Differential equations and finite groups

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Dedicated to the memory of B. Dwork

Introduction

Let a finite group G and a representation $\rho : G \to \mathrm{GL}(n, \overline{\mathbf{Q}})$ be given. The aim of this paper is to construct a linear differential equation L (either in matrix form or in scalar form) over the differential field $k := \overline{\mathbf{Q}}(z)$, with derivation $\frac{d}{dz}$, such that the differential Galois group acting upon the solution space of L is isomorphic to the given representation ρ.
Suppose that a Galois extension $K \supset k$ with Galois group G is given. In the first section we will construct from these data the required differential equation. The method is in fact a simple application of Tannakian categories.
The problem of constructing Galois extensions of $k = \overline{\mathbf{Q}}(z)$ with a prescribed Galois group (and maybe additional ramification data) is well known to be a difficult one. It seems that no efficient general algorithm is known, although a theoretical algorithm can be given. In the second section we start from the other side and suppose that a differential equation with a finite differential Galois group is given. Then we will give a construction (algorithm) which produces the Picard-Vessiot field. In this way certain Galois extensions of k are produced.
The first two sections of this paper can be summarized as follows: Let a finite group G and a faithful irreducible G-module V over k be given. There is a constructive bijection between the Galois extensions with group G (if properly counted) and the (isomorphy classes of) differential modules over k with solution space isomorphic to the G-module V.

We give now an outline of the theory and the algorithms, presented in more detail in the later sections, for the construction of differential equations for special finite groups. Let the finite group G and an irreducible faithful representation V over $\overline{\mathbf{Q}}$ be given. Since it is rather difficult to construct (algebraically) the Galois extensions of k with group G, we will use complex analytic tools for the *existence* of suitable Galois extensions of k. In order to compare the Galois extensions of k and $\mathbf{C}(z)$, we fix a ramification locus for the extensions. For the actual computations we will only consider Galois extensions of k which are at most ramified above the points $0, 1, \infty$. To every Galois extension $K \supset k$, with group G and at most ramified above the points $0, 1, \infty$, one associates the extension $K_{\mathbf{C}} := K \otimes_{\overline{\mathbf{Q}}} \mathbf{C} \supset \mathbf{C}(z)$ which is again Galois with group G and at most ramified above the points $0, 1, \infty$. According to a theorem of Grothendieck, this yields a bijection between the two sets of Galois extensions.

For the study of the Galois extensions of $\mathbf{C}(z)$, unramified outside $0, 1, \infty$, one considers the fundamental group π_1 of $\mathbf{C} \setminus \{0, 1\}$. The Galois extensions, that we want to know, correspond to surjective homomorphism $h : \pi_1 \to G$ (counted with some care). Since π_1 is generated by three elements a_0, a_1, a_∞ with the only relation $a_0 a_1 a_\infty = 1$, one finds that h is given by an *admissible triple* (g_0, g_1, g_∞) for G, i.e., g_0, g_1, g_∞ generate G and $g_0 g_1 g_\infty = 1$. We note that our choice of the branch locus imposes that the finite group G is generated by two elements. Each (equivalence class of an) admissible triple determines a Galois extension $K \supset k$. In general, it seems impossible to calculate K explicitly! (Compare however with [M]). Instead of trying to compute K we collect all the analytic information of $K_{\mathbf{C}}$, or better of the algebraic curve C corresponding to K and its complexification $C_{\mathbf{C}}$ as Riemann surface.

In the sequel we prefer to work with scalar differential equations (or operators) L instead of matrix differential equations or differential modules. The reason is that L is determined by fewer coefficients than a matrix differential equation. The field K (determined by a given admissible triple) is the Picard-Vessiot extension of a scalar differential equation L, that we want to compute. The solution space W of L in K is a linear subspace of K over $\overline{\mathbf{Q}}$ on which G acts as the fixed representation V. Any choice of a G-invariant linear subspace $W \subset K$ (over $\overline{\mathbf{Q}}$), such that W and V are isomorphic as G-modules determines a unique monic operator $L \in k[\partial]$ which has the required properties.

It is natural to choose $W \subset K$ in the ring of functions which are regular outside the ramification points. That still leaves us with many possibilities for W. The basis idea for narrowing down the choice for W, already present in [Hu], is to consider a subspace $W \subset K$ such that Wdz is a subspace of the vector space $H^0(C, \Omega)$ of the holomorphic differentials on C. In order to find the possible W's, one has to know $H^0(C, \Omega)$ as G-module. We will derive an explicit formula for the character $\mathcal{D}$ of this G-module in terms of the admissible triple. Using this formula one observes that in many cases $H^0(C, \Omega)$ contains a G-submodule isomorphic to the given V. In other cases one has to admit differential forms with a few well specified poles.

Let $W \subset K$ with the required properties for Wdz be chosen. Local calculations at the points $0, 1, \infty$ determine the exponents, modulo integers, at those three points. The embedding of Wdz in the space of holomorphic differentials (or differentials with prescribed poles) produces lower bounds for the exponents at $0, 1, \infty$. We make the *assumption* that L has no apparent singularities. In other words, we assume that L is a scalar Fuchsian differential operator with only $0, 1, \infty$ as singular points. The Fuchs relation for exponents yields upper bounds for them. Other upper bounds on the exponents can be derived from the Weierstrass gap formula. We conclude that for a fixed admissible triple, we have a finite number of possibilities for the exponents.

Suppose that the exponents for L are chosen. If L has degree two, then L is determined by its exponents. One could test whether L has the correct differential Galois group with known methods [H-W],[U-W],[H-R-U-W]. However, in all cases that we consider it turns out that $H^0(C, \Omega)$ contains the given (irreducible faithful) representation precisely once. Thus the subspace $W \subset K$ and the corresponding scalar differential equation L of order two are uniquely determined by our approach. Moreover the lower bounds for the exponents of L add up to 1, which means that we have no choice for the exponents of L and that L cannot have apparent singularities. In other words, the theory proves that L has the correct differential Galois group.
If L has degree three, then there is one degree of freedom left in L, called the *accessory parameter*. Suppose that some exponent difference at one of points $0, 1, \infty$ happens to be an integer. The condition that there is no logarithmic term in the local solutions at that point determines a polynomial equation for the accessory parameter and thus determines L. In other cases one has to work with the accessory parameter as a variable and use that a suitable symmetric power of L has a rational solution. This produces a polynomial equation for the accessory parameter and L is determined. Finally the L that we found can be tested to have the correct differential Galois group with the

known algorithms [H-W],[U-W],[H-R-U-W]. Again in many cases this verification is superfluous, since the theory proves that L with the prescribed exponents exists and is unique.
The methods developed in this paper are not limited to differential equations of order three. However, for a Fuchsian scalar equation L of order n, with $0, 1, \infty$ as singular points and prescribed exponents, there are $n^2/2 - 3n/2 + 1$ accessory parameters to be computed.
We note that a given admissible triple does correspond with a unique differential module of the required type. This module is Fuchsian, i.e., without apparent singularities, because we have supposed that the G-module V is irreducible. In some cases no suitable scalar differential equation L is found (within the range of exponents that we consider here). We conclude that either our range of search for the exponents is not large enough or that the assumption that L has no apparent singularities is not valid. We refer to [Bo] for details on this matter.

The calculations have been carried out for certain finite subgroups of SL(2) and SL(3). The subgroups of SL(2) that we consider are the three primitive groups with notations $A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}$. Our results are complete and the relation with the well known list of Schwarz for hypergeometric differential equations is given.
For the eight finite primitive subgroups of SL(3) our computational results can be seen as an analogue of this Schwarz list. Our results are in principle complete. However, since the list is too long we present here interesting examples for each of the eight primitive subgroups of SL(3).

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1 From Galois extensions to differential equations

We will need some terminology. A *differential field* F is a field equipped with a differentiation, written as $f \mapsto f'$, satisfying the rules $(f+g)' = f'+g'$ and $(fg)' = f'g + fg'$. The field of constants of F is the set $\{f \in F |\ f' = 0\}$. The *ring of differential operators over* F is the skew polynomial ring $F[\partial]$, with its structure given by the formula $\partial f = f\partial + f'$. A differential module M over F is a left module over $F[\partial]$, which is finite dimensional as vector space over F. Equivalently, a differential module $M = (M, \partial)$ over F is a finite dimensional vector space over F, equipped with an additive map $\partial : M \to M$ satisfying $\partial(fm) = f'm + f\partial m$ for all $m \in M,\ f \in F$. Choose a basis $e_1, \ldots, e_m$ of M over F and let A denote the matrix of ∂ with respect to this basis. Then the module M corresponds with the matrix differential equation $y' + Ay = 0$. An element $c \in M$ is called *a cyclic vector* if $c, \partial c, \ldots, \partial^{m-1}c$ forms a basis of M over F. The minimal monic differential operator $L = \partial^m + a_{m-1}\partial^{m-1} + \cdots + a_1\partial + a_0 \in F[\partial]$ is a scalar differential equation (or operator) associated to M. The differential module M is called *trivial* if there is a basis $e_1, \ldots, e_m$ over F such that all $\partial e_i = 0$. One considers the category Diff_F of all differential modules over F. A morphism $f : M_1 \to M_2$ between differential modules is a linear map satisfying $f \circ \partial = \partial \circ f$. In the category Diff_F one can make all "constructions of linear algebra", i.e., kernels, images, submodules, tensor products, symmetric powers et cetera. As an example the tensor product $M_1 \otimes M_2$ is defined as the tensor product of the two linear spaces over F equipped with the map ∂ given by the formula $\partial(m_1 \otimes m_2) = (\partial m_1) \otimes m_2 + m_1 \otimes (\partial m_2)$. The category Diff_F is known to be a neutral Tannakian category, if the field of constants C of F is an algebraically closed field of characteristic 0. This means that Diff_F is isomorphic to the category Repr_H of the finite dimensional representations of a certain affine group scheme H over C. We refer for this to [De-M] and [De].

The differential field which concerns us here is $k := \overline{\mathbf{Q}}(z)$ with differentiation $f \mapsto f' := \frac{df}{dz}$. Its field of constants is $\overline{\mathbf{Q}}$. For any finite extension $K \supset k$, there is a unique extension to K of the differentiation of k. Let a Galois extension $K \supset k$ with Galois group G be given. We consider a category $\mathrm{Diff}_{K/k}$, which is the full subcategory of Diff_k whose objects are the differential modules M over k such that the differential module $K \otimes M$ over K is trivial. We note that ∂ on $K \otimes M$ is given by the formula $\partial(a \otimes m) = a' \otimes m + a \otimes (\partial m)$ for all $a \in K$ and $m \in M$. The full subcategory $\mathrm{Diff}_{K/k}$ of Diff_k is closed under all "constructions of linear algebra". This implies that $\mathrm{Diff}_{K/k}$ is also a neutral Tannakian category and is isomorphic to a category of finite dimensional representations over $\overline{\mathbf{Q}}$ of a certain group scheme. In this case the group scheme is just the finite group G and Repr_G is the category of the finite dimensional of G over $\overline{\mathbf{Q}}$. For the convenience of the reader and for the sake of the explicit constructions of differential equations we will give an explicit proof.

Theorem 1.1 *The tensor category* $\mathrm{Diff}_{K/k}$ *is equivalent to the category* Repr_G.

Proof. We define first a functor $\alpha : \mathrm{Diff}_{K/k} \to \mathrm{Repr}_G$. For an object $M \in \mathrm{Diff}_{K/k}$ we consider $K \otimes M$. By assumption $K \otimes M$ is trivial. This implies that the $\overline{\mathbf{Q}}$-vector space $V := ker(\partial, K \otimes M)$ has the property that the canonical map $K \otimes_{\overline{\mathbf{Q}}} V \to K \otimes M$ is an isomorphism. The action of G on K induces an action of G on $K \otimes M$ given by the formula $g(f \otimes m) = g(f) \otimes m$. This action commutes with ∂ on $K \otimes M$. Thus V is invariant under G and we find a G-action on V. In other words V is a G-module and belongs to Repr_G. One defines $\alpha(M) = V$. It is clear how to define α for morphisms and it is easy to verify the following assertions:

1. The functor α commutes with tensor products.
2. The $\overline{\mathbf{Q}}$-linear map $\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(\alpha(M_1), \alpha(M_2))$ is an isomorphism.

For the proof of the equivalence we will also need a functor $\beta : \mathrm{Repr}_G \to \mathrm{Diff}_{K/k}$. From the construction of β that we will give, it is easily deduced that $\alpha \circ \beta$ and $\beta \circ \alpha$ are canonical isomorphic to the identity on the categories Repr_G and $\mathrm{Diff}_{K/k}$.
Let $V \in \mathrm{Repr}_G$ of dimension m be given. The K-vector space $N := K \otimes_{\overline{\mathbf{Q}}} V$ is given a G-action by $g(f \otimes v) = g(f) \otimes g(v)$ and a derivation ∂ by the formula $\partial(f \otimes v) = f' \otimes v$. The action of G commutes with ∂. Write $M := N^G$, i.e., the set of the G-invariant elements of N. Then M is clearly a finite dimensional vector space over k. Since ∂ commutes with the G-action on N, we have that $\partial(M) \subset M$. Thus M is a differential module over k. We claim that there

exists a basis $e_1, \ldots, e_m$ of N over K consisting of G-invariant elements. This follows from general theorems (e.g. $H^1(G, \mathrm{GL}(m, K))$ is trivial, [S]), but we will give here a constructive proof. Take a normal basis $a_1, \ldots, a_n$ of K over k, i.e., an element $a \in K$ such that $\{g(a)|\ g \in G\} = \{a_1, \ldots, a_n\}$ is a basis. Then $R := \overline{\mathbf{Q}}a_1 \oplus \cdots \oplus \overline{\mathbf{Q}}a_n$ is the regular representation of G over $\overline{\mathbf{Q}}$. The set of the G-invariants of $R \otimes V$ is a vector space over $\overline{\mathbf{Q}}$ of dimension m. This follows from the fact that every irreducible representation of G is present in R as often as the dimension of this representation. Let $e_1, \ldots, e_m$ be a basis of $(R \otimes V)^G$, then it is easily seen that $e_1, \ldots, e_m$ is also a basis of N over K consisting of G-invariant elements. The elements $e_1, \ldots, e_m$ can be calculated as the generators of the kernel of the operator $-id + \frac{1}{|G|}\sum_{g\in G} g$ on $R \otimes V$. The formula for ∂ on N produces then a matrix differential equation with the required properties. After choosing a cyclic element, one also finds a scalar differential equation over k with the required properties. $\square$

Remarks

(1) The above theorem is valid in the general situation of a Picard-Vessiot extension K/k of differential fields, with algebraically closed field of constants C. The group G is now the differential Galois group, which is a linear algebraic group over C. Further Repr_G is the category of the algebraic representations of G on finite dimensional vector spaces over C. With the same definition of $\mathrm{Diff}_{K/k}$ one has an equivalence of the tensor categories $\mathrm{Diff}_{K/k}$ and Repr_G.

(2) Concerning the proof of theorem 1.1. One does not need to construct a normal basis of K over k. In fact the operator $-id + \frac{1}{|G|}\sum_{g\in G} g$ acts as a k-linear operator on $N := K \otimes V$. With linear algebra over k one can also calculate a basis for its kernel. We note further that it suffices to make the calculations for the irreducible representations of G.

Examples

(1) Consider the polynomial $P := T^4 - (4T - 3)\frac{1}{1+3z^2}$ over $\mathbf{Q}(z)$. Its splitting field $K \supset \mathbf{Q}(z)$ has Galois group A_4. Consider the 3-dimensional vector space $V = \{\sum_{i=1}^4 \lambda_i e_i|\ \sum \lambda_i = 0\}$. An element $\sigma \in A_4$ acts on V by $\sigma(\sum \lambda_i e_i) = \sum \lambda_i e_{\sigma(i)}$. Let $\alpha_i, i = 1, 2, 3, 4$ denote the roots of P in K. The element $f := \sum_{i=1}^4 \alpha_i \otimes e_i \in K \otimes V$ is clearly invariant under A_4. Moreover $\partial^j f = \sum \alpha_i^{(j)} \otimes e_i$ for $j = 0, 1, 2, 3$. These elements are linearly dependent over the field $\mathbf{Q}(z)$. The resulting equation $(\partial^3 + a_2\partial^2 + a_1\partial + a_0)f = 0$ (with all $a_i \in \mathbf{Q}(z)$) is the

order three differential equation over $\mathbf{Q}(z)$ associated to the Galois extension $K \supset \mathbf{Q}(z)$ and the representation V of A_4. The equation is

$$L(y) = y''' + \frac{18z}{1+3z^2}\, y'' + \frac{115+729z^2}{12(1+3z^2)^2}\, y' + \frac{27z}{4(1+3z^2)^2}\, y$$

(2) A_5 is the Galois group of $K/\overline{\mathbf{Q}}(z)$, where K is the splitting field of the polynomial $P := T^5 - (5T-4)\frac{5z^2-1}{z^2}$. The group A_5 has two irreducible representations of degree 3. They are permuted by the automorphism of $\mathbf{Q}(\sqrt{5})/\mathbf{Q}$, which sends $\sqrt{5}$ to $-\sqrt{5}$. We haven choosen here for a "contravariant" method to obtain the differential equation L for the given data. This means that we produce a three dimensional subspace $V \subset K$ where A_5 acts as the given representation and compute then the corresponding differential operator L over $\overline{\mathbf{Q}}(z)$ of order three with solution space V. Let $f_1, \ldots, f_5$ denote the five roots of P in K and let $W \subset K$ be the four dimensional vector space over $\overline{\mathbf{Q}}$ generated by the f_i. The third symmetric power of W contains a unique subspace V of dimension three and with the required A_5-action. A basis e_1, e_2, e_3 for V in terms of homogeneous polynomials of order three in the f_i is easily found. The derivatives of the e_i are obtained from the equation P in an obvious way. This determines the operator $L = \partial^3 + A_2\partial^2 + A_1\partial + A_0$ satisfying $L(e_i) = 0$ for $i = 1, 2, 3$. The coefficients A_1, A_2, A_3 are found as rational expressions in the f_i. However the A_i are in $\overline{\mathbf{Q}}(z)$ with known denominators and bounds on the degrees of the numerators. An evaluation at the point $z = 0$ (i.e., using Puiseux series in z) determines the A_i. One finds the equation

$$L(y) = y''' + \frac{6}{z}\, y'' + \frac{3}{100}\frac{5000z^4 - 1525z^2 - 8\sqrt{5}z + 225}{z^2(5z^2-1)^2}\, y' +$$

$$\frac{3}{100}\frac{20\sqrt{5}z^3 - 1075z^2 + 12\sqrt{5}z - 25}{z^3(5z^2-1)^3}\, y$$

2 From differential equations to Galois extensions

Let a scalar differential operator L over $k := \overline{\mathbf{Q}}(z)$ be given with a (known) finite differential Galois group G. According to [C-S] there is an algorithm for the Picard-Vessiot field K of L, since G is in particular also a reductive group. For a finite group G acting upon a solution space V, the method is essentially the following. One calculates a basis of the algebra $\overline{\mathbf{Q}}[V]^G$ of G-invariant elements in the symmetric algebra $\overline{\mathbf{Q}}[V]$. At a suitable point, say $z=0$, one makes an explicit calculation for the solution space $\tilde{V}$ inside (a finite extension of) the differential field $\overline{\mathbf{Q}}((z))$. The (as yet unknown) map $V \to \tilde{V}$ induces a map of $\overline{\mathbf{Q}}[V]^G$ to the subfield $\overline{\mathbf{Q}}(z)$. This, with additional data on denominators and numerators of the images of the G-invariant elements, is going to determine the map $k[V] \to \overline{\mathbf{Q}}((z))$. The image of this map is the Picard-Vessiot field of the equation.

However, in our situation where L is constructed from analytic data, the algorithm for the Picard-Vessiot field can be simplified. We will sketch an efficient method for an order three Fuchsian equation L, determined by an admissible triple (g_0, g_1, g_∞) for G and a faithful irreducible representation $G \subset \mathrm{GL}(V)$ of dimension 3 (over $\overline{\mathbf{Q}}$). In sections 3,4 and 5 the Galois coverings of $\mathbf{P}^1$, at most ramified above $0, 1, \infty$, and corresponding differential equations will be given in more detail.

We assume that at least one of the three elements g_0, g_1, g_∞ has three distinct eigenvalues (this will be the case for all our examples). For notational convenience we suppose that g_0 has three distinct eigenvalues and corresponding eigenvectors $v_1, v_2, v_3 \in V$. Let $k[V]$ (and $\overline{\mathbf{Q}}[V]$) denote the symmetric algebra of V over k (and over $\overline{\mathbf{Q}}$). Let e_0 be the order of g_0. It can be seen that g_0 is (conjugated to) the local monodromy for L at 0 (see proposition (5.1)) and thus the solution space V_0 of $L(y)=0$ in $\overline{\mathbf{Q}}((z^{1/e_0}))$ has dimension 3. Let $\lambda_1 < \lambda_2 < \lambda_3$ denote the three exponents at 0 and let $S_1, S_2, S_3 \in V_0$ be defined by $S_i = z^{\lambda_i}(1 + *z + *z^2 + \cdots)$. Then S_1, S_2, S_3 are three eigenvectors for the action of g_0 on V_0. The Picard-Vessiot field K of L will be identified with the subfield of $\overline{\mathbf{Q}}((z^{1/e_0}))$ generated over $\overline{\mathbf{Q}}(z)$ by S_1, S_2, S_3. The G-equivariant isomorphism $\psi : V \to V_0$ is determined up to a constant, since the representation is irreducible. The map ψ is also equivariant for the actions of g_0 on V and V_0. Thus (for a suitable numbering) one has that $\psi(v_i) = c_i S_i$ for certain elements $c_1, c_2, c_3 \in \overline{\mathbf{Q}}^*$. If one wants to fix ψ one can normalize the c_1, c_2, c_3 by imposing $c_3 = 1$. For the actual calculation of the c_1, c_2, c_3 we have to weaken the assumption that ψ is G-equivariant as follows. The map ψ is supposed to satisfy:

(a) $\psi v_i = c_i S_i$ for $i = 1, 2, 3$ and $c_i \in \overline{\mathbf{Q}}^*$.

(b) Let ψ satisfy condition (a). Then ψ extends in a unique way to a k-algebra homomorphism $k[V] \to \overline{\mathbf{Q}}((z^{1/e_0}))$, which we will also call ψ. Then we require that ψ maps $\overline{\mathbf{Q}}[V]^G$ to the subfield $\overline{\mathbf{Q}}(z)$ of $\overline{\mathbf{Q}}((z^{1/e_0}))$.
Let the G-equivariant map be given by the triple $(d_1, d_2, 1)$. Then can easily verify that any (c_1, c_2, c_3) of the form $(dd_1(e^{2\pi i\lambda_1})^n, dd_2(e^{2\pi i\lambda_2})^n, d(e^{2\pi i\lambda_3})^n)$, with $d \neq 0$ and $n = 0, \ldots, e_0 - 1$, satisfies condition (b). In general we expect that those are all the possibilities for (c_1, c_2, c_3). In any case, the condition that G is an irreducible subgroup of $\mathrm{GL}(V)$ will imply that modulo scalars, the set of solutions for (c_1, c_2, c_3) is finite.

Suppose that c_1, c_2, c_3 satisfying (a) and (b) above are known (and chosen). Then there are various ways to produce the field K. One way would be to calculate the kernel $\underline{m}$ of the surjective homomorphism $\psi : k[V] \to K \subset \overline{\mathbf{Q}}((z^{1/e_0}))$. For a basis of "invariants" F, i.e., elements of $\overline{\mathbf{Q}}[V]^G$ generating this algebra, one calculates the image $\psi(F)$ as rational function lying in k. The ideal $I \subset k[V]$ generated by the $F - \psi(F)$ has finite codimension in $k[V]$ and is contained in $\underline{m}$. With Gröbner theory, one can produce the maximal ideals containing I and choose the correct maximal ideal above I. Another way of presenting K would be to give an equation for an element, generating K over k. For this purpose one takes a $v \in V$ which has an G-orbit of length $\#G$. The image $\psi(v)$ generates K over k. The equation of $\psi(v)$ over k can be found by calculating the the elementary symmetric functions of $\{\psi(gv) | g \in G\}$, which are known as elements of $\overline{\mathbf{Q}}((z^{1/e_0}))$, as elements of k. Finally, one can take an element $v \in V$ such that its stabilizer H is maximal among the subgroups of G which do not contain a proper normal subgroup. As above one can again calculate the minimal equation of $\psi(v)$ over k. The field K is the splitting field of this polynomial equation.

Now we give an algorithm for finding the c_1, c_2, c_3 which satisfy (a) and (b) above. Consider a homogeneous element F of degree d in the ring of invariants $\overline{\mathbf{Q}}[V]^G$. Write $F = \sum_{|\alpha|=d} f_\alpha v^\alpha$, where we have used multi-index notation: $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \sum \alpha_i$ and $v^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3}$. The coefficients f_α are in $\overline{\mathbf{Q}}$. Since F is invariant under g_0, one finds that $f_\alpha \neq 0$ implies that $\sum \alpha_i \lambda_i \in \mathbf{Z}$. The condition that $\psi(F) \in k$ will be made more explicit by calculating the possible orders of $\psi(F)$ at the points $0, 1, \infty$. For the point 0 this goes as follows. The image $\psi(F) = \sum_\alpha c^\alpha f_\alpha S^\alpha$ (again with multi-index notation) is a sum of terms S^α with $\sum \alpha_i \lambda_i \in \mathbf{Z}$ and $\sum \alpha_i = d$. Thus

$ord_0(\psi(F))$ is greater or equal to the minimum of the numbers $\sum \alpha_i\lambda_i$, with the restrictions $\sum \alpha_i\lambda_i \in \mathbf{Z}$, all $\alpha_i \in \mathbf{Z}_{\geq 0}$ and $\sum \alpha_i = d$. The lower bounds for $ord_1(\psi(F))$ and $ord_\infty(\psi(F))$ are similar. We note that $\psi(F)$ can only have poles at $0, 1, \infty$. Write n_0, n_1, n_∞ for the lower bounds thus obtained. Then $H := z^{-n_0}(z-1)^{-n_1}\psi(F)$ has order ≥ 0 at 0 and 1 and its order at ∞ is $\geq n_\infty - n_0 - n_1$. If $M := -n_\infty + n + 0 + n_1$ is < 0 then $\psi(F) = 0$. For $M \geq 0$ one has that H is a polynomial of degree $\leq M$. The expression H is known as element of $\overline{\mathbf{Q}}[c_1, c_2, c_3][[z]]$. The vanishing of the coefficients of z^n for $n > M$ produces an infinite set of homogeneous equations on the c_i, all of degree d.

We are left with the problem of determining how many equations are needed. More precisely, we want to determine an integer N such that for any $c_1, c_2, c_3 \in \overline{\mathbf{Q}}^*$ the vanishing of the coefficients of z^n for $n = M+1, \ldots, N$ implies the vanishing of all coefficients. Consider values for c_1, c_2, c_3 and a polynomial P in z of degree $\leq M$, such that the order of $f := H - P$, as element of $\overline{\mathbf{Q}}[[z]]$ is $> N$. The element $f \in K$ is seen as rational function on the curve C corresponding to K. There are $\frac{|G|}{e_0}$ points on C above 0. For one of them we have that the order of f is $> e_0 N$. For the others, we have that the order is $\geq e_0(-n_0 + d \min E_0)$. Here E_a, for $a = 0, 1, \infty$ denotes the set of the exponents at the points $0, 1, \infty$. For the $\frac{|G|}{e_1}$ points of C above 1 there is one point with order of f is ≥ 0. For the other points one has that the order of f is $\geq e_1(-n_1 + d \min E_1)$. Finally for one point above ∞ one has that the order is $\geq e_\infty(n_0 + n_1 + n_\infty)$. For the other $\frac{|G|}{e_\infty} - 1$ points one has that the order of f is $\geq e_\infty(n_0 + n_1 + d \min E_\infty)$. The rational function f has no other poles. If N is chosen such that the total sum of the orders is positive, then N has the required property. A calculation yields the following estimate:

$$Ne_0 \geq -e_0 n_0 - e_1 n_1 - e_2 n_2 + d(e_0 \min E_0 + e_1 \min E_1 + e_\infty \min E_\infty)$$
$$-d|G|(\min E_0 + \min E_1 + \min E_\infty)$$

Since $d \min E_a \leq n_a$ for $a = 0, 1, \infty$, one could also work with the weaker inequality

$$N \geq \frac{d|G|}{e_0}(-\min E_0 - \min E_1 - \min E_\infty).$$

Some variations

(1) Suppose that L is given and that we only know that the differential Galois group H is a subgroup of G. Then the above algorithm works equally

well. One can determine values for c_1, c_2, c_3. A priori, $< g_0 > \subset H \subset G$. If one wants to exclude the possibility $H \neq G$, then one has to compute an H-invariant element $F \in \overline{\mathbf{Q}}[V]$ which is not G-invariant and show that $\psi(F) \notin k$. It suffices to do this for conjugacy classes of maximal subgroups H with the property $H \neq G$ and $< g_0 > \subset H$.
(2) Suppose that L contains an unkown accessory parameter μ and let the finite group $G \subset \mathrm{GL}(V)$ be given. The above algorithm still works in the sense that the S_1, S_2, S_3 have now coefficients involving μ. We will see in (5.1) that L has the form $L_0 + \frac{\mu}{z^2(z-1)^2}$ where L_0 does not depend on μ. From this it is easily deduced that $S_i = z^{\lambda_i}(1 + \sum_{n>0} a_{i,n} z^n)$ where the $a_{i,n}$ are polynomials in μ of degree n. The assumption that the differential Galois group of L is contained in the given finite group G produces a set of polynomial equations involving c_1, c_2, μ. Those equations will determine values for c_1, c_2, c_3 and μ.

3 Galois coverings of $\mathbf{P}^*$

Put $\mathbf{P}^* = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ as variety over $\overline{\mathbf{Q}}$. We want to construct Galois coverings $C \to \mathbf{P}^1$, which are only ramified above $\{0, 1, \infty\}$ and have Galois group G. We start by working over the field of complex numbers $\mathbf{C}$. An embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$ is fixed.
The fundamental group π_1 of $\mathbf{P}^*_{\mathbf{C}}$ with base point, say, 1/2 is generated by three elements a_0, a_1, a_∞ and has the only relation $a_0 a_1 a_\infty = 1$. The a_0, a_1 are represented by circles through 1/2 around 0 and 1 with suitable orientations. The third element a_∞ is represented by a suitable loop through 1/2 around ∞. Complex conjugation acts as the automorphism on π_1, which sends a_0, a_1, a_∞ to $a_0^{-1}, a_1^{-1}, a_1 a_0$. On the universal covering $U \to \mathbf{P}^*_{\mathbf{C}}$ the fundamental group π_1 acts as the group of automorphisms. A Galois covering $D \to \mathbf{P}^1 \otimes \mathbf{C}$ with group G and unramified above $\mathbf{P}^* \otimes \mathbf{C}$ corresponds to a surjective homomorphism $h : \pi_1 \to G$. The map h is given by a triple $(h(a_0), h(a_1), h(a_\infty)) = (g_0, g_1, g_\infty)$ of elements in G such that G is generated by g_0, g_1, g_∞ and $g_0 g_1 g_\infty = 1$. We will call such a triple an *admissible triple*. We note that we have to restrict ourselves in the sequel to finite groups G generated by two elements.
Two triples $(g_0, g_1, g_\infty), (\tilde{g}_0, \tilde{g}_1, \tilde{g}_\infty)$ yield isomorphic coverings of $\mathbf{P}^1 \otimes \mathbf{C}$ if and only if the kernels of the corresponding (surjective) homomorphisms $h, \tilde{h}$:

$\pi_1 \to G$ are the same. We will call two admissible triples $(g_0, g_1, g_\infty), (\tilde{g}_0, \tilde{g}_1, \tilde{g}_\infty)$ equivalent if there is an automorphism A of the group G such that $(Ag_0, Ag_1, Ag_\infty) = (\tilde{g}_0, \tilde{g}_1, \tilde{g}_\infty)$. From the above it follows that the set of the (isomorphism classes of) Galois coverings of $\mathbf{P}^1_{\mathbf{C}}$ with group G and unramified outside $\{0, 1, \infty\}$ is in a 1-1 correspondence with the set of equivalence classes of the admissible triples for G.
Every covering $C_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ (with group G and unramified outside $\{0, 1, \infty\}$) has a unique model C over $\overline{\mathbf{Q}}$. The Galois group of $\overline{\mathbf{Q}}$ acts in a natural way on the finite étale coverings of $\mathbf{P}^*$ and thus on the set of the equivalence classes of admissible triples. Let $[e_0, e_1, e_\infty]$ denote the set of equivalence classes of admissible triples (g_0, g_1, g_∞) for G such that the order of g_p is e_p for $p = 0, 1, \infty$. We will call the set $[e_0, e_1, e_\infty]$ a *branch type*. The set $[e_0, e_1, e_\infty]$ is invariant under the action of the Galois group of $\overline{\mathbf{Q}}$. The Galois action is however "unknown". The only non trivial element τ of this Galois group for which we know the action is the automorphism induced by the complex conjugation and the given embedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}$. The action of τ on admissible triples is given by $(g_0, g_1, g_\infty) \mapsto (g_0^{-1}, g_1^{-1}, g_1 g_0)$.

We note that the fundamental problem for the "construction" of Galois coverings of $\mathbf{P}^1$, like the ones above, is that the analytic construction does not lead to a determination of the algebraic curve $C \to \mathbf{P}^1$. Nevertheless we will try to calculate as much as possible about C, corresponding to an admissible triple (g_0, g_1, g_∞), as we can.

Lemma 3.1 *Let $C \to \mathbf{P}^1$ be the Galois covering corresponding to (the equivalence class of) the admissible triple (g_0, g_1, g_∞) for the group G.*
(1) *The ramification indices e_0, e_1, e_∞ of the points $\{0, 1, \infty\}$ are the orders of the elements g_0, g_1, g_∞.*
(2) *The genus $g(C)$ of the curve C is given by the formula*

$$g(C) = 1 + \frac{1}{2}|G|(1 - \frac{1}{e_0} - \frac{1}{e_1} - \frac{1}{e_\infty}).$$

We will omit the obvious proof.

3.1 Some examples of coverings of $\mathbf{P}^*$

The points $0, 1, \infty$ on $\mathbf{P}^1$ can be permuted by an automorphism of $\mathbf{P}^1$. We may restrict ourselves therefore to admissible triples with $e_0 \leq e_1 \leq e_\infty$. Moreover, in the case of two or three equal values among the e_0, e_1, e_∞, some permutations of the points $0, 1, \infty$ act on the branch type $[e_0, e_1, e_\infty]$ and we will only write down the equivalence classes of admissible triples up to those permutations. We consider in this subsection some special cases. The data for the branch types of the three finite primitive subgroups of SL(2) and the eight finite primitive subgroups of SL(3) wil be given (in some detail) in sections 7 and 8.

3.1.1 *The n-cyclic coverings of* $\mathbf{P}^1$*, unramified above* $\mathbf{P}^*$.
They are well known to be given by the affine equation $y^n = z^a(z-1)^b$ with $0 \leq a, b$ and g.c.d.$(a, b, n) = 1$. Those coverings are of minor interest here. We note that the equations $y^6 = z^3(z-1)$, $y^4 = z^2(z-1)$ and $y^3 = z(z-1)$ have branch type $[2, 3, 6]$, $[2, 4, 4]$ and $[3, 3, 3]$ and the genus of the corresponding curve is 1.

3.1.2. *The Galois coverings of* $C \to \mathbf{P}^1$*, unramified above* $\mathbf{P}^*$ *and with genus* $g(C) = 0$.
The condition $g(C) = 0$ is equivalent to $\frac{1}{e_0} + \frac{1}{e_1} + \frac{1}{e_\infty} > 1$. The possibilities for the e_0, e_1, e_∞ are:

1. $2, 2, n$ with $n \geq 2$. The group G is the dihedral group $D_n \subset \mathrm{PSL}(2)$.

2. $2, 3, 3$ and G is $A_4 \subset \mathrm{PSL}(2)$.

3. $2, 3, 4$ and G is $S_4 \subset \mathrm{PSL}(2)$.

4. $2, 3, 5$ and G is $A_5 \subset \mathrm{PSL}(2)$.

For each case, the set of equivalence classes $[e_0, e_1, e_\infty]$ consists of one element (Compare [B-D]). The inverse image in SL(2) of each of those groups will be denoted by $D_n^{\mathrm{SL}(2)}, A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}$. The differential equations, to be computed, for irreducible representations of D_n, A_4, S_4, A_5 are easily deduced from differential equations for the groups $D_n^{\mathrm{SL}(2)}, A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}$. We will give one example:

Consider a 3-dimensional irreducible representation V of A_5 and the corresponding order three differential operator L. There exists a covering $C \to \mathbf{P}^1$ (in fact more than one according to the list of section 7), unramified outside $0, 1, \infty$ and with group $A_5^{\mathrm{SL}(2)}$, such that C/Z has genus 0. Here Z denotes the center of $A_5^{\mathrm{SL}(2)}$. For $A_5^{\mathrm{SL}(2)}$ one considers an irreducible representation W of dimension two, such that $V \cong sym^2 W$. Let M denote the order two differential equation corresponding to the covering $C \to \mathbf{P}^1$ and the representation W. Then the order three scalar differential equation $sym^2 M$ must be equivalent to L above. This is a consequence of the Tannakian approach, i.e., theorem 1.1.

3.1.3. *The Galois coverings* $C \to \mathbf{P}^1$, *unramified outside* $0, 1, \infty$ *and with genus* $g(C) = 1$.
The condition $g(C) = 1$ is equivalent to $\frac{1}{e_0} + \frac{1}{e_1} + \frac{1}{e_\infty} = 1$. The possibilities for the branch type are: $[2, 3, 6]$, $[2, 4, 4]$ and $[3, 3, 3]$. Choose a point e on C as the neutral element for the group structure on C. The automorphism group $\mathrm{Aut}(C)$ has C as normal subgroup. The quotient $\mathrm{Aut}(C)/C$ can be identified with $\mathrm{Aut}(C, e)$, the group of the automorphisms of C which fix e. The latter group has order 6,4 or 2. The finite group $G \subset \mathrm{Aut}(C)$ that we are looking for must have the property that $C \to C/G \cong \mathbf{P}^1$ is ramified above three points. A small calculation shows that the image of G in $\mathrm{Aut}(C, e)$ can only have order 6,4 or 3. Further $F := G \cap C$ must be a normal subgroup of G. In other words $C \to C/G \cong \mathbf{P}^1$ factors as $C \to C' = C/F \to \mathbf{P}^1$. The first map is unramified and C' is again an elliptic curve. The second map $C' \to \mathbf{P}^1$ is cyclic of order 6,4 or 3 and is given by the above equations $y^6 = z^3(z-1)$, $y^4 = z^2(z-1)$ or $y^3 = z(z-1)$.

4 Galois action on differential forms

4.1 Riemann-Hurwitz-Zeuthen with group action

We consider the general situation: $\pi : X \to Y$ a (ramified) Galois covering between two curves with Galois group G over an algebraically closed field F of characteristic 0. The usual Riemann-Hurwitz-Zeuthen formula gives the dimension of $H^0(X, \Omega_X)$, the space of the holomorphic differential forms on

X, in terms of the dimension of $H^0(Y, \Omega_Y)$ and the ramification data. The action of G on X is supposed to be a right action. The induced left action of G on $H^0(X, \Omega_X)$ makes this space into a G-module. The structure of this G-module is what we want to know. Let $\mathcal{D}$ denote the character of $H^0(X, \Omega_X)$, then we want to obtain a formula for $\mathcal{D}$ in terms of the ramification data. For the formulation of the first easy result we have to introduce some *notations*:

Let H denote a finite group. For any character χ of H we denote by χ^* the dual character, i.e., the character of the dual representation. The trivial character and the regular character of H are denoted by $triv_H$ and r_H. For two finite groups $H \subset G$ and a character χ of H we write $Ind_H^G \chi$ for the induction of the character χ of H to a character of G. The scalar product for function on the group H is denoted by $< \, , \, >_H$ or $< \, , \, >$.
Let Y_{ram} denote the set of the ramified points of Y. For every $y \in Y_{ram}$ we choose a point $x \in X$ above y and define H_y as the decomposition group x, i.e., the stabilizer of x in the group G. The conjugacy class of the cyclic group H_y does not depend on the choice of x. Finally the genus of Y is denoted by g_Y.

Proposition 4.1 *The sum $\mathcal{D} + \mathcal{D}^*$ is equal to*

$$2 \cdot triv_G + (2g_Y - 2 + \#Y_{ram}) \cdot r_G - \sum_{y \in Y_{ram}} Ind_{H_y}^G triv_{H_y}$$

Proof. We will work in the abelian group $K_0(G) = \mathbf{Z}\chi_1 + \mathbf{Z}\chi_2 + \cdots + \mathbf{Z}\chi_h$, where $\chi_1, \ldots, \chi_h$ are all irreducible characters of G. A finite dimensional left G-module V over F will be identified with its character in $K_0(G)$.
The exact sequence of sheaves

$$0 \to \pi^* \Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

induces the equality in $K_0(G)$

$$H^0(\Omega_{X/k}) - H^1(\Omega_{X/k}) = H^0(\pi^* \Omega_{Y/k}) - H^1(\pi^* \Omega_{Y/k}) + H^0(\Omega_{X/Y}).$$

The sheaf $\Omega_{Y/k}$ can be identified with $O_Y(D)$ where D is a canonical divisor on Y of degree $2g_Y - 2$. Thus $\pi^* \Omega_{Y/k}$ can be identified with $O_X(\pi^* D)$.
For $g_Y = 0$ one has an exact sequence

$$0 \to O_X(\pi^* D) \to O_X \to \mathcal{Q} \to 0,$$

for $g_Y = 1$ one has $D = 0$ and for $g_Y > 1$ there is an exact sequence

$$0 \to O_X \to O_X(\pi^* D) \to \mathcal{Q} \to 0.$$

The sheaf $\mathcal{Q}$ is a skyscraper sheaf and $H^0(\mathcal{Q})$ is a multiple of r_G. The Serre duality for the cohomology groups of sheaves on X is G-equivariant. For a G-module W we will denote the dual module by W^*. It follows that $H^0(\pi^*\Omega_{Y/k}) - H^1(\pi^*\Omega_{Y/k})$ is equal to

$$(-H^0(\Omega_{X/k})^* + H^1(\Omega_{X/k})^*) + (2g_Y - 2) \cdot r_G.$$

Consider now a ramified point $x \in X$ above a ramified point $y \in Y$ with ramification index $e = e_y$, inertia group $H_y =< h_x >$ and local parameter t such that $h_x(t) = \zeta t$ with ζ a primitive eth root of unity. Then $\Omega_{X/Y,x}$ has basis $dt, tdt, \ldots, t^{e-2}dt$ with the obvious h_x action. Thus $\Omega_{X/Y,x}$ as a H_y-module is equal to $r_{H_y} - triv_{H_y}$. Moreover $\sum_{z,\, \pi(z)=y} \Omega_{X/Y,z}$ as G-module is $Ind_{H_y}^G \Omega_{X/Y,x}$. We conclude that $\sum_{z,\, \pi(z)=y} \Omega_{X/Y,z} = r_G - Ind_{H_y}^G triv_{H_y}$. Combining the above information one finds the formula of the proposition. □

We note that the proposition does not give a formula for $\mathcal{D}$ itself. Sometimes the more complicated "classical" formula for $\mathcal{D}$ is needed. This will be given here. For any $y \in Y_{ram}$ we have chosen a point x above y. Let t_x be a local parameter at x. For any $h \in H_y$ (the stabilizer of x) one defines a constant $\lambda_y(h) \in F^*$ by $h^*(t_x) = \lambda_y(h) t_x + * t_x^2 + \cdots$ (in the local ring of x). The map $h \mapsto \lambda_y(h) \in F^*$ is a character of H_y. The order of H_y is denoted by e_y and is equal to the ramification index of y (i.e., the ramification index of any point above y).

Theorem 4.2 *The character $\mathcal{D}$ of $H^0(X, \Omega_X)$ is equal to*

$$triv_G + (g_Y - 1) \cdot r_G + \sum_{y \in Y_{ram}} \frac{1}{e_y} \sum_{d=1}^{e_y - 1} d \cdot Ind_{H_y}^G \lambda_y^{-d}.$$

The literature [C-W],[K],[N] gives this formula with λ_y^d instead of λ_y^{-d}. This difference is due to our choice of the right action of G on X. It is an exercise to show that this theorem implies the previous proposition. In the appendix we will give a modern proof of theorem 4.2, based on notes by Bas Edixhoven.

4.2 Application to coverings of $\mathbf{P}^*$

We want to specialize the proposition and the theorem for the case of a Galois covering $f : C \to \mathbf{P}^1$, with group G, unramified outside $0, 1, \infty$ and given by an admissible triple (g_0, g_1, g_∞). It is clear from the analytic situation that for a suitable choice of points in C above $0, 1, \infty$ one has $H_j =< g_j >$ for $j \in \{0, 1, \infty\}$. This information suffices for the calculation of $\mathcal{D} + \mathcal{D}^*$, according to the proposition. In order to apply the theorem we have to compute the characters λ_j for $j \in \{0, 1, \infty\}$. This is done in the next lemma.

Lemma 4.3 *For $j \in \{0, 1, \infty\}$, the character λ_j is given by $\lambda_j(g_j) = e^{2\pi i/e_j}$.*

Proof. The admissible triple (g_0, g_1, g_∞) is given and from this a (topological or complex analytic) covering $f : C^*_{\mathbf{C}} \to \mathbf{P}^*_{\mathbf{C}}$ is uniquely determined. The action of the fundamental group of $\mathbf{P}^*_{\mathbf{C}}$ on $C^*_{\mathbf{C}} = C_{\mathbf{C}} \setminus$ the ramification points is determined (up to conjugation). It can be described as follows. Fix an element $c_0 \in f^{-1}(1/2)$. For every element $c \in f^{-1}(1/2)$ there is a unique element $g \in G$ with $c_0 g = c$. Take a closed path λ through $1/2$. Then λ lifts uniquely to a path $\lambda^* : [0, 1] \to C^*_{\mathbf{C}}$ with $\lambda^*(0) = c_0$. The end point $\lambda^*(1)$ lies again in $f^{-1}(1/2)$ and can be written as $c_0 g$ with $g \in G$. This defines the homomorphism of the fundamental group to G corrresponding to the admissible triple.
The preimage $f^{-1}(\{z \in \mathbf{C} |\ |z| < 1\}$ is known to be the disjoint union $V_1, \ldots, V_m$ with $m = \frac{|G|}{e_0}$. Each V_j is isomorphic to an open disc and the restriction of f to $V_j \to \{z \in \mathbf{C} |\ |z| < 1\}$ has, for a suitable identification of V_j with $\{z \in \mathbf{C} |\ |z| < 1\}$, the form $z \mapsto z^{e_0}$. Suppose that $c_0 \in V_1$. The action of g_0 is obtained by lifting the loop a_0 around 0. It is clear that $g_0(c_0) \in V_1$. Thus the "center" of V_1 is a fixed point for g_0. Moreover, the action of g_0 on V_1 has, in terms of the suitable coordinate of V_1, the form $z \mapsto e^{2\pi i/e_0} z$. This shows that $\lambda_0(g_0) = e^{2\pi i/e_0}$. The formulas for λ_1 and λ_∞ are verified in a similar way. □

Corollary 4.4 *Let the Galois covering $f : C \to \mathbf{P}^1$ (with Galois group G and unramified outside $0, 1, \infty$) be given by the admissibe triple (g_0, g_1, g_∞). Then:*
(1) $\mathcal{D} + \mathcal{D}^* = 2 \cdot triv_G + r_G - \sum_{j=0,1,\infty} Ind^G_{<g_j>} triv_{<g_j>}$.

(2) *Let* λ_j *be the character of the above lemma, then*

$$\mathcal{D} = triv_G - r_G + \sum_{j=0,1,\infty} \sum_{i=1}^{e_j - 1} \frac{i}{e_j} Ind_{<g_j>}^G \lambda_j^{-i}.$$

4.3 Differential forms with poles

We continue with the notations of subsection 4.2. Let a faithful irreducible representation ρ of G over $\overline{\mathbf{Q}}$ with character χ be given and suppose that this representation is not present in the G-module $H^0(C, \Omega)$. Then our aim is to consider differential forms with poles in such a way that ρ is present in the new situation.
Choose $j \in \{0, 1, \infty\}$ and define the divisor D as $\sum_{f(p)=j} p$. Let $\Omega(D)$ denote the sheaf of the differential forms on C having poles of order at most 1 at the points of $f^{-1}(j)$. The exact sequence

$$0 \to \Omega \to \Omega(D) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the skyscaper sheaf given by its stalks: $\mathcal{Q}_p = \overline{\mathbf{Q}}$ for $f(p) = j$ and $\mathcal{Q}_p = 0$ for other points p. This leads to an exact sequence of G-modules

$$0 \to H^0(\Omega) \to H^0(\Omega(D)) \to \sum \mathcal{Q}_p \to H^1(\Omega) \to 0.$$

The space $\sum \mathcal{Q}_p$ can be identified as G-module with the functions $h : f^{-1}(j) \to \overline{\mathbf{Q}}$. We note that G acts on the right on $f^{-1}(j)$. The group $< g_j >$ is the stabilizer of some point $f^{-1}(j)$. Then $f^{-1}(j)$ can be identified with the coset $< g_j > \backslash G$. It follows that the G-module $\sum \mathcal{Q}_p$ is isomorphic to the induction of the trivial representation of $< g_j >$ to G.
Using that the sum of the residues of a differential form is 0 and that $H^1(\Omega) = \overline{\mathbf{Q}}$ one finds that $V := H^0(\Omega(D))/H^0(\Omega)$ consists of the elements of $\sum \mathcal{Q}_p$ with total sum 0. In other words, $\sum \mathcal{Q}_p = V \oplus triv$.
Let χ be the character corresponding to the representation ρ. Let $\mathcal{D}^+$ denote the character of the space $H^0(C, \Omega(D))$. Then

$$\mathcal{D}^+ = \mathcal{D} + Ind_{<g_j>}^G (triv_{<g_j>}) - triv_G \text{ and}$$

$$< \chi | \mathcal{D}^+ > = < \chi|_{<g_j>} | triv >_{<g_j>} = \frac{1}{e_j} \sum_{n=0}^{e_j - 1} \chi(g_j^n)$$

and is equal to the multiplicity of 1 as eigenvalue of $\rho(g_j)$. Thus it is easily verified whether ρ occurs now in the representation $H^0(C, \Omega(D))$.

Corollary 4.5 *Let χ denote the character of an irreducible faithful representation of G. The character of the dual representation is denoted by χ^*. If neither χ nor χ^* is present in the G-module $H^0(C, \Omega)$ then χ (and χ^*) is present in the G-module $H^0(C, \Omega(f^{-1}(j)))$ for some $j \in \{0, 1, \infty\}$.*

Proof. We note that $< \chi, \mathcal{D}^* >=< \chi^*, \mathcal{D} >$. From part (1) of corollary 4.4 one concludes that $< \chi + \chi^*, \mathcal{D} >= \chi(e) - \sum_{j=0,1,\infty} < \chi, Ind^G_{<g_j>} triv_{<g_j>} >$. The assumption that $< \chi, \mathcal{D} >=< \chi^*, \mathcal{D} >= 0$ implies that $< \chi, Ind^G_{<g_j>} triv_{<g_j>} >\neq 0$ for some j. The last expression is the multiplicity of 1 as eigenvalue for $\rho(g_j)$. The matrix $\rho^*(g_j)$ has the same multiplicity for the eigenvalue 1. $\square$

Examples 4.6

(1) Let an irreducible, finite primitive $G \subset \mathrm{SL}(2)$ be given and let the given embedding have character χ. We note that $\chi = \chi^*$. Thus

$$< \chi, \mathcal{D} >= 1 - 1/2 \sum_{j=0,1,\infty} < \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>}$$

Further $< \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>}$ is the multiplicity of 1 as eigenvalue for $g_j \in G \subset \mathrm{SL}(2)$. Since $g_j \neq 1$ and the determinant of g_j is 1, we conclude that $< \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>}= 0$ in all cases. We conclude that $< \chi|\mathcal{D} >= 1$ and that $H^0(C, \Omega)$ contains the representation χ once.

(2) Similarly, for an irreducible finite $G \subset \mathrm{GL}(3)$ one has

$$< \chi, \mathcal{D} > + < \chi^*|\mathcal{D} >= 3 - \sum_{j=0,1,\infty} < \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>} .$$

Again $< \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>}$ is the multiplicity of 1 as eigenvalue for $g_j \in G \subset \mathrm{SL}(3)$. Since $g_j \neq 1$ and the determinant of g_j is 1, we conclude that $< \chi|_{<g_j>}, triv_{<g_j>} >_{<g_j>}$ is 0 or 1.
If the value 1 occurs for every $j \in \{0, 1, \infty\}$ then $< \chi|\mathcal{D} >=< \chi^*, \mathcal{D} >= 0$. Then χ is contained precisely once in $H^0(\Omega_C(\pi^{-1}j))$ for any $j \in \{0, 1, \infty\}$.

If the value 1 occurs for at most two values of $j \in \{0, 1, \infty\}$, then one of the two $< \chi | \mathcal{D} >, < \chi^*, \mathcal{D} >$ is strictly positive.

We note that χ^* is also the complex conjugated of the character χ. Thus $\chi \neq \chi^*$ is equivalent to χ is not real. Suppose that χ is not real and that $< \chi, \mathcal{D} >= 0$. The embedding $G \subset \mathrm{SL}(3, \mathbf{C})$ with character χ gives by complex conjugation an embedding with the character χ^*. From this ones finds an automorphism A of G which interchanges χ and χ^*. After changing the admissible triple in question by the automorphism A one finds an admissible triple with $< \chi, \mathcal{D} > \neq 0$.

5 The exponents

5.1 Generalities on scalar Fuchsian equations

Consider the scalar Fuchsian equation

$$L(y) := y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y^{(1)} + a_n y = 0,$$

with singular points $p_1, \ldots, p_s \in \mathbf{C}$ and possibly ∞. We suppose that for each singular point p the set E_p of exponents consists of n elements. The nth exterior power of this equation reads $f' = -a_1 f$. Let $y_1, \ldots, y_n$ be a basis of solutions of the original equation, then the Wronskian w is a non zero solution of $f' = -a_1 f$. It is easily seen that the order μ_i of w at p_i is the $-(n-1)n/2 + \sum E_{p_i}$ (i.e.,$-(n-1)n/2$ plus the sum of the exponents of L at p_i). The order μ_∞ for w at ∞ is $(n-1)n/2 + \sum E_\infty$. Further $-a_1 dz$ has simple poles and the residues are μ_i at p_i and μ_∞ at ∞. The sum of the residues $-a_1 dz$ is zero. This implies the well known Fuchs relation:

$$\sum_i \sum E_{p_i} + \sum E_\infty = \frac{(s-1)(n-1)n}{2}.$$

Suppose further that the differential Galois group of the equation lies in $\mathrm{SL}(n)$. Then $w' = -a_1 w$ has a non trivial solution in $\mathbf{C}(z)$. The residues of $a_i dz$ are integers and thus for all $p \in \{p_1, \ldots, p_k, \infty\}$ one has $\sum E_p$ is an integer.

We specialize now to the situation of a Fuchsian scalar equation L of order two with singular points $\{0,1,\infty\}$. The form of L is

$$y^{(2)}+(\frac{a_0}{z}+\frac{a_1}{z-1})y^{(1)}+(\frac{b_0}{z^2}+\frac{b_1}{(z-1)^2}+\frac{b_2}{z(z-1)})y=0, \text{ and one has}$$

$$t(t-1)+a_0t+b_0=\prod_{\alpha\in E_0}(t-\alpha),$$

$$t(t-1)+a_1t+b_1=\prod_{\alpha\in E_1}(t-\alpha),$$

$$t(t+1)-(a_0+a_1)t+(b_0+b_1+b_2)=\prod_{\alpha\in E_\infty}(t-\alpha).$$

The three polynomials in t are the indicial equations at $0,1,\infty$. One observes that L is determined by E_0,E_1,E_∞. Further $\sum_{j=0,1,\infty}\sum E_j=1$. The differential Galois group is a subgroup of SL(2) if and only if $\sum E_j$ is an integer for $j=0,1,\infty$.

For a third order Fuchsian differential equations (with singular points $0,1,\infty$) we will use the normalized form

$$L=\partial^3+(\frac{a_0}{z}+\frac{a_1}{z-1})\partial^2+(\frac{b_0}{z^2}+\frac{b_1}{(z-1)^2}+\frac{b_2}{z(z-1)})\partial$$

$$+\frac{c_0}{z^3}+\frac{c_1}{(z-1)^3}+\frac{c_2(z-1/2)}{z^2(z-1)^2}+\frac{\mu}{z^2(z-1)^2}$$

The indicial equations at $0,1,\infty$ are:

$$t(t-1)(t-2)+a_0t(t-1)+b_0t+c_0=\prod_{\alpha\in E_0}(t-\alpha),$$

$$t(t-1)(t-2)+a_1t(t-1)+b_1t+c_1=\prod_{\alpha\in E_1}(t-\alpha),$$

$$t(t+1)(t+2)-(a_0+a_1)t(t+1)+(b_0+b_1+b_2)t-(c_0+c_1+c_2)=$$

$$=\prod_{\alpha\in E_\infty}(t-\alpha).$$

Thus $a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2$ are determined from the exponent sets E_0, E_1, E_∞. We will call μ the *accessory parameter*. Thus L is determined by the exponents and the value of the accessory parameter. The dual L^* of L has the data $-a_0, -a_1, b_0+2a_0, b_1+2a_1, b_2, -c_0-2a_0-2b_0, -c_1-2a_1-2b_1, -c_2-2b_2, -\mu$. The indicial equations for L^* at $0, 1, \infty$ are obtained from the indicial equations for L by the substitutions $t \mapsto -t+2, -t+2, -t-2$.
The substitution $z \mapsto 1-z$ applied to L, with exponents sets E_0, E_1, E_∞ and accessory parameter μ, produces a Fuchsian equation M with exponents sets E_1, E_0, E_∞ and accessory parameter $-\mu$.

5.2 Restrictions on the exponents

The following data are given: a finite group $G \in \mathrm{GL}(n)$; (g_0, g_1, g_∞) an admissible triple for G which determines a covering $C \to \mathbf{P}^1$, unramified outside $0, 1, \infty$; the function field of C is $K \supset k = \overline{\mathbf{Q}}(z)$; V a G-invariant subspace of K such that G-module V is isomorphic to the given embedding $G \subset \mathrm{GL}(n)$. Let L denote the monic nth order differential operator in $k[\partial]$ with solution space V. For the points $j \in \{0, 1, \infty\}$ one denotes by E_j the set of exponents of L at the point j.

Proposition 5.1 *Write $E_j = \frac{1}{e_j}\{b_1 < \cdots < b_n\}$. Then the eigenvalues (counted with multiplicity) of $g_j \in G \subset \mathrm{GL}(n)$ are $\{\zeta_j^{b_1}, \ldots, \zeta_j^{b_n}\}$ with $\zeta_j = e^{2\pi i/e_j}$.*

Proof. Consider a point $a_j \in C$ above j which is fixed under g_j. Let t be a local parameter at a_j such that $g_j(t) = \zeta_j t$. Then t^{e_j} is a local parameter at j on $\mathbf{P}^1$. The field K is embedded in the quotient field $\overline{\mathbf{Q}}((t))$ of $\overline{\mathbf{Q}}[[t]]$ and thus V is considered as a subspace of $\overline{\mathbf{Q}}((t))$. There is a basis $v_1, \ldots, v_n$ of V with $v_i = t^{b_i}(1 + *t + *t^2 + \cdots)$ for all i. The action of g_j on $\overline{\mathbf{Q}}((t))$ is given by $g_j(t) = \zeta_j t$. This implies that the matrix of g_j on V with respect to the basis $v_1, \ldots, v_n$ has an upper trianglular form. In particular the eigenvalues of g_j on V are $\{\zeta_j^{b_1}, \ldots, \zeta_j^{b_n}\}$. $\square$

Remarks
The proposition can be rephrased as: The exponent sets E_0, E_1, E_∞ are determined, modulo integers, by the admissible triple for $G \subset \mathrm{GL}(n)$. In [S-U3],

lemma 2.4, a similar result is proven and conditions on the exponents are derived from this. The present proposition 5.1 is sharper in the sense that the three local monodromies and the covering are related by means of the admissible triple (g_0, g_1, g_∞).

We translate the exponent sets in terms of differential forms. The solution space $V \subset K$ of the differential equation L is now chosen such that Vdz lies in $H^0(C, \Omega)$ or $H^0(C, \Omega(f^{-1}(j))$ for some $j \in \{0, 1, \infty\}$. Consider again points $a_0, a_1, a_\infty \in C$ above $0, 1, \infty$. Consider at a_0 the formal local parameter t satisfying $t^{e_0} = z$. Then the set of exponents is $E_0 := \{\frac{1}{e_0} ord_{a_0}(y) | y \in V,\ y \neq 0\}$. From $ydz = e_0 t^{e_0 - 1} ydt$ it follows that

$$E_0 = -1 + \frac{1}{e_0} + \frac{1}{e_0}\{ord_{a_0}(\omega)|\ \omega \in Vdz,\ \omega \neq 0\}.$$

Similarly, the set E_1 of exponents at 1 is

$$E_1 = -1 + \frac{1}{e_1} + \frac{1}{e_1}\{ord_{a_1}(\omega)|\ \omega \in Vdz,\ \omega \neq 0\}.$$

Finally

$$E_\infty = 1 + \frac{1}{e_\infty} + \frac{1}{e_\infty}\{ord_{a_\infty}(\omega)|\ \omega \in Vdz,\ \omega \neq 0\}.$$

The formulas give a *lower bound for the exponents*, since the $\omega \in Vdz$ are either holomorphic or have a pole of order at most one for one of the a_j. The formula $\sum_{j=0,1,\infty} \sum E_j = \frac{n(n-1)}{2}$ gives an *upper bound for the exponents.*

The *Weierstrass gap theorem* provides another upper bound for the exponents. We will give this formula with a slight variation of [G-H]. Let C be a curve of genus $g > 2$. A point $c \in C$ will be called *ordinary* if $\{ord_c(\omega)|\ \omega \in \Omega_C(C),\ \omega \neq 0\}$ is equal to $\{0, 1, \ldots, g-1\}$. The other points are called *Weierstrass points.* Let c be a Weierstrass point with

$$\{ord_c(\omega)|\ \omega \in \Omega_C(C),\ \omega \neq 0\} = \{b_1, \ldots, b_g\}$$

and $b_0 < b_1 < \ldots < b_{g-1}$. The *weight* $W(c)$ of c is defined by $W(c) = \sum_{i=0,\ldots,g-1}(b_i - i)$. The Weierstrass gap theorem states that $\sum_{c \in C} W(c) = (g-1)g(g+1)$. We note further that any automorphism of the curve C

preserves Weierstrass point and their weight.
In our special situation one finds for the points a_0, a_1, a_∞ above the points $0, 1, \infty$ the inequality:

$$\frac{|G|}{e_0}W(a_0) + \frac{|G|}{e_1}W(a_1) + \frac{|G|}{e_\infty}W(a_\infty) \leq (g-1)g(g+1).$$

If all the Weierstrass points lie above one of the points $0, 1, \infty$, then one would have an equality in the above formula. It is clear that the inequality implies an upper bound for the exponents set E_0, E_1, E_∞.

6 Some features of the scalar differential equations

6.1 Tests for the differential Galois group

Let a given differential equation L of order two or three (over $k = \overline{\mathbf{Q}}(z)$) be suspected to have a prescribed finite differential Galois group G (e.g., the examples to be constructed with the theory and algorithms of this paper). For equations of *order two* and differential Galois group contained in SL(2), one can use one the many algorithms improving Kovacic's algorithm, e.g. [H-W],[U-W]. The main idea is to search for "invariants of degrees 4,6,8,12", i.e., rational solutions of the corresponding symmetric powers of the equation. However, as remarked in the introduction, the equations of order two that we construct are guaranteed to have the correct differential Galois group.

For the third order equations that are constructed here, we have explained in section 2 a (rather efficient) method to find the value(s) of the accessory parameter and the group by making good use of the construction data. In many cases the theory guarantees that the constructed equation has the correct differential Galois group. In other cases, namely where we make a guess for the exponents, one has to combine this with the following more general method: *For an irreducible third order equation L (containing possibly an accessory parameter) with differential Galois group $G \subset$ SL(3), a search for (semi-) invariants of degrees 2,3,4,6,9,12 will determine the group.*
For the precise statement we refer to [H-R-U-W]. In particular for a finite

primitive $G \subset \mathrm{SL}(3)$ there is an invariant of degree 6, except for the group $H_{216}^{\mathrm{SL}(3)}$ where there is an invariant of degree 9.
The existence of invariants of a certain degree will in general produce polynomial equations for the accessory parameter.

6.2 The equation for the accessory parameter

Let L be a scalar Fuchsian equation of order three in normalized form, with unknown accessory parameter μ. We try to determine the value(s) μ for which the differential Galois group could be finite. We make here the *assumption* that for some $j \in \{0, 1, \infty\}$ the set E_j contains two elements with difference $m \in \mathbf{Z}$, $m > 0$. We note that this situation occurs if and only if g_j has multiple eigenvalues.

Lemma 6.1 *Assume that the differential Galois group of L is finite then μ satisfies a polynomial equation over $\mathbf{Q}$ of degree m.*

Proof. For notational convenience we suppose $j = 0$ and by assumption $\lambda, \lambda+m \in E_0$ with m a positive integer. The assumption that the differential Galois group of L is finite implies that there are three Puiseux series at $z = 0$, solutions of $L = 0$. One of these has the form $z^\lambda g$ with $g = 1+c_1z+c_2z^2+\cdots \in \mathbf{Q}[[z]]$. There is a formula

$$L(z^t) = P_0(t)z^{t-3} + P_1(t)z^{t-2} + P_2(t)z^{t-1} + P_3(t)z^t + \ldots,$$

where the P_i are polynomials in t and μ. In fact P_0 does not contain μ and the other P_i have degree 1 in μ. An evaluation of the equation $L(z^\lambda(1+c_1z+c_2z^2+\cdots)) = 0$ produces a set of linear equations for the coefficients c_i. In order to have a solution, a determinant must be zero. This determinant is easily seen to be a polynomial in μ of degree m. $\square$

Explicit formulas:
$P_0(t) = t(t-1)(t-2) + a_0t(t-1) + b_0t + c_0$
$P_1(t) = -a_1t(t-1) - b_2t - c_2/2 + \mu$
$P_2(t) = -a_1t(t-1) + (b_1 - b_2)t + 2\mu$
$P_3(t) = -a_1t(t-1) + (2b_1 - b_2)t + c_2/2 + 3\mu$
If $\lambda, \lambda+1$ are exponents at 0, then $P_1(\lambda) = 0$.

If $\lambda, \lambda+2$ are exponents at 0, then $P_1(\lambda)P_1(\lambda+1) - P_0(\lambda+1)P_2(\lambda) = 0$
If λ and $\lambda+3$ are exponents at 0 then the determinant of the matrix $\begin{pmatrix} P_1(\lambda) & P_0(\lambda+1) & 0 \\ P_2(\lambda) & P_1(\lambda+1) & P_0(\lambda+2) \\ P_3(\lambda) & P_2(\lambda+1) & P_1(\lambda+2) \end{pmatrix}$ is zero.

6.3 Apparent singularities and families of equations

We restrict ourselves to order three equations and a finite primitive subgroup $G \subset \mathrm{SL}(3)$. The following data are fixed:
An admissible triple for G, giving rise to a covering $f : C \to \mathbf{P}^1$ with function field $K \supset k$. A faithful unimodular irreducible representation of degree 3 be given with character χ.
We give here an outline of the search for Fuchsian equations L.

6.3.1 *Suppose that χ appears* once *in $H^0(C, \Omega)$ (or in $H^0(C, \Omega(f^{-1}0))$).*
The solution space $V \subset K$, with $Vdz \subset H^0(C, \Omega)$ (or $Vdz \subset H^0(C, \Omega(f^{-1}0))$) is unique. The corresponding order three differential equation L is also unique. The differential module corresponding to L is known to be Fuchsian (See [Bo]). The scalar differential equation L need not be Fuchsian. A singular point, different from $0, 1, \infty$, will be called an *apparent singularity*. Let f_1, f_2, f_3 be a basis of V, then the coefficients of L are uniquely determined by $L(f_i) = 0$ for $i = 1, 2, 3$. One finds in fact three inhomogeneous equations for the A_0, A_1, A_2 and the determinant of the system is w, the Wronskian of f_1, f_2, f_3. Let O denote the integral closure of $\overline{\mathbf{Q}}[z, \frac{1}{z(z-1)}]$ in K. Then O is invariant under differentiation and $V \subset O$. We conclude that an apparent singularity of L must be a zero of w (different from $0, 1$). More information will follow from the possibilities for the exponent sets. The data determine *lower bounds*, written as $E_0(min), E_1(min), E_\infty(min)$ for the exponent sets. The actual possiblities for the exponent sets, for a supposed Fuchsian L, are obtained from the minimal bounds by adding non negative integers to the minimal data such that the final sum is equal to 3. One considers the following situations:
(1) *The sum of all numbers in $E_0(min), E_1(min), E_\infty(min)$ is 3.*
Then $E_0(min), E_1(min), E_\infty(min)$ are the actual exponent sets. There is no apparent singularity and there is only one value possible for the accessory

parameter. (Indeed, L exists and is unique). If one finds indeed only one value for μ, then the theory which we developed, guarantees that L has the correct differential Galois group and representation. Moreover, by uniquenes, μ is rational.
(2) *The sum of all numbers in* $E_0(min), E_1(min), E_\infty(min)$ *is 2.*
The actual exponent sets that we have to consider are obtained by adding 1 to any of the numbers (however keeping the restriction on exponent sets). For each case one has to use a method for obtaining a suitable μ and verify that the constructed L has the correct differential Galois group. At most one case and at most one value for μ will give a positive answer. (Indeed, L is unique).
(3) If in (2) no L is found then this proves that L is not Fuchsian. By counting one find that there must be one apparent singular point $c \neq 0, 1, \infty$ and that the exponent set for c is $E_c = \{0, 1, 3\}$. The equation L can actually be determined from these data.
(4) If the sum of all numbers in $E_0(min), E_1(min), E_\infty(min)$ is ≤ 1, then there are many possibilities for the exponent sets of the supposed Fuchsian L. One proceeds as in (2).
(5) If for (4) no L is found then L has one or two apparent singularities.

6.3.2 *Suppose now that* χ *appears* twice *in* $H^0(C, \Omega)$ *(or in* $H^0(C, \Omega(f^{-1}0)))$*.* Then the solution space V for our problem is not unique. In fact one has a family of solution spaces $\{V_t |\ t \in \mathbf{P}^1\}$ and a family of third order equations L_t. We can make this more explicit by writing $t = (t_0 : t_1) \in \mathbf{P}^1$ and by choosing bases f_1, f_2, f_3 and g_1, g_2, g_3 of two different choices for the subspace $V \subset K$ such that the linear map which sends each f_i to g_i is G-equivariant. Then V_t has basis $t_0 f_1 + t_1 g_1, t_0 f_2 + t_1 g_2, t_0 f_3 + t_1 g_3$. One considers the following situations:
(1) *The numbers in* $E_0(min), E_1(min), E_\infty(min)$ *add up to 3.*
Then $E_0(min), E_1(min), E_\infty(min)$ are the actual values for the exponents. Thus one finds a rational map $\mathbf{P}^1 \to \mathbf{A}^1$, which assigns to a value for t the corresponding value of the accessory parameter. Such a rational map does not exist and the data cannot occur.
(2) *The numbers* $E_0(min), E_1(min), E_\infty(min)$ *add up to* 2.
Suppose that the family L_t does not involve an apparent singularity. Then each L_t determines exponent sets E_0, E_1, E_∞ and a value for the accessory parameter. As in (1), this is not possible. Thus for general t, the equation

L_t has a one apparent singularity $c = c(t)$. The map $t \mapsto c(t)$ cannot be constant, since otherwise one would find a non constant morphism form $\mathbf{P}^1$ to the affine set of accessory parameters. Thus $t \mapsto c(t)$ gives rise to a morphism $\tilde{c} : \mathbf{P}^1 \to \mathbf{P}^1$. The Wronskian of the three generators of the space V_t can be seen to have the form $z^{-3+\sum E_0(min)}(z-1)^{-3+\sum E_1(min)}(az+b)$, where a, b are homogeneous in t_0, t_1 of degree 3. Thus $c(t) = \frac{-b}{a}$ and $\tilde{c}$ has degree 3. In particular, $0, 1, \infty$ are in the image of $\tilde{c}$ and one finds the existence of Fuchsian equations L with exponent sets obtained by adding 1 to numbers in $E_0(min)$ or $E_1(min)$ or $E_\infty(min)$. This can also be seen in a more direct way. Suppose, to begin with, that the exponents E_0 at 0 do not differ by integers. For the bases f_1, f_2, f_3 and g_1, g_2, g_3 of the two copies of V in $H^0(C, \Omega)$ we take eigenvectors with respect to the action of g_0, i.e., the local monodromy at 0. For any $t = (t_0 : t_1)$ the leading coefficient of $t_0 f_1 + t_1 g_1$ is homogeneous linear in t_0, t_1. Thus there is a unique value for t such that the order of $t_0 f_1 + t_1 g_1$ is at least 1 higher than the given lower bound. This argument proves that by adding $+1$ to any of the numbers in E_0, one obtains the exponent sets for a Fuchsian equation L. Suppose now that in E_0 integer exponents differences occur, say $E_0 = \{\lambda_1, \lambda_1 + 1, \lambda_2\}$ with $\lambda_2 - \lambda_1$ not an integer. Then one cannot add $+1$ to λ_1. It seems that one is missing one possible eqution for L. However by adding $+1$ to $\lambda_1 + 1$ and using lemma 6.1, one finds two values for μ. A local calculation at the point 0 shows that both values actually lead to an L with the correct differential Galois group. We conclude that we find nine equivalent Fuchsian equations in this situation.
(3) *The numbers in $E_0(min), E_1(min), E_\infty(min)$ add up to $3 - m < 2$.*
Something similar happens. The wronskian can be seen to be:
$z^{-3+\sum E_0(min)}(z-1)^{-3+\sum E_1(min)}(a_m z^m + \cdots + a_1 z + a_0)$, where $a_m, \ldots, a_0$ are homogeneous in t_0, t_1 of degree 3. For a given value of t, the zeros of the polynomial $(a_m z^m + \cdots + a_1 z + a_0)$ are the candidates for the apparent singularities. In general there seems to be no reason to expect an equation L_t which has no apparent singuarities at all.

6.3.3 *Suppose now that χ appears* three times *in $H^0(C, \Omega)$ $(H^0(C, \Omega(f^{-1}0)))$.* This gives rise to a $\mathbf{P}^2$-family of differential equations L_t. The sum of all numbers in $E_0(min), E_1(min), E_\infty(min)$ have to add up to a number $N \leq 1$. The number of apparent singularities in a general L_t is $3-N$. We will discuss the particularly interesting case $N = 1$. The wronskian of L_t has the form $z^*(z-1)^*(a_2 z^2 + a_1 z + a_0)$, with fixed powers of z and $z - 1$. The a_0, a_1, a_2

are homogeneous in $t = (t_0 : t_1 : t_2)$ of degree 3. The map $\mathbf{P}^2 \to \mathbf{P}^2$, given by $t \mapsto (a_2 : a_1 : a_0)$, is finite of degree 9. The number of values t such that L_t has no apparent singularities is 45. Indeed, arguments as in the above case with a $\mathbf{P}^1$-family, prove that by adding $+1$ twice to any of the elements in $E_0(min), E_1(min), E_\infty(min)$ will give a unique Fuchsian equation with the required properties (provided that there are no integer exponent differences). In the case of integer exponents differences, a local calculation shows that the counting remains valid.

In section 8, we will produce examples for all the situations that we have discussed here.

7 Second order equations

The finite primitive subgroups of SL(2) are: the icosahedral group $A_5^{SL_2}$ of order 120, the octahedral $S_4^{SL_2}$ of order 48 and the tetrahedral group $A_4^{SL_2}$ of order 24. Each branch type happens to consist of a single equivalence class of admissible triples. We will moreover work modulo the permutation of the points $0, 1, \infty$.

7.1 Schwarz' list and Klein's theorem

The hypergeometric differential equations with finite monodromy group (or equivalently finite differential Galois group) were studied by H.A. Schwarz [Schw]. The idea is the following. One considers a Fuchsian differential equation L of order two with set of exponents E_0, E_1, E_∞ at the singular point $0, 1, \infty$. As always $\sum E_0 + \sum E_1 + \sum E_\infty = 1$ and the assumption that the exponents correspond to a hypergeometric equation is equivalent to $0 \in E_0$ and $0 \in E_1$. Since we are looking for finite monodromy groups we may restrict ourselves to the case of rational exponents. Let λ, μ, ν denote the absolute values of the differences of the exponents at $0, 1, \infty$. Further $\lambda' \in (0, 1)$ is defined by $\lambda' \equiv \pm\lambda$ modulo 2. The numbers μ' and ν' are defined in a similar way. Moreover, one may transform a triple λ', μ', ν' by replacing two items a of the triple by $1 - a$. Finally λ'', μ'', ν'' is the same as λ', μ', ν' but ordered such that $\lambda'' \geq \mu'' \geq \nu''$. The *list of Schwarz* is the set of (fifteen) triples λ'', μ'', ν'' such that the corresponding hypergeometric equation has a finite irreducible differential Galois group. The "*Schwarz triples*" for the group $A_4^{\mathrm{SL}(2)}$ are $(1/2, 1/3, 1/3)$ and $(2/3, 1/3, 1/3)$; for the group $S_4^{\mathrm{SL}(2)}$ they are $(1/2, 1/3, 1/4)$ and $(2/3, 1/4, 1/4)$ and for $A_5^{\mathrm{SL}(2)}$ they are $(1/2, 1/3, 1/5)$, $(2/5, 1/3, 1/3)$, $(2/3, 1/5, 1/5)$, $(1/2, 2/5, 1/5)$, $(3/5, 1/3, 1/5)$, $(2/5, 2/5, 2/5)$, $(2/3, 1/3, 1/5)$, $(4/5, 1/5, 1/5)$, $(1/2, 2/5, 1/3)$, $(3/5, 2/5, 1/3)$.

Suppose for convenience that the λ, μ, ν above are already in $(0, 1)$. The monodromy group G of the equation is a subgroup of GL(2) and let G^{proj} denote its image in PGL(2). Under the assumption that $1 < \lambda + \mu + \nu < 1 + \min(\lambda, \mu, \nu)$ there is a spherical triangle Δ in the complex sphere $\mathbf{P}^1(\mathbf{C})$ with angles $(\lambda\pi, \mu\pi, \nu\pi)$. The group associated to Δ is the one generated by the reflections in the three sides of Δ. The group G^{proj} is the subgroup of this group consisting of the elements which are a product of an even number of reflections. An elementary spherical triangle is a triangle with all angles of the form $\frac{\pi}{n}$ and $n \in \mathbf{Z}$, $n > 1$. Thus the spherical elementary triangles are given by the angles $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n})$ (and $n \geq 2$), $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})$ and $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5})$. The above triangle Δ gives rise to a finite group G^{proj} if and only if Δ can be dissected into elementary spherical triangles of the same type. The last statement and some combinatorics leads to the list of Schwarz. For a modern survey of the Schwarz theory we refer to [Be].

The problem that we have to study is the connection of Schwarz' classification with the order two equations L with differential Galois group $G \in \{D_n^{\mathrm{SL}(2)}, A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}\}$ that we will produce here. Let $K \supset k = \overline{\mathbf{Q}}(z)$ denote the Picard-Vessiot field. Take a basis f_0, f_1 of the solution space of L in K and define $K^- = k(\frac{f_1}{f_0})$. Since the derivative of $\frac{f_1}{f_0}$ is equal to $\frac{1}{f_0^2}$ (and lies in K^-) one finds that K^- is the fixed field of the center of G and has index two in K. Shifting each set of exponents E_0, E_1, E_∞ by a rational constant, changes of course the equation L and the Picard-Vessiot field K. A basis of solutions for the new equation is $z^a(z-1)^b f_0, z^a(z-1)^b f_1$ for suitable rational a, b. The subfield K^- does not change under this operation. The monodromy classified by Schwarz is the one of the function $\frac{f_1}{f_0}$, where f_0, f_1 are two independent solutions of a hypergeometric equation. Thus we may suppose that our L is hypergeometric. We conclude that the list of Schwarz classifies the Galois coverings of $\mathbf{P}^*$ with Galois groups $G^{proj} \in \{D_n, A_4, S_4, A_5\}$ with sometimes additional data from the inbedding of G^{proj} into PSL(2) which

comes from the equation L. A detailed analysis shows that for the groups A_4 and S_4 the list of Schwarz classifies only the coverings. For the group A_5 the situation is more complicated. The coverings of $\mathbf{P}^*$ with group isomorphic to A_5 can be listed as:
branch type 2,3,5, genus 0. Appears twice in the list of Schwarz.
branch type 2,5,5, genus 4. Appears once in the list of Schwarz.
branch type 3,5,5, genus 5. Appears twice in the list of Schwarz.
branch type 3,5,5, genus 9. There are two such coverings. One appears once in the list of Schwarz, the other twice.
branch type 5,5,5, genus 13. Appears twice in the list of Schwarz.
We note that we work also modulo permutations of the points $0, 1, \infty$.

A modern survey of order two differential equations with finite differential Galois group and especially Klein's theorem on this subject is the inspiring paper [B-D]. Klein's theorem is stated there as follows:

For each of the groups $G \in \{D_n^{\mathrm{SL}(2)}, A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}\}$ there is a standard differential equation St_*, normalized as $y'' + (\frac{A}{t^2} + \frac{B}{t(t-1)} + \frac{C}{(t-1)^2})y = 0$ with suitable constants A, B, C. For any order two differential equation L over $k = \overline{\mathbf{Q}}(z)$ with differential Galois group G there is a $\overline{\mathbf{Q}}$-linear homomorphism $\phi : \overline{\mathbf{Q}}(t) \to k$ of fields such that $\phi_* St_*$ is equivalent to the given L.

The link with our list of equations is the following. For each of the groups, the standard differential equation St_* is present. The other equations are derived from this standard equation by a map ϕ which has the rather special property that $\phi_* St_*$ has again $0, 1, \infty$ as the only singular points. There are very few ϕ's possible as is seen from the list.

7.2 The equations

For each of the groups $A_4^{\mathrm{SL}(2)}, S_4^{\mathrm{SL}(2)}, A_5^{\mathrm{SL}(2)}$ we give the branch types. It turns out that each branch type contains only one equivalence class of admissible triples and moreover the branch type determines the group. Thus the branch types are in bijection with the Galois extensions of k (unramified outside $0, 1, \infty$ and with Galois group isomorphic to one of the considered groups). The first group has only one irreducible faithful representation of degree 2. The other groups have two such representations. Thus the first group gives a differential equation for each branch type and the other groups produce two differential equations per branch type. For each equation L we give the exponent sets E_0, E_1, E_∞; we list the branch type, the genus of the field K, the genus of the field K^-, the character, and the triple of the list of Schwarz.

7.2.1 The tetrahedral group $A_4^{\mathrm{SL}(2)}$

The group has 7 conjugacy classes $conj_1, \ldots, conj_7$. They correspond to elements of order $1, 2, 3, 3, 4, 6, 6$. There is only one faithful (unimodular) character of degree 2, denoted by χ_4. The two values $0 \leq \lambda < 1$ such that the $e^{2\pi i\lambda}$ are the eigenvalues for the representation corresponding to χ_4 are given for each conjugacy class: $(\frac{0,0}{1}, \frac{1,1}{2}, \frac{1,2}{3}, \frac{1,2}{3}, \frac{1,3}{4}, \frac{1,5}{6}, \frac{1,5}{6})$. This information leads to unique data for the exponents and the equations.

branch type	genera	char	exp 0	exp 1	exp ∞	Schwarz triple
3,3,4	2,0	χ_4	-2/3,-1/3	-2/3,-1/3	5/4,7/4	1/2,1/3,1/3
3,3,6	3,1	χ_4	-2/3,-1/3	-2/3,-1/3	7/6,11/6	2/3,1/3,1/3
3,4,6	4,0	χ_4	-2/3,-1/3	-3/4,-1/4	7/6,11/6	1/2,1/3,1/3
4,6,6	6,0	χ_4	-3/4,-1/4	-5/6,-1/6	7/6,11/6	1/2,1/3,1/3
6,6,6	7,1	χ_4	-5/6,-1/6	-5/6,-1/6	7/6,11/6	2/3,1/3,1/3

7.2.2 The octahedral group $S_4^{\mathrm{SL}(2)}$

This group has 8 conjugacy classes $conj_1, \ldots, conj_8$. They correspond to elements of order $1, 2, 3, 4, 4, 6, 8, 8$. There are two faithful (unimodular) irreducible representations of degree 2. Their characters are denoted by χ_4 and χ_5 and the eigenvalues of these characters are given for each conjugacy class:

$$\chi_4 : (\frac{0,0}{1}, \frac{1,1}{2}, \frac{1,2}{3}, \frac{1,3}{4}, \frac{1,3}{4}, \frac{1,5}{6}, \frac{1,7}{8}, \frac{3,5}{8})$$

$$\chi_5 : (\frac{0,0}{1}, \frac{1,1}{2}, \frac{1,2}{3}, \frac{1,3}{4}, \frac{1,3}{4}, \frac{1,5}{6}, \frac{3,5}{8}, \frac{1,7}{8})$$

Thus there is an automorphism of $S_4^{\mathrm{SL}(2)}$ which permutes the classes $conj_7, conj_8$. For the admissible triple, representing the unique element of the branch type we make the choice that the number of times that $conj_7$

occurs is greater than or equal to the number of times that $conj_8$ is present. This information suffices to calculate all exponents and equations.

branch t.	genera	char	exp 0	exp 1	exp ∞	Schwarz triple
3,4,8	8,0	χ_4	-2/3,-1/3	-3/4,-1/4	9/8,15/8	1/2,1/3,1/4
		χ_5	-2/3,-1/3	-3/4,-1/4	11/8,13/8	1/2,1/3,1/4
3,8,8	11,3	χ_4	-2/3,-1/3	-5/8,-3/8	9/8,15/8	2/3,1/4,1/4
		χ_5	-2/3,-1/3	-7/8,-1/8	11/8,13/8	2/3,1/4,1/4
4,6,8	13,0	χ_4	-3/4,-1/4	-5/6,-1/6	11/8,13/8	1/2,1/3,1/4
		χ_5	-3/4,-1/4	-5/6,-1/6	9/8,15/8	1/2,1/3,1/4
6,8,8	15,3	χ_4	-5/6,-1/6	-5/8,-3/8	11/8,13/8	2/3,1/4,1/4
		χ_5	-5/6,-1/6	-7/8,-1/8	9/8,15/8	2/3,1/4,1/4

7.2.3 The icosahedral group $A_5^{\mathrm{SL}(2)}$

There are 9 conjugacy classes $conj_1, \ldots, conj_9$ corresponding to elements of orders $1, 2, 3, 4, 5, 5, 6, 10, 10$. There are two faithful irreducible (unimodular) representations of degree 2. As before, their characters χ_2 and χ_3 are given on the conjugacy classes by the two eigenvalues:

$$\chi_2 : (\frac{0,0}{1}, \frac{1,1}{2}, \frac{1,2}{3}, \frac{1,3}{4}, \frac{2,3}{5}, \frac{1,4}{5}, \frac{1,5}{6}, \frac{3,7}{10}, \frac{1,9}{10})$$

$$\chi_3 : (\frac{0,0}{1}, \frac{1,1}{2}, \frac{1,2}{3}, \frac{1,3}{4}, \frac{1,4}{5}, \frac{2,3}{5}, \frac{1,5}{6}, \frac{1,9}{10}, \frac{3,7}{10})$$

Thus there is an automorphism of $A_5^{\mathrm{SL}(2)}$ which permutes the two pairs of conjugacy classes $conj_5, conj_6$ and $conj_8, conj_9$. For the admissible triple representing the unique element of the branch type there are several choices. One can deduce from the table which choice has been made.

branch t.	genera	char	exp 0	exp 1	exp ∞	Schwarz triple
3,3,10	15,5	χ_2	-2/3,-1/3	-2/3,-1/3	11/10,19/10	2/3,1/3,1/5
		χ_3	-2/3,-1/3	-2/3,-1/3	13/10,17/10	2/5,1/3,1/3
3,4,5	14,0	χ_2	-2/3,-1/3	-3/4,-1/4	7/5,8/5	1/2,1/3,1/5
		χ_3	-2/3,-1/3	-3/4,-1/4	6/5,9/5	1/2,2/5,1/3
3,4,10	20,0	χ_2	-2/3,-1/3	-3/4,-1/4	11/10,19/10	1/2,1/3,1/5
		χ_3	-2/3,-1/3	-3/4,-1/4	13/10,17/10	1/2,2/5,1/3
3,5,5	17,9	χ_2	-2/3,-1/3	-3/5,-2/5	6/5,9/5	3/5,1/3,1/5
		χ_3	-2/3,-1/3	-4/5,-1/5	7/5,8/5	3/5,1/3,1/5
3,5,6	19,5	χ_2	-2/3,-1/3	-3/5,-2/5	7/6,11/6	2/3,1/3,1/5
		χ_3	-2/3,-1/3	-4/5,-1/5	7/6,11/6	2/5,1/3,1/3
3,5,10	23,9	χ_2	-2/3,-1/3	-3/5,-2/5	11/10,19/10	2/3,1/5,1/5
		χ_3	-2/3,-1/3	-4/5,-1/5	13/10,17/10	3/5,2/5,1/3
3,10,10	29,9	χ_2	-2/3,-1/3	-9/10,-1/10	13/10,17/10	3/5,1/3,1/5
		χ_3	-2/3,-1/3	-7/10,-3/10	11/10,19/10	3/5,1/3,1/5
4,5,5	22,4	χ_2	-3/4,-1/4	-3/5,-2/5	6/5,9/5	1/2,2/5,1/5
		χ_3	-3/4,-1/4	-4/5,-4/5	7/5,8/5	1/2,2/5,1/5
4,5,6	24,0	χ_2	-3/4,-1/4	-3/5,-2/5	7/6,11/6	1/2,1/3,1/5
		χ_3	-3/4,-1/4	-4/5,-1/5	7/6,11/6	1/2,2/5,1/3
4,5,10	28,4	χ_2	-3/4,-1/4	-3/5,-2/5	13/10,17/10	1/2,2/5,1/5
		χ_3	-3/4,-1/4	-4/5,-1/5	11/10,19/10	1/2,2/5,1/5
4,6,10	30,0	χ_2	-3/4,-1/4	-5/6,-1/6	11/10,19/10	1/2,1/3,1/5
		χ_3	-3/4,-1/4	-5/6,-1/6	13/10,17/10	1/2,2/5,1/3
4,10,10	34,4	χ_2	-3/4,-1/4	-9/10,-1/10	13/10,13/10	1/2,2/5,1/5
		χ_3	-3/4,-1/4	-7/10,-3/10	11/10,19/10	1/2,2/5,1/5
5,5,6	27,9	χ_2	-3/5,-2/5	-3/5,-2/5	7/6,11/6	2/3,1/5,1/5
		χ_3	-4/5,-1/5	-4/5,-1/5	7/6,11/6	3/5,2/5,1/3
5,5,10	31,13	χ_2	-3/5,-2/5	-3/5,-2/5	11/10,19/10	4/5,1/5,1/5
		χ_3	-4/5,-1/5	-4/5,-1/5	13/10,17/10	2/5,2/5,2/5
5,6,10	33,9	χ_2	-3/5,-2/5	-5/6,-1/6	13/10,17/10	3/5,1/3,1/5
		χ_3	-4/5,-1/5	-5/6,-1/6	11/10,19/10	3/5,1/3,1/5
6,6,10	35,5	χ_2	-5/6,-1/6	-5/6,-1/6	11/10,19/10	2/3,1/3,1/5
		χ_3	-5/6,-1/6	-5/6,-1/6	13/10,17/10	2/5,1/3,1/3
6,10,10	39,9	χ_2	-5/6,-1/6	-9/10,-1/10	11/10,19/10	2/3,1/5,1/5
		χ_3	-5/6,-1/6	-7/10,-3/10	13/10,17/10	3/5,2/5,1/3
10,10,10	43,13	χ_2	-9/10,1/10	-9/10,-1/10	11/10,19/10	4/5,1/5,1/5
		χ_3	-7/10,3/10	-7/10,-3/10	13/10,17/10	2/5,2/5,2/5

8 Third order equations

8.1 The finite primitive subgroups of $\mathrm{SL}(3)$

The well known list (see [Bl]) is:
1. The simple group G_{168} of order 168, which is isomorphic to $\mathrm{PSL}(2, \mathbf{F}_7)$.
2. $G_{168} \times C_3$, where C_3 is the center of $\mathrm{SL}(3)$.
3. A_5.
4. $A_5 \times C_3$.
5. $H_{216}^{SL_3}$ of order 648. The image of this group in $\mathrm{PSL}(3)$ is the *Hessian group* of order 216.
6. The normal subgroup $H_{72}^{SL_3} \subset H_{216}^{SL_3}$ of order 216.
7. The group $F_{36}^{SL_3} \subset H_{72}^{SL_3}$ of order 108.
8. The Valentiner Group $A_6^{SL_3}$ of order 1080. The image of this group in $\mathrm{PSL}(3)$ is $A_6^{SL_3}/Z(A_6^{SL_3}) \cong A_6$.

8.2 The groups G_{168} and $G_{168} \times C_3$

8.2.1 The data for G_{168}

There are six conjugacy classes $conj_1, \ldots, conj_6$, they correspond to elements of order $\{1, 2, 3, 4, 7, 7\}$. There are two irreducible characters of degree three, called χ_2, χ_3. Both are faithful and unimodular. The three values $0 \leq \lambda < 1$ such that $e^{2\pi i \lambda}$ are the eigenvalues for the representation are given for each conjugacy class as follows:

$$\chi_2 : (\frac{0,0,0}{1}, \frac{0,1,1}{2}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{3,5,6}{7}, \frac{1,2,4}{7})$$

$$\chi_3 : (\frac{0,0,0}{1}, \frac{0,1,1}{2}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{1,2,4}{7}, \frac{3,5,6}{7})$$

The character χ_3 is the dual of χ_2. This implies that the two characters are permuted under an automorphism A of G_{168}. The same automorphism of G_{168} permutes the conjugacy classes $conj_5$ and $conj_6$. For any group G we define $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inner}(G)$, i.e., the group of all autmorphisms of G divided out by the normal subgroup of the inner automorphisms. The character χ_2 corresponds to an embedding $G_{168} \subset \mathrm{SL}(3, \mathbf{Q}(e^{2\pi i/7}))$. This induces

a homomorphism $\mathrm{Gal}(\mathbf{Q}(e^{2\pi i/7})/\mathbf{Q}) \to \mathrm{Aut}(G_{168})$. This homomorphism induces a bijection $\mathrm{Gal}(\mathbf{Q}(\sqrt{-7})/\mathbf{Q}) \to \mathrm{Out}(G_{168})$.
We will give all the data that we calculated for the branch types. We introduce some terminology. The *conjugacy triple* i, j, k *of an admissible triple* (g_0, g_1, g_∞) is defined by: the conjugacy classes of g_0, g_1, g_∞ are $conj_i, conj_j, conj_k$. Further, an *invariant of degree d for a group* $H \subset \mathrm{GL}(V)$ *and a differential equation L with Picard-Vessiot field K* means a (non-zero) homogeneous element in $\overline{\mathbf{Q}}[V]^H$, such that its image in K belongs to $k = \overline{\mathbf{Q}}(z)$. In order to give this a clear meaning, one has to specify the $\overline{\mathbf{Q}}$-algebra homomorphism from $\overline{\mathbf{Q}}[V]$ to K. In our situation we will consider a point $a \in \{0, 1, \infty\}$ and a $< g_a >$-equivariant isomorphism $\psi : V \to V_a \subset K \subset \overline{\mathbf{Q}}((z^*))$, where V_a denotes the solution space at a in the algebraic closure of the completion of k at the point a. The chosen ψ extends in a unique way to a k-algebra homomorphism $\psi : k[V] \to K$.

(1). *Branch type* $[2, 3, 7]$ consists of one element, represented by the conjugacy triple 2,3,5. The genus of the curve is 3. The character $\mathcal{D}$ coincides with χ_3. For this character, the lower bounds for the exponents add up to 3, so they are the actual exponents. From the exponent difference 1 at $z = 0$ one obtains all the data for L: $-1/2, 0, 1/2||-2/3, -1/3, 0||8/7, 9/7, 11/7||\mu = 12293/24696$. This equation was in fact found by Hurwitz [Hu]. Our theoretical considerations provide an "overkill" since the corresponding covering is well known. It is the Klein curve in $\mathbf{P}^2$ given by the homogeneous equation $x_0x_1^3 + x_1x_2^3 + x_3x_0^3 = 0$, having automorphism group G_{168}, or in another terminology, it is the modular curve $X(7)$ with automorphism group $\mathrm{PSL}(2, \mathbf{F}_7)$. The Weierstrass gap theorem implies that the points above ∞ are the only Weierstrass points and they have weight 1.

(2). *Branch type* $[2, 4, 7]$ consists of one element; conjugacy triple 2,4,5; genus 10; $< \chi_2, \mathcal{D} >= 0$, $< \chi_3, \mathcal{D} >= 1$; for χ_3 the lower bounds for the exponents add up to 3 and are the actual bounds; at $z = 0$ there is an exponent difference 1. This leads to the data $-1/2, 0, 1/2||-3/4, -1/4, 0||8/7, 9/7, 11/7||\mu = 5273/10976$.

(3). *Branch type* $[2, 7, 7]$ consists of one element; conjugacy triple 2,5,5; genus 19; $< \chi_2, \mathcal{D} >= 0$, $< \chi_3, \mathcal{D} >= 2$; for χ_3 the lower bounds for the exponents add up to 2; there is an integer exponent difference at $z = 0$. Thus we have

a $\mathbf{P}^1$-family for χ_3 and nine scalar Fuchsian equations, all equivalent over $\overline{\mathbf{Q}}$. The data for the exponents and μ are:
$-1/2, 1, 1/2|| -6/7, -5/7, -3/7||8/7, 9/7, 11/7||\mu = 1045/686$
$-1/2, 0, 3/2|| -6/7, -5/7, -3/7||8/7, 9/7, 11/7||\mu = 2433/1372 \pm 3/392\sqrt{21}$
$-1/2, 0, 1/2|| \quad 1/7, -5/7, -3/7||8/7, 9/7, 11/7||\mu = 1317/2744$
$-1/2, 0, 1/2|| -6/7, \quad 2/7, -3/7||8/7, 9/7, 11/7||\mu = 1205/2744$
$-1/2, 0, 1/2|| -6/7, -5/7, \quad 4/7||8/7, 9/7, 11/7||\mu = 1149/2744$
$-1/2, 0, 1/2|| -6/7, -5/7, -3/7||15/7, 9/7, 11/7||\mu = 3375/2744$
$-1/2, 0, 1/2|| -6/7, -5/7, -3/7||8/7, 16/7, 11/7||\mu = 3263/2744$
$-1/2, 0, 1/2|| -6/7, -5/7, -3/7||8/7, 9/7, 18/7||\mu = 3207/2744$

(4). *Branch type* $[3, 3, 4]$ consists of two elements; both conjugacy triples are 3,3,4; genus 8; no equations found.

(5). *Branch type* $[3, 3, 7]$ consists of one element; conjugacy triple 3,3,5; genus 17; $< \chi_2, \mathcal{D} >= 0$, $< \chi_3, \mathcal{D} >= 1$; for χ_3 the lower bounds for the exponents add up to 2. One adds $+1$ to one of the exponents and tries to find an invariant of degree 4 for G_{168}. The only succesful case is $-2/3, -1/3, 0|| -2/3, -1/3, 0||9/7, 11/7, 15/7||\mu = 0$. One concludes that the differential Galois group G of this equation is a subgroup of G_{168}. There are two subgroups H_1 and H_2 of G_{168}, with orders 21 and 24, such that any proper subgroup of G_{168} is contained in a conjugate of H_1 or H_2. The group H_1 has an invariant of degree 3 and H_2 has another invariant of degree 4. One can verify that no invariants of those types are present here and the conclusion is that the differential Galois group is G_{168}. We note that G contains elements of order 3 and 7, so this also excludes H_2.

(6). *Branch type* $[3, 4, 4]$ consists of one element; conjugacy triple 3,4,4; genus 15. By adding poles of order 1 at the points above 0, one finds that χ_3 occurs with multiplicity 1. For χ_3, the lower bounds for the exponents add up to 2. One adds $+1$ two any of the eight exponents and finds only one case $-1, -1/3, 1/3|| -3/4, -1/4, 0||5/4, 7/4, 2||\mu = 3/16$, where there is an invariant of degree 4. This proves that the differential Galois group H of this equation is a subgroup of G_{168}. It is a proper subgroup, since there is another invariant of degree 4.
The character χ_2 leads to the same data for a possible equation L. Thus we found no equation here for G_{168}.

(7). *Branch type* $[3,4,7]$ consists of two elements; both conjugacy triple are 3,4,5; in both cases $<\chi_2,\mathcal{D}>=0$, $<\chi_3,\mathcal{D}>=1$; no equations are found.

(8). *Branch type* $[3,7,7]$ consists of two elements; conjugacy triples 3,6,6 and 3,6,5; genus 33. For the *first* case $<\chi_2,\mathcal{D}>=2$, $<\chi_3,\mathcal{D}>=0$. For χ_2 we have a $\mathbf{P}^1$-family of equations and the lower bound on the exponents add up to 1. One has to add $+1$ to two of the exponents and search for an invariant of degree 4 in the 45 cases. The only possible data are: $-1/3,1/3,1||-6/7,-5/7,-3/7||8/7,9/7,11/7||\mu=3385/3087$. As in (5) one verifies that the differential Galois group is G_{168}.
For the *second* case $<\chi_2,\mathcal{D}>=1$, $<\chi_3,\mathcal{D}>=1$. For χ_2 data for G_{168} are found: $-2/3,-1/3,0||-6/7,-5/7,-3/7||10/7,13/7,19/7||\mu=830/1029$. For χ_3 data for G_{168} are found:
$-2/3,-1/3,0||-4/7,-1/7,5/7||8/7,9/7,11/7||\mu=-1846/3087$. As in (5) one verifies in both cases that the differential Galois group is G_{168}.

(9). *Branch type* $[4,4,4]$ consists of two elements; both conjugacy triple are 4,4,4; genus 22; no equations found.

(10). *Branch type* $[4,4,7]$ consists of one element; conjugacy triple 4,4,6; genus 31; $<\chi_2,\mathcal{D}>=1$, $<\chi_3,\mathcal{D}>=0$. For χ_2, the sum of the lower bounds on the exponents is 2. One has to add $+1$ to any of the exponents and search for an invariant of degree 4. The only possible data
$-3/4,-1/4,0||-3/4,-1/4,0||9/7,11/7,15/7||\mu=0$ are verified, as in (5), to produce the differential Galois group G_{168}.

(11). *Branch type* $[4,7,7]$ consists of two elements; both conjugacy triples are 4,6,5; genus 40; $<\chi_2,\mathcal{D}>=1$, $<\chi_3,\mathcal{D}>=1$; no equations found.

(12). *Branch type* $[7,7,7]$ consists of one element; conjugacy triple 5,5,5; genus 49; $<\chi_2,\mathcal{D}>=0$, $<\chi_3,\mathcal{D}>=3$. For χ_3 we have a $\mathbf{P}^2$-family and the lower bounds of the exponents add up to 0. The candidates for exponents are obtained by adding $+1$ three times to the lower bounds. For the (many) candidates one imposes an invariant of degree 4 and one uses the method of (5) for the verification. The lower bounds for the exponents are $-6/7,-5/7,-3/7||-6/7,-5/7,-3/7||8/7,9/7,11/7$. We will give the results in table form, indicating the places where the +1's are added and the values for μ.

3									123/49
1	2								75/49
1		2							107/49
			3						-123/49
			1	2					-75/49
			1		2				-107/49
						3			0
						1	2		0
						1		2	0
1			1			1			0
1				1			1		-4/49
1					1			1	-6/49
	1		1				1		4/49
	1			1		1			0
		1	1					1	6/49
		1			1	1			0

The symmetries in this table can be explained by the action of S_3 on $\mathbf{P}^*$.

8.2.2 The group $G_{168} \times C_3$

The differential modules M of dimension 3 over k with group G_{168} and Picard-Vessiot field $K_{168} \supset k$, such that this extension is only ramified above the points $0, 1, \infty$ are given above in the form of one or more cyclic vectors c for M and the minimal equation $L = \partial^3 + A_2\partial^2 + A_1\partial + A_0 \in k[\partial]$. There are four distinct cyclic extensions $K_3 \supset k$ of degree 3, at most ramified above the points $0, 1, \infty$. They are given by the equations $y^3 = z(z-1)$, $y^3 = z$, $y^3 = z - 1$ and $y^3 = z(z-1)^2$. The corresponding differential modules are ke with $\partial e = ae$, and $a = \frac{1/3}{z} + \frac{1/3}{z-1}, \frac{1/3}{z}, \frac{1/3}{z-1}$ and $\frac{1/3}{z} + \frac{2/3}{z-1}$. The differential module $M \otimes_k ke$ has differential Galois group $K_{168} \otimes K_3$. All differential modules with differential Galois group $G_{168} \times C_3$ are obtained in this way. The new module has $c \otimes e$ as cyclic vector. The minimal equation for the cyclic vector $c \otimes e$ is obtained from L by the shift $\partial \mapsto \partial - a$. In this way one finds all the data for the group $G_{168} \times C_3$.

8.3 The groups A_5 and $A_5 \times C_3$

As in subsection 8.2.2, the data for the group $A_5 \times C_3$ can easily be derived from those for A_5. In the sequel we investigate the data for A_5. There are 5 conjugacy classes, denoted by $conj_1, \ldots, conj_5$. They correspond to elements of order $1,2,3,5,5$. There is an automorphism of A_5 permuting the classes $conj_4$ and $conj_5$. There are two irreducible (faithful, unimodular) characters χ_2, χ_3 of dimension 3. Their values on the conjugacy classes are given as the sum of the three eigenvalues:
$\chi_2 : \frac{0,0,0}{1}, \frac{0,1,1}{2}, \frac{0,1,2}{3}, \frac{0,1,4}{5}, \frac{0,2,3}{5}$ $\chi_3 : \frac{0,0,0}{1}, \frac{0,1,1}{2}, \frac{0,1,2}{3}, \frac{0,2,3}{5}, \frac{0,1,4}{5}$.
Both characters are equal to their own dual. For each admissible triple (g_0, g_1, g_∞) the eigenvalue 1 occurs for each g_j. It follows that $<\chi_2, \mathcal{D}>=<\chi_3, \mathcal{D}>= 0$ in all cases. This has as consequence that we have to search for a solution space V such that Vdz consists of differential forms with certain poles. The choice $Vdz \subset H^0(C, \Omega(f^{-1}0))$ leads to a unique V and a unique Fuchsian differential module with group A_5. However, the corresponding scalar equation may have apparent singularities. We note also that the second symmetric power of the order two equations for $A_5^{\mathrm{SL}(2)}$ produce Fuchsian equations for A_5 of order three. The exponents of a Fuchsian equation obtained in this way have the form $-a, 0, a|| - b, 0, b|| - c + 1, 1, 1 + c$ with $0 < a, b, c < 1$. The possibilities for a, b, c are of course determined by the chosen admissible triple. One observes that this type of Fuchsian equations has more poles at the points above 0 and 1. In the following table, we have chosen for each possible Picard-Vessiot field and the two characters χ_2, χ_3, a corresponding Fuchsian equation from the abundance of equations having more poles. We note that in some of the cases of the table a verification has been made to exclude the possibility that the differential Galois group is $\mathrm{PSL}(2) \subset \mathrm{SL}(3)$.

branch type	genus	char	exp 0	exp 1	exp ∞	μ
2,3,5	0	χ_2	-1,-1/2,1/2	-2/3,-1/3,0	6/5,9/5,2	43/225
		χ_3	-1,-1/2,1/2	-2/3,-1/3,0	7/5,8/5,2	52/225
2,5,5	4	χ_2	-1,-1/2,1/2	-4/5,-1/5,0	7/5,8/5,2	1/5
		χ_3	-1,-1/2,1/2	-3/5,-2/5,0	6/5,9/5,2	1/5
3,3,5	5	χ_2	-1/3,0,1/3	-1/3,0,1/3	3/5,1,7/5	0
		χ_3	-1/3,0,1/3	-1/3,0,1/3	1/5,1,9/5	0
3,5,5(1)	9	χ_2	-1,-2/3,-1/3	-3/5,-2/5,0	6/5,9/5,3	-16/25
		χ_3	-1,-2/3,-1/3	-4/5,-1/5,1	7/5,8/5,2	-9/5
3,5,5(2)	9	χ_2	-1,-1/3,1/3	-4/5,-1/5,0	6/5,9/5,2	4/25
		χ_3	-1,-1/3,1/3	-3/5,-2/5,0	7/5,8/5,2	6/25
5,5,5	13	χ_2	-1,-3/5,3/5	-3/5,-2/5,0	7/5,8/5,2	6/25
		χ_3	-1,-1/5,1/5	-4/5,-1/5,0	6/5,9/5,2	4/25

8.4 The group $H_{216}^{\mathrm{SL}(3)}$

The group $G = H_{216}^{\mathrm{SL}(3)}$ has 24 conjugacy classes corresponding to elements of order $1, 2, 3, 3, 3, 3, 3, 4, 6, 6, 9, 9, 9, 9, 9, 9, 12, 12, 18, 18, 18, 18, 18, 18$. There are 6 irreducible faithful unimodular characters of degree 3, called $\chi_{8,9,10,11,12,13}$. The Galois group of $\mathbf{Q}(e^{2\pi i/9})/\mathbf{Q}$ maps isomorphically to $\mathrm{Out}(G)$. This Galois group, which is a cyclic group of order 6, acts transitively on the 6 characters mentioned above. The number of isomorphism classes of coverings of $\mathbf{P}^*$ with group isomorphic to is 20. For each one of them there are 6 Fuchsian differential modules. Some of those 120 differential modules will be represented by more than one Fuchsian scalar equation, some maybe by none. We will present here the data found for one of the two elements in the branch type $[9, 12, 18]$. The characters $\chi_{8,9,10,11,12,13}$ appear with multiplicity $0, 1, 1, 2, 3, 2$ in the character $\mathcal{D}$ of the holomorphic differential forms on the corresponding curve of genus 244.

For χ_8 we have not found an equation.

For χ_9 *the lower bounds on the exponents at* $0, 1, \infty$ *are* $-7/9, -1/9, 8/9|| - 7/12, -1/3, -1/12||10/9, 23/18, 29/18$. They add up to 3 and so they are the actual sets of exponents. There is an exponent difference 1 at $z = 0$. This leads to the value $\mu = \frac{2105}{2592}$.

For χ_{10} the lower bounds on the exponents at $0, 1, \infty$ are $-7/9, -4/9, 2/9|| - 7/12, -1/3, -1/12||23/18, 16/9, 35/18$. They add up to 3 and are the actual exponents. The exponent difference 1 at $z = 0$ leads to $\mu = \frac{245}{2592}$.

For χ_{11} one finds a $\mathbf{P}^1$-family and the lower bounds on the exponents at $0, 1, \infty$: $-5/9, -2/9, 7/9|| - 11/12, -2/3, -5/12||19/18, 11/9, 31/18$ add up to 2. The actual exponents are obtained by adding $+1$ to each of the nine exponents, with the exception of the exponent $-2/9$, because there is an integer exponent difference at $z = 0$. Working from right to left with this adding of $+1$ (i.e., first $+1$ to the last exponent of E_∞ and so on) one finds the following values for μ:

$$\frac{1427}{1296}, \frac{1625}{1296}, \frac{1763}{1296}, \frac{835}{1296}, \frac{1517}{2592}, \frac{763}{1296}, \mu_1, \mu_2, \frac{1295}{1296},$$

where $\mu_{1,2}$ are the two roots of the irreducible equation $\frac{9}{112}\mu^2 - \frac{5203}{16128}\mu + \frac{1687855}{5225472}$. All the equations are equivalent over $\overline{\mathbf{Q}}(z)$.

For χ_{12} one finds a $\mathbf{P}^2$-family and the lower bounds on the exponents at $0, 1, \infty$: $-8/9, -5/9, 4/9|| - 11/12, -2/3, -5/12||19/18, 25/18, 14/9$ add up to 1. The actual exponents are obtained by adding twice $+1$ to exponents, with the exception of the exponent $-5/9$ (because of the integer exponent difference at $z = 0$). This leads to 45 equations which are equivalent over the field $\overline{\mathbf{Q}}(z)$. The values for μ are easily calculated by means of lemma 6.1. The most interesting among the 45 cases are those where the integer difference of the exponents at $z = 0$ are 2 and 3.
All the quadratic equations for μ (i.e., with exponent difference 2 at $z = 0$) that one finds are irreducible. The exponent difference 3 occurs for the exponent sets $-8/9, -5/9, 22/9|| - 11/12, -2/3, -5/12||19/18, 25/18, 14/9$ and the equation for μ is the irreducible polynomial

$$\frac{3}{1120}\mu^3 - \frac{3683}{161280}\mu^2 + \frac{6690323}{104509440}\mu - \frac{5343433}{90699264} = 0.$$

For χ_{13} one finds a $\mathbf{P}^1$-family and the lower bounds on the exponents at $0, 1, \infty$: $-8/9, -2/9, 1/9|| - 11/12, -2/3, -5/12||25/18, 31/18, 17/9$ add up to 2. The actual exponents are obtained by adding $+1$ to any of them with the exception of the exponent $-8/9$. We do this adding $+1$, as before, from right to left and find the values for μ:

$$\frac{1343}{1296}, \frac{1301}{1296}, \frac{1325}{1296}, \frac{-227}{1296}, \frac{-175}{1296}, \frac{133}{1296}, \mu_1, \mu_2, \frac{2417}{1296},$$

where $\mu_{1,2}$ are the two roots of the irreducible equation $\frac{9}{16}\mu^2 - \frac{5071}{2304}\mu + \frac{3211535}{1492992}$. All the equations are equivalent over the field $\overline{\mathbf{Q}}(z)$.

8.5 The group $H_{72}^{\mathrm{SL}(3)}$

The number of conjugacy classes is 16 and there are 2 irreducible faithful unimodular characters of degree 3. There is an isomorphism of the Galois group of $\mathbf{Q}(e^{2\pi i/9})/\mathbf{Q}$ to $\mathrm{Out}(G)$. There are 11 coverings of $\mathbf{P}^*$ with Galois group isomorphic to $H_{72}^{\mathrm{SL}(3)}$. No equations found.

8.6 The group $F_{36}^{\mathrm{SL}(3)}$

The number of conjugacy classes is 14. They corespond to elements of order $1, 2, 3, 3, 3, 3, 4, 4, 6, 6, 12, 12, 12, 12$. There are two irreducible faithful unimodular characters of degree 3, called χ_6 and χ_{10}. The values λ such that $e^{2\pi i\lambda}$ are eigenvalues of the corresponding matrices for χ_6 and χ_{10} are:

$$(\frac{1,1,1}{1}, \frac{0,1,1}{2}, \frac{2,2,2}{3}, \frac{1,1,1}{3}, \frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{1,1,4}{6}, \frac{2,5,5}{6}, \frac{1,4,7}{12}, \frac{5,8,11}{12}, \frac{1,4,7}{12}, \frac{5,8,11}{12})$$

$$(\frac{1,1,1}{1}, \frac{0,1,1}{2}, \frac{1,1,1}{3}, \frac{2,2,2}{3}, \frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{2,5,5}{6}, \frac{1,1,4}{6}, \frac{5,8,11}{12}, \frac{1,4,7}{12}, \frac{5,8,11}{12}, \frac{1,4,7}{12})$$

χ_{10} is the dual of χ_6. The representation χ_6 produces an embedding of the group in $\mathrm{SL}(3, \mathbf{Q}(e^{2\pi i/12}))$. The resulting homomorphism of $\mathrm{Gal}(\mathbf{Q}(e^{2\pi i/12})/\mathbf{Q})$ to the group $\mathrm{Out}(G)$ turns out to be an isomorphism. There are 12 coverings of $\mathbf{P}^*$ with group isomorphic to $F_{36}^{\mathrm{SL}(3)}$. There are many scalar Fuchsian equations attached to this group. We only give some examples:

The branch type $[6, 12, 12]$ has two elements. For the first one, the character χ_{10} has multiplicity 1 in $\mathcal{D}$. For χ_{10}, the lower bounds for the exponents at $0, 1, \infty$ are $-2/3, -1/6, 5/6|| - 11/12, -2/3, -5/12||17/12, 5/3, 23/12$. Since they add up to 3, they are the actual exponents. The value $\mu = \frac{1345}{864}$ follows from the exponent difference 1 at $z = 0$.

For the second element of the branch type $[6,12,12]$, the character χ_{10} has multiplicity 3 in $\mathcal{D}$. For this character, we have a $\mathbf{P}^2$-family of differential equations. The lower bounds on the exponents at $0,1,\infty$ are:
$-5/6,-1/3,1/6||-11/12,-2/3,-5/12||13/12,4/3,21/12$. These numbers add up to 1. The result is a collection of 45 differential equations. The values for μ are easily obtained from the integer exponents difference at $z=0$.

8.7 The Valentiner group $A_6^{\mathrm{SL}(3)}$.

The group $G = A_6^{\mathrm{SL}(3)}$ has 17 conjugacy classes $conj_1, \ldots, conj_{17}$ corresponding to elements of orders $1,2,3,3,3,3,4,5,5,6,6,12,12,15,15,15,15$. There are 4 irreducible characters $\chi_2,\chi_3,\chi_4,\chi_5$ of degree 3, all of them faithful and unimodular. The eigenvalues corresponding to the 4 characters are roots of unity which are represented as $e^{2\pi i\lambda}$ with $0 \le \lambda < 1$. We give a table for all the values of the λ's for the characters $\chi_2,\chi_3,\chi_4,\chi_5$. For the first four conjugacy classes they are (of course) $\frac{0,0,0}{1}, \frac{0,1,1}{2}, \frac{1,1,1}{3}, \frac{2,2,2}{3}$. For the other conjugacy classes they are:

$$\frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{0,2,3}{5}, \frac{0,1,4}{5}, \frac{1,1,4}{6}, \frac{2,5,5}{6}, \frac{1,4,7}{12}, \frac{5,8,11}{12}, \frac{1,4,10}{15}, \frac{2,5,8}{15}, \frac{7,10,13}{15}, \frac{5,11,14}{15}$$

$$\frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{0,1,4}{5}, \frac{0,2,3}{5}, \frac{1,1,4}{6}, \frac{2,5,5}{6}, \frac{1,4,7}{12}, \frac{5,8,11}{12}, \frac{7,10,13}{15}, \frac{5,11,14}{15}, \frac{1,4,,10}{15}, \frac{2,5,8}{15}$$

$$\frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{0,2,3}{5}, \frac{0,1,4}{5}, \frac{2,5,5}{6}, \frac{1,1,4}{6}, \frac{5,8,11}{12}, \frac{1,4,7}{12}, \frac{5,11,14}{15}, \frac{7,10,13}{15}, \frac{2,5,8}{15}, \frac{1,4,10}{15}$$

$$\frac{0,1,2}{3}, \frac{0,1,2}{3}, \frac{0,1,3}{4}, \frac{0,1,4}{5}, \frac{0,2,3}{5}, \frac{2,5,5}{6}, \frac{1,1,4}{6}, \frac{5,8,11}{12}, \frac{1,4,7}{12}, \frac{2,5,8}{15}, \frac{1,4,10}{15}, \frac{5,11,14}{15}, \frac{7,10,13}{15}$$

The character χ_2 corresponds to a certain embedding $G \subset \mathrm{SL}(3,\mathbf{Q}(e^{2\pi i/60}))$. This induces a homomorphism of the Galois group of $\mathbf{Q}(e^{2\pi i/60})/\mathbf{Q}$ to $\mathrm{Out}(G)$, which turns out to be an isomorphism. There are 98 coverings of $\mathbf{P}^*$ with group isomorphic to $A_6^{\mathrm{SL}(3)}$ and so there are 4×98 Fuchsian differential modules attached to this group and probably many more Fuchsian scalar equations. We give just two "easy" examples:

The branch type $[2,5,15]$ has only one element. The character χ_3 occurs with multiplicity 1 in $\mathcal{D}$ and the corresponding lower bounds for the exponents at $0,1,\infty$ are $-1/2,0,1/2||-3/5,-2/5,0||16/15,19/15,5/3$. Those numbers add up to 3 and so they are the actual exponents. The exponent difference

1 at $z = 0$ leads to $\mu = \frac{541}{1080}$.

The branch type $[6, 15, 15]$ has two elements. For one of them and for χ_2 one finds a single equation with data
$-5/6, -1/3, 1/6|| -2/3, -4/15, -1/15||4/3, 26/15, 29/15||\mu = 133/1350$.
For χ_3 one finds a $\mathbf{P}^2$-family with lower bounds for the exponents
$-5/6, -1/3, 1/6|| - 13/15, -2/3, -7/15||17/15, 4/3, 23/15$ adding up to 1. This results in 45 equations. The values for μ follows easily from the integer exponents difference at $z = 0$.
For χ_4 one finds a $\mathbf{P}^1$-family with lower bounds for the exponents
$-2/3, -1/6, 5/6|| - 14/15, -11/15, -1/3||16/15, 19/15, 5/3$ which add up to 2. There results 9 equations for $A_6^{\mathrm{SL}(3)}$.

9 Appendix

9.1 The holomorphic Lefschetz formula

We note that the holomorphic Lefschetz fixed point formula, as given in [G-H], differs by a sign from the formula proved here. This is due our choice of right action instead of left action. The proof presented here is due to Bas Edixhoven.

Proposition 9.1 *Let X/k be a curve (smooth, irreducible, complete) of genus $\neq 0$ over an algebraically closed field of characteristic 0. Let $g \neq 1$ be an automorphism of X. The set of fixed points of g is denoted by X^g. For every fixed point $x \in X^g$ one can choose a formal local parameter t_x such that $g^*t_x := t_x \circ g$ is equal to $\zeta_x t_x$ for some root of unity ζ_x. Then the trace $tr(g^*, \Omega_C(C))$ of the induced action of g on the space of the holomorphic differential form is equal to $1 + \sum_{x \in X^g} \frac{1}{\zeta_x^{-1} - 1}$.*

Proof. The formula is easily verified for an elliptic curve X. Hence we will assume in the sequel that the genus of X is > 1. The element g has then finite order n and the existence of the local parameter t_x with the required property is easily seen. Consider the (possibly ramified) covering $\pi : X \to Y = X/ < g >$. The exact sequence of sheaves on X,

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

induces the long exact sequence

$$0 \to H^0(Y, \Omega_{Y/k} \otimes_{O_Y} \pi_* O_X) \to H^0(X, \Omega_{X/k}) \to \Omega_{X/Y}$$

$$\to H^1(Y, \Omega_{Y/k} \otimes_{O_Y} \pi_* O_X) \to H^1(X, \Omega_{X/k}) \to 0,$$

since $\Omega_{X/Y}$ is a skyscraper sheaf. Further $H^1(X, \Omega_{X/k}) = H^0(X, O_X)^* \cong k$ with the trivial g-action. We will introduce some notations:
$H^* = H^0 \ominus H^1$, $tr(g^*, H^*) = tr(g^*, H^0) - tr(g^*, H^1)$, $< g >$ is the finite cyclic group generated by g, $< g >^*$ is its dual, i.e., the group of the characters on $< g >$ with values in k^*.
The sheaf $\pi_* O_X$ decomposes as a direct sum of line bundles L_χ for $\chi \in < g >^*$. The line bundle L_χ is defined by $L_\chi(U) = \{f \in O_X(\pi^{-1}U) | \, g^* f = \chi(g^{-1}) f\}$. In particular, $L_{triv} = O_Y$. Then

$$H^*(Y, \Omega_{Y/k} \otimes_{O_Y} \pi_* O_X) = \oplus_{\chi \in <g>^*} H^*(Y, \Omega_{Y/k} \otimes_{O_Y} L_\chi).$$

Riemann Roch implies

$$tr(g^*, H^*(Y, \Omega_{Y/k} \otimes_{O_Y} \pi_* O_X) =$$

$$\sum_\chi \chi(g^{-1})(\dim H^0(\Omega_{Y/k} \otimes L_\chi) - \dim H^1(\Omega_{Y/k} \otimes L_\chi)) =$$

$$\sum_\chi \chi(g^{-1})(\deg L_\chi + g(Y) - 1) = \sum_{\chi \in <g>^*} \chi(g^{-1}) \deg L_\chi.$$

We have to calculate now the degrees of L_χ. For a point $x \in X$ we write $Stab_x \subset < g >$ for the stabilizer subgroup at x. The cyclic group $Stab_x$ has order e_x and is generated by g^{d_x} where $n = e_x \cdot d_x$. There exists a formal local parameter t_x at x such that $h^* t_x$ is a suitable constant multiple of t_x. Define the character $\chi_x : Stab_x \to k^*$ by the formula $h^* t_x = \chi_x(h^{-1}) t_x$ for all $h \in Stab_x$. For $\chi \in < g >^*$ we define the integer $a_{\chi,x} \geq 0$ to be minimal such that $\chi|_{Stab_x}$ (the restriction of χ to $Stab_x$) is equal to $\chi_x^{a_{\chi,x}}$.
Define the sheaf $\mathcal{Q}$ by the exact sequence exact:

$$0 \to L_\chi^{\otimes ord(L_\chi)} \to O_Y \to \mathcal{Q} \to 0.$$

The sheaf $\mathcal{Q}$ is a skyscraper sheaf and the dimension of its space of global sections is the sum of the dimension of its stalks at the ramified points of X. For every ramified point y of Y (i.e., there lies a ramified point of X above y) one chooses a point x above y and one finds the formula

$$\deg L_\chi = \frac{-1}{ord(L_\chi)} \sum_{y \in Y} \frac{a_{\chi,x} ord(\chi)}{e_x} = -\sum_{y \in Y} \frac{a_{\chi,x}}{e_x}.$$

Thus is sufficient information about the term $tr(g^*, H^*(Y, \Omega_{Y/k} \otimes_{O_Y} \pi_* O_X)$. The next term $tr(g^*, \Omega_{X/Y})$ is equal to $\sum_{x \in X} tr(g^*, (\Omega_{X/Y})_x)$. For $x \notin X^g$ this local term at x is seen to be zero. For $x \in X^g$, the local contribution is -1.

Combining the results obtained so far one has

$$tr(g^*, \Omega_{X/k}(X)) = 1 - \#(X^g) - \sum_{\chi \in <g>^*} \chi(g^{-1}) \cdot (\sum_{y \in Y} \frac{a_{\chi,x}}{e_x}).$$

Interchanging the summation leads to the formula

$$tr(g^*, \Omega_{X/k}(X)) = 1 - \frac{1}{n} \sum_{x \in X^g} (\sum_{i=1}^{n} i \zeta_x^i) = 1 + \sum_{x \in X^g} \frac{1}{\zeta_x^{-1} - 1}.$$

□

9.2 The G-module of holomorphic differentials

Consider the following situation: $X \to Y$ is a (ramified) Galois covering of curves (non singular, irreducible, complete, over $\overline{\mathbf{Q}}$) with group G. The group G is supposed to act on the right on X. This induces a left action of G on the space of holomorphic differentials $\Omega_X(X)$. We are interested in the irreducible components of this G-module. Let $\mathcal{D}$ denote the character of the G-module $\Omega_X(X)$. The following formula yields the information that we are looking for.

Theorem 9.2 *Let χ be any character of G. Then*

$$< \chi|\mathcal{D} > = < \chi|triv\ > + \chi(e)(g(Y) - 1) +$$

$$\sum_{y \in Y} \frac{1}{e_y} \sum_{i=1,\ldots,e_y-1} i < \chi|_{Stab_x}|\chi_x^{-i} >_{Stab_x},$$ *where:*

- *e is the neutral element of G, $triv$ is the trivial character of G and $g(Y)$ is the genus of Y.*

- *The sum $\sum_{y\in Y}$ is extended over the points of Y which are ramified.*

- *e_y is the ramification index of the point y (i.e., the ramification index of any point x lying above y).*

- *For every ramified point $y \in Y$ one chooses a point $x \in X$ above y.*

- *$Stab_x$ is the stabilizer (in G) of the point x. This is a cyclic group of order e_y.*

- *$\chi|_{Stab_x}$ is the restriction of the character χ to the subgroup $Stab_x$.*

- *Let t be a local parameter at x. Then any element $g \in Stab_x$ acts on t as $g^*t = t \circ g = \sum_{n\geq 1} a_n(g)t^n$. The character χ_x of $Stab_x$ is defined by $\chi_x(g) = a_1(g)$. Since χ_x is injective, all characters of $Stab_x$ are $\{\chi_x^j | 0 \leq j < e_y\}$.*

- *$< \mid >_{Stab_x}$ denotes the scalar product of characters on the group $Stab_x$.*

- *Choose a fixed e_yth root of unity $\zeta = e^{2\pi i/_y}$. Let $h \in Stab_x$ satisfy $\chi_x(h) = \zeta$. Then the sum (corresponding to a fixed y and i with $1 \leq i \leq e_y - 1$) can be rewritten as $\frac{1}{e_y}\sum_{0\leq j\leq e_y-1}\chi(h^j)\zeta^{ij}$.*

Proof. (After Bas Edixhoven)
According to the proposition we have for $g \in G$, $g \neq 1$ that $\mathcal{D}(g) = 1 + \sum_{x\in X^g}\frac{1}{\zeta_{g,x}^{-1}-1}$ where $g^*t_x = \zeta_{g,x}t_x$ for any $x \in X^g$ and t_x a suitable local formal parameter at x. Define the (possibly virtual) character $\mathcal{D}_* = \mathcal{D} - triv$. Then $\mathcal{D}_*(g) = \sum_{x\in X^g}\mathcal{D}_x^*(g)$ with $(\mathcal{D}_*)_x(g) := \frac{1}{\zeta_{g,x}^{-1}-1}$. Let χ be any (possibly virtual) character of G. Then $< \chi|\mathcal{D}_* >=$

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\mathcal{D}_*(g)} = \frac{1}{|G|}\{\chi(e)(g(X)-1) + \sum_{g\neq e}\chi(g)\sum_{x\in X^g}\overline{(\mathcal{D}_*)_x(g)}\}$$

$$= \frac{1}{|G|}\{\chi(e)(g(X)-1) + \sum_{x\in X}\sum_{e\neq g\in Stab_x}\chi(g)\overline{\mathcal{D}_*(g)}\};$$

the double sum is constant on the orbit of x and thus we find an equality

$$= \frac{1}{|G|}\chi(e)(g(X)-1) + \sum_{y\in Y}\frac{1}{e_y}\sum_{e\neq g\in Stab_x}\chi(g)\overline{(\mathcal{D}_*)_x(g)},$$

with some $x \in X$ choosen above each ramified $y \in Y$,

$$= \frac{\chi(e)(g(X)-1)}{|G|} + \sum_{y\in Y}\frac{1}{e_y}\sum_{e\neq g\in Stab_x}\chi(g)\frac{1}{e_y}\sum_{i=1}^{e_y-1} i\chi_x(g)^i,$$

since $\overline{(\mathcal{D}_*)_x(g)} = \frac{1}{\chi_x(g)-1} = \sum_{i=1}^{e_y-1} i\chi_x(g)^i$ and the former expression is thus equal to

$$= \frac{\chi(e)(g(X)-1)}{|G|} + \sum_{y\in Y}\frac{1}{e_y}\sum_{i=1}^{e_y-1} i\frac{1}{e_y}\sum_{e\neq g\in Stab_x}\chi(g)\overline{\chi_x^{-i}(g)};$$

the last sum is equal to $<\chi|_{Stab_x}|\chi_x^{-i}>_{Stab_x} - \frac{\chi(e)}{e_y}$ and the total expression is equal to

$$\frac{\chi(e)(g(X)-1)}{|G|} - \chi(e)\sum_{y\in Y}\frac{1}{e_y}\frac{(e_y-1)e_y/2}{e_y} + \sum_{y\in Y}\sum_{i=1}^{e_y-1}\frac{i}{e_y} < \chi|_{Stab_x}|\chi_x^{-i}>_{Stab_x}$$

$$= \frac{\chi(e)(g(X)-1)}{|G|} - \frac{\chi(e)}{2}\sum_{y\in Y}(1-\frac{1}{e_y}) + \sum_{y\in Y}\frac{1}{e_y}\sum_{i=1}^{e_y-1} i < \chi|_{Stab_x}|\chi_x^{-i}>_{Stab_x}.$$

Using the Riemann-Hurwitz-Zeuthen formula and replacing $\mathcal{D}_*$ by $\mathcal{D} - triv$ one finally finds the required formula $<\chi|\mathcal{D}>=$

$$<\chi|triv> + \chi(e)(g(Y)-1) + \sum_{y\in Y}\frac{1}{e_y}\sum_{i=1,\dots,e_y-1} i < \chi|_{Stab_x}|\chi_x^{-i}>_{Stab_x}.$$

□

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