{irreducible factors } \eta \text{ of } \zeta \text{ with } \delta(\eta)=1\}} & \text{if } \zeta \text{ is minimal} \\ 0 & \text{otherwise} \end{cases}$$

Let $D(\zeta)$ enumerate the decreasing chains in the réseau $[e,\zeta]_\prec$. By Theorem 2.11(6), an increasing chain in ζ becomes a decreasing chain in ζ^{-1}. The following result is now immediate.

Corollary 6.19. *For $\zeta \in \mathcal{B}_\infty$, $\chi(\zeta) = I(\zeta) = D(\zeta)$.*

7. The Pieri-type formula

Recall that for $\zeta \in \mathcal{B}_n$, $\delta(\zeta) = 1$ if $a > 0$ implies $\zeta(a) > 0$ and $\delta(\zeta) = 0$ otherwise.

Lemma 7.1. *Suppose $\zeta \in \mathcal{B}_n$ with $\delta(\zeta) = 1$. Then there is a quadratic form f on V with $f|_K \equiv 0$ for every $K \in \mathcal{Y}_\zeta$.*

Proof. We may suppose that $\mathrm{supp}(\zeta) = [n]$; otherwise pull back the quadratic form along the map $V \hookrightarrow V \oplus (H_2)^{\oplus t}$ ($t = n - \#\mathrm{supp}(\zeta)$) of Section 4. Let $E_\bullet, E'_\bullet$ be opposite isotropic flags. Since $E_\bullet, E'_\bullet$ are opposite flags, we have $V = E'_{\overline{1}} \oplus E_{\overline{1}}$. Let β be the alternating form on V and define a symmetric bilinear form q on V by

$$q(x,y) \;:=\; \beta(x^+, y^-) + \beta(y^+, x^-),$$

where v^+, v^- are the projections of $v \in V$ to the summands $E'_{\overline{1}}$ and $E_{\overline{1}}$. This form is non-degenerate. Let f be the associated quadratic form.

Let $u \in \mathcal{B}_n$ with $u \leq_0 \zeta u$ and ζu a Grassmannian permutation, as in Remark 2.7. Then there is a $k \in [n]$ with $i \leq k \Rightarrow \zeta u(i) \leq \overline{1}$ and $i > k \Rightarrow \zeta u(i) \geq 1$. Since $\delta(\zeta) = 1$, $u(i) \leq \overline{1}$ for $i \leq k$ and $u(i) \geq 1$ for $i > k$. Let $F_\bullet \in Y_u E_\bullet \bigcap Y_{\omega_0 \zeta u} E'_\bullet$. Then

$$\dim F_{\overline{1}} \bigcap E_{\overline{1}} \;\geq\; \#\{l \geq 1 \mid u(l) \leq \overline{1}\,\} \;=\; k$$

and

$$\dim F_{\overline{1}} \bigcap E'_{\overline{1}} \;\geq\; \#\{l \geq 1 \mid \zeta u(l) \geq 1\} \;=\; n - k.$$

Thus $F_{\overline{1}} = F_{\overline{1}} \bigcap E_{\overline{1}} \oplus F_{\overline{1}} \bigcap E'_{\overline{1}}$. Since $F_{\overline{1}}$ is isotropic for the alternating form β, this decomposition shows it is isotropic for the symmetric form q, proving the lemma. ∎

Corollary 7.2. *Suppose $\zeta = \zeta_1 \cdots \zeta_s$ is the factorization of ζ into irreducible permutations. Suppose exactly r of the ζ_i have $\delta(\zeta_i) = 1$. Then there exist r linearly independent quadratic forms $f_1, \ldots, f_r$ on V with $f_i|_K \equiv 0$ for every $K \in \mathcal{Y}_\zeta$.*

Proof. By Theorem 3.4(1), there exist symplectic spaces $V_1, \ldots, V_s$ with $V = V_1 \oplus \cdots \oplus V_s$ such that $K \in \mathcal{Y}_\zeta$ may be written as $K = K_1 \oplus \cdots \oplus K_s$, where, for $i = 1, \ldots, s$, $K_i = K \bigcap V_i$ and $K_i \in \mathcal{Y}_{\zeta'_i}$, with $\zeta'_i \in \mathcal{B}_{n_i}$ shape equivalent to ζ_i. (Here, $\dim V_i = 2n_i$.) We may assume that the ζ_i are ordered so that $\delta(\zeta_i) = 1$ for $i = 1, \ldots, r$ and $\delta(\zeta_i) = 0$ for $i > r$. By Lemma 7.1, for each $i = 1, \ldots, r$, there is a quadratic form f_i on V_i vanishing on $K_i \in \mathcal{Y}_{\zeta_i'}$. The pullback of f_i along the orthogonal projection $V \twoheadrightarrow V_i$ is a quadratic form on V, also denoted f_i. These are the desired forms. ▀

The special Schubert variety $\Upsilon_{(m)}$ of $Sp_{2n}\mathbb{C}/P_0$ consists of those Lagrangian subspaces which meet a fixed $(n+1-m)$-dimensional isotropic subspace M of V. In what follows, write Υ_M for this Schubert variety, and assume M is in general position in V.

For $\zeta \in \mathcal{B}_n$, define

$$\chi(\zeta) \;=\; \begin{cases} 2^{\#\{\text{irreducible factors } \eta \text{ of } \zeta \text{ with } \delta(\eta)=1\}} & \text{if } \zeta \text{ is minimal} \\ 0 & \text{otherwise} \end{cases}$$

For $\zeta \in \mathcal{B}_n$, define $\mathcal{W}_\zeta$ to be the zero locus of the forms $f_1, \dots, f_r$ of Corollary 7.2, a complete intersection. Then $\mathcal{W}_\zeta$ is the cone over a subvariety of $\mathbb{P}V$ of degree 2^r. If ζ is minimal, then this degree is $\chi(\zeta)$.

Corollary 7.3. *Let $\zeta \in \mathcal{B}_n$ with $\mathcal{L}(\zeta) = m$ and* $\operatorname{supp}(\zeta) = [n]$. *Let M be a general isotropic $(n+1-m)$-dimensional subspace of V. Then $M \bigcap \mathcal{W}_\zeta = \{0\}$ unless ζ is minimal, and in that case, $M \bigcap \mathcal{W}_\zeta$ is $\chi(\zeta)$ reduced lines.*

Proof. Since $Sp_{2n}\mathbb{C}$ acts transitively on $\mathbb{P}V$, Kleiman's theorem on the transversality of a general translate [22] will imply the corollary if we show

$$\dim M + \dim \mathcal{W}_\zeta \;\leq\; n+1, \tag{14}$$

with equality only if ζ is minimal.

By Corollary 7.2, $\dim \mathcal{W}_\zeta = 2n - r$, where r counts the irreducible factors η of ζ with $\delta(\eta) = 1$. Since $n = \#\operatorname{supp}(\zeta)$, Corollary 6.5 implies that $\dim M = n+1-\mathcal{L}(\zeta) \leq 1 + \sum_i \delta(\zeta_i) = 1 + r$, with equality only if each ζ_i is a minimal cycle, establishing (14). ▀

Theorem 7.4. *Suppose $\zeta \in \mathcal{B}_n$ is minimal and* $\operatorname{supp}(\zeta) = [n]$. *Then a general line $\langle v \rangle$ in $\mathcal{W}_\zeta$ determines a unique $K \in \mathcal{Y}_\zeta$ with $v \in K$.*

We deduce the Pieri-type formula from Theorem 7.4. First, define $\theta(\zeta) = 0$ if ζ is not minimal, and for ζ minimal, set

$$\theta(\zeta) \;:=\; 2^{\#\{\text{irreducible factors of } \zeta\} - 1}.$$

Recall that b^ζ_m was the structure constant corresponding to c^ζ_m for $So_{2n+1}\mathbb{C}/B$.

Theorem D. (Pieri-type Formula) *Let $\zeta \in \mathcal{B}_\infty$ with $\mathcal{L}(\zeta) = m$. Then $c^\zeta_m = \chi(\zeta)$ and $b^\zeta_m = \theta(\zeta)$.*

By Corollaries 6.13 and 6.19, this implies the chain-theoretic version of the Pieri-type formula (Theorem A).

Proof. Let $s(\zeta)$ be the number of sign changes in a permutation $\zeta \in \mathcal{B}_\infty$. Since $s(v_m) = 1$ and $s(\zeta) = s(\zeta u) - s(u)$ if $u \leq_0 \zeta u$, Equation (4) implies that $b^\zeta_m = 2^{s(\zeta)-1} c^\zeta_m$. Since, for a minimal cycle ζ, $s(\zeta) + \delta(\zeta) = 1$, we have

$$\theta(\zeta) \;=\; 2^{s(\zeta)-1}\chi(\zeta).$$

Thus it suffices to show $c^\zeta_m = \chi(\zeta)$.

By Theorem B(2), replacing ζ by a shape equivalent permutation if necessary, we may assume that $\mathrm{supp}(\zeta) = [n]$. Let M be a general isotropic $(n+1-m)$-dimensional subspace of V. By the projection formula, $c^\zeta_m = \deg(\Upsilon_M \bigcap \mathcal{Y}_\zeta)$. Since $K \in \Upsilon_M \bigcap \mathcal{Y}_\zeta$ implies that K meets $M \bigcap \mathcal{W}_\zeta$ non-trivially, we see that

$$c^\zeta_m \;=\; d \cdot \deg(M \bigcap \mathcal{W}_\zeta),$$

where d counts the $K \in \mathcal{Y}_\zeta$ which contain a general line of $\mathcal{W}_\zeta$, and the degree is taken in $\mathbb{P}V$. By Theorem 7.4, $d = 1$, which completes the proof. ∎

Reduction of Theorem 7.4 to the case of ζ a minimal cycle.

Let $\zeta = \zeta_1 \cdots \zeta_s$ be the irreducible factorization of ζ. In the notation of the proof of Corollary 7.2, a general $0 \neq v \in \mathcal{W}_\zeta$ has the form $v = v_1 \oplus \cdots \oplus v_s$, where $0 \neq v_i \in \mathcal{W}_{\zeta'_i}$ for $i = 1, \ldots, s$. Moreover $v \in K \in \mathcal{Y}_\zeta$ if and only if $v_i \in K_i \in \mathcal{Y}_{\zeta'_i}$. Thus is suffices to prove Theorem 7.4 for ζ a minimal cycle. We do this in the following sections.

7.1. **Case $\delta(\zeta) = 1$.**

Theorem 7.5. *Let $\zeta \in \mathcal{B}_n$ with $\mathrm{supp}(\zeta) = [n]$ and $\mathcal{L}(\zeta) = n-1$ so that ζ is a minimal cycle with $\delta(\zeta) = 1$. Then, for a general $0 \neq v \in \mathcal{W}_\zeta$, there is a unique $K \in \mathcal{Y}_\zeta$ with $v \in K$.*

Proof. Define $\eta \in \mathcal{S}_\infty$ by $\iota(\eta) = \zeta$. Set $k := \#\{a \mid a < \eta(a)\}$. Recall the notation of Section 5.3: Let $L, L^\perp$ be complementary Lagrangian subspaces of V, which are identified as linear duals. Define the map $\Phi_k : \mathbf{G}_k(L) \to Sp_{2n}\mathbb{C}/B$ by $H \mapsto (H + H^\perp)$, where $H^\perp \subset L^\perp$ is the annihilator of H. Define $\pi_k : \mathbb{F}\ell(L) \to \mathbf{G}_k(L)$ by $E_\bullet \mapsto E_k$. By Corollary 5.11, $\Phi_k : \mathcal{X}_\eta \xrightarrow{\sim} \mathcal{Y}_\zeta$ where $\mathcal{X}_\eta := \pi_k(X_u E_\bullet \bigcap X_{(\eta u)^\vee} E'_\bullet)$.

Schubert varieties Ω_ϱ of the Grassmannian $\mathbf{G}_k(L)$ are indexed by ordinary partitions (*weakly* decreasing sequences) $\varrho : n-k \geq \varrho_1 \geq \cdots \geq \varrho_k \geq 0$ [20]. We show

$$\Phi_k^{-1}(\{K \in Sp_{2n}\mathbb{C}/B \mid v \in K\}) \;=\; \Omega_{(n-k,1^{k-1})}, \tag{15}$$

where $(n-k, 1^{k-1})$ is the hook-shaped partition with first row $n-k$ and first column k. It follows from the projection formula that

$$\deg\Big(\mathcal{Y}_\zeta \bigcap \{K \mid v \in K\}\Big) \;=\; \deg\Big(\mathcal{X}_\eta \bigcap \Omega_{(n-k,1^{k-1})}\Big)$$

which is $\deg(\mathfrak{S}_w \cdot \mathfrak{S}_{\omega_0 \eta w} \cdot \pi_k^* S_{(n-k,1^{k-1})})$, the product in $H^*\mathbb{F}\ell L$. By [31, Theorem 8], this counts the chains in the k-Bruhat order

$$w \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-1}} \eta w$$

with $\beta_1 > \beta_2 > \cdots > \beta_k < \beta_{k+1} < \cdots < \beta_{n-1}$. The conclusion follows by Lemma 6.8.

To show (15), suppose $H \in \mathbf{G}_k(L)$ and with $v \in H + H^\perp$. Let v^- be the projection of v into L and v^+ its projection to $L^\perp$. Then $v = v^- \oplus v^+$ and $v^- \in H$ and $v^+ \in H^\perp$ so that $H \subset (v^+)^\perp$. Thus $\Phi_k^{-1}(\{K \mid v \in K\}) = \{H \mid v^- \in H \text{ and } H \subset (v^+)^\perp\}$, which is just the Schubert variety $\Omega_{(n-k,1^{k-1})}$. ∎

7.2. **Case $\delta(\zeta) = 0$.**

Theorem 7.6. *Let $\zeta \in \mathcal{B}_n$ with $\delta(\zeta) = 0$ and $\mathcal{L}(\zeta) = n$. Then $c_n^\zeta = 1$.*

This completes the proof of Theorem 7.4, as $c_n^\zeta = 1$ counts the $K \in \mathcal{Y}_\zeta$ which meet a generic line in V. We use some geometric constructions to reduce the computation of c_n^ζ to the cohomology of the classical flag manifold. Let $L, L^\perp$ be complementary Lagrangian subspaces of V with $L^\perp$ identified with the linear dual of L as in Section 5.3. For $1 \leq k \leq n$, set

$$\mathbb{F}_k \ := \ \{F_{k-1} \subset F_k \subset L \mid \dim F_i = i\},$$

a variety of partial flags in L.

Let $\wp : \mathbb{F}\ell(L) \to \mathbb{F}_k$ be the projection. Then the projections π_{k-1}, π_k of $\mathbb{F}\ell(L)$ to $\mathbf{G}_{k-1}(L)$, $\mathbf{G}_k(L)$ factor through $\wp$. Let π_{k-1}, π_k also denote the projections of $\mathbb{F}_k$ to $\mathbf{G}_{k-1}(L)$, $\mathbf{G}_k(L)$. Let $Lag(V)$ denote $Sp_{2n}\mathbb{C}/P_0$, the Grassmannian of Lagrangian subspaces of V. Consider the incidence variety Γ:

$$\begin{tikzcd}[column sep=small] & \Gamma := \{(F_{k-1}, F_k, K) \mid F_{k-1} \oplus F_k^\perp \subsetneq K \subsetneq F_k \oplus F_{k-1}^\perp\} \arrow[dl, "g"'] \arrow[d, "f"] \\ \mathbb{F}_k & Lag(V) \end{tikzcd}$$

Then Γ is a $\mathbb{P}^1$-bundle over $\mathbb{F}_k$ and f is genericaly 1-1: The image of f consists of those K with $\dim K \bigcap L \geq k-1$ and $\dim K \bigcap L^\perp \geq n-k$, which is an intersection of Schbert varieties. Thus a generic K in this intersection determines $g(f^{-1}(K)) = (K \bigcap L, (K \bigcap L^\perp)^\perp)$ uniquely.

For $j \in [n]$ and a (not necessarily strict) partition ϱ with $n-j \geq \varrho_1 \geq \cdots \geq \varrho_j \geq 0$, let $\sigma_\varrho \in H^*\mathbf{G}_j(L)$ be the Schubert class associated to the partition ϱ, as in [20]. We show:

Lemma 7.7. $g_* f^* q_n \ = \ \pi_k^* \sigma_{(n-k, 1^{k-1})} + \pi_{k-1}^* \sigma_{(n-k+1, 1^{k-2})}$.

Let $\zeta \in \mathcal{B}_n$ be minimal with $\delta(\zeta) = 0$, $\mathcal{L}(\zeta) + n$, and $\mathrm{supp}(\zeta) = [n]$. We construct a minimal permutation $\eta \in \mathcal{S}_n$ with $||\eta|| = n-1$, a $k \in [n]$, and $w \in \mathcal{S}_n$ with $\eta w \lhd_k w$, and $\eta w \not\lhd_{k-1} w$. Let $\mathcal{X}_{\eta^{-1}}$ be $\wp(X_{\eta w} \bigcap X'_{w^\vee})$. Then $[\mathcal{X}_{\eta^{-1}}] = \wp_*(\mathfrak{S}_\eta w \cdot \mathfrak{S}_{w^\vee})$. We show

Lemma 7.8. $[\mathcal{Y}_\zeta] \ = \ f_* g^* [\mathcal{X}_{\eta^{-1}}]$.

Theorem 7.6 follows from these Lemmas.

$$\begin{aligned} c_n^\zeta \ = \ \deg([\mathcal{Y}_\zeta] \cdot q_n) \ &= \ \deg(f_* g^* [\mathcal{X}_{\eta^{-1}}] \cdot q_n) \\ &= \ \deg([\mathcal{X}_{\eta^{-1}}] \cdot g_* f^* q_n), \end{aligned}$$

by the projection formula and Lemma 7.8. By Lemma 7.7, this is

$$\deg\left(\mathfrak{S}_{\eta w} \cdot \mathfrak{S}_{w^\vee} \cdot (\pi_k^* \sigma_{(n-k, 1^{k-1})} + \pi_{k-1}^* \sigma_{(n-k+1, 1^{k-2})})\right).$$

Since $\eta w \not\lhd_{k-1} w$, only the first term is non-zero. By [31, Theorem 8] and Lemma 6.8, this degree is 1. ∎

Proof of Lemma 7.7 The class $q_n \in H^*Lag(L)$ is represented by the Schubert variety

$$\Upsilon_v \;:=\; \{K \in Lag(V) \mid v \in K\} \;=\; \{K \mid \beta(v,K) \equiv 0\},$$

where $0 \neq v \in V$ and β is the alternating form. Then $g_*f^*q_n$ is represented by $g(f^{-1}\Upsilon_v)$ which is

$$\{F_{k-1} \subset F_k \mid \exists K \text{ with } v \in K \text{ and } F_{k-1} \oplus F_k^\perp \subsetneq K \subsetneq F_k \oplus F_{k-1}^\perp\}.$$

Since $V = L \oplus L^\perp$, we may write a general v uniquely as $v = w \oplus u$ with $w \in L$ and $u \in L^\perp$ and so $g(f^{-1}\Upsilon_v)$ is a subset of

$$\{F_{k-1} \subset F_k \mid w \in F_k\} \bigcap \{F_{k-1} \subset F_k \mid F_{k-1} \subset u^\perp\}. \tag{16}$$

This is an intersection of Schubert varieties (in general position if v is general) of codimensions $n-k$ and $k-1$, respectively. These Schubert varieties have classes $\pi_k^*\sigma_{(n-k)}$ and $\pi_{k-1}^*\sigma_{(1^{k-1})}$. Since Υ_v has codimension n, $g(f^{-1}\Upsilon_v)$ equals this intersection if the map $g : f^{-1}\Upsilon_v \to g(f^{-1}\Upsilon_v)$ is finite. Thus

$$g_*f^*q_n \;=\; d(\pi_k^*\sigma_{(n-k)} \cdot \pi_{k-1}^*\sigma_{(1^{k-1})}),$$

where d is the degree of the map $g : f^{-1}\Upsilon_v \to g(f^{-1}\Upsilon_v)$ (which is 0 if the map is not finite).

To compute d, let $F_{k-1} \subset F_k$ satisfy $w \in F_k$ and $F_{k-1} \subset u^\perp$ with $F_k \not\subset u^\perp$ and $w \notin F_{k-1}$. Then $F_k \oplus F_{k-1}^\perp \not\subset v^\perp$, and so $f(g^{-1}(F_{k-1}, F_k)) = F_k \oplus F_{k-1}^\perp \bigcap v^\perp$, which shows $d = 1$. Lastly, the Pieri-type formula [31] in $H^*\mathbb{F}_k$ shows

$$\pi_k^*\sigma_{(n-k)} \cdot \pi_{k-1}^*\sigma_{(1^{k-1})} \;=\; \pi_k^*\sigma_{(n-k),1^{k-1})} + \pi_{k-1}^*\sigma_{(n-k+1,1^{k-2})}. \qquad \blacksquare$$

Proof of Lemma 7.8 We first make some definitions. Replacing ζ by ζ^{-1} if necessary, we may assume that $\alpha > 0$ is the unique number with $t_\alpha\zeta \prec \zeta$, by Lemma 6.6. Since $\delta(t_\alpha\zeta) = 1$, we define $\eta \in \mathcal{S}_n$ by $\iota(\eta) = t_\alpha\zeta$. Set

$$k \;:=\; \#\{i \mid i > \eta(i)\} \;=\; \{i > 0 \mid \zeta(i) > i\}.$$

Let $u \in \mathcal{B}_n$ satisfy $u \leq_0 \zeta u$ with ζu a Grassmannian permutation (*cf.* Remark 2.7). Since $\delta(\zeta) = 0$, $\zeta u(n-k+2) > 0 > \zeta u(n+1-k)$. Define $j \geq k$ by $\zeta u(n+1-j) = \overline{\alpha}$. Then $u(n+1-j) > 0$. Let $w \in \mathcal{S}_n$ be defined by

$$w(i) \;=\; \begin{cases} u(n+1-i) & i < k \\ u(n+1-j) & i = k \\ \overline{u(i-k)} & k < i < n+1+k-j \\ \overline{u(i+1-k)} & n+1+k-j \leq i \leq n \end{cases}. \tag{17}$$

We claim that $\eta w \lhd_k w$. Since $\mathrm{supp}(\eta) = [n]$, we cannot have both $\eta w \lhd_k w$ and $\eta w \lhd_{k-1} w$. Since $u <_0 \iota(\eta)u$, condition (1) of Proposition 2.1 is satisfied. For condition (2), as ζu is a Grassmannian permutation,

$$\eta w(1) > \cdots > \eta w(k-1) \quad \text{and} \quad \eta w(k+1) > \cdots > \eta w(n),$$

so we only need show $i < k$ with $\eta w(i) < \eta w(k) = \alpha$ implies $w(i) < w(k)$. Let $l = n+1-i > n+1-j$. Then $\eta w(i) = \iota(\eta)u(l) < \iota(\eta)u(n+1-j) = \alpha$, which implies $u(l) < u(n+1-j)$ and hence $w(i) < w(k)$, as $u <_0 \iota(\eta)u$.

Example 7.9. *Let $\zeta = \langle 1,2,3,5,7,6,\overline{4}] \in \mathcal{B}_7$. Then $\alpha = 4$ and $t_4\zeta = \langle 1,2,3,5,7,6,4\rangle \prec \zeta$ so that $\eta = (1,2,3,5,7,6,4) \in \mathcal{S}_7$. Here, $k = 3$. If we set $u = \overline{5}\,\overline{3}6\overline{2}\,\overline{1}47$, then $\zeta u = 7\,\overline{5}\,\overline{4}\,\overline{3}\,\overline{2}16$ is a Grassmannian permutation and $u <_0 \zeta u$. We see that $j = 5 \geq 3 = k$ and $w = 746\,5321$ so that $\eta w = 614\,7532$ and $\eta w \lhd_3 w$.*

Lemma 7.8 is a consequence of the following construction:

Lemma 7.10. *Let ζ, u, η, w,and k be as above. Then there exists a commutative diagram*

$$\begin{tikzcd}
& (\wp^*\Gamma)|_{X_{\eta w}\bigcap X'_{w^\vee}} \arrow[dl, "g"'] \arrow[d, "\wp"] \arrow[rr, dashed, "h"] & & Y'_u\bigcap Y_{\omega_0\zeta u} \arrow[d, "\pi"] \\
X_{\eta w}\bigcap X'_{w^\vee} \arrow[d, "\wp"'] & \Gamma|_{\mathcal{X}_{\eta^{-1}}} \arrow[dl, "g"] \arrow[rr, "f"] & & \mathcal{Y}_\zeta \\
\mathcal{X}_{\eta^{-1}} & & &
\end{tikzcd}$$

where the maps $h, f, \wp$, and π are isomorphisms on Zariski dense sets.

Since $g^{-1}(\mathcal{X}_{\eta^{-1}}) = \Gamma|_{\mathcal{X}_{\eta^{-1}}}$ and $\mathcal{Y}_\zeta$ have the same dimension, and the maps are generically 1-1, $f(g^{-1}(\mathcal{X}_{\eta^{-1}})) = \mathcal{Y}_\zeta$, which proves Lemma 7.8.

Proof. We first define an injective map $h : (\wp^*\Gamma)|_{X^\circ_{\eta w}} \to Y_{\omega_0\zeta u}$, then show the restriction of h to $(\wp^*\Gamma)|_{X_{\eta w}\bigcap X'_{w^\vee}}$ has image contained in $Y'_u\bigcap Y_{\omega_0\zeta u}$. Since the maps π and $\wp$ are generically 1-1, by the analog of Theorem 3.3 (Theorem 5.1.4 of [3]) for $\mathbb{F}\ell(L)$, the concluson follows.

Let $E_\bullet \in \mathbb{F}\ell(L)$ be a complete flag and let $(F_\bullet, K) \in \wp^*\Gamma|_{X^\circ_{\eta w}E_\bullet}$, so that $F_\bullet \in X^\circ_{\eta w}E_\bullet$ and $K \in Lag(V)$ satisfy

$$F_{k-1} \oplus F_k^\perp \ \subsetneq\ K \ \subsetneq\ F_k \oplus F_{k-1}^\perp.$$

Define $h(F_\bullet, K) \in Sp_{2n}\mathbb{C}/B$ by

$$h(F_\bullet, K)_{\overline{\imath}} := \begin{cases} F_{n+1-i} & n \geq i \geq n+2-k \\ F_{k-1} + F^\perp_{i+k-2} & n+1-k \geq i \geq n+2-j \\ K\bigcap(L + E_{n+\zeta u(i)}) & n+1-j \geq i \end{cases}.$$

We show this defines a flag in $Y_{\omega_0\zeta u}\varphi_n E_\bullet$. Note first that $h(F_\bullet, K)_{\overline{\imath}} = \varphi_{k-1}(F_\bullet)_{\overline{\imath}}$ for $i \geq n+2-j$. Since $\epsilon_{k-1}(\eta w)(i) = \omega_0\zeta u(i)$ for $\geq n+2-j$, Lemma 5.9 shows that

$$\dim(\varphi_n E_\bullet)_a \bigcap h(F_\bullet, K)_{\overline{\imath}} \ \geq\ \#\{l \geq i \mid a \geq \omega_0\eta u(i)\},$$

for $i \geq n+2-j$.

We show this for $i \leq n+1-j$, which shows $h(F_\bullet, K) \in Y_{\omega_0\zeta u\varphi_n E_\bullet}$. Since ζu is a Grassmannian permutation, and, for $a > 0$ $(\varphi_n E_\bullet)_a = L + E^\perp_{n-a}$, we need only show that

$$\dim K \bigcap (L + E^\perp_{n+\zeta u(i)}) \;\geq\; n+1-i \tag{18}$$

for $i \leq n+1-j$ (as $\zeta u(i) < 0$ in this range).

It suffice to show this for the dense subset of $(F_\bullet, K)$ with $K \bigcap L = F_{k-1}$. Then $F^\perp_{k-1}$ is the image of K under the projection $V \twoheadrightarrow L$. Thus, for $a > 0$

$$\begin{aligned}\dim K \bigcap (L + E^\perp_{n-a}) &= k-1+\dim F^\perp_{k-1} \bigcap E^\perp_{n-a}\\ &= a + \#\{k-1 \geq l \mid \eta w(l) \geq a+1\}\end{aligned}$$

If $k-1 \geq l$, then $\eta w(l) = \zeta u(n+1-l)$ and these exhaust the positive values of ζu. If $i \leq n+1-j$ and we set $a = \overline{\zeta u(i)}$, we see that $\dim K \bigcap (L + E^\perp_{n+\zeta u(i)})$ is $\overline{\zeta u(i)} + \#\{l \mid \zeta u(l) \geq \overline{\zeta u(i)} + 1\}$. Since $\zeta u \in \mathcal{B}_n$ is a Grassmannian permutation,

$$\{\overline{\zeta u(i)}, \ldots, n\} \;=\; \{\overline{\zeta u(i)}, \ldots, \overline{\zeta u(1)}\} \coprod \{\zeta u(l) \mid \zeta u(l) > \overline{\zeta u(i)}\}.$$

Thus $n+1-\overline{\zeta u(i)} = i + \#\{l \mid \zeta u(l) \geq \overline{\zeta u(i)}+1\}$ and so $\dim K \bigcap (L + E^\perp_{n+\zeta u(i)}) = n+1-i$, which completes the proof that $h(F_\bullet, K) \in Y_{\omega_0\zeta u\varphi_n E_\bullet}$ for $(F_\bullet, K) \in g^{-1}X^\circ_{\eta w}E_\bullet$.

We do a useful calculation before we finish the proof of Lemma 7.10.

Lemma 7.11. *If $F_\bullet \in X^\circ_{\eta w}E_\bullet$ and $1 \leq i \leq n+1-j$, then $F^\perp_k \bigcap E^\perp_{n+\zeta u(i)} = F^\perp_{k+i-1}$.*

Proof. Similar to the last paragraph, $\dim F^\perp_{k-1} \bigcap E^\perp_{n+\zeta u(i)} = n+1-i-k = \dim F^\perp_{k+i-1}$. Thus, we need only show $F^\perp_{k+i-1} \subset F^\perp_{k-1} \bigcap E^\perp_{n+\zeta u(i)}$.

Note that

$$\dim F_{k+i-1} \bigcap E_{n+\zeta u(i)} \;=\; \#\{k+i-1 \geq l \mid \eta w(l) \geq \overline{\zeta u(i)} + 1\}$$

Since $\eta w(k+1) > \cdots > \eta w(n)$ and $\eta w(k+i) = \overline{\zeta u(i)}$, this equals

$$\#\{k+i > l \mid \eta w(l) > \eta w(k+i)\} \;=\; n - \eta w(k+i) = n + \zeta u(i),$$

which shows $E_{n+\zeta u(i)} \subset F_{k+i-1}$. Since $F_k \subset F_{k+i-1}$, this completes the proof. ∎

We complete the proof of Lemma 7.10, showing that if we further require $F_\bullet \in X'_{w^\vee}$, then $h(F_\bullet, K) \in Y'_u$. Let $E'_\bullet$ be a flag opposite to $E_\bullet$ in L. We show $h(F_\bullet, K) \in Y_u\varphi_0 E'_\bullet$ for $(F_\bullet, K) \in (\wp^*\Gamma)|_{X_{\eta w} \bigcap X'_{w^\vee}}$. Note first that $F_\bullet$ satisfies

$$\dim E'_a \bigcap F_b \;=\; \#\{b \geq l \mid w(l) \leq a\}.$$

As before

$$\dim(\varphi_0 E'_\bullet)_a \bigcap h(F_\bullet, K)_{\bar{\imath}} \;\geq\; \#\{i \leq l \mid a \geq u(l)\} \tag{19}$$

for $i > n+1-j$ since $\epsilon_{k-1}(w^\vee)$ and u agree in this range, as do $h(F_\bullet, K)_{\bar{\imath}}$ and $(\varphi_{k-1}F_\bullet)_{\bar{\imath}}$. For $i \leq n+1-j$, it suffices to establish (19) for the dense subset of those $(F_\bullet, K)$ with $K \bigcap L^\perp = F^\perp_k$.

Suppose $a > 0$. Since $K \bigcap L^\perp = F_k^\perp$, F_k is the image of K under the projection $V \twoheadrightarrow L$. Similarly, F_k is the image of $K \bigcap (L + E^\perp_{n+\overline{\zeta u(i)}}) = h(F_\bullet, K)_{\overline{\imath}}$ under this projection. Since the kernel of this is $F_k^\perp \bigcap E^\perp_{n+\zeta u(i)} = F^\perp_{k+i-1}$ and $(\varphi_0 E'_\bullet)_a = (L^\perp + E'_a)$,

$$\begin{aligned}\dim(\varphi_0 E'_\bullet)_a \bigcap h(F_\bullet, K)_{\overline{\imath}} &= \dim F^\perp_{k+i-1} + \dim F_k \bigcap E'_a,\\ &= n+1-k-i+\#\{k \geq l \mid w(l) \leq a\},\\ &= n+1-i-\#\{k \geq l \mid w(l) > a\}.\end{aligned}$$

Since $\{w(1), \ldots, w(k)\} = \{u(n), \ldots, u(n+2-k), u(n+1-j)\}$ are the positive values of u and $i \leq n+1-j$, this is

$$n+1-i-\#\{i \leq l \mid u(l) > a\} \;=\; \#\{i \leq l \mid u(l) \leq a\},$$

which shows (19).

Now suppose $a < 0$. Then $(\varphi E'_\bullet)_a = (E'_{\overline{a}-1})^\perp \subset L^\perp$ and so

$$\begin{aligned}(\varphi_0 E'_\bullet)_a \bigcap h(F_\bullet, K)_{\overline{\imath}} &= F_k^\perp \bigcap E^\perp_{n+\overline{\zeta u(i)}} \bigcap (E'_{\overline{a}-1})^\perp\\ &= F^\perp_{k+i-1} \bigcap (E'_{\overline{a}-1})^\perp.\end{aligned}$$

This has dimension

$$n-k-i+2-\overline{a}+\#\{k+i-1 \geq l \mid w(l) \leq \overline{a}-1\} \;=\; \#\{k+i \leq l \mid w(l) \geq \overline{a}\}.$$

The values $\overline{w(l)}$ for $l \geq k+i$ are simply the negative values $u(l)$ for $l \geq i$. Thus this is $\#\{i \leq l \mid u(l) \leq a\}$, which completes the proofs of Lemma 7.10 and the Pieri-type formula. ∎

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Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA

E-mail address, Nantel Bergeron: `bergeron@mathstat.yorku.ca`

URL, Nantel Bergeron: `http://www.math.yorku.ca/bergeron`

Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin 53706-1388, USA

E-mail address, Frank Sottile: `sottile@math.wisc.edu`

URL, Frank Sottile: `http://www.math.wisc.edu/~sottile`