A CHARACTERIZATION OF SEMIMODULAR LATTICES WHOSE COMPARABILITY GRAPHS ARE WEAKLY CONNECTED

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Abstract. Let L be a semimodular lattice of rank d that is not a chain ($d \in \mathbb{N}_0$). It is shown that the comparability graph of L is $(d+1)$-connected if and only if L has no simplicial elements (where $z \in L$ is simplicial if the elements comparable to z form a chain).

A theorem of T. Hibi and N. Terai for finite distributive lattices, and a theorem of Hibi for finite modular lattices (suggested by R. P. Stanley), are obtained as corollaries: To wit, if a finite modular or distributive lattice of rank d contains a complemented rank 3 interval, then the lattice is $(d+1)$-connected.

The following structural results are obtained: If L is a rank d semimodular lattice and $L \setminus C$ is disconnected, where $C \subseteq L$ has at most d elements, then C is a d-element chain. Moreover, all but at most one of the components of $L \setminus C$ are bottom-rooted trees whose leaves are simplicial elements. If L is modular, and a and b minimal elements from different components of $L \setminus C$, then either a or b is simplicial. Further, either (1) C is the set of elements greater or less than a simplicial element, or (2) $L \setminus C$ is the disjoint union of a top-rooted tree and a bottom-rooted tree. If L is distributive, then either (1) holds or (2') L is the ordinal sum of a chain, a product of two chains, and another chain. It is proved that a distributive lattice of rank d, with poset of join-irreducibles P, is $(d+1)$-connected if and only if P contains a three-element antichain or a four-element crown (provided L is not a chain).

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1. Introduction.

In many graph-theoretic applications, from highways to electrical circuits, one is interested in how many elements one must remove from a graph before it becomes disconnected.

In the context of lattice theory, it is natural to consider graphs arising from partial orders and lattices, *comparability graphs*. The vertices of a comparability graph are the elements of an ordered set. Two elements are joined by an edge if they are comparable in the poset.

Three classes of lattices that are studied by both lattice theorists and algebraic combinatorialists are the classes of *distributive*, *modular*, and *semimodular* lattices. T. Hibi and N. Terai have shown that the comparability graph of a finite distributive lattice of rank $d \in \mathbb{N}_0$ is d*-connected* (meaning that if one removes up to $d-1$ elements, the graph is still connected); and it is $(d+1)$-connected if it contains a rank 3 interval that is a Boolean lattice. (See [8], Theorem 3.3.)

At the suggestion of R. P. Stanley, Hibi has extended this theorem to finite modular lattices ([7], Theorem 0.1): A finite modular lattice of rank $d \in \mathbb{N}_0$ is $(d+1)$-connected if it contains a rank 3 interval that is complemented. Heavy machinery from homological algebra is used to prove the aforementioned theorems.

By contrast, using only elementary lattice theory, we prove a stronger result (Theorem 3.7): A rank d semimodular lattice — finite or infinite — is $(d+1)$-connected if and only if (1) it is a chain, or else (2) it contains no simplicial elements (where an element is *simplicial* if the set of elements greater or less than it forms a chain, called a *simplicial* chain). Hence *any* rank d modular lattice is $(d+1)$-connected, provided it contains a complemented rank 3 interval (Corollary 3.10).

Moreover, if $C \subseteq L$ is a subset with d or fewer elements whose removal disconnects the comparability graph of L (where d is the rank of the semimodular lattice), then C is a d-element chain, and all but at most one of the components of $L \setminus C$ are trees whose leaves are simplicial elements (Proposition 4.2). If L is modular, then, given a pair of minimal elements from different components of $L \setminus C$, one of them must be simplicial (Proposition 4.3). Further, either C is simplicial, or else $L \setminus C$ has exactly two components, a top-rooted tree and a bottom-rooted tree, the leaves of both of which are simplicial (Proposition 4.4). If L is distributive, then either C is simplicial, or else $L \setminus C$ has exactly two components, both chains, and L is the ordinal sum of a chain, a product of two chains, and another chain (Proposition 4.5). We also prove that a finite distributive lattice has no simplicial elements if and only if its poset of join-irreducibles contains a three-element antichain or a four-element crown (Theorem 4.6).

Our lattice-theoretic approach has several advantages over the homological one: We obtain stronger results using simpler methods. Our theorems apply to infinite lattices, which the homological method cannot handle. We also obtain structural information about the lattice (and the disconnecting subset) in the event it is d- but not $(d+1)$-connected. Such structural information would be difficult if not impossible to obtain using homological tools.

Our results also answer the natural question that arises whenever a theorem is proved by indirect means: Is there a direct proof? This sort of question has been asked by K. Baclawski and A. Björner. Having obtained certain results for finite lattices using homological methods, they ask if there is a direct proof, which might extend to infinite lattices ([1], p. 336). It is a question of aesthetics and epistemology: While homological methods are sufficient to establish *that* a theorem is true, direct methods tell us *why* it is true.

Our aim is not to assert the primacy of lattice-theoretic over other methods of investigation; quite the opposite. For, in general, homological arguments *are* superior to direct ones. By establishing what can be done lattice-theoretically, we are actually encouraging algebraic combinatorialists to forge ahead into unexplored territory: We come not to bury Cæsar, but to praise him.

2. Definitions, Notation, and Basic Results.

A basic reference is [3].

Let $G = (V, E)$ be a graph with vertex set V and edge set $E \subseteq V \times V$. It is *connected* if, for all v, $v' \in V$, there exist $k \in \mathbb{N}_0$ and $v_0, \ldots, v_k \in V$ such that $v_0 := v$, $v_k := v'$, and (v_0, v_1), (v_1, v_2), $\ldots$, $(v_{k-1}, v_k) \in E$. It is *n-connected* (where $n \in \mathbb{N}_0$) if the restriction of G to the vertices $V \setminus C$ is connected whenever $C \subseteq V$ has fewer than n elements.

Let P be a poset. The *comparability graph* of P is the graph with vertex set P and edge set $\{\, (p, p') \in P \times P \mid p \leqslant p' \text{ or } p \geqslant p' \,\}$. The poset is *connected* if its comparability graph is connected. A *component* of P is a maximal connected subset. A poset is *disconnected* if it has two or more components.

For all $p \in P$, a poset, let

$$\downarrow p := \{\, p' \in P \mid p' \leqslant p \,\}$$

and let

$$\uparrow p := \{\, p' \in P \mid p' \geqslant p \,\}.$$

For all p, $p' \in P$ such that $p \leqslant p'$, the *(closed) interval* $[p, p']$ is the set $\{\, q \in P \mid p \leqslant q \leqslant p' \,\}$; $[p, p')$ is the set $\{\, q \in P \mid p \leqslant q < p' \,\}$. A *down-set* is a subset D of P such that $\downarrow d \subseteq D$ whenever $d \in D$.

The least element of a poset is denoted by 0, if it exists; the greatest element by 1, if it exists.

A *chain* is a totally ordered set. From now on we will restrict ourselves to posets whose chains are finite.

An element of a poset is *simplicial* if the set of elements greater or less than it forms a chain, which we also call *simplicial* (cf. [5], §4.1; [6], §6). A chain C has *rank* d if its cardinality $^{\#}C$ is $d+1$, in which case we write $C = \mathbf{d+1}$ ($d \in \mathbb{N}_0$). A poset has *rank* d ($d \in \mathbb{N}_0$) if every maximal chain has rank d; the *rank* of an element of the poset is the rank $r(p)$ of $\downarrow p$.

A *bottom-rooted tree* is a poset with 0 such that, for every element p, $\downarrow p$ is a chain; dually we define *top-rooted trees*. [N. B. We call these posets "trees" because their *Hasse* diagrams (as opposed to their comparability graphs) are trees in the graph-theoretic sense.] An *antichain* is a poset such that no two distinct elements are comparable. The four-element *crown* is the poset $\{a, b, c, d\}$ in which $a, b < c, d$, and no other comparabilities hold but the necessary ones (Figure 1).

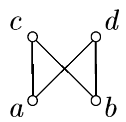

Figure 1. The four-element crown.

Let P be a poset. We write $p \lessdot p'$ for p, $p' \in P$ if $p < p'$ and $p \leqslant q < p'$ implies $p = q$ for all $q \in P$; we say p is a *lower cover* of p' and p' an *upper cover* of p. An element with a unique lower cover is *join-irreducible*; an element with a unique lower cover and a unique upper cover is *doubly irreducible*.

The *coalesced ordinal sum* of a poset P with 1 and a poset Q with 0 is the poset $P \boxplus Q$ obtained by identifying the 1 of P with the 0 of Q (Figure 2).

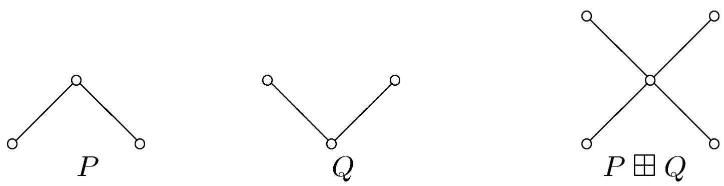

Figure 2. The posets P, Q, and $P \boxplus Q$.

The *product* $P \times Q$ of P and Q is the poset with underlying set

$$\{\, (p, q) \mid p \in P,\ q \in Q \,\}$$

in which $(p, q) \leqslant (p', q')$ if $p \leqslant p'$ and $q \leqslant q'$ (Figure 3).

A *lattice* is a non-empty poset in which every pair of elements $\{a, b\}$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. A lattice L with 0

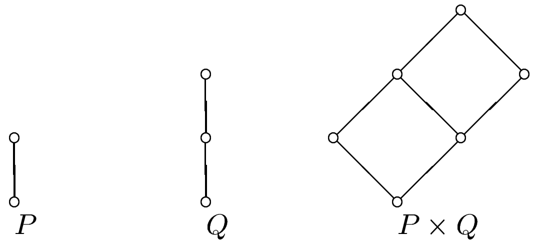

Figure 3. The posets P, Q, and $P \times Q$.

and 1 is *complemented* if, for all $a \in L$, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. A lattice with 0 is *atomistic* if every element is a least upper bound of a subset of the elements covering 0.

A lattice L is *distributive* if, for all a, b, $c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. A lattice is *Boolean* if it is distributive and complemented. An element b in a lattice L is *left modular* if, for all a, $c \in L$ such that $a \leqslant c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$; L is *modular* if every element is left modular. A lattice in which every join-irreducible element is left modular is a *generalized matroid lattice*. A poset or lattice is *(upper) semimodular* if, whenever two elements have a common lower cover, they have a common upper cover; *lower semimodularity* is defined dually. A lattice is *geometric*, or a *matroid lattice*, if it is semimodular and atomistic (equivalently, a lattice is geometric if it is an atomistic generalized matroid lattice). See [9], §29.

Every distributive lattice of finite rank is finite and isomorphic to the lattice of down-sets of its poset of join-irreducibles; every finite Boolean lattice is a power set lattice, that is, $\mathbf{2}^n$ for some $n \in \mathbb{N}_0$. A corollary of Priestley duality is that the category of finite distributive lattices, with $\{0, 1\}$-preserving homomorphisms, is dually equivalent to the category of finite posets with order-preserving maps ([3], 8.17, 8.21, and 8.24).

A modular lattice is distributive if and only if it does not contain M_3 as a sublattice, where M_3 is the poset $\{0, a, b, c, 1\}$ in which $0 < a, b, c < 1$ and no other comparabilities hold but the necessary ones (Figure 4; [3], Theorem 6.10).

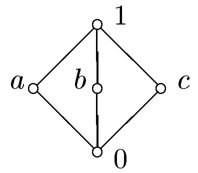

Figure 4. The lattice M_3.

In a modular lattice, $[a \wedge b, a]$ is isomorphic to $[b, a \vee b]$ for all elements a and b. Hence, a modular lattice is both upper and lower semimodular. A modular lattice of finite rank is complemented if and only if it is atomistic ([2], Chapter I, Theorems 13 and 14; Chapter IV, Theorem 6).

A semimodular lattice with no infinite chains has finite rank, and, for all elements a and b, we have the following rank inequality:

$$r(a) + r(b) \geqslant r(a \wedge b) + r(a \vee b)$$

([2], Chapter II, Theorem 15); hence, if a covers $a \wedge b$, then $a \vee b$ covers b.

In a finite modular lattice, a chain $0 =: x_0 < x_1 < \cdots < x_s := 1$ of rank $s \in \mathbb{N}_0$ is *fundamental* if $[x_{i-1}, x_i]$ is complemented for $i = 1, \ldots, s$, and $[x_{i-1}, x_{i+1}]$ is non-complemented for $i = 1, \ldots, s-1$ ([7], §1).

3. Semimodular Lattices Whose Comparability Graphs Are Weakly Connected.

In this section, we prove that every semimodular lattice of finite rank d is d-connected, and, if it is not a chain, it is $(d+1)$-connected just when it has no simplicial elements (Theorem 3.7). As a corollary, we obtain the infinitary analogue of Hibi's theorem for finite modular lattices ([7], Theorem 0.1): A finite or infinite modular lattice of rank d is $(d+1)$-connected if it has a complemented rank 3 interval (Corollary 3.10).

For Claims 3.1–3.6, we establish the following notation: Let L be a semimodular lattice of rank d ($d \in \mathbb{N}_0$). Let $C \subseteq L$ be a subset such that ${}^{\#}C \leqslant d$ and assume $L \setminus C$ is disconnected.

Let $a \in L \setminus C$ have minimal rank in $L \setminus C$ and let $b \in L \setminus C$ be a minimal element in another component. Hence $[0, a) \cup [0, b) \subseteq C$. Clearly $[a \vee b, 1] \subseteq C$.

Claim 3.1. *The set C equals $[0, a) \cup [0, b) \cup [a \vee b, 1]$, and $[0, a]$, $[0, b]$, and $[a \vee b, 1]$ are chains. Also, $r(a) + r(b) = r(a \wedge b) + r(a \vee b)$.*

Proof. The claim follows from the rank inequality and counting. □

Claim 3.2. *The element a covers $a \wedge b$ (which therefore is its unique lower cover).*

Proof. Assume not. Then there exist $\bar{a}$, $\bar{b} \in L$ such that $a \wedge b \lessdot \bar{a} < a$ and $a \wedge b \lessdot \bar{b} < b$. Clearly $\bar{a} \vee \bar{b} \nleqslant a, b$, so $\bar{a} \vee \bar{b} \notin C$.

Let $a^* := a \vee \bar{b}$ and let $b^* := \bar{a} \vee b$; then $a \lessdot a^* < a \vee b$ and $b \lessdot b^* < a \vee b$, so that a^*, $b^* \notin C$.

Hence $a \leqslant a^* \geqslant \bar{a} \vee \bar{b} \leqslant b^* \geqslant b$ is a path in $L \setminus C$, a contradiction. □

As a consequence of Claim 3.2, we get the following:

Claim 3.3. *For any $x \in L \setminus C$ not in the same component as a, $a \wedge x \lessdot a$.* $\square$

Claims 3.1 and 3.2 imply the following:

Claim 3.4. *For any $x \in L \setminus C$ not in the same component as a, $x \lessdot a \vee x \in C$; for all x, $y \in L \setminus C$ not in the same component as a, $r(x) = r(y)$ implies $a \vee x = a \vee y$.* $\square$

Claim 3.5. *For all $x \in L \setminus C$ not in the same component as a, $\downarrow x$ is a chain.*

Proof. Assume not. Then there exist distinct $u, v, w \in \downarrow x \setminus C$ such that $u \gtrdot v, w$. By Claim 3.4, $v, w \lessdot a \vee v = a \vee w \in C$, which therefore equals $v \vee w = u \notin C$, a contradiction. $\square$

Finally, from Claim 3.5, we get the statement below:

Claim 3.6. *The maximal elements of $L \setminus C$ not in the same component as the element a are simplicial.* $\square$

Applying the above claims, we deduce the main result.

Theorem 3.7. *Let L be a semimodular lattice of rank d ($d \in \mathbb{N}_0$). Then the comparability graph of L is d-connected; and it is $(d+1)$-connected if and only if either L has no simplicial elements or else L is a chain.* $\square$

Lemma 3.8. *If a modular lattice has a simplicial element, then so does every interval.*

Proof. Let z be simplicial in the modular lattice L, and let x, $y \in L$ be such that $x \leqslant y$. Let $t := (x \vee z) \wedge y = x \vee (z \wedge y)$. By modularity, $[x, t]$ is isomorphic to $[x \wedge z, y \wedge z]$ and $[t, y]$ to $[x \vee z, y \vee z]$, all of which are chains. $\square$

Lemma 3.9. *A complemented modular lattice of rank $d \in \mathbb{N}_0$ contains a simplicial element if and only if $d \leqslant 2$.*

Proof. Such a lattice is atomistic. $\square$

The following is the generalization of [7], Theorem 0.1 to infinite lattices.

Corollary 3.10. *Let L be a modular lattice of rank d ($d \in \mathbb{N}_0$). Then the comparability graph of L is $(d+1)$-connected if L has a complemented interval of rank 3.* $\square$

Remark. Our Theorem 3.7 should be contrasted with Hibi's criterion ([7], Corollary 1.4):

Theorem (Hibi)**.** *A finite modular lattice of rank d is $(d+1)$-connected if and only if it has no fundamental chain of rank $d-1$ ($d \in \mathbb{N}_0$).*

The fundamental chain criterion, however, appears cumbersome to check. Hibi himself requires several lattice-theoretic lemmas before he can deduce his result involving rank 3 intervals; whereas, our simplicial element criterion is easy to check, and the rank 3 result follows almost immediately.

A concrete example showing the utility of the simplicial element criterion, and the fact that our Theorem 3.7 is stronger than the theorems of Hibi and Hibi–Terai cited in §1, is $\mathbf{2}^2 \boxplus \mathbf{2}^2$ (Figure 5):

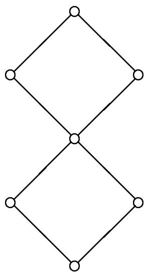

Figure 5. The distributive lattice $\mathbf{2}^2 \boxplus \mathbf{2}^2$.

It is a rank 4 distributive lattice, and it obviously has no simplicial elements, so by Theorem 3.7 it is 5-connected. The theorems of Hibi and Hibi–Terai cited in §1 do not apply, since none of the rank 3 intervals are Boolean.

Example 2.4 of [7] shows that Hibi's rank 3 interval criterion fails to guarantee $(d+1)$-connectedness for semimodular lattices: Consider the lattice $\mathbf{2}\times\mathbf{2}\times\mathbf{3}$; add a new element that covers a rank 1 point of $\mathbf{2}\times\mathbf{2}\times\mathbf{3}$ possessing two upper covers, and that is covered by the rank 3 element of $\mathbf{2}\times\mathbf{2}\times\mathbf{3}$ possessing three lower covers (Figure 6).

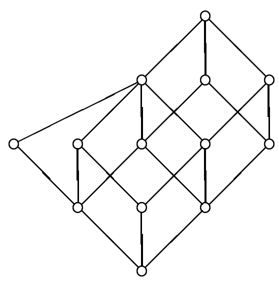

Figure 6. A semimodular lattice with a Boolean interval of rank 3.

The result is a rank 4 semimodular lattice that contains a Boolean interval of rank 3, but it is not 5-connected. Of course, our Theorem 7 continues to apply.

As the following example from §1 of [7] shows, the necessary and sufficient conditions of our theorem (and, evidently, Hibi's) fail for semimodular *posets*: Consider two four-element crowns that share a maximal element; adjoin to the poset a 0 and a 1 (Figure 7).

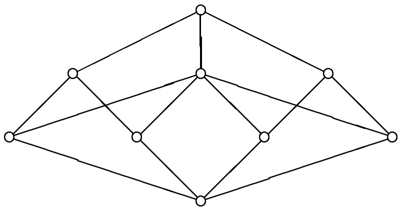

Figure 7. A semimodular poset with no simplicial elements.

The rank 3 poset is semimodular and has no simplicial elements, but it is not 4-connected.

Slightly more general than Lemma 3.8 is the following fact: If a lattice has a simplicial left modular element, then every interval has a simplicial left modular element. Therefore we can extend Corollary 3.10 in the following way:

Let L be a generalized matroid lattice of rank d ($d \in \mathbb{N}_0$). Then the comparability graph of L is $(d+1)$-connected if L has an interval of rank 3 that is geometric.

4. Structural Results.

In this section we show that the disconnecting set C of §3 must be a chain, and that, except in very special cases, it is simplicial (Propositions 4.1, 4.4, and 4.5). (Note that C is simplicial if and only if $L \setminus C$ has a one-element component.)

We also determine the structure of the components of $L \setminus C$ (Proposition 4.2). Finally, we characterize $(d+1)$-connected distributive lattices of rank d in terms of their poset of join-irreducibles (Theorem 4.6); the theorem of Hibi and Terai is an easy corollary (Corollary 4.7).

The results are all simple consequences of Claims 3.1–3.6.

For Propositions 4.1–4.5, we establish the following notation: Let L be a semimodular lattice of rank d ($d \in \mathbb{N}_0$). Let $C \subseteq L$ be a subset such that ${}^{\#}C \leqslant d$, and assume that $L \setminus C$ is disconnected. Let $\mathrm{gap}(C)$ be the (unique) rank number not represented in C.

Proposition 4.1. *The set C is a d-element chain with elements from all ranks but* $\mathrm{gap}(C)$. □

Proposition 4.2.

(1) *With at most one exception, all of the components of $L \setminus C$ are bottom-rooted trees whose roots have rank* $\mathrm{gap}(C)$. *The exceptional component is the one containing the element of lowest rank in $L \setminus C$.*

(2) *If a component of $L \setminus C$ contains no elements of rank less than* $\mathrm{gap}(C)$, *then that component is a bottom-rooted tree whose leaves are simplicial elements.* □

Illustration. Let $M := \{0, a, b, v, w, x, y, z, 1\}$ be the poset where

$$0 < a, b;\ a < v, w, x < 1;\ b < x, y, z < 1;$$

and no other comparabilities hold but the necessary ones (Figure 8).

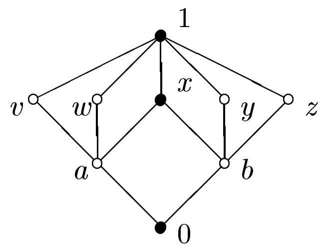

Figure 8. The semimodular lattice M.

Let $C := \{0, x, 1\}$ (the filled-in vertices of Figure 8). Then M is a rank 3 semimodular lattice that is not 4-connected: $M \setminus C$ has two components that are both bottom-rooted trees with roots of rank 1 and leaves that are simplicial ($\{a, v, w\}$ and $\{b, y, z\}$). The set C is a three-element chain missing an element of rank 1.

Proposition 4.3. *If L is a modular lattice, and a, $b \in L \setminus C$ are minimal elements from distinct components, then either a or b is simplicial.* □

Remark. The statement of Proposition 4.3 fails for semimodular lattices, as the example of Figure 8 reveals: neither a nor b is simplicial.

Proposition 4.4. *Suppose L is modular and C is not simplicial.*

Then $L \setminus C$ has exactly two components, a bottom-rooted tree T and a top-rooted tree U, with roots of rank $\mathrm{gap}(C)$. *Every element of T is covered by an element of C; every element of U covers an element of C.* □

Illustration. Let $N := \{0, a, b, u, v, w, x, y, z, 1\}$ be the poset where

$$0 < a, u, v < w;\ v < b;\ w, b < x < 1;\ b < y, z < 1;$$

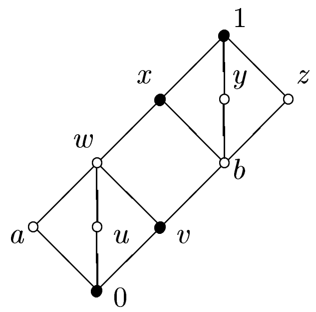

Figure 9. The rank 4 modular lattice N.

and no other comparabilities hold but the necessary ones (Figure 9).

Let $C := \{0, v, x, 1\}$ (the filled-in vertices of Figure 9). Then N is a rank 4 modular lattice that is not 5-connected: C is not simplicial, and $N \setminus C$ has two components, a bottom-rooted tree $T = \{b, y, z\}$ and a top-rooted tree $U := \{a, u, w\}$.

Proposition 4.5. *Suppose L is distributive and C is not simplicial. Then:*

1. *$L \setminus C$ is the disjoint union of two chains, and*
2. *L is isomorphic to $\mathbf{r} \boxplus (\mathbf{s} \times \mathbf{2}) \boxplus \mathbf{t}$ for some r, s, $t \in \mathbb{N}$.* □

Illustration. In Figure 10, we see $R := \mathbf{2} \boxplus (\mathbf{4} \times \mathbf{2}) \boxplus \mathbf{3}$.

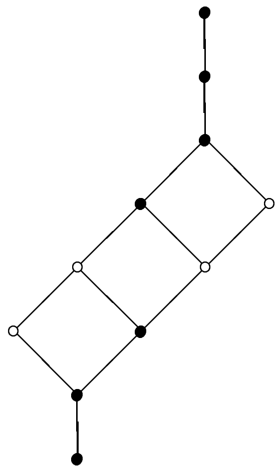

Figure 10. The rank 7 distributive lattice R.

Here C contains everything but the rank 3 elements and the doubly irreducible elements of ranks 2 and 4. (The elements of C are the filled-in vertices of Figure 10.)

We see R is a rank 7 distributive lattice that is not 8-connected, and C is not simplicial.

In light of Theorem 3.7 and Priestley duality, it is reasonable to characterize the distributive lattices with simplicial elements in terms of their posets of join-irreducibles.

Theorem 4.6. *Let L be a rank d distributive lattice that is not a chain ($d \in \mathbb{N}_0$). Let P be its poset of join-irreducibles.*

The following are equivalent:

1. *L is d- but not $(d+1)$-connected;*
2. *P may be covered by two chains A and B, between which the only comparabilities are of the form $a < b$ ($a \in A$, $b \in B$);*
3. *P contains no subposet isomorphic to a three-element antichain or a four-element crown (Figure 11);*
4. *L has no homomorphic image isomorphic to $\mathbf{2}^3$ or $\mathbf{2}^2 \boxplus \mathbf{2}^2$ (Figure 12).*

Figure 11. The three-element antichain and the four-element crown.

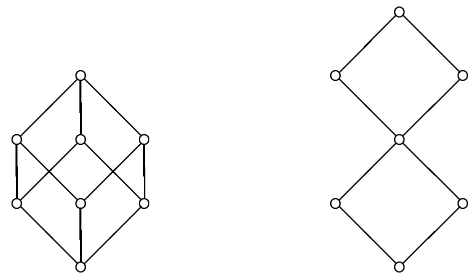

Figure 12. The lattices $\mathbf{2}^3$ and $\mathbf{2}^2 \boxplus \mathbf{2}^2$.

Proof. Let (1) hold. By Theorem 3.7, L has a simplicial element p. Since L is not a chain, $p \in P$, and $A := \downarrow p$ (in P) is a chain. As $\uparrow p$ (in L) is a chain, $B := P \setminus \downarrow p$ cannot have a two-element antichain; hence it is a chain. Thus (2) holds.

Now let (2) hold. As P can be covered by two chains, it cannot contain a three-element antichain. If it contained a four-element crown, there would be a relation of the form $a > b$ for some $a \in A$ and $b \in B$. Thus (3) holds.

Finally, let (3) hold. By Dilworth's Theorem ([4], Theorem 1.1), P may be partitioned into two chains, A and B. Without loss of generality, P has two maximal elements, a and b, and two minimal elements. Clearly, either $\downarrow a = A$ or $\downarrow b = B$, so that a or b is simplicial in L. Hence (1) holds by Theorem 3.7.

Statements (3) and (4) are equivalent by Priestley duality. $\square$

As an illustration, we deduce the next result (which also follows from Corollary 3.10).

Corollary 4.7 ([8], Theorem 3.3)**.** *Let L be a distributive lattice of rank d ($d \in \mathbb{N}_0$). Then the comparability graph of L is $(d+1)$-connected if L has a rank 3 interval that is a Boolean lattice (Figure 13).*

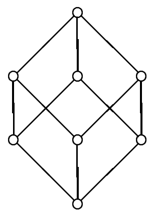

Figure 13. The rank 3 Boolean lattice.

Proof. Since L is the lattice of down-sets of a finite poset, if L has a rank 3 Boolean interval, then that poset must have a three-element antichain. The result follows from Theorem 4.6. $\square$

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