Positively curved surfaces with no tangent support plane

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Abstract

We discuss a characterization of positively curved surfaces M with the property that at each point the tangent plane to M is not a support plane for the entire surface. A one parameter family of examples which have special relevance with respect to the characterization is also given. Each member of this family is a smooth embedded surface in $\mathbb{R}^3$ that is topologically a disk, has everywhere positive Gauss curvature, but has none of its tangent planes as a support plane.

Introduction

Throughout this paper M denotes a smooth positively curved immersion into $\mathbb{R}^3$ of a compact connected surface which may have a (smooth) boundary.

The condition of positive curvature implies that M is *strictly locally convex,* i.e., the plane T_pM tangent to M at p is a support plane for some (intrinsic) neighborhood N of p, and $N \cap T_pM = \{p\}$. A well known theorem of Hadamard [5] asserts that if $\partial M = \phi$, then M is (globally) the boundary of a (strictly) convex region in $\mathbb{R}^3$. Various authors have generalized Hadamard's theorem. We mention in particular that do Carmo and Lima [2] draw essentially the same conclusion for complete, non-compact surfaces, though in this case the region in $\mathbb{R}^3$ is non-compact.

Recently Mohammad Ghomi extended Hadamard's theorem to the case when $\partial M \neq \phi$. When does such a surface lie in the boundary of a strictly convex region?

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One sees immediately the following necessary condition. For each $p \in \partial M$ the tangent plane T_pM is a support plane for ∂M, and $\partial M \cap T_pM = \{p\}$. We say that ∂M is *convex compatible* if this condition holds component-wise, i.e., for each component C of ∂M and each $p \in C$ we have that T_pM is a support plane for C and $C \cap T_pM = \{p\}$. Ghomi proves this latter condition is sufficient for M to lie in the boundary of a convex region.

Theorem 1 (Ghomi [4]) *A compact, positively curved surface M has convex compatible boundary if and only if M is a subset of the boundary of a convex region in $\mathbb{R}^3$.*

In particular, every tangent plane to M is a support plane for M.

In the case $\partial M \neq \phi$ we denote the convex hull of ∂M by K. Following Ghomi, we say ∂M is *convex* if $\partial M \subset \partial K$. This condition clearly holds for many positively curved surfaces. In particular, if M "bulges out" so that each interior point of M is in $\mathbb{R}^3 \backslash K$, then ∂M is convex, n.b., Figure 1. It was observed in [4] that this condition is

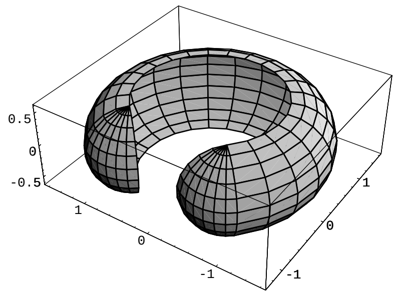


Figure 1: Positively curved surface with boundary.

too weak to imply M lies in the boundary of a convex region. (This also is illustrated in Figure 1.) Convexity of the boundary is enough, however, to imply the existence of at least one tangent support plane. This follows from a recent characterization of positively curved surfaces with convex boundary.

Theorem 2 (Alexander-Ghomi [3]) *∂M is convex if and only if*

$$\operatorname{int} M \subset \mathbb{R}^3 \backslash K. \tag{1}$$

Thus, if ∂M is convex, then for some $p \in \operatorname{int} M$, we have $d(p, K) = \max_q d(q, K)$. It follows that T_pM is a support plane for M.

The same reasoning shows that *if M has no tangent support plane,* then

$$\operatorname{int} M \subset \operatorname{int} K. \tag{2}$$

Under the same assumption on M,

$$T_pM \cap \operatorname{int} K \neq \phi \text{ for all } p \in \partial M. \tag{3}$$

Otherwise, T_pM is a support plane for K and hence for M by (2).

Since (1) is known to hold for a reasonably large class of surfaces and (2) is quite the opposite property, one might guess that at least one of (2) and (3) is always violated. Presumably, such considerations led to the conjecture in [4] that every positively curved immersion M has at least one tangent support plane. We describe below a counterexample which is an embedded topological disk.

Notice that (2) and (3) characterize the positively curved surfaces with no tangent support plane. More generally, these surfaces are characterized by the condition that for some $A \subset \operatorname{int} M$,

$$A \subset \operatorname{int} K \quad \text{and} \quad T_pM \cap \operatorname{int} K \neq \phi \text{ for all } p \in M \backslash A. \tag{4}$$

From this observation we obtain the following lemma.

Lemma 3 *If M is a positively curved immersion with no tangent support plane, and $\tilde{M} \subset M$ is a surface with boundary such that the convex hull $\tilde{K}$ of $\partial\tilde{M}$ is K, then $\tilde{M}$ has no tangent support plane. In particular, if $B \subset \partial M$ has convex hull K and $B \subset \tilde{M}$, then the conclusion holds.*

Aside from simplifying the discussion below, this lemma shows the importance of the convex hull K. In our example, the set B that generates K consists of ten small arcs which, with obvious modifications, may be made arbitrarily small.

It should also be remarked that there are simpler examples than the one described below. (David Hoffman and Jim Hoffman have suggested two such constructions. See the schematic diagrams at the end of the paper.) We felt, however, that our example illustrated dramatically the significance of the convex hull K of ∂M.

Now we show that our remarks are not vacuous.

Examples

The starting point for our construction is the following lemma which is verified by an elementary calculation.

Lemma 4 *Let $\gamma = \gamma(s)$ be a smooth planar curve in $\{(x, y, 0)\} \subset \mathbb{R}^3$ with nonvanishing curvature $\kappa = \kappa(s) = |\ddot{\gamma}|$ and unit normal $n = \ddot{\gamma}/|\ddot{\gamma}|$.*

Let $T^+(s)$ and $T^-(s)$ define a domain $\Sigma = \{(s,t) : T^-(s) < t < T^+(s)\}$.

Let $M^+ = \max_s T^+$ and $M^- = \min_s T^-$, and assume that $f : [M^-, M^+] \to [0, \infty)$ is a smooth function satisfying $f(0) = 0$ and $f'' > 0$. Assume also that $f \circ T^+(s), f \circ T^-(s) < 1/\kappa(s)$ for all s.

Then $X(s,t) = \gamma(s) + fn + te_3$ gives a well defined positively curved immersion M of Σ into $\mathbb{R}^3$ where $e_3 = (0, 0, 1)$.

We apply Lemma 4 with γ given by the "spirograph curve"

$$\gamma(\theta) = (3\cos\theta + \rho\cos(3\theta/2), 3\sin\theta - \rho\sin(3\theta/2))$$

determined by a point rigidly fixed at a distance ρ from the center of a circle of radius 2 which is rolling inside a circle of radius 5; see Figure 3(a). We assume initially that $2 < \rho \leq 5/2$. An elementary calculation then shows

$$\kappa(\theta) = \frac{3\rho^2 - 8 - 2\rho\cos(5\theta/2)}{3(\rho^2 + 4 - 4\rho\cos(5\theta/2))^{3/2}} > 0.$$

Let $T^{\pm}(\theta) = \pm[A + \epsilon - (A - \epsilon)\cos(5\theta/2)]$ where $0 < \epsilon \leq 1/2\sqrt{\kappa(0)} < A \leq 1/2\sqrt{\kappa(2\pi/5)}$. This choice defines a strip Σ as in Figure 2(a).

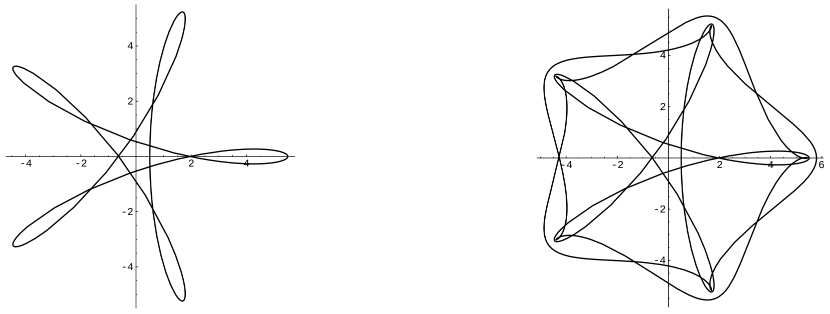
(a) (b)

Figure 2: Strip between two periodic curves.

Finally, we take $f(t) = t^2$, and taking account of periodicity in Lemma 3, we obtain an annular immersion parameterized by

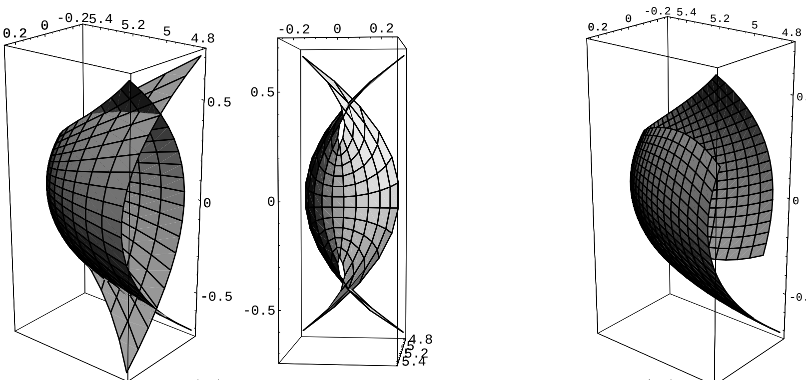
$$X(\theta, t) = \gamma + t^2 n + te_3.$$

The resulting surface M is composed of five (positively curved) taco shells (see Figure 5). Unlike standard flat taco shells, these curl at the corners. Because of



(a) (b)

Figure 3: Spirograph curve and the projection of ∂M.



(a) (b)

Figure 4: (a) Where two tacos meet ($-\pi/8 \le \theta \le \pi/8$). (b) The desingularization.

this, they may be smoothly concatenated so that their backs lie successively along the spirograph curve, and their wings are directed outward. The projection into the x, y-plane of ∂M is shown in Figure 3(b), and a closeup of the corners where two taco shells join is shown in Figure 4(a)

The convex hull K of ∂M is generated by ten arcs on the extreme edges of the wings. These arcs are quite small when A is large (say ≥ 2), though Figure 5 corresponds to a considerably lower value of A. The fact that the arcs are small can be used to give a proof that M has no tangent support plane as follows.

The largest z-values (in magnitude) are obtained when κ is at its minima and $t = T^{\pm}(\theta)$. These boundary values are found to be at $\theta_k = 2\pi/5 + 4\pi k/5$ for $k \in \mathbb{Z}$

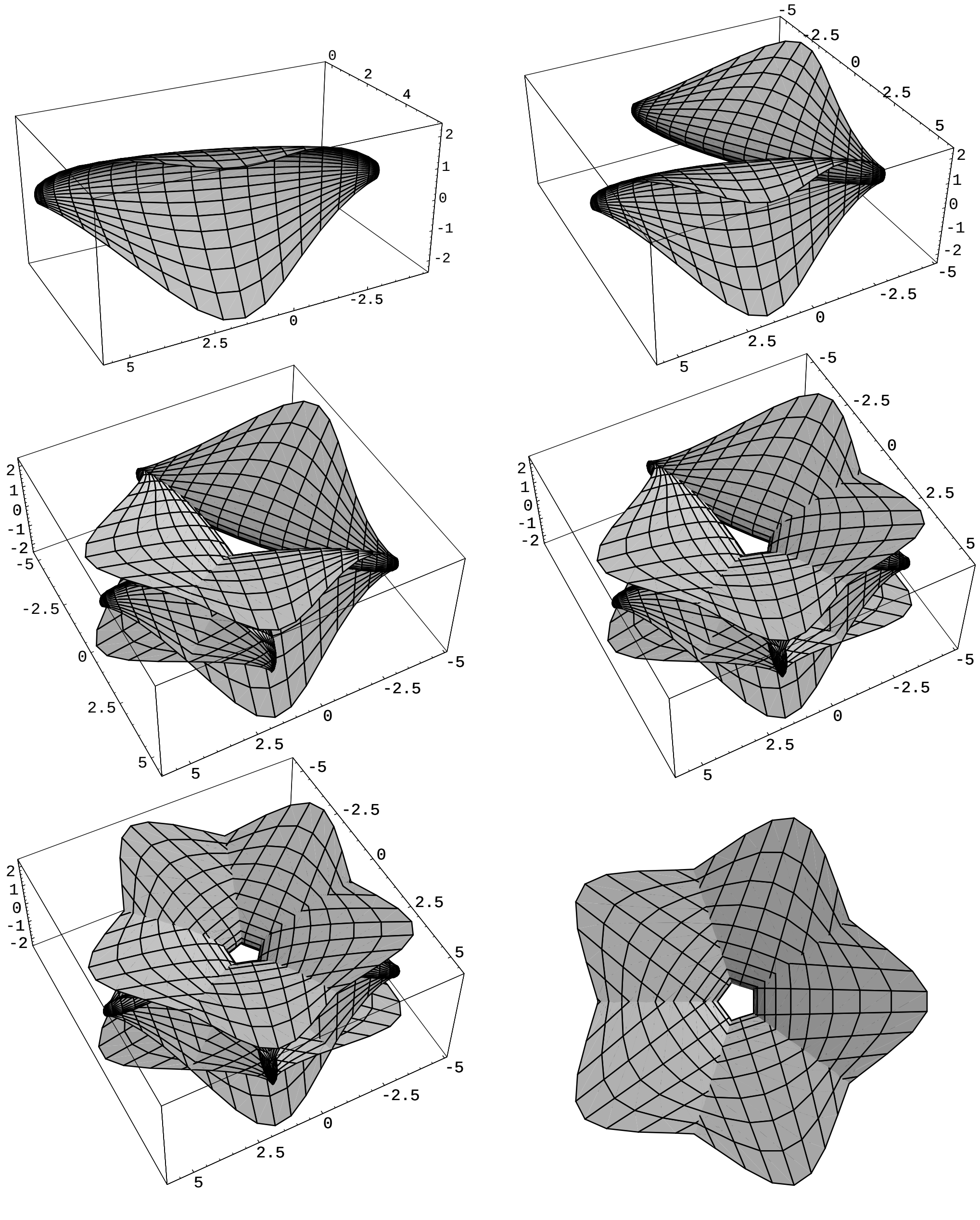


Figure 5: Construction of the immersed annular example.

and are given by the ten points

$$X(\theta_k, T^{\pm}(\theta_k)) = \gamma(\theta_k) + T^{+}(\theta_k)^2 n \pm T^{+}(\theta_k) e_3, \qquad k = 0, \pm 1, \pm 2.$$

The convex hull K_0 of $\{X(\theta_k, T^{\pm}(\theta_k))\}$ is a pentagonal prism, and $K_0 \subset K$. One can verify directly that, for large enough values of A, most interior points of M are in $\operatorname{int} K_0$. More precisely, if $A \geq 2$, then the condition $|\theta - \theta_k| > \pi/8$ for $k = 0, \pm 1, \pm 2$ implies

$$\begin{aligned} |\gamma(\theta) + t^2 n| &\leq |\gamma(\theta)| + T^{+}(\theta)^2 \\ &< r_0 \equiv (3 - \rho + 4A^2)\cos(\pi/5). \end{aligned} \tag{5}$$

The number r_0 in this inequality is the radius of the circular cylinder inscribed in K_0. Thus, $X(\theta, t) \in \operatorname{int} K_0$.

For the remaining points $p = X(\theta, t) = \gamma + t^2 n + t e_3$ with $|\theta - \theta_k| \leq \pi/8$ for some $k = 0, \pm 1, \pm 2$, we show that $T_p M \cap \operatorname{int} K_0 \neq \phi$. The line through p with tangent direction $X_t = 2tn + e_3$ comes closest to the origin at the point

$$q = \gamma - t^2 \left(\frac{1 + 4\gamma \cdot n}{4t^2 + 1} \right) n + 2t \left(\frac{t^2 - \gamma \cdot n}{4t^2 + 1} \right) e_3 \in T_p M.$$

If we specialize to $\rho = 5/2$, then it follows that $0 \leq \gamma \cdot n \leq 1/2$ and, hence, that the third component of q is in the open interval $(T^-(\theta), T^+(\theta))$. Consequently, we only need to check that the projection

$$q_0 = \gamma - t^2 \left(\frac{1 + 4\gamma \cdot n}{4t^2 + 1} \right) n$$

lies in the pentagonal base of K_0. It is easily checked under our present assumptions, $\rho = 5/2$, $A \geq 2$, and $|\theta - \theta_k| \leq \pi/8$, that $|\gamma| \leq 3$. From this it is easy to see that $|q_0| < r_0$ where r_0 is given in (5). Thus, $q \in T_p M \cap \operatorname{int} K_0$.

By the characterization (4), we conclude that M has no tangent support plane.

By removing portions of Σ as indicated in Figure 2(b), we obtain an embedded disk which, since the dark curves correspond to a set B whose convex hull is K, has no tangent support plane by Lemma 3. Two views of this surface are shown in Figure 6.

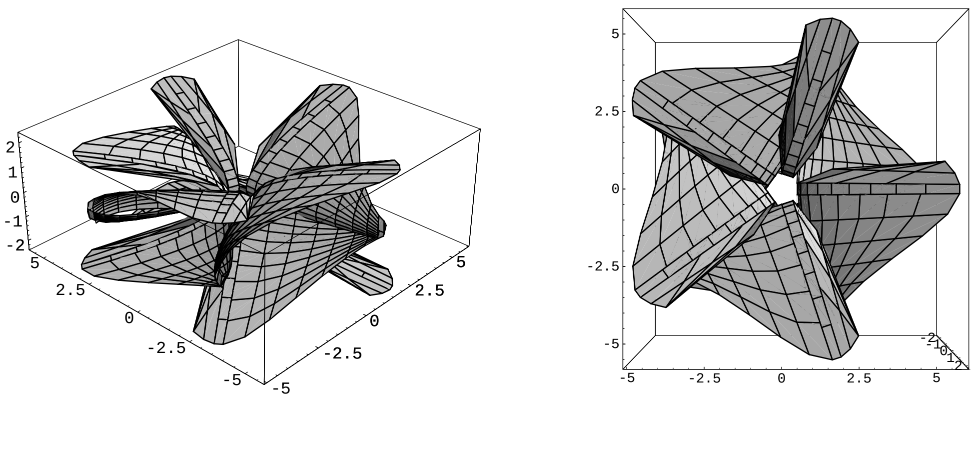


Figure 6: An embedded topological disk.

David Hoffman and Jim Hoffman [6] have given other examples (Figure 7). David's example consists of two portions of spheres connected by a strip of positive curvature like that described in Lemma 4. Jim's is a positively curved ribbon that nearly lies on a catenoid.

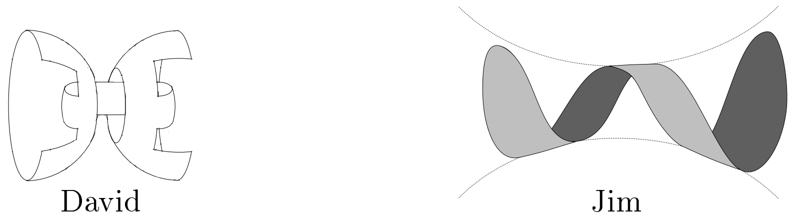


Figure 7: Other examples.

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