A QUASI-O-MINIMAL GROUP WITHOUT THE EXCHANGE PROPERTY

OLEG BELEGRADEK, FRANÇOISE POINT, AND FRANK O. WAGNER

Abstract. We construct an example of a quasi-o-minimal group without the Exchange Property.

A structure $\langle M, <, \dots \rangle$ is called *quasi-o-minimal* [1] if in any structure elementarily equivalent to it the definable subsets are exactly the Boolean combinations of 0-definable subsets and intervals. A complete theory T is said to have the *Exchange Property* if, for any model $\mathcal{M}$ of T and for any $X \subseteq \mathcal{M}$ and b, $c \in \mathcal{M}$,

$$\text{if } b \in \mathrm{acl}(X \cup \{c\}) \text{ but } b \notin \mathrm{acl}(X), \text{ then } c \in \mathrm{acl}(X \cup \{b\}).$$

In general, in contrast to o-minimal theories, the theory of a quasi-o-minimal structure need not satisfy the Exchange Property; an easy counterexample can be found in [1]. However, that example does not carry a group structure; in fact it is also shown there that any quasi-o-minimal expansion of $\langle \mathbb{Z}, +, < \rangle$ does have the Exchange Property. In this note we shall prove that the theory of the expansion $\mathcal{A}$ of the lexicographically ordered group $\mathbb{Z} \times \mathbb{Z}$ by the unary operation $(x, y) \mapsto (0, x)$ and the constant $(1, 0)$ is quasi-o-minimal and does not have the Exchange Property. It suffices to prove that the theory of a certain definitional expansion $\mathcal{M}$ of $\mathcal{A}$ is quasi-o-minimal and does not have the Exchange Property.

Let $\mathcal{M}$ be the expansion of $\mathcal{A}$ in the signature

$$\mathcal{L} = \{<, +, -, 0, 1', 1'', D_n, f\}_{n \geq 1},$$

where $-$ is subtraction, $1'$ and $1''$ are interpreted as $(0, 1)$ and $(1, 0)$, respectively; f is interpreted as the map $(x, y) \mapsto (0, x)$, and $D_n(a)$ means 'n divides a', for any positive integer n. Clearly, $\mathcal{M}$ is a definitional expansion of $\mathcal{A}$.

We shall find a set of first order $\mathcal{L}$-sentences T, each of which holds in $\mathcal{M}$, and show that the theory T admits quantifier elimination. We shall prove that $\mathcal{M}$ is embeddable into any model of T. Therefore T is complete and hence is an axiom system for $\mathrm{Th}(\mathcal{M})$. Using quantifier elimination for T we shall deduce quasi-o-minimality of $\mathcal{M}$. Even though in $\mathcal{M}$ itself the Exchange Property trivially holds (as every element of

---

Date: September 30, 1998.

1991 *Mathematics Subject Classification.* Primary: 03C67; Secondary: 20E36, 06F15, 03C60.

Work partially done while the authors were visiting the Mathematical Sciences Research Institute, Berkeley. Research at MSRI is supported in part by NSF grant DMS-9701755.

Point is a Senior Research Associate at the "Fonds National de la Recherche Scientifique".

Wagner is supported by a Heisenberg-Stipendiat der Deutschen Forschungsgemeinschaft (Wa 899/2-1).

$\mathcal{M}$ is definable), we shall construct a model of T in which the Exchange Property fails.

Let T be the following set of first-order $\mathcal{L}$-sentences:

1. the axioms of linearly ordered abelian groups;
2. f is an endomorphism of the ordered group;
3. $\mathrm{Ker}(f) = \mathrm{Rng}(f)$;
4. $1'$ is the least positive element;
5. $f(1'') = 1'$;
6. for any element a, there are unique integers i, j with $0 \leq i, j < n$ such that a and $i1'' + j1'$ are congruent modulo n;
7. $\forall x\, (D_n(x) \leftrightarrow \exists y\, ny = x)$.

Here 6 and 7 are axiom schemes in which n runs over all positive integers.

Clearly, $\mathcal{M}$ is a model of T. From now on, let G be an arbitrary model of T. In particular, G is an expansion of an ordered abelian group, and f is an order-preserving group endomorphism of G. Let K be the kernel of f in G; it is easy to see that K is a convex pure subgroup of G.

For integers i, j, we will denote the $\mathcal{L}$-term $i1'' + j1'$ by c_{ij}. Axioms 3 and 5 imply $f(1') = 0$, and so $f(c_{ij}) = c_{0i}$ in G.

Theorem 1. *$\mathcal{M}$ embeds naturally into G via the map $\alpha : (i, j) \mapsto c_{ij}$.*

Proof. Clearly, α preserves $+$, $-$, and 0; using Axiom 6, it preserves D_n, for every $n > 0$. Obviously, α preserves $1'$ and $1''$.

To show that α preserves strict order it suffices to prove that α maps the positive elements into the positive elements. Suppose $(i, j) > (0, 0)$; then either $i > 0$, or $i = 0$ and $j > 0$. If $i > 0$ we have $f(c_{ij}) = i1' > 0$, by Axiom 4, and hence $c_{ij} > 0$. If $i = 0$ and $j > 0$ we have $c_{ij} = j1' > 0$ by Axiom 4.

Since the map α preserves strict order, it is injective. Finally, α preserves f because

$$\alpha(f(i, j)) = \alpha(0, i) = c_{0i} = f(c_{ij}) = f(\alpha(i, j)).$$

$\square$

Theorem 2. *T admits quantifier elimination.*

Proof. For the proof we need a series of lemmata.

Lemma 3. *Every $\mathcal{L}$-term $t(x_1, \ldots, x_k)$ is T-equivalent to a term of the form*

$$\sum_{s=1}^{k} (n_s x_s + m_s f(x_s)) + c_{ij},$$

where i, j and n_s, m_s for $1 \leq s \leq k$ are integers.

Proof of Lemma 3. By induction on the complexity of the term t, using the identities $f(x \pm y) = f(x) \pm f(y)$, $f(1') = 0$, and $f(1'') = 1'$. $\square$

Lemma 4. *For a positive integer n, the following equivalences hold in T:*

(i) $nx + mf(x) = u \ \equiv\ n^2x = nu - mf(u)$,
(ii) $nx + mf(x) < u \ \equiv\ nf(x) < f(u) \vee (nf(x) = f(u) \wedge n^2x < nu - mf(u))$,

(iii) $nx + mf(x) > u \;\equiv\; nf(x) > f(u) \vee (nf(x) = f(u) \wedge n^2x > nu - mf(u))$.

Proof of Lemma 4. (i) As $f^2 = 0$, we have

$$\begin{aligned} nx + mf(x) = u \equiv\ & n^2x + mnf(x) = nu \wedge nf(x) = f(u) \\ \equiv\ & n^2x + mf(u) = nu \wedge nf(x) = f(u) \\ \equiv\ & n^2x = nu - mf(u). \end{aligned}$$

(ii) Suppose $nx + mf(x) < u$. Then $nf(x) \leq f(u)$. If $nf(x) = f(u)$, then $n^2x < nu - mf(u)$.

Conversely, first suppose $nf(x) < f(u)$. Then $f(nx + mf(x)) < f(u)$, whence $nx + mf(x) < u$. Second, suppose $nf(x) = f(u)$ and $n^2x < nu - mf(u)$. Then again $nx + mf(x) < u$.

(iii) Similar to (ii). □

Lemma 5. *Every $\mathcal{L}$-formula of the form $\exists x\phi(x,\bar{y})$, where ϕ is quantifier-free, is T-equivalent to an $\mathcal{L}$-formula of the form $\bigvee_i(\psi_i(\bar{y}) \wedge \exists x \bigwedge_j \phi_{ij}(x,\bar{y}))$, where ψ_i is quantifier-free and each ϕ_{ij} is of one of the forms:*

1. $nx + mf(x) = v$,
2. $nx + mf(x) < v$,
3. $v < nx + mf(x)$.

Here v is an $\mathcal{L}$-term in variables $\bar{y}$, and n, m are integers.

Proof of Lemma 5. As for any positive integers n, m and any term t the formula $D_m(t)$ is T-equivalent to $D_{nm}(nt)$, we may assume that for all subformulae of ϕ of the form $D_m(t)$ the numbers m are the same.

First we prove that for every positive integer m any $\mathcal{L}$-formula of the form $D_m(t)$ or $\neg D_m(t)$, where t is an $\mathcal{L}$-term in $x, \bar{y}$, is T-equivalent to a positive Boolean combination of formulae of the forms $D_m(p)$ or $D_m(x - c_{ij})$, where p is an $\mathcal{L}$-term in $\bar{y}$. As $\neg D_m(t)$ is T-equivalent to

$$\bigvee\{D_m(t - c_{ij}) : 0 \leq i, j < m,\ (i,j) \neq (0,0)\}$$

by Axioms 6 and 7, we need only consider formulae $D_m(t)$. By Lemma 3, we may assume that t has the form $nx + kf(x) + s(\bar{y})$, where $s(\bar{y})$ is some term in $\bar{y}$. Axioms 6 and 7 imply that the formula $D_m(nx + kf(x) + s(\bar{y}))$ is T-equivalent to

$$\bigvee_{0 \leq i,j < m} (D_m(x - c_{ij}) \wedge D_m(nc_{ij} + kf(c_{ij}) + s(\bar{y})).$$

Now, using Lemma 3, the equivalences of the theory of linear orderings

$$u \neq v \equiv u < v \vee v < u,$$

$$u \not< v \equiv u = v \vee v < u,$$

and standard equivalences of the first order logic, it is easy to show that $\exists x\phi(x,\bar{y})$ is T-equivalent to an $\mathcal{L}$-formula of the form $\bigvee_i(\psi_i(\bar{y}) \wedge \exists x \bigwedge_j \phi_{ij}(x,\bar{y}))$, where ψ_i is quantifier-free and each ϕ_{ij} is of one of the forms 1–3, or of the form

4. $D_k(x - c_{pq})$.

We may assume that $\bigwedge_j \phi_{ij}$ contains at most one conjunct of the form 4, because $D_k(x-t) \wedge D_k(x-t')$ is T-equivalent to $D_k(x-t) \wedge D_k(t-t')$.

Suppose $\bigwedge_j \phi_{ij}$ is of the form $\theta(x,\bar{y}) \wedge D_k(x-c_{pq})$. Then $\exists x \bigwedge_j \phi_{ij}$ is T-equivalent to $\exists x \theta(c_{pq}+kx,\bar{y})$. Clearly, $\theta(c_{pq}+kx,\bar{y})$ is T-equivalent to a conjunction of formulae of the forms 1–3.

So $\exists x \phi(x,\bar{y})$ is T-equivalent to a formula of desired form. □

Now, in order to prove Theorem 2, it suffices to show that any formula $\exists x \bigwedge_i \phi_i(x,\bar{y})$, where each ϕ_i is of one of the forms 1–3, is equivalent to a quantifier-free $\mathcal{L}$-formula. By Lemma 4 we may assume that in each ϕ_i exactly one of the coefficients n,m is nonzero. So we may assume ϕ_i to be of one of the forms

(i) $nx = v$,
(ii) $nf(x) = v$,
(iii) $nx < v$,
(iv) $nf(x) < v$,
(v) $nx > v$,
(vi) $nf(x) > v$,

where $n > 0$. Since $y = z$ is T-equivalent to $my = mz$, and $y < z$ is T-equivalent to $my < mz$, for any positive m, we may assume that the coefficient n in ϕ_i does not depend on i. Further, we may assume that, for each of the six forms, at most one of ϕ_i is of this form. This last observation is seen easily, e.g. in case (iii) from the T-equivalence of $nx < u \wedge nx < u'$ and

$$(nx < u \wedge u \leq u') \vee (nx < u' \wedge u' \leq u).$$

Lemma 6. *For every $n < \omega$ there is a quantifier-free $\mathcal{L}$-formula $\vartheta_n(y,z)$ which is T-equivalent to $\exists x\,(y < nx < z)$.*

Proof. The assertion is obvious for $n = 0$, so assume $n \geq 1$. Put $\vartheta_n := \phi \vee \psi$, where

$$\phi(y,z) := \bigvee_{j=1}^{n} (D_n(y + j1') \wedge y + j1' < z)$$

$$\psi(y,z) := \bigvee_{i=1}^{n-1} \bigvee_{j=1}^{n} (D_n(y + c_{ij}) \wedge 2(y + c_{ij}) < 2z + 1'').$$

Suppose $y < nx < z$ for elements x, y, z in a model of T. By Axiom 6 there are unique $0 \leq i < n$ and $0 < j \leq n$ such that $y + c_{ij}$ is divisible by n. If $y + c_{ij} < z$, then $\vartheta_n(y,z)$ holds (either ϕ or ψ, depending on whether $i = 0$ or not). Otherwise $0 < z - nx \leq y + c_{ij} - nx < y + c_{ij} - y = c_{ij}$; since $j \leq n$ and $y + c_{ij} - nx$ is divisible by n, necessarily $i > 0$ by Axiom 4. But $f(y + c_{ij} - nx)$ is also divisible by n, and

$$f(y + c_{ij} - z) \leq f(y + c_{ij} - nx) \leq f(y + c_{ij} - y) = i1';$$

as $i < n$, we get $f(y + c_{ij} - nx) \leq 0$ by Axiom 4. It follows that $2f(y + c_{ij} - z) \leq 0$, whence $2(y + c_{ij} - z) < 1''$ and $\psi(y,z)$ holds.

Conversely, suppose $\vartheta_n(y,z)$ for some y,z in a model of T. If $\phi(y,z)$ holds, we immediately obtain $\exists x\,(y < nx < z)$. Suppose $\psi(y,z)$ holds, and let $0 < i < n$ and

$0 < j \leq n$ be such that $D_n(y+c_{ij}) \wedge 2(y+c_{ij}) < 2z+1''$ holds. Put $u := y+c_{ij}-z+1'$. If $u \leq 0$, then $y + c_{ij} < z$ and we are done. Suppose $u > 0$. We have $2f(u) \leq 1'$, whence $f(u) \leq 0$ by Axiom 4, so $f(nu) \leq 0$ and $nu < 1''$. Clearly $y + c_{ij} - nu$ is divisible by n; furthermore

$$y < y + 1'' - nu \leq y + c_{ij} - nu = z - 1' - (n-1)u < z,$$

and we are done again. □

Remark. The result of Lemma 6 can be also extracted from [2]. There, Weispfennig gives an explicit axiom set T_0 for the theory of the $(\mathcal{L} \setminus \{f\})$-reduct of $\mathcal{M}$ and proves quantifier elimination for T_0; it can then be checked that T proves T_0. However, in order to make the proof of Theorem 2 self-contained, we have prefered to give a direct argument.

Lemma 7. *If among the formulae ϕ_i there is a formula of the form* (i) *or* (ii)*, then $\exists x \bigwedge_i \phi_i(x, \bar{y})$ is T-equivalent to a quantifier-free $\mathcal{L}$-formula.*

Proof of Lemma 7. First suppose that among the formulae ϕ_i there is a formula of the form (i). The formula $\exists x \bigwedge_i \phi_i(x, \bar{y})$ is of the form $\exists x(nx = v \wedge \psi(nx, \bar{y}))$, where ψ is a quantifier-free formula, and so is T-equivalent to the quantifier-free $\mathcal{L}$-formula $D_n(v) \wedge \psi(v, \bar{y})$.

Now suppose that among the formulae ϕ_i there is no formula of the form (i); so one of the formulae ϕ_i is of the form (ii). We may assume that among the formulae ϕ_i there is no formula of the forms (iv), (vi). Indeed, say, $nf(x) = v \wedge nf(x) < u$ is T-equivalent to $nf(x) = v \wedge v < u$.

Thus, we assume that among the ϕ_i exactly one is of the form (ii), all others are of the forms (iii) or (v), and there is at most one of each of these forms. So it suffices to prove that the formula $\phi := \exists x\, (nf(x) = v \wedge u < nx < w)$ is T-equivalent to the quantifier-free $\mathcal{L}$-formula

$$\theta := D_n(v) \wedge f(v) = 0 \wedge f(u) \leq v \leq f(w) \wedge \vartheta_n(u, w),$$

where ϑ_n is the quantifier-free formula obtained in Lemma 6. Here we allow $u = -\infty$ or $w = \infty$ (and mean that the corresponding inequality and ϑ_n are omitted; in these cases the proofs are just simplier).

As $f^2 = 0$, clearly ϕ implies θ. Conversely, suppose θ holds. Since $\mathrm{Ker}(f)$ is a pure subgroup, and $\mathrm{Ker}(f) = \mathrm{Rng}(f)$, there is y with $v = nf(y)$, due to the first two conjuncts of θ. If $f(u) < v < f(w)$, then $u < ny < w$ and ϕ holds. Suppose $v = f(u)$. Axiom 6 implies that $u + c_{ij}$ is divisible by n, for some $0 \leq i < n$ and $1 \leq j \leq n$. Then $f(u + c_{ij}) = v + i1'$ is also divisible by n, and so $i = 0$ by Axiom 4. Therefore n divides $u + j1'$. We have $u + j1' < w$; otherwise, due to $\vartheta_n(u, w)$, there is z with $u + j1' > nz > u$, and the interval $(0, j1')$ contains the element $u + j1' - nz$ divisible by n, in contradiction to Axiom 4. As $f(u + j1') = v$ and $v < u + j1' < w$, the formula ϕ holds. The case $v = f(w)$ is similar. □

Lemma 8. *If each ϕ_i is of one of the forms* (iii)–(vi) *then $\exists x \bigwedge_i \phi_i(x, \bar{y})$ is T-equivalent to a quantifier-free $\mathcal{L}$-formula.*

Proof of Lemma 8. First note that we may assume that among the ϕ_i there is no formula of the forms (iii) and (v). Indeed, the formula $y < z$ is T-equivalent to

$$(f(y) < f(z)) \vee (f(y) = f(z) \wedge y < z).$$

Therefore, say, $\exists x(\theta \wedge nx < u)$ is T-equivalent to

$$\exists x(\theta \wedge nf(x) < f(u)) \vee \exists x(\theta \wedge nf(x) = f(u) \wedge nx < u).$$

The second disjunct of the latter formula is T-equivalent to a quantifier-free $\mathcal{L}$-formula by Lemma 7.

So it suffices to prove that the formula $\phi := \exists x(u < nf(x) < w)$ is T-equivalent to the quantifier-free formula

$$\theta := f(u) \leq 0 \leq f(w) \wedge \vartheta_n(u, w),$$

where again we allow $u = -\infty$ or $w = \infty$, and ϑ_n is the formula from Lemma 6.

As $f^2 = 0$, clearly ϕ implies θ. Conversely, assume θ. If $u < 0 < w$ then ϕ obviously holds; so we may assume that $0 \leq u$ or $w \leq 0$.

Suppose $0 \leq u$; then $f(u) = 0$. If $f(w) = 0$ then ϕ holds because $\vartheta_n(u, w)$ is true, and $\mathrm{Ker}(f) = \mathrm{Rng}(f)$ is a convex pure subgroup. Suppose $f(w) \neq 0$. Axiom 6 implies that $y = u + 1' + c_{ij}$ is divisible by n for some $0 \leq i, j < n$; since $f(y) = f(u) + i1' = i1'$ is also divisible by n, we get $i = 0$ by Axiom 4, and $f(y) = f(u) = 0$. As $\mathrm{Ker}(f) = \mathrm{Rng}(f)$ is a pure subgroup, $y = nf(x)$, for some x. Since $f(y) = 0$ and $\mathrm{Ker}(f)$ is convex, $u < y < w$, and so the formula ϕ holds. The case $w \leq 0$ is similar. □

The proof of Theorem 2 is completed. □

From Theorems 1 and 2 we immediately obtain

Theorem 9. *The theory T is complete.*

Theorem 10. *The theory T is quasi-o-minimal. Moreover, every definable set in a model G of T is a Boolean combination of intervals and 0-definable sets of the form*

$$c_{ij} + \{x : D_k(nx + mf(x))\}.$$

Remark. The set $\{x : D_k(nx + mf(x))\}$ is a 0-definable subgroup, so the definable sets in G are exactly the Boolean combinations of intervals and cosets of definable subgroups.

Proof. By Lemmata 3, 4, and quantifier elimination for T, it suffices to prove the result for the subsets of G which are defined by the formulae of the following forms:

(a) $nx = a$,
(b) $nx > a$,
(c) $nf(x) = a$,
(d) $nf(x) > a$,
(e) $D_k(nx + mf(x) + a)$.

Here $a \in G$, and k, n, m are integers, $k > 0$.

Let $\mathcal{L}_0 = \mathcal{L} \setminus \{f\}$, and $\mathcal{M}_0$ be the $\mathcal{L}_0$-reduct of $\mathcal{M}$. It had been proven in [1] that every definable set in a model G_0 of the $\mathcal{L}_0$-theory of $\mathcal{M}_0$ is a Boolean combination of intervals and 0-definable sets of the form $c_{ij} + nG_0$. So in cases (a) and (b) the desired result holds.

Consider case (c). If $a \notin \mathrm{Rng}(f)$ then $nf(x) = a$ defines the empty set. Suppose $a = f(b)$. Then in G the formula $nf(x) = a$ is equivalent to $f(nx - b) = 0$. Note that in $\mathcal{M}$ (and so in G, due to completeness of T) the formula $f(x) = 0$ is equivalent to the $\mathcal{L}_0$-formula $-1'' < 2x < 1''$. Therefore $nf(x) = a$ is equivalent to $-1'' < 2(nx - b) < 1''$, and the result follows from case (b).

Consider case (d). We may assume that both of the sets $\{x : nf(x) > a\}$ and $\{x : nf(x) \le a\}$ are non-empty. Then $f(a) = 0$, as we see by applying f to the inequalities. As $\mathrm{Ker}(f) = \mathrm{Rng}(f)$, we have $a = f(b)$ for some b. Then $nf(x) > a$ is equivalent in G to $nx > b \wedge nf(x) \ne a$, and the result follows from cases (b) and (c).

Finally, consider case (e). As the difference of any two elements in the set $\{x : D_k(nx + mf(x) + a)\}$ lies in the subgroup $\{x : D_k(nx + mf(x))\}$, and that subgroup has the set $\{c_{ij} : 0 \le i, j < k\}$ as coset representatives by Axiom 6, the formula $D_k(nx + mf(x) + a)$ defines either the empty set, or one of those cosets. $\square$

Theorem 11. *The theory T does not have the Exchange Property.*

Proof. Let B be the lexicographically ordered group $\mathbb{Q} \times \mathbb{Z}$, and C be the lexicographically ordered group $B \times B$. Expand C to an $\mathcal{L}$-structure G by defining $D_n(x)$ as 'n divides x', putting $f(x, y) = (0, x)$ for $x, y \in B$, and setting $1' = ((0,0),(0,1))$ and $1'' = ((0,1),(0,0))$. It is easy to see that G is a model of T.

Let $a = (\alpha, n)$, where $\alpha \in \mathbb{Q} \setminus \{0\}$ and $n \in \mathbb{Z}$; then $a \in B$. Let $b = (0, a)$ and $c = (a, 0)$; then $b, c \in G$. We have $f(c) = b$, whence $b \in \mathrm{acl}(c)$ in G. We will show that in G we have $c \notin \mathrm{acl}(b)$, even though b is not algebraic.

First we prove that b is non-algebraic. For each positive rational β, the map

$$g_\beta : (\gamma, m) \mapsto (\beta\gamma, m)$$

is an automorphism of the ordered group B. It is easy to check that the map

$$h_\beta : (x, y) \mapsto (g_\beta(x), g_\beta(y))$$

is an automorphism of the structure G. If $\beta \ne \gamma$ then $g_\beta(a) \ne g_\gamma(a)$ and so

$$h_\beta(b) = (0, g_\beta(a)) \ne (0, g_\gamma(a)) = h_\gamma(b).$$

Thus the $\mathrm{Aut}(G)$-orbit of b is infinite, and hence b is not algebraic.

Now we prove that $c \notin \mathrm{acl}(b)$. For each positive rational β, the map

$$q_\beta : (\gamma, m) \mapsto (\beta\gamma, 0)$$

is an endomorphism of the ordered group B, and $q_\beta(0,1) = (0,0)$. It is easy to check that the map

$$s_\beta : (x, y) \mapsto (x, y + q_\beta(x))$$

is an automorphism of the structure (G, b). If $\beta \ne \gamma$ then $q_\beta(a) \ne q_\gamma(a)$ and so

$$s_\beta(c) = (a, q_\beta(a)) \ne (a, q_\gamma(a)) = s_\gamma(c).$$

Thus the $\mathrm{Aut}(G,b)$-orbit of c is infinite, and hence $c \notin \mathrm{acl}(b)$. □

Remark. Since $\sigma_\beta : (x,y) \mapsto (x, g_\beta(y))$ is an automorphism of C (i.e. the pure ordered group) with $f(\sigma_\beta(x,y)) = (0,x)$ and $\sigma_\beta(f(x,y)) = (0, g_\beta(x))$, it is clear that f is not definable in the pure group language.

.

References

[1] Oleg Belegradek, Ya'akov Peterzil and Frank Wagner, *Quasi-o-minimal structures*, to appear in the Journal of Symbolic Logic.

[2] Volker Weispfenning, *Elimination of quantifiers for certain ordered and lattice-ordered groups*, Bull. Soc. Math. Belg., Ser. B, 33 (1981), 131–155.

Oleg Belegradek, Kemerovo University, Kemerovo 650043, Russia
E-mail address: beleg@kaskad.uucp.stanet.ru

Françoise Point, Institut de Mathématique et Informatique, Université de Mons-Hainaut, Le Pentagone, 6, avenue du Champ de Mars, B-7000 Mons, Belgium
E-mail address: point@logique.jussieu.fr

Frank Wagner, Mathematical Institute, University of Oxford, 24–29 St Giles', Oxford OX1 3LB, UK
E-mail address: wagner@maths.ox.ac.uk