Rotations of the three-sphere and symmetry of the Clifford Torus

John McCuan* and Lafe Spietz

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Abstract

We describe decomposition formulas for rotations of $\mathbb{R}^3$ and $\mathbb{R}^4$ that have special properties with respect to stereographic projection. We use the lower dimensional decomposition to analyze stereographic projections of great circles in $\mathbb{S}^2 \subset \mathbb{R}^3$. This analysis provides a pattern for our analysis of stereographic projections of the Clifford torus $\mathcal{C} \subset \mathbb{S}^3 \subset \mathbb{R}^4$. We use the higher dimensional decomposition to prove a symmetry assertion for stereographic projections of $\mathcal{C}$ which we believe we are the first to observe and which can be used to characterize the Clifford torus among embedded minimal tori in $\mathbb{S}^3$—though this last assertion goes beyond the scope of this paper. An effort is made to intuitively motivate all necessary concepts including rotation, stereographic projection, and symmetry.

Introduction

It is known (and intuitively believable) that the spheres in $\mathbb{R}^3$ are characterized by being the only compact surfaces with planes of symmetry whose normals exhaust all possible directions. That is, *if S is a compact surface and, for each unit vector $\mathbf{n}$ in $\mathbb{R}^3$, there is some plane Π with normal $\mathbf{n}$ such that S is invariant under reflection in Π, then S is a sphere.* Technically, S could be some collection of concentric spheres, but this can be fixed by requiring explicitly that S be connected, i.e., a single surface. The point is that spheres can be characterized by their (reflectional) symmetry.

In this paper we describe an analogous symmetry condition for a certain surface, the Clifford torus. Our task is complicated by the fact that this torus is located in

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the three-sphere, $\mathbb{S}^3$, where our intuition from $\mathbb{R}^3$ is of limited use. For this reason we will employ a certain transformation, *stereographic projection,* that allows us to realize $\mathbb{S}^3$ (at least most of it) in the Euclidean space $\mathbb{R}^3$. In fact, our symmetry condition will apply, more properly, to the stereographic projections of the Clifford torus. Furthermore, our symmetry condition is substantially more complicated than the simple one above for spheres, and we discuss at some length why it is a natural one.

Another objective of the paper is to give what we consider novel, geometrically based, expositions of several well known topics. (Some of these are mentioned briefly below.) From this point of view, we offer an introduction to $\mathbb{S}^3$ that we hope is a geometric counterpart to the algebraic treatment in, for example, [12].

The paper is organized as follows. In the following §1 we review stereographic projection of $\mathbb{S}^2$ and discuss a decomposition formula for rotations of $\mathbb{S}^2 \subset \mathbb{R}^3$. We show, in particular, that projections of rotations of great circles are circles in $\mathbb{R}^2$ whose size and position are given in terms of the parameters of the decomposition. This discussion is somewhat artificial because it is easy to show that the projection of essentially any circle in $\mathbb{S}^2$ is a circle in $\mathbb{R}^2$ whose center and radius are easy to calculate. This is the case, however, owing to the fact that a circle in $\mathbb{S}^2$ is the intersection of an affine subspace (a plane) with $\mathbb{S}^2$. This luxury is not afforded us by the Clifford torus, and the decomposition technique presented here will be used with considerable advantage in the more complicated higher dimensional case. Furthermore, our discussion of stereographic projection of $\mathbb{S}^2$ is used in §2 to give an exposition of symmetry for planar sets, and it provides intuition for the higher dimensional stereographic projection considered in §3.

We also give considerable attention to building up intuition about *rotations*, especially rotations of $\mathbb{R}^4$ since we consider $\mathbb{S}^3$ as a subset of $\mathbb{R}^4$. Recall that rotations of $\mathbb{R}^2$ (centered at zero) can be represented by matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

That is, given a rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ there is a matrix M of the form given above such that for each vector $\mathbf{x}$, we have that $R(\mathbf{x})$ is given by the matrix multiplication $M\mathbf{x}$. Through such a representation we are immediately presented with two algebraic facts, namely that R is *linear* and that $\det M = 1$. It is also easy to check from this representation that R is *orthogonal*, i.e., it preserves the orthonormality of bases. Many authors define a *rotation* of $\mathbb{R}^3$ to be an orthogonal linear transformation corresponding to a matrix of determinant 1. This definition is concise and computationally convenient, but we find it unintuitive. It can perhaps be argued (via the parallelogram rule) that linearity is an intuitive assumption, but the role played by the determinant is difficult to see geometrically for rotations of $\mathbb{R}^3$, much less for rotations of $\mathbb{R}^4$. In an appendix to this paper we take the point of view that rotations are *rigid motions* (i.e., distance preserving transformations) that fix the origin and result from *smooth "homogeneous" motions.* We then prove linearity, orthogonality, and representation by matrices of determinant 1 (thereby showing the two definitions are equivalent). This appendix may be read at any time, but we recommend reading it after §1 and before §3.

In §2 we introduce *circles of Apollonius* and show that their symmetry as a family of planar curves is a natural generalization of the symmetry exhibited by concentric circles. We incorporate in this discussion an explanation of why symmetry and reflection about circles are natural generalizations of symmetry and reflection about lines. Furthermore, we observe that the generalized symmetry exhibited by circles of Apollonius is described naturally in terms of a *line of centers* l. More precisely, given a family of circles of Apollonius $\mathcal{A}$, we can find, for each point $\mathbf{x}$ on l, an *orthogonal Steiner circle* C with center $\mathbf{x}$ and the property that every circle in $\mathcal{A}$ is symmetric with respect to C. Finally, we observe that this formulation of generalized symmetry can be easily extended to surfaces in $\mathbb{R}^3$. This sets the stage for our main (and we believe original) result which is roughly as follows. *Given any stereographic projection Q of the Clifford torus, there is a line l in $\mathbb{R}^3$, and for each point $\mathbf{x}$ on l there is a sphere S centered at $\mathbf{x}$ so that Q is symmetric with respect to S.* See Figure 5. A careful statement and proof, which include conditions on the radii of the spheres of symmetry, are given in §3. The proof follows, in outline, the discussion of §1 and §2.

From a wider perspective, the Clifford torus and its stereographic projections are interesting surfaces primarily due to curvature considerations that are beyond the scope of this paper. More primitively, they are interesting because they are critical points for certain functionals—which they are *believed* to minimize. For further information, see [3, 6, 10, 11]. In this framework, the Clifford torus plays a role in $\mathbb{S}^3$ similar to that of the plane *and* the sphere in $\mathbb{R}^3$. It has Gauss curvature and mean curvature zero like the plane, and it has constant mean curvature and is compact like the sphere. The symmetry properties shown in this paper may be considered as a first step in getting a feel for curvature of surfaces in $\mathbb{S}^3$, and while the main result stated above may seem curious at first sight, it is completely analogous to the observation that any sphere in $\mathbb{R}^3$ has a plane of reflective symmetry with any given normal direction.

We extend our thanks to Ed Bueler, David Hoffman, and Silvio Levy who gave us useful comments that greatly improved the exposition.

1 Stereographic Projection

In this section we consider the two-dimensional sphere $\mathbb{S}^2$ as a subset of $\mathbb{R}^3$:

$$\mathbb{S}^2 = \{\mathbf{x} = (x, y, z) : |\mathbf{x}| = 1\}.$$

As usual $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$.

We are interested in a map called *stereographic projection* that sends $\mathbb{S}^2$ (except for one point) into the x, y-plane, and we are interested in how the image of a certain set changes as the sphere is rotated. To be precise, stereographic projection π given by

$$\pi(\mathbf{x}) = \frac{1}{1-z}(x, y) \tag{1}$$

maps $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$ onto $\mathbb{R}^2 = \{(x, y, 0)\}$ in a one-to-one fashion.

Exercise 1 *Show that π is one-to-one and onto.*

The map (1) has a convenient geometric interpretation:

> Each point $\mathbf{p} \in \mathbb{S}^2 \backslash \{(0, 0, 1)\}$ determines a unique line passing through $\mathbf{p}$ and $(0, 0, 1)$. The line l, in turn, intersects the x, y-plane in a unique point $\mathbf{q}$. We set $\pi(\mathbf{p}) = \mathbf{q}$.

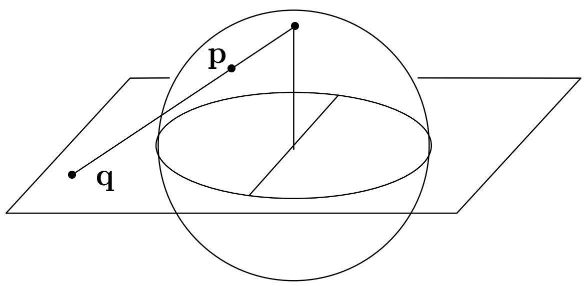

Figure 1: Stereographic Projection.

Exercise 2 **(i)** *Using this geometric statement as a definition, derive formula* (1) *for π.*

(ii) *The geometric definition makes sense for any point in $\mathbb{R}^3$ with third component not equal to 1. In this way π may be extended from $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$ to $\mathbb{R}^3 \backslash \{(x, y, 1)\}$. We call this extended map $\bar{\pi}$. Does formula* (1) *still apply?*

Next we consider the equator circle

$$C = \{\mathbf{x} \in \mathbb{S}^2 : x^2 + y^2 = 1\} = \{\mathbf{x} : x^2 + y^2 = 1, z = 0\}$$

in the sphere. The stereographic projection $\pi(C)$ of C is particularly simple—it is C itself.

What happens if we first rotate the sphere and then stereographically project? Say we rotate about the y-axis for example—by an angle ϕ. If we call this rotation R^y_ϕ, then $R^y_\phi(C) = \{(x\cos\phi, y, x\sin\phi) : x^2 + y^2 = 1\}$, and the stereographic projection is

$$\pi \circ R^y_\phi(C) = \left\{ \left(\frac{x\cos\phi}{1 - x\sin\phi}, \frac{y}{1 - x\sin\phi} \right) : x^2 + y^2 = 1 \right\}. \tag{2}$$

This set in the plane is (perhaps) not so easy to recognize.

On page 19 of [1], however, Lars Ahlfors gave a nice way to look at it: $R^y_\phi(C)$ is the intersection of a plane $P = \{\mathbf{x} : x\sin\phi - z\cos\phi = 0\}$ with $\mathbb{S}^2$. If we could find an inverse map $\pi^{-1} : \mathbb{R}^2 \to \mathbb{S}^2$, then

$$\pi \circ R^y_\phi(C) = \{\mathbf{a} = (a, b) : \pi^{-1}(\mathbf{a}) \in P\}.$$

Stereographic projection on $\mathbb{S}^2 \backslash \{(0,0,1)\}$ does have an inverse (though the extended map $\bar{\pi}$ does not—why?). The formula for the inverse is

$$\pi^{-1}(\mathbf{a}) = \frac{1}{|\mathbf{a}|^2 + 1}(2a, 2b, |\mathbf{a}|^2 - 1).$$

Exercise 3 *Derive this formula for* π^{-1}.

The statement $\pi^{-1}(\mathbf{a}) \in P$ now translates into an equation:

$$-2a\tan\phi \ + a^2 + b^2 = 1.$$

Such an equation, as we know, represents a *circle*,

$$(a - \tan\phi)^2 + b^2 = \sec^2\phi,$$

with center $(\tan\phi, 0)$ and radius $r = |\sec\phi|$.

If you think about what happens to C when the sphere is rotated, it is fairly clear that the simple rotation R^y_ϕ is typical. In particular, $\pi \circ R(C)$ should be a circle for any rotation R of $\mathbb{S}^2$. One way to make this precise is to decompose an arbitrary rotation R into simple coordinate rotations like R^y_ϕ. The following theorem gives such a decomposition.

Theorem 4 *Any rotation R is the composition of three rotations—one about the z-axis, one about the y-axis, and another about the z-axis. Thus, there are angles θ, ϕ, and ψ such that*

$$R = R^z_\psi \circ R^y_\phi \circ R^z_\theta.$$

Proof. The simple rotations R^z_θ, R^y_ϕ, and R^z_ψ, like rotations of $\mathbb{R}^2$, can be represented by matrices. For example,

$$R^z_\psi(\mathbf{x}) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^3$. Consequently, these rotations are *linear* transformations of $\mathbb{R}^3$. Moreover, we show in the appendix that *all rotations are linear.* Because of this, we only need to find θ, ϕ, and ψ such that $R^z_\psi \circ R^y_\phi \circ R^z_\theta$ and R agree on an orthonormal basis.

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$. Notice that for any θ we have $R^z_\theta(\mathbf{e}_3) = \mathbf{e}_3$ so that $R^z_\psi \circ R^y_\phi \circ R^z_\theta(\mathbf{e}_3) = R^z_\psi \circ R^y_\phi(\mathbf{e}_3)$. Thus, we want to find ϕ and ψ such that, in particular,

$$R(\mathbf{e}_3) = R^z_\psi \circ R^y_\phi(\mathbf{e}_3) = \begin{pmatrix} -\cos\psi\sin\phi \\ -\sin\psi\sin\phi \\ \cos\phi \end{pmatrix} \tag{3}$$

where the second equality is by direct calculation. On the other hand, $R(\mathbf{e}_3)$ is some unit vector (u_1, u_2, u_3), and since $|u_3| \le 1$, we can find an angle ϕ with $\cos\phi = u_3$. If $|u_3| = 1$, then $\sin\phi = 0$ and (3) holds. Otherwise, $\sin\phi \ne 0$ and $(u_1/\sin\phi)^2 + (u_2/\sin\phi)^2 = 1$. Thus, for some angle ψ, $\cos\psi = -u_1/\sin\phi$, $\sin\psi = -u_2/\sin\phi$, and again (3) holds.

We have then for any θ, $R^z_\psi \circ R^y_\phi \circ R^z_\theta(\mathbf{e}_3) = R(\mathbf{e}_3)$. It remains to choose θ so that $R^z_\psi \circ R^y_\phi \circ R^z_\theta(\mathbf{e}_j) = R(\mathbf{e}_j)$ for $j = 1$ and 2.

Let $S = R^z_\psi \circ R^y_\phi$ and note that $S^{-1} \circ R(\mathbf{e}_3) = \mathbf{e}_3$. Therefore, $S^{-1} \circ R$ induces a rotation R_θ (by some angle θ) on $\mathbb{R}^2$. That is, $S^{-1} \circ R(\mathbf{e}_j) = R_\theta(\mathbf{e}_j)$ for $j = 1$ and 2. Since R_θ extends to a rotation R^z_θ on $\mathbb{R}^3$ we are done. $\Box$

We can now apply Theorem 4 as follows.

$$\begin{aligned} \pi \circ R(C) &= \pi \circ R^z_\psi \circ R^y_\phi \circ R^z_\theta(C) \\ &= R^z_\psi \circ \pi \circ R^y_\phi(C). \end{aligned}$$

This set is the circle $(a - \tan\phi)^2 + b^2 = \sec^2\phi$ rotated by an angle ψ, i.e., it is the circle with center $(\cos\psi\tan\phi, \sin\psi\tan\phi)$ and radius $r = |\sec\phi|$.

Exercise 5 *Show that any circle in $\mathbb{S}^2$ stereographically projects to a circle in $\mathbb{R}^2$, and any circle in $\mathbb{R}^2$ is the projection of a circle in $\mathbb{S}^2$. Hint: A circle in $\mathbb{S}^2$ is the intersection of a plane in $\mathbb{R}^3$ with $\mathbb{S}^2$.*

We will need one other fact about the stereographic projection $\pi : \mathbb{S}^2 \to \mathbb{R}^2$:

Lemma 6 *If C_1 and C_2 are two (smooth) curves in $\mathbb{S}^2$ that intersect at a point $\mathbf{p}$ in an angle γ, then the image curves $\pi(C_1)$ and $\pi(C_2)$ intersect at $\pi(\mathbf{p})$ in the same angle γ.*

This property is expressed by saying that π is *conformal*. Henry Wente told us a short proof of Lemma 6 which we have included in an appendix. For another proof, we refer the reader to the classic book [5] by David Hilbert.

2 Symmetry

When we described the stereographic projection of a rotation of the equator circle ($\pi \circ R(C)$) in the last section, we ignored the unpleasant possibility that C had been rotated onto $(0,0,1)$—where stereographic projection is not defined. This happens, of course, when the angle ϕ of the preceding section is $\pi/2$. Our analysis in that case is flawed since $\tan\phi$ and $\sec\phi$ are not defined, and in fact the image of $R(C)$ (aside from the point $(0,0,1)$) is then a *line* in $\mathbb{R}^2$. It is one of our objectives in this section to address this apparent difficulty. Our second and main objective is to generalize in a natural way our intuitive notion of symmetry, so that we can introduce the symmetry assertion of the main theorem.

The analysis of stereographic projection of circles that pass through $(0,0,1)$ is simple. For example, if we take $\phi = \pi/2$, then in place of (2) we have

$$\begin{aligned}\pi \circ R^y_{\pi/2}(C\backslash\{(1,0,0)\}) &= \left\{\left(0, \frac{y}{1-x}\right) : x^2+y^2=1,\ x \neq 1\right\} \\ &= \left\{\left(0, \pm\sqrt{\frac{1+x}{1-x}}\right) : -1 \le x < 1\right\}.\end{aligned}$$

The last set is clearly the line $x = 0$, since $(1+x)/(1-x)$ maps $[-1,1)$ monotonely onto $[0,\infty)$. The more general cases can be handled similarly. What we really wish to emphasize, however, is the following: *because circles in $\mathbb{S}^2$ that pass through $(0,0,1)$ are geometrically identical to other circles that do not, it is natural for our purposes to view straight lines in $\mathbb{R}^2$ as circles with (infinite radius and) one point at ∞.*

For starters, this viewpoint allows Exercise 5 to make sense as stated. More importantly it illustrates how sets and structures in $\mathbb{S}^2$ can provide insight for terminology and constructions in $\mathbb{R}^2$. We proceed further along this line presently.

The symmetry of a circle in $\mathbb{R}^2$ is perhaps most easily described in terms of its center. Given a point $\mathbf{p}$ in a circle A with center $\mathbf{a}$, A is generated by rotating $\mathbf{p}$ about $\mathbf{a}$. The same circle can also be generated by reflecting $\mathbf{p}$ about each of the lines through $\mathbf{a}$. This latter characterization will be the one of interest to us.

Definition 7 *A set $A \subset \mathbb{R}^2$ has* Euclidean reflectional symmetry with respect to a point $\mathbf{a} \in \mathbb{R}^2$ *if, for each line E passing through $\mathbf{a}$, we have $\psi_E(A) = A$, where $\psi_E : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about E.*

Exercise 8 *Show that any such set (with Euclidean reflectional symmetry) is a union of concentric circles with center $\mathbf{a}$.*

If, as we have suggested, lines should be considered simply as circles with infinite radius, then it is natural to ask for a definition of symmetry in which reflection about lines is replaced with *reflection about circles.* Extrapolating directly from the definition above we might try to replace the family of symmetry lines passing through $\mathbf{a}$ with a family of *symmetry circles* passing through a common point $\mathbf{a}$. Unfortunately, the full geometric situation is not completely evident from considering the plane alone. Again we turn to stereographic projection. The inverse image of each line through $\mathbf{a}$ is a circle in $\mathbb{S}^2$ passing through $\pi^{-1}(\mathbf{a})$ and $(0, 0, 1)$. (There are two points of intersection.) If one then rotates slightly this family of circles in $\mathbb{S}^2$ and stereographically projects, a family of circles in $\mathbb{R}^2$ is obtained that pass through two distinct points $\mathbf{a}_1$ and $\mathbf{a}_2$, as shown on the left in Figure 4. Note that this family also contains the line determined by $\mathbf{a}_1$ and $\mathbf{a}_2$. We call the circles that pass through two given points $\mathbf{a}_1$ and $\mathbf{a}_2$ *Steiner symmetry circles.* We use these circles to generalize the symmetry lines in Definition 7.

It remains to specify, for each circle S passing through $\mathbf{a}_1$ and $\mathbf{a}_2$, a transformation $\psi_S : \mathbb{R}^2 \to \mathbb{R}^2$ which we will call *reflection about S.* Again we look to the sphere for intuition. What transformation of $\mathbb{S}^2$ corresponds to reflection about a line in $\mathbb{R}^2$? If E is a line in $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, then the inverse image of E is some circle C in $\mathbb{S}^2$ passing through $(0, 0, 1)$. Furthermore, there is a unique point $\mathbf{c} = (c_1, c_2, 1)$ such that the segments connecting $\mathbf{c}$ to C form a right circular cone (tangent to $\mathbb{S}^2$), n.b., Figure 2. The point $\mathbf{c}$ can now be used to define a transformation $\Psi_C : \mathbb{S}^2 \to \mathbb{S}^2$:

> For each point $\mathbf{p} \in \mathbb{S}^2$, the line determined by $\mathbf{p}$ and $\mathbf{c}$ intersects $\mathbb{S}^2$ in a set $\{\mathbf{p}, \mathbf{q}\}$. We set $\Psi_C(\mathbf{p}) = \mathbf{q}$.

Note that this definition is much like the geometric definition of stereographic projection. The reader can check (and we will show below) that Ψ_C corresponds to the reflection ψ_E in the sense that $\psi_E = \pi \circ \Psi_C \circ \pi^{-1}$. This construction also works if

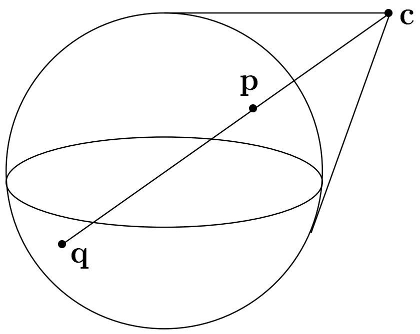

Figure 2: Reflection on $\mathbb{S}^2$.

E passes through $\mathbf{0}$—though in that case the transformation Ψ_C is simply given by reflection about the plane determined by C.

Notice that the fact C passes through $(0,0,1)$ is not required for the geometric definitions of Ψ_C to make sense. That is, for any circle C in $\mathbb{S}^2$, if C is not a great circle, it defines a cone point[1] $\mathbf{c}$, and the definition above gives a transformation Ψ_C of $\mathbb{S}^2$ that is geometrically identical (modulo rotation) to one that corresponds to Euclidean reflection. For great circles we use the alternative construction.

Going back to $\mathbb{R}^2$, we may start with any circle (or straight line) S, take $C = \pi^{-1}(S)$ and apply the construction described above to obtain a transformation $\psi_S = \pi \circ \Psi_C \circ \pi^{-1}$ of $\mathbb{R}^2 \backslash \{\mathbf{a} = \bar{\pi}(\mathbf{c})\}$. This is the transformation we call *reflection about* S.

We proceed to derive a formula for $\psi_S : \mathbb{R}^2 \backslash \{\mathbf{a}\} \to \mathbb{R}^2$. Notice first that if C' is a circle in $\mathbb{S}^2$ that meets C in right angles at points $\mathbf{p}_1$ and $\mathbf{p}_2$, then C' lies in the plane determined by $\mathbf{c}$, $\mathbf{p}_1$ and $\mathbf{p}_2$, and one sees from this that $\Psi_C(C') = C'$. It follows moreover, since π is an angle and circle preserving transformation, that any circle S' in $\mathbb{R}^2$ which is orthogonal to S is mapped by ψ_S into itself. Thus, consider a point $\mathbf{p}$ inside S as shown in Figure 3. Let S' be the line determined by the center of S and $\mathbf{p}$. There is also a circle S'' which passes through $\mathbf{p}$, meets S orthogonally, and has its center on S'. Since $\psi_S(S' \backslash \{\mathbf{a}\}) = S' \backslash \{\mathbf{a}\}$, $\psi_S(S'') = S''$, and $\psi_S(\mathbf{p}) \neq \mathbf{p}$, it follows that $\psi_S(\mathbf{p}) = \mathbf{p}'$ is the other point of intersection of S' and S''. Moreover, if $\mathbf{a}$ is the center of S and $\mathbf{q} \in S \cap S''$, then triangles $\mathbf{aqp}$ and $\mathbf{aqp}'$ are similar. It follows that $|\mathbf{p} - \mathbf{a}||\mathbf{p}' - \mathbf{a}| = \rho^2$ where ρ is the radius of S. The same reasoning applies if $\mathbf{p}$ lies outside of S, and we obtain the formula

$$\mathbf{p}' = \psi_S(\mathbf{p}) = \rho^2 \frac{\mathbf{p} - \mathbf{a}}{|\mathbf{p} - \mathbf{a}|^2} + \mathbf{a} \tag{4}$$

for reflection about the circle S in $\mathbb{R}^2$ with center $\mathbf{a}$ and radius ρ. This same discussion applied to the case when C passes through $(0,0,1)$ and $S = E$ is a straight line provides a proof of the assertion made above that Ψ_C corresponds to ψ_E.

[1] Or horizon point. See [5] for other interesting properties of this point. In particular, $\bar{\pi}(\mathbf{c})$ is the center of $\pi(C)$.

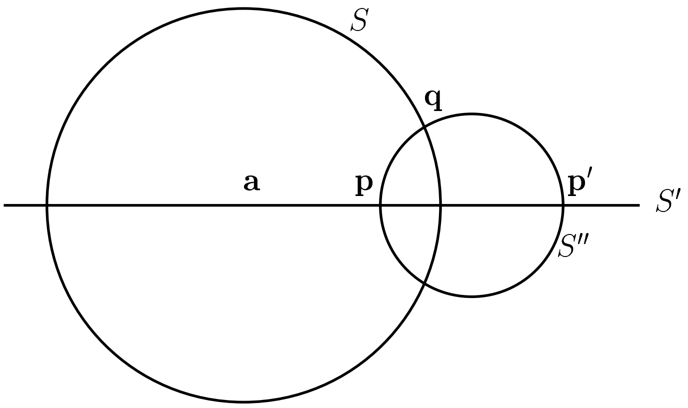

Figure 3: Mapping Circles

Finally we have the following

Definition *A set* $A \subset \mathbb{R}^2$ *has* generalized reflectional symmetry *if there are two distinct points* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *such that for each Steiner circle* S *passing through* $\mathbf{a}_1$ *and* $\mathbf{a}_2$*,* $\psi_S(A) = A$ *where* ψ_S *is reflection about* S*.*

We may allow one of the points $\mathbf{a}_1$ or $\mathbf{a}_2$ to be at ∞, in which case the circles S are all the lines passing through the other point. Furthermore, it can be shown (in analogy to Exercise 8) that *if* A *has generalized reflectional symmetry, then* A *is a union of circles, each of which is orthogonal to all the circles through* $\mathbf{a}_1$ *and* $\mathbf{a}_2$. We will not need this fact, but it is a special case of an assertion proved in [8]. These circles are called *circles of Apollonius*, and once it is known that they are circles, it is easy to show the following. (See Figure 4.)

Lemma 9 *The circles of Apollonius determined by the Steiner circles through* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *in* $\mathbb{R}^2$ *are disjoint and are in one to one correspondence with their centers which comprise the line* E *passing through* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *except for the segment between* $\mathbf{a}_1$ *and* $\mathbf{a}_2$*. Let* $\mathbf{m}_0 = (\mathbf{a}_1 + \mathbf{a}_2)/2$*. The circle of Apollonius with center* $\mathbf{a} \in \mathbb{R}^2$ *has radius* $r = \sqrt{d^2 - \rho_0^2}$ *where* $d = |\mathbf{a} - \mathbf{m}_0|$ *and* $\rho_0 = |\mathbf{a}_1 - \mathbf{a}_2|/2$*.*

The centers of the Steiner circles also form a line l, and once we know the point $\mathbf{m}_0 = (\mathbf{a}_1 + \mathbf{a}_2)/2$ on l and the reference distance $\rho_0 = |\mathbf{a}_1 - \mathbf{a}_2|/2$, we can express generalized reflectional symmetry without reference to $\mathbf{a}_1$ or $\mathbf{a}_2$ as follows.

Definition 10 *A set* Q *has* generalized reflectional symmetry along a line l *if, for each point* $\mathbf{a} \in l$*, we have* $\psi_S(Q) = Q$*, where* ψ_S *is given by* (4) *with* $\rho = \sqrt{d^2 + \rho_0^2}$ *and* $d = |\mathbf{a} - \mathbf{m}_0|$*.*

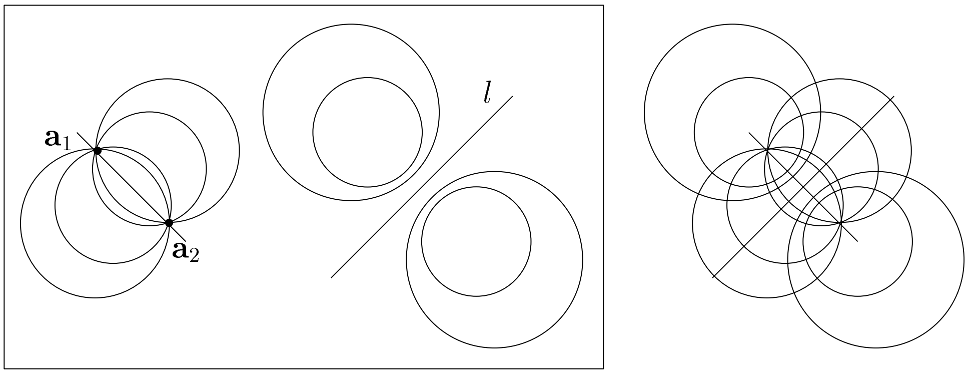

Figure 4: Steiner circles and circles of Apollonius.

Notice finally that formula (4) makes sense for points $\mathbf{a}$ and $\mathbf{p}$ in $\mathbb{R}^3$ and gives a generalization of reflection about circles to reflection about spheres. Thus, this statement of generalized symmetry can be applied to sets $Q \subset \mathbb{R}^3$.

In the next section we use this formulation and take Q to be a stereographic projection of the Clifford torus.

3 Stereographic projection of the Clifford Torus

The unit sphere in $\mathbb{R}^4$ is the three-dimensional space

$$\mathbb{S}^3 = \{\mathbf{x} = (x, y, z, w) : |\mathbf{x}| = 1\}.$$

Because we are used to visualizing things that are described by three Euclidean coordinates (i.e., things in $\mathbb{R}^3$), it is often difficult to see what objects look like in $\mathbb{S}^3$. For this reason, it is convenient to use a stereographic projection $\pi : \mathbb{S}^3 \backslash \{(0,0,0,1)\} \to \mathbb{R}^3$. The formula for such a map is similar to the one for $\mathbb{S}^2$:

$$\pi(\mathbf{x}) = \frac{1}{1-w}(x, y, z),$$

and a similar geometric description applies as well.

We are interested in a particular geometric object in $\mathbb{S}^3$ called the *Clifford torus*:

$$\mathcal{C} = \{\mathbf{x} : x^2 + y^2 = 1/2 = z^2 + w^2\}.$$

The stereographic projection $\pi(\mathcal{C})$ of the Clifford torus is particularly nice because of its *symmetry*.

Exercise 11 **(i)** *Show that $\pi(\mathcal{C})$ is rotationally symmetric with respect to the z-axis in $\mathbb{R}^3 = \{(x, y, z, 0)\}$.*

(ii) *What is the intersection of $\pi(\mathcal{C})$ with the x, z-plane?*

From Exercise 11 it is clear that $\pi(\mathcal{C})$ is described by its intersection with the half planes $\Pi_\theta = \{(r \cos\theta, r \sin\theta, z) : r > 0\}$. In fact, it is enough to know only $\pi(\mathcal{C}) \cap \Pi_0$.

As with circles in $\mathbb{S}^2$, rotating the three-sphere (i.e., moving $\mathcal{C}$ around in $\mathbb{S}^3$) changes the stereographic projection. Since the rotated surface is geometrically identical to $\mathcal{C}$ however, one might expect that some kind of symmetry of the projection is preserved. In fact, we show the following.

Theorem 12 *Let R be any rotation of $\mathbb{S}^3 \subset \mathbb{R}^4$. The surface $Q = \pi \circ R(\mathcal{C})$ has generalized symmetry in the sense described in the last section.*

We will, as we did in §1, consider first a particular rotation and then show that that rotation is typical via a decomposition formula (Theorem 14) for general rotations. With hindsight from the decomposition formula, we consider the rotation R^{xw}_ψ of the x, w-plane corresponding to the matrix

$$\begin{pmatrix} \cos\psi & 0 & 0 & -\sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\psi & 0 & 0 & \cos\psi \end{pmatrix}.$$

Proposition 13 *Let $Q = Q(\psi) = \pi \circ R^{xw}_\psi(\mathcal{C})$.*

(i) *If $\psi = 0$, then $Q = Q_0$ has generalized reflectional symmetry along the z-axis.*

(ii) *If $0 < |\psi| \leq \pi/2$, then Q has generalized reflectional symmetry along the vertical line $(x, y) = (\tan\psi, 0)$ and along the horizontal line $(x, z) = (-\cot\psi, 0)$.*

(iii) *Let R be the rotation of $\mathbb{R}^3$ about the x-axis by $\pi/2$. Then*

(a) *$Q(\psi + \pi/2) = R(Q(\psi))$, and*

(b) *$Q(\psi + \pi) = Q(\psi)$, for all ψ.*

In particular, $Q = Q(\psi)$ has generalized reflectional symmetry for all ψ.

Theorem 14 *Any rotation R of $\mathbb{S}^3 \subset \mathbb{R}^4$ is a composition*

$$R = R_0 \circ R^{xw}_\psi \circ R^{zw}_\phi \circ R^{xy}_\theta \tag{5}$$

where R_0 is the trivial extension[2] to $\mathbb{R}^4$ of a rotation of $\mathbb{R}^3 = \{(x, y, z, 0)\}$ and R^{xy}_θ is the trivial extension[2] to $\mathbb{R}^4$ of a rotation of $\mathbb{R}^2 = \{(x, y, 0, 0)\}$ etc..

[2]For a precise definition see the appendix on rotations.

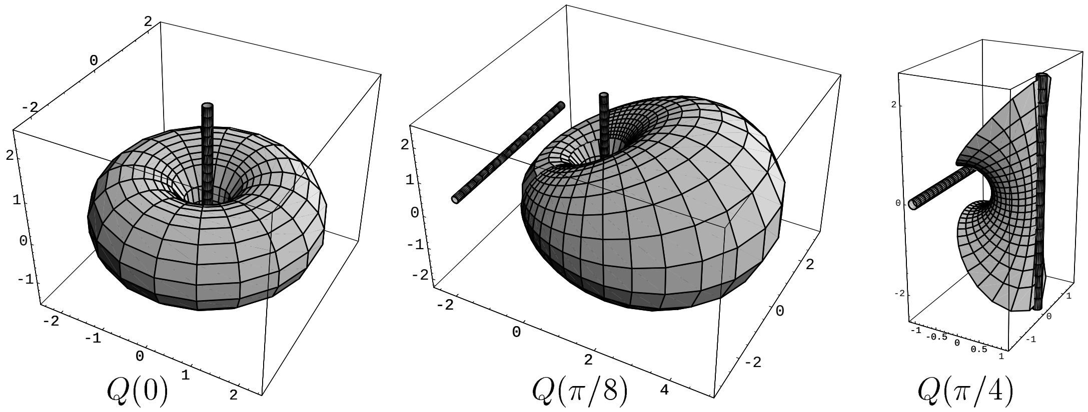

Figure 5: Stereographic projections and lines of centers.

Before we prove Proposition 13 and Theorem 14 we will show that they imply Theorem 12. It is easy to check (see Lemma 17 below) that rotations of the x, y and z, w-planes leave $\mathcal{C}$ invariant. Furthermore, if S is any set in $\mathbb{S}^3 \backslash \{(0,0,0,1)\}$ and R_0 is a rotation of $\{(x,y,z,0)\}$ as above, then $\pi \circ R_0(S) = R_0 \circ \pi(S)$. To see this, let $R_0(x,y,z,0) = (x', y', z', 0)$ and note that

$$\begin{aligned}
\pi \circ R_0(x,y,z,w) &= \pi(x', y', z', w) \\
&= \frac{1}{1-w}(x', y', z', 0) \\
&= R_0 \left(\frac{1}{1-w}(x,y,z,0) \right) \\
&= R_0 \circ \pi(x,y,z,w).
\end{aligned}$$

Thus, using the decomposition formula (5) we have

$$\begin{aligned}
\pi \circ R(\mathcal{C}) &= \pi \circ R_0 \circ R_\psi^{xw} \circ R_\phi^{zw} \circ R_\theta^{xy}(\mathcal{C}) \\
&= R_0 \circ \pi \circ R_\psi^{xw}(\mathcal{C}).
\end{aligned}$$

The set $\pi \circ R_\psi^{xw}(\mathcal{C})$ is described in Proposition 13. The rotation R_0 only changes that description by a rigid rotation in $\mathbb{R}^3$. In particular, $\pi \circ R(\mathcal{C})$ has generalized reflectional symmetry. $\square$

Proof of Proposition 13. Part (i) follows from Exercise 11 where one finds that Definition 10 is satisfied with $\mathbf{m}_0 = \mathbf{0} \in \mathbb{R}^3$ and $\rho_0 = 1$. Therefore, the sphere of symmetry with center $\mathbf{a} = (0,0,c)$ which we denote by $S = S_\rho(\mathbf{a})$ has radius

$$\rho = \sqrt{1 + c^2}. \tag{6}$$

For $0 < |\psi| \le \pi/2$ we will show that $C = \pi^{-1}(S)$ determines a transformation Ψ_C of $\mathbb{S}^3$. This construction is analogous to the discussion in §2 of circles $C \subset \mathbb{S}^2$. Moreover, we will again have the correspondence $\psi_S = \pi \circ \Psi_C \circ \pi^{-1}$ where ψ_S is the reflection about S. Furthermore, a geometrically identical transformation $\Psi_{\tilde{C}}$ will be determined by $\tilde{C} = R_\psi^{xw}(C)$, and $\Psi_{\tilde{C}}$ will correspond to reflection about the *sphere* $\tilde{S} = \pi(\tilde{C})$. The spheres $\tilde{S}$ thus corresponding to the spheres $S = S_\rho(\mathbf{a})$ will be symmetry spheres for $Q = Q(\psi)$ that satisfy Definition 10. Of course, at this point we do not even know that $\tilde{S}$ is a sphere. We give now a precise higher dimensional version of Exercise 5 which will establish this fact. For the statement we use the notation $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ when $\mathbf{n} = (n_1, n_2, n_3, n_4)$ and the notation $(\mathbf{n}, n_4) = (n_1, n_2, n_3, n_4)$ when $\mathbf{n} = (n_1, n_2, n_3)$.

Lemma 15 *Let* $\Pi = \{\mathbf{x} = (x, y, z, w) : \mathbf{n} \cdot \mathbf{x} = e\}$ *be a three-plane in* $\mathbb{R}^4$. *If* $(0,0,0,1) \notin \Pi$, *then*

$$\pi(\Pi \cap \mathbb{S}^3) = \left\{ \mathbf{a} = (a,b,c) : \left| \mathbf{a} - \frac{\underline{\mathbf{n}}}{n_4 - e} \right|^2 = \frac{n_4 + e}{n_4 - e} + \frac{|\underline{\mathbf{n}}|^2}{(n_4 - e)^2} \right\} \tag{7}$$

(which is a sphere). If $(0,0,0,1) \in \Pi$, *then*

$$\pi(\Pi \cap \mathbb{S}^3 \backslash \{(0,0,0,1)\}) = \{\mathbf{a} : \underline{\mathbf{n}} \cdot \mathbf{a} = n_4\} \tag{8}$$

(which is a plane).

On the other hand, let $S = \{\mathbf{a} : |\mathbf{a} - \mathbf{a}_0|^2 = \rho^2\}$ *be a sphere in* $\mathbb{R}^3$. *Then* $\pi^{-1}(S) = \Pi \cap \mathbb{S}^3$ *where*

$$\Pi = \{\mathbf{x} : (-2\mathbf{a}_0, \rho^2 - |\mathbf{a}_0|^2 + 1) \cdot \mathbf{x} = \rho^2 - |\mathbf{a}_0|^2 - 1\}. \tag{9}$$

Let $P = \{\mathbf{a} : \mathbf{n} \cdot \mathbf{a} = e\}$ *be a plane in* $\mathbb{R}^3$. *Then* $\pi^{-1}(P) \cup \{(0,0,0,1)\} = \Pi \cap \mathbb{S}^3$ *where*

$$\Pi = \{\mathbf{x} : (\mathbf{n}, e) \cdot \mathbf{x} = e\}. \tag{10}$$

Since we have given explicit equations, Lemma 15 follows from simple substitution using the formulas for π and π^{-1}, and we omit the proof. Note that equations (7) and (8) allow degenerate cases corresponding to $\Pi \cap \mathbb{S}^3 \subset \{(0,0,0,1)\}$. In our applications below however, we will know that $\Pi \cap \mathbb{S}^3$ is nontrivial.

Recall that $S = S_\rho(\mathbf{a})$ is a sphere of symmetry for Q_0. One sees from (9) and (6) that $C = \pi^{-1}(S) = \Pi \cap \mathbb{S}^3$ where $\Pi = \{\mathbf{x} : (\mathbf{a}, -1) \cdot \mathbf{x} = 0\}$. Note that Π passes through $\mathbf{0} \in \mathbb{R}^4$, i.e., C is a *great sphere* in $\mathbb{S}^3$. Let $\mathbf{n} = (\mathbf{a}, -1)$ be the normal to Π. We consider the reflection Ψ_C of $\mathbb{S}^3$ about Π defined by

$$\Psi_C(\mathbf{x}) = \mathbf{x} - \frac{2\mathbf{x} \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n}. \tag{11}$$

To see that $\psi_S = \pi \circ \Psi_C \circ \pi^{-1}$, we extend the discussion of Figure 3 in §2. Let $S' = \{\mathbf{a}' : \mathbf{n}' \cdot (\mathbf{a}' - \mathbf{a}) = 0\}$ be a plane orthogonal to S. According to (10) we have $\pi^{-1}(S') \cup \{(0,0,0,1)\} = \Pi' \cap \mathbb{S}^3$ where $\Pi' = \{\mathbf{x} : (\mathbf{n}', \mathbf{n}' \cdot \mathbf{a}) \cdot \mathbf{x} = \mathbf{n}' \cdot \mathbf{a}\}$. Since $(\mathbf{n}', \mathbf{n}' \cdot \mathbf{a}) \cdot \mathbf{n} = (\mathbf{n}', \mathbf{n}' \cdot \mathbf{a}) \cdot (\mathbf{a}, -1) = 0$, we see that $\Psi_C(\Pi') = \Pi'$. Consequently, $\pi \circ \Psi_C \circ \pi^{-1}(S' \backslash \{\mathbf{a}\}) = S' \backslash \{\mathbf{a}\}$. It follows similarly that $\pi \circ \Psi_C \circ \pi^{-1}(S'') = S''$ for any sphere S'' orthogonal to S.

It then follows that formula (4) gives the value of $\pi \circ \Psi_C \circ \pi^{-1}(\mathbf{p})$, and hence that $\psi_S = \pi \circ \Psi_C \circ \pi^{-1}$. To see this, we can apply the discussion of Figure 3 in §2 where we interpret S, S' and S'' as $S_\rho(\mathbf{a})$, a plane (orthogonal to the paper), and a sphere respectively. Technically, we should also introduce the plane of the paper S''' which can be used to show that $\mathbf{p}' \in S'''$.

There is nothing special about $S = S_\rho(\mathbf{a})$ in this reasoning, except that its inverse image is a great sphere. We have actually shown the following.

Lemma 16 *Let S be any sphere or plane in $\mathbb{R}^3$ whose inverse image $C = \pi^{-1}(S)$ is determined by a three-plane $\Pi = \{\mathbf{x} \cdot \mathbf{n} = 0\}$ through $\mathbf{0} \in \mathbb{R}^4$. Then the reflection ψ_S about S is given by $\psi_S = \pi \circ \Psi_C \circ \pi^{-1}$ where Ψ_C is the reflection about Π given by* (11).

We are now in a position to finish the proof of Proposition 13. The rotation $\tilde{C} = R^{xw}_\psi(C)$ is also a great sphere in $\mathbb{S}^3$, and for $0 < |\psi| \le \pi/2$, the plane $\tilde{\Pi} = R^{xw}_\psi(\Pi) = \{\mathbf{x} : (\sin\psi, 0, c, -\cos\psi) \cdot \mathbf{x} = 0\}$ does not contain $(0,0,0,1)$. Thus, we have from (7) that $\pi(\tilde{C})$ is the sphere

$$\tilde{S} = \{\tilde{\mathbf{a}} : |\tilde{\mathbf{a}} - (\tan\psi, 0, c/\cos\psi)|^2 = (1 + c^2)/\cos^2\psi\}.$$

If we set $\tilde{\mathbf{m}}_0 = (\tan\psi, 0, 0)$ and $\tilde{\rho_0} = 1/\cos^2\psi$, we see that $Q = \pi \circ R^{xw}_\psi(\mathcal{C})$ satisfies Definition 10 since

$$\begin{aligned}
\psi_{\tilde{S}}(Q) &= \pi \circ \Psi_{\tilde{C}} \circ \pi^{-1} \ \circ \ \pi \circ R^{xw}_\psi(\mathcal{C}) \\
&= \pi \circ R^{xw}_\psi \circ \Psi_C \circ R^{xw}_{-\psi} \ \circ \ R^{xw}_\psi(\mathcal{C}) \\
&= \pi \circ R^{xw}_\psi \ \circ \ \pi^{-1} \circ \pi \circ \Psi_C \circ \pi^{-1} \circ \pi(\mathcal{C}) \\
&= \pi \circ R^{xw}_\psi \ \circ \ \pi^{-1} \circ \psi_S \circ \pi(\mathcal{C}) \\
&= \pi \circ R^{xw}_\psi(\mathcal{C}) = Q.
\end{aligned}$$

We have established that Q has a vertical line $(x, y) = (\tan\psi, 0)$ of generalized reflectional symmetry for $0 < |\psi| \le \pi/2$.

The planes $P_\phi = \{\mathbf{a} : (-\sin\phi, \cos\phi, 0) \cdot \mathbf{a} = 0\}$ are also planes of reflective symmetry for Q_0. Applying Lemma 16 and Lemma 15 much as we have done above with the spheres $S_\rho(\mathbf{a})$ we find that for $0 < |\psi| \le \pi/2$ the spheres

$$\begin{aligned}\tilde{S} &= \pi \circ R_\psi^{xw} \circ \pi^{-1}(P_\phi) \\ &= \{\mathbf{a} : |\mathbf{a} - (-\cot\psi, \cot\phi/\sin\psi, 0)|^2 = (1+\cot^2\phi)/\sin^2\psi\}\end{aligned}$$

are spheres of symmetry along the horizontal line $(x, z) = (-\cot\psi, 0)$ which satisfy Definition 10 with $\tilde{\mathbf{m}}_0 = (-\cot\psi, 0, 0)$ and $\tilde{\rho}_0 = 1/\sin^2\psi$. This finishes the proof of statement (ii).

The first identity in statement (iii) follows from explicit calculation and the following observation.

Lemma 17 *For any function $f : \mathbb{R}^4 \to \mathbb{R}^4$ we have $\{f(x, y, z, w) : \mathbf{x} \in \mathcal{C}\} = \{f(x', y', z', w') : \mathbf{x} \in \mathcal{C}\}$ where x' is $\pm x$ and y' is $\pm y$ (or possibly x' is $\pm y$ and y' is $\pm x$), and similarly z' is $\pm z$ and w' is $\pm w$ (or possibly z' is $\pm w$ and w' is $\pm z$).*

Statement (iiib) follows from (iiia). $\square$

Proof of Theorem 14. We use again the fact that rotations are precisely those linear transformations that correspond to orthogonal matrices of determinant 1. Let M be the matrix representing R. Let $N = N(\theta, \phi, \psi)$ be the unknown matrix representing $R_\psi^{xw} \circ R_\phi^{zw} \circ R_\theta^{xy}$, and let N_0 be the unknown matrix representing R_0. We then need to show $M = N_0 N$.

We know that N_0 has the form

$$N_0 = \begin{pmatrix} & & & 0 \\ & \star & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

where $\star$ represents a rotation matrix for $\mathbb{R}^3$. From this we see that the last row of $N_0 N$ and the last row of N are the same. This last row is given by $N^{\mathrm{T}}\mathbf{e}_4$ (where T indicates the transpose), and we need to have $M^{\mathrm{T}}\mathbf{e}_4 = N^{\mathrm{T}}\mathbf{e}_4$.

Lemma 18 *There exist angles θ, ϕ, and ψ such that $M^{\mathrm{T}}\mathbf{e}_4 = N^{\mathrm{T}}\mathbf{e}_4$.*

Proof. Since M is orthogonal, so is $M^{-1} = M^{\mathrm{T}}$. Therefore, the columns of M^{T} (i.e., the rows of M) form an orthonormal basis—see [4, pp. 127–129]. In particular, $M^{\mathrm{T}}\mathbf{e}_4 = (m_{41}, m_{42}, m_{43}, m_{44})$ is some unit vector.

On the other hand, by direct calculation we see that

$$N^{\mathrm{T}}\mathbf{e}_4 = (\sin\psi\cos\theta, -\sin\psi\sin\theta, \cos\psi\sin\phi, \cos\psi\cos\phi).$$

Since $m_{41}^2 + m_{42}^2 \leq 1$, there is some angle ψ with $\sin^2\psi = m_{41}^2 + m_{42}^2$. If $\sin\psi \neq 0$, then we can find θ with $\cos\theta = m_{41}/\sin\psi$ and $\sin\theta = -m_{42}/\sin\psi$. If $\sin\psi = 0$, then $m_{41} = m_{42} = 0$. In either case, our choice of θ and ψ implies that the first two coordinates of $M^{\mathrm{T}}\mathbf{e}_4$ and $N^{\mathrm{T}}\mathbf{e}_4$ agree. Since $m_{43}^2 + m_{44}^2 = 1 - m_{41}^2 + m_{42}^2 = \cos^2\theta$, we can choose ϕ, much as we chose θ, and have the last two coordinates match. □

To prove Theorem 14 it remains to specify N_0. Let $\mathbf{m}_j = (m_{j1}, m_{j2}, m_{j3}, m_{j4})$ be the jth row of M and $\mathbf{n}_j$ be the jth row of N for $j = 1, 2, 3, 4$, and consider

$$\begin{aligned} MN^{-1} &= \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \\ \mathbf{n}_4 \end{pmatrix} (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{m}_4) \\ &= \begin{pmatrix} & & & \mathbf{m}_1\cdot\mathbf{m}_4 \\ & \star & & \mathbf{m}_2\cdot\mathbf{m}_4 \\ & & & \mathbf{m}_3\cdot\mathbf{m}_4 \\ \mathbf{n}_4\cdot\mathbf{n}_1 & \mathbf{n}_4\cdot\mathbf{n}_2 & \mathbf{n}_4\cdot\mathbf{n}_3 & \mathbf{n}_4\cdot\mathbf{m}_4 \end{pmatrix} \\ &= \begin{pmatrix} & & & 0 \\ & \star & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \mathbf{m}_4\cdot\mathbf{m}_4 \end{pmatrix} \end{aligned}$$

which is of the form (12). Thus, we let N_0 be the rotation matrix on the right and clearly we have $M = N_0 N$. □

Epilogue

We observed in §2 that stereographic projections of rotations of the equator circle in $\mathbb{S}^2$ are circles. Exercise 5 points out that other circles in $\mathbb{S}^2$ have this property as well. That is, we have *not* characterized the rotations of the equator circle C. Nevertheless, C and its rotations (the great circles) are "balanced" on the surface of $\mathbb{S}^2$ in a way that the other circles are not. This "balance" is expressed precisely by saying the *geodesic curvature* is zero or simply that these curves are *geodesics*. It is this balance that justifies the specific attention we have given to the great circles.

In a similar way, the Clifford torus is "balanced" in the three-sphere $\mathbb{S}^3$, because its *mean curvature* is zero, i.e., it is a *minimal surface*. While it can be shown that every geodesic curve in $\mathbb{S}^2$ is (part of) a great circle, there is a great variety of minimal surfaces in $\mathbb{S}^3$. There are spheres and tori and surfaces of genus two (two holed tori), etc.. In fact, Lawson [7] has given examples of closed minimal surfaces in $\mathbb{S}^3$ of every *topological genus*, i.e., tori with any number of holes. On the other hand, Bryant [3] has shown that every embedded (non self-intersecting) minimal surface in $\mathbb{S}^3$ that is topologically spherical stereographically projects to a standard Euclidean sphere. A similar characterization for minimal tori is not known, but it is believed that, up to a rigid rotation of $\mathbb{S}^3$, the Clifford torus $\mathcal{C}$ is the unique embedded minimal torus.

For this reason, symmetry properties of $\mathcal{C}$ as shown above are of great interest. We remark finally that the only closed surfaces in $\mathbb{R}^3$ possessing the symmetry shown above for stereographic projections of $\mathcal{C}$ are topologically spherical or toroidal [8]. This symmetry, moreover, has geometric consequences as well, and we hope to give, in another paper, an elementary introduction to the curvature of surfaces in $\mathbb{S}^3$ and prove that $\mathcal{C}$ *is the unique minimal torus possessing such symmetry.*

Appendix: Rotations

We seek below to give an intuitive introduction to the family of *rotations* of Euclidean space $\mathbb{R}^n$, $n \geq 3$. Our starting point is with the distance preserving transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy

$$|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|. \tag{13}$$

We refer to all such transformations as *rigid motions*, and our objective is to determine which rigid motions should be called *rotations.*

Notice first of all that *translations* are rigid motions. That is, for any fixed vector $\mathbf{a}$, the transformation defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ satisfies (13). It is a fundamental algebraic fact that up to a translation every rigid motion is *linear.*

Theorem 19 *If $T_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion, then T defined by $T(\mathbf{x}) = T_0(\mathbf{x}) - T_0(\mathbf{0})$ is a linear transformation.*

Proof.[3] Recall that T is linear if $T(a\mathbf{x}) = aT(\mathbf{x})$ and $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}^1$. Recall also the *triangle inequality* in $\mathbb{R}^n$:

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \textit{ with equality only if } \mathbf{x} = \lambda\mathbf{y} \textit{ for some } \lambda \geq 0.$$

See [9, Exercise 1-2] for a proof.

Because T preserves distance and fixes the origin,

$$|T(\mathbf{x})| = |\mathbf{x}|, \ |T(a\mathbf{x})| = |a|\,|\mathbf{x}|, \ |T(a\mathbf{x}) - T(\mathbf{x})| = |a - 1|\,|\mathbf{x}|. \tag{14}$$

On the other hand, by the triangle inequality

$$|T(\mathbf{x}) - T(a\mathbf{x})| + |T(a\mathbf{x})| \geq |T(\mathbf{x})|. \tag{15}$$

[3] This discussion is considered in a more general setting in [2, pp. 1–6].

If $0 \le a \le 1$, one can use (14) to check that equality holds in (15). Consequently, for some $\lambda \ge 0$

$$T(\mathbf{x}) - T(a\mathbf{x}) = \lambda T(a\mathbf{x}),$$

or $T(\mathbf{x}) = (1+\lambda)T(a\mathbf{x})$. Taking the norm of both sides we see that $T(a\mathbf{x}) = aT(\mathbf{x})$.

By exchanging $|T(a\mathbf{x})|$ and $|T(\mathbf{x})|$ in (15), and following the same line of reasoning, one sees that $T(a\mathbf{x}) = aT(\mathbf{x})$ also for $1 < a$.

Finally, if $a < 0$, the same reasoning applied to the inequality

$$|T(a\mathbf{x})| + |T(\mathbf{x})| \ge |T(\mathbf{x}) - T(a\mathbf{x})|$$

yields again that $T(a\mathbf{x}) = aT(\mathbf{x})$.

Next consider $T(\mathbf{x}+\mathbf{y})$. In fact, let $a < 0$ and note that

$$|T(\mathbf{x}+a\mathbf{y}) - T(\mathbf{x})| + |T(\mathbf{x}) - T(\mathbf{x}+\mathbf{y})| \ge |T(\mathbf{x}+a\mathbf{y}) - T(\mathbf{x}+\mathbf{y})|.$$

The left side is $|a\mathbf{y}| + |\mathbf{y}| = (1-a)\mathbf{y}$, and the right side is $|a\mathbf{y} - \mathbf{y}| = (1-a)\mathbf{y}$. Since they are equal, there exists $\lambda \ge 0$ with $T(\mathbf{x}+a\mathbf{y}) - T(\mathbf{x}) = \lambda(T(\mathbf{x}) - T(\mathbf{x}+\mathbf{y}))$. It is easy to see that $\lambda = -a$, so

$$T(\mathbf{x}+a\mathbf{y}) = T(\mathbf{x}) + a(T(\mathbf{x}+\mathbf{y}) - T(\mathbf{x})).$$

Subtracting $aT(\mathbf{y}) = T(a\mathbf{y})$ from both sides and rearranging we get

$$a[T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y}))] = T(\mathbf{x}+a\mathbf{y}) - T(a\mathbf{y}) - T(\mathbf{x}).$$

Therefore,

$$\begin{aligned} |a|\,|T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y}))| &\le |T(\mathbf{x}+a\mathbf{y}) - T(a\mathbf{y})| + |T(\mathbf{x})| \\ &= 2|\mathbf{x}|. \end{aligned}$$

Notice that the right side is a fixed value, but $|a|$ on the left may be taken as *large* as we like. The only way the inequality can continue to hold is if

$$|T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y}))| = 0,$$

i.e., $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$. □

From now on, we assume our rigid motions satisfy

$$T(\mathbf{0}) = \mathbf{0}. \tag{16}$$

Thus, we are only going to consider rotations about the origin—since other rotations only differ from these by a translation. Furthermore, we can refer to the matrix M which corresponds to a rigid motion T (which will be the matrix of T with respect to the standard basis unless stated otherwise).

From Theorem 19 we can easily prove

Corollary 20 *If* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *form an orthonormal basis, then* $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ *form an orthonormal basis as well.*

Proof. Once linearity is established, the preservation of orthonormal bases follows by expressing the inner product in terms of the norm $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. In fact,

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(|\mathbf{x} + \mathbf{y}|^2 - |\mathbf{x}|^2 - |\mathbf{y}|^2).$$

It follows, using the linearity, that $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. In particular, $T(\mathbf{e}_i) \cdot T(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form an orthonormal basis. □

Any linear transformation that preserves orthonormality of bases is called an *orthogonal* transformation. The main properties of orthogonal transformations and the matrices that represent them may be found in [4, pp. 127–129]. In particular, the inverse matrix M^{-1} that represents T^{-1} is the transpose matrix M^T of M. It follows from this and the product formula for determinants that $(\det M)^2 = 1$, or since M is a real matrix, that $\det M = \pm 1$. As pointed out in the introduction, the additional condition

$$\det M = 1 \tag{17}$$

is often used to distinguish T as a rotation.

So far we have used the intuitive condition (13) and the normalization (16) to derive some algebraic facts. We now return to our intuition concerning rotations and ask for a precise condition which, in conjunction with (13), will express our intuitive idea of what defines a rotation. This is not so easy, but we should keep in mind that such a condition is likely to be equivalent to (17).

Our first approach might be to give an intuitive (yet precise) definition of *orientation*, and then try to connect a condition concerning *orientation preserving* transformations with (17). The reader may be surprised to find, as we were, that our physical intuition concerning this approach is limited to two dimensions. To see this, take a piece of paper and draw an orthonormal basis on it dark enough so that you can see it from the back side of the paper (see Figure 6). Now draw a dot on the desk to represent the origin $\mathbf{0}$. Any rigid motion of the plane (i.e., the paper) which fixes the origin must map $\mathbf{e}_1$ to some unit vector $(\cos\theta, \sin\theta)$—which you can draw on the desk. Now there are two obvious ways to rigidly move the paper so that it lies flat on the desk and $\mathbf{e}_1$ lies on top of $(\cos\theta, \sin\theta)$. Intuitively, if the paper is facing up, the motion is a rotation. If the paper is facing down, it is not. That is, whether or not a rigid motion is a rotation is determined by how we "orient" the paper before placing it on the desk.

It is very difficult however (if not impossible) for us to rigidly move a physical representation of $\mathbb{R}^3$ (like a wooden block) so as to change its orientation.

It turns out that the only way to define *orientation* of bases for $\mathbb{R}^3$ is, one way or another, to append an additional dimension. One feels, however, that we *do* have an intuitive idea of what constitutes a rotation of $\mathbb{R}^3$ independent of additional dimensions.

A second approach might be based on the idea of a *rotation axis.* Indeed, every rotation of $\mathbb{R}^3$ has 1 as an eigenvalue so that it does have a fixed vector, $\mathbf{x} = R(\mathbf{x})$, which can be used to define an axis of rotation (See [4, pg. 291, Corollary 33.3]). Unfortunately, there is no such rotation axis for non-trivial rotations of $\mathbb{R}^2$, and there need not be one for rotations of $\mathbb{R}^4$.

A third approach (since we are getting frustrated) could be to use the idea of *decomposition* as in Theorems 4 and 14—except in reverse. That is, we could define an *elementary rotation* to be a rotation of just one coordinate two-plane, i.e., a transformation R_ψ^{kj} corresponding to a matrix of the form

$$\begin{array}{c} \\ k \to \\ \\ j \to \\ \\ \end{array} \begin{pmatrix} I_{k-1} & & & & \\ & \cos\psi & & -\sin\psi & \\ & & I_{j-k-1} & & \\ & \sin\psi & & \cos\psi & \\ & & & & I_{n-j} \end{pmatrix}$$

where I_m denotes an $m \times m$ identity matrix and there are zeros filling all the spaces. Then we could define a *rotation* to be a composition of elementary rotations. After pondering this, however, it is not at all clear that a composition of rotations should be a rotation. In fact, the decomposition in Theorem 4 is not intuitively a rotation of $\mathbb{R}^3$—it is the composition of three rotations, one executed after another *in time.* And here is the key. A *rotation* is a transformation which can be realized as a physical rigid motion (parameterized by time) that is the *same* motion at each instant of time. To make this statement precise is fairly easy.

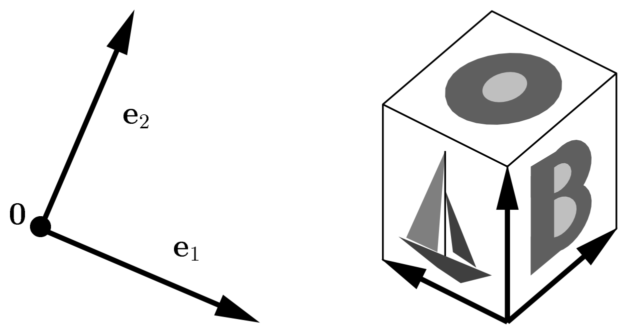

Figure 6: Moving orthonormal bases.

Definition 21 *A rigid motion R is a* rotation *if there is a smoothly parameterized family of rigid motions $R_0(t)$ such that $R_0(0) = \mathrm{id}_{\mathbb{R}^n}$ and, for each $m = 1, 2, 3, \ldots$,*

$$R_0(1/m)^m = \underbrace{R_0(1/m) \circ \cdots \circ R_0(1/m)}_{m \text{ times}} = R.$$

Notice that $R_0(t)$ for $t \in (0, 1)$ is not explicitly required to be a rotation (so the definition is not circular). One might be worried however that the definition allows transformations of determinant -1 which we don't want as rotations in $\mathbb{R}^2$. It is easy to show that this does not happen, but it turns out that the most difficult thing to see is that condition (17) does not allow transformations that Definition 21 excludes. Nevertheless, we have the following.

Proposition 22 *If $R : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion represented by the matrix M, then the following are equivalent.*

(i) *R is a rotation.*

(ii) *There exist elementary rotations $R_1, R_2, \ldots, R_k$ such that $R = R_1 \circ \cdots \circ R_k$.*

(iii) $\det M = 1$.

(iv) *With respect to some basis R is represented by a matrix of the form*

$$\begin{pmatrix} I_k & & & \\ & R_{\theta_1} & & \\ & & \ddots & \\ & & & R_{\theta_{(n-k)/2}} \end{pmatrix}$$

where I_k is a $k \times k$ identity matrix and $R_{\theta_1}, \ldots, R_{\theta_{(n-k)/2}}$ are 2×2 rotation matrices.

Notice that condition (ii) was mentioned above as a condition that came somewhat short in expressing our intuitive idea of a rotation. It is however a useful condition, of which Theorems 4 and 14 are particular instances, so we have included it. We will use the following exercise and lemma to show that it is part of the equivalence.

Exercise 23 *Show that elementary rotations are rotations and have determinant* 1.

Lemma 24 *Given any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ of the same length, there is a rotation Q, which is a composition of elementary rotations R_θ^{kl}, such that $Q\mathbf{v} = \mathbf{w}$.*

Proof. We first note that it is enough to prove the lemma for $\mathbf{v} = \mathbf{e}_n$ and $\mathbf{w} = \mathbf{u}$ an arbitrary *unit* vector. To see this simply note that $\tilde{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ and $\tilde{\mathbf{w}} = \mathbf{w}/|\mathbf{w}|$ are unit vectors. Thus, if we can find Q_1 and Q_2 (compositions of elementary rotations) with $Q_1(\mathbf{e}_n) = \tilde{\mathbf{v}}$ and $Q_2(\mathbf{e}_n) = \tilde{\mathbf{w}}$, we can take $Q = Q_2 \circ Q_1^{-1}$ and it is easily checked that $Q(\mathbf{v}) = \mathbf{w}$.

We prove that $\mathbf{e}_n$ can be "coordinate rotated" to $\mathbf{u}$ by induction. The initial case, $n = 2$ follows from the fact that in $\mathbb{R}^2$ any unit vector $\mathbf{u}$ can be represented by $(\cos\theta, \sin\theta)$ for some angle θ.

For $n > 2$, let $\mathbf{u} = (u_1, \ldots, u_n)$ and $u_n = \cos\phi$. It follows that for some $\mathbf{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$

$$R_\phi^{1n}(\mathbf{e}_n) = (v_1, \ldots, v_{n-1}, u_n).$$

Also, by Exercise 23, R_ϕ^{1n} preserves length, so $|\mathbf{v}| = |\mathbf{w}|$ where $\mathbf{w} = (u_1, \ldots, u_{n-1})$. By induction, there is a composition Q of elementary rotations of $\mathbb{R}^{n-1}$ such that $Q(\mathbf{v}) = \mathbf{w}$. Notice that Q extends to a composition of elementary rotations of $\mathbb{R}^n$, and we have $Q \circ R_\phi^{1n}(\mathbf{e}_n) = \mathbf{u}$. This completes the induction and the proof of Lemma 24. □

Proof of Proposition 22. That (i) implies (iii) follows from the product formula for determinants applied to $R = R_0(1/2) \circ R_0(1/2)$. That (iii) implies (iv) is Theorem 30.5 in [4, pg. 270] where rotations are viewed as orthogonal transformations of determinant 1. Condition (i) follows from (iv) by taking $R_0(t)$ to be the transformation corresponding (in the same basis) to the matrix

$$\begin{pmatrix} I_k & & & \\ & R_{t\theta_1} & & \\ & & \ddots & \\ & & & R_{t\theta_{(n-k)/2}} \end{pmatrix}.$$

Condition (ii) we deal with separately. It is clear from the product formula for determinants and Exercise 23 that (ii) implies (iii). We obtain the reverse implication by induction. As discussed above, if $n = 2$, then $R(\mathbf{e}_1)$ must be some unit vector $(\cos\theta, \sin\theta)$. By orthogonality $R(e_2) = \pm(-\sin\theta, \cos\theta)$. Only the $+$ sign is compatible with (iii).

For $n \geq 3$, let M be the orthogonal matrix representing R. Let $\mathbf{v} = R(\mathbf{e}_n)$. According to Lemma 24, there is a composition Q of elementary rotations such that $Q\mathbf{v} = \mathbf{e}_n$. Thus, $Q \circ R$, fixes $\mathbf{e}_n$, and if N is the matrix representing Q we have

$$NM = \begin{pmatrix} & & & 0 \\ & U & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{18}$$

where U is an $(n-1)\times(n-1)$ rotation matrix. By Exercise 23 and the multiplication formula for determinants $\det U = \det(NM) = \det M = 1$. Therefore, by induction $U = U_1 \cdots U_k$ for some elementary rotation matrices $U_1, \ldots, U_k$. These rotations extend linearly to elementary rotations of $\mathbb{R}^n$ represented by matrices N_j of the form (18) with U_j in place of U. Hence, $M = N^{-1}N_1 \cdots N_k$ is a product of elementary rotation matrices. □

We note that the construction just used to extend the elementary rotation matrices can be used to extend any rotation to a rotation on a higher dimensional space. To be precise, if R is a rotation of $\mathbb{R}^k$ with standard basis elements $\mathbf{e}_1, \ldots, \mathbf{e}_k$, $k < n$, and $J = \{j_1 < \cdots < j_k\}$ is a subset of k indices from $\{1, \ldots, n\}$, then one obtains a rotation $\tilde{R}$ of $\mathbb{R}^n$ with standard basis elements $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n$ by setting $\tilde{R}(\tilde{\mathbf{e}}_j) = \tilde{\mathbf{e}}_j$ if $j \notin J$ and $\tilde{R}(\tilde{\mathbf{e}}_{j_l}) = \sum b_m \tilde{\mathbf{e}}_{j_m}$ where $R(\mathbf{e}_l) = \sum b_m \mathbf{e}_m$. The rotation $\tilde{R}$ is called a *trivial extension* of R.

Appendix: Conformality of Stereographic Projection

Here we give a short proof of Lemma 6. Let us first assume that the two curves C_1 and C_2 intersect at the point $\mathbf{p} = (0,0,-1)$ in an angle $\gamma(C_1, C_2, \mathbf{p})$. Let T_j be a unit tangent vector to C_j at p_j for $j = 1$ and 2, and let $\tilde{C}_j$ be the intersection (circle) of the plane Π_j containing T_j and $(0,0,1)$ with $\mathbb{S}^2$. Clearly we have $\gamma(C_1, C_2, \mathbf{p}) = \gamma(\tilde{C}_1, \tilde{C}_2, \mathbf{p})$ is the angle between Π_1 and Π_2.

On the other hand, $\pi(\mathbf{p}) = (0,0)$, and $\pi(\tilde{C}_j)$ is the intersection (line) of Π_j with the x,y-plane for $j = 1$ and 2. Thus, the angle of intersection of $\pi(\tilde{C}_1)$ and $\pi(\tilde{C}_2)$ is again the angle between Π_1 and Π_2. We have shown that π is conformal at $\mathbf{p} = (0,0,-1)$.

If the point of intersection $\mathbf{p}$ is any point other than $(0,0,-1)$ in $\mathbb{S}^2\backslash\{(0,0,1)\}$, we again take tangents T_j to C_j at $\mathbf{p}$ and let $\tilde{C}_j$ be the intersection (circle) determined by the plane Π_j containing T_j and $(0,0,-1)$ for $j = 1$ and 2. Here $\pi(\tilde{C}_1)$ and $\pi(\tilde{C}_2)$ are circles that intersect in two points $\mathbf{q}_1 = \pi(\mathbf{p})$ and $\mathbf{q}_2 = (0,0)$. Since circles (on the sphere and in the plane) intersect in equal angles at their two points of intersection, we have

$$\begin{aligned}
\gamma(\pi(C_1), \pi(C_2), \pi(\mathbf{p})) &= \gamma(\pi(\tilde{C}_1), \pi(\tilde{C}_2), \pi(\mathbf{p})) \\
&= \gamma(\pi(\tilde{C}_1), \pi(\tilde{C}_2), (0,0)) \\
&= \gamma(\tilde{C}_1, \tilde{C}_2, (0,0,-1)) \\
&= \gamma(\tilde{C}_1, \tilde{C}_2, \mathbf{p}) \\
&= \gamma(C_1, C_2, \mathbf{p}). \square
\end{aligned}$$

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John McCuan
Mathematics Department
University of California, Berkeley
Berkeley, CA 94720
johnm@math.berkeley.edu