Ultimate Polynomial Time

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Abstract

The class $\mathcal{UP}$ of 'ultimate polynomial time' problems over $\mathbb{C}$ is introduced; it contains the class $\mathcal{P}$ of polynomial time problems over $\mathbb{C}$.

The τ-Conjecture for polynomials implies that $\mathcal{UP}$ does not contain the class of non-deterministic polynomial time problems definable without constants over $\mathbb{C}$. This latest statement implies that $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

A notion of 'ultimate complexity' of a problem is suggested. It provides lower bounds for the complexity of structured problems.

1 Introduction

A model of Computation and Complexity over a ring was developed in [2] and [1], generalizing the classical $\mathcal{NP}$-completeness theory [3]. Of particular interest is the model of Complexity over the ring $\mathbb{C}$ of complex numbers.

In the model of complexity over $\mathbb{C}$, a machine is allowed to input, to output and to store complex numbers, to compute polynomials and to branch on equality (See the textbook [1] for background). This model shares some of the features of the classical (Turing) model of computation (There is a discussion in [7]). It is known [5, 9] that the hypothesis $\mathcal{BPP} \not\supseteq \mathcal{NP}$ in the Turing setting implies $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$. ($\mathcal{BPP}$ stands for Bounded

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Probability Polynomial Time. If $\mathcal{BPP}$ would happen to contain $\mathcal{NP}$, then there would be polynomial time randomized algorithms for such tasks as factorizing large integers or breaking most modern cryptographic systems).

In [8, 1, 10], the hypothesis $\mathcal{P} \neq \mathcal{NP}$ over the Complex numbers was related to a number-theoretical conjecture. Define a straight-line program as a list

$$s_0 = 1 \ , \ s_1 = x \ , \ s_2 \ , \ \cdots \ , \ s_\tau$$

where s_i is, for $i \geq 2$, either $s_j + s_k$, $s_j - s_k$ or $s_j s_k$, for some $j, k < i$. Each s_i is thus a polynomial in x. The straight-line program is said to *compute* the polynomial $s_\tau(x)$.

Given a polynomial $f \in \mathbb{Z}[x]$, the quantity $\tau(f)$ is defined as the smallest τ such that there exists a straight-line program $s_0, \cdots, s_\tau$ computing $f(x)$. For instance, $\tau(x^{2^n} - 1) = 2 + n$. Similarly, if $g \in \mathbb{Z}[x_1, \cdots, x_n]$, then $\tau(g)$ is the minimal length of a straight-line program $s_0 = 1, s_1 = x_1, \cdots, s_n = x_n, s_{n+1}, \cdots, s_\tau = g(x)$.

The τ Conjecture for Polynomials . *There is a constant $a > 0$ such that for any univariate polynomial $f \in \mathbb{Z}[x]$,*

$$n(f) < \tau(f)^a$$

where $n(f)$ is the number of integer zeros of f, without multiplicity.

It is known [1] that the τ-Conjecture for polynomials implies $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$. A main step towards this result is the fact that, if the τ-Conjecture is true, then the polynomials

$$p_d(x) = (x-1)(x-2)\cdots(x-d)$$

are *ultimately hard to compute.* This means that there cannot be constants a and b such that, for any degree d, for some non-zero polynomial f (depending on d), we would have

$$\tau\ (\ p_d(x) f(x)\ ) < a\, (\log_2 d)^b$$

Therefore, all non-zero multiples of p_d are hard to compute, hence the wording ultimately hard.

The goal of this paper is to define a new complexity class $\mathcal{UP}$, of *ultimate polynomial time* problems. This class will contain $\mathcal{P} \cap \mathcal{K}$, where $\mathcal{P}$ is the class of problems decidable in polynomial time and $\mathcal{K}$ is the class of problems definable without constants (See [4] and Definition 1 below). Moreover:

Theorem 1. *The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are true:*

(a) The τ-conjecture for polynomials.

(b) $\forall d$, p_d is ultimately hard to compute.

(c) $\mathcal{UP} \not\supseteq \mathcal{NP} \cap \mathcal{K}$ over $\mathbb{C}$.

(d) $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

The implication (a) $\Rightarrow$ (b) $\Rightarrow$ (d) appears in [1], the hypothesis (c) in-between is new. It is at least as likely as the τ-conjecture, while still implying $\mathcal{P} \neq \mathcal{NP}$.

We will also show a $\mathcal{NP}$-hardness result for the class $\mathcal{UP}$: there is a structured problem $(HN, HN^{\text{yes}}) \in \mathcal{NP} \cap \mathcal{K}$, such that:

Theorem 2. $\mathcal{UP} \not\supseteq \mathcal{NP} \cap \mathcal{K}$ *over $\mathbb{C}$ if and only if $(HN, HN^{\text{yes}}) \notin \mathcal{UP}$ over $\mathbb{C}$.*

The problem (HN, HN^{yes}) is precisely the (structured) Hilbert Nullstellensatz, known to be $\mathcal{NP}$-complete over $\mathbb{C}$ ([1]).

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2 Background and Notations

Recall from [1] that $\mathbb{C}^\infty$ is the disjoint union

$$\mathbb{C}^\infty = \bigsqcup_{i=0,1,\cdots} \mathbb{C}^i$$

This means that there is a well-defined *size* function,

$$\begin{array}{rccl} \text{Size}: & \mathbb{C}^\infty & \to & \mathbb{N} \\ & x & \mapsto & \text{Size}(x) = i \text{ such that } x \in \mathbb{C}^i \end{array}$$

A decision problem X is a subset of $\mathbb{C}^\infty$. It is in the class $\mathcal{P}$ if and only if there is a machine M over $\mathbb{C}$, that terminates for any input x in time bounded by a polynomial on $\mathrm{Size}(x)$, and such that

$$M(x) = 0 \;\Leftrightarrow\; x \in X$$

where $M(x)$ is the result of running M with input x. Without loss of generality we may assume that $M(x) \in \{0; 1\}$.

Under some circumstances, it is possible to assume that the machine M above has only coefficients 0 or 1 (This is called a *constant-free* machine). However, one may have to replace problem X over $\mathbb{C}$ by problem $X \cap \mathbb{Z}$ over $\mathbb{Z}$, with unit cost. (This is the contents of Propositions 3 and 9 of Chapter 7 of [1]). In order to avoid this technical complication and keep the same problem over $\mathbb{C}$, we will follow another approach to Elimination of Constants.

This approach was introduced by Koiran in [4]. The idea is to consider only machines for a subclass of problems. This subclass will contain most of the interesting examples, while precluding pathological cases such as $X = \{\pi\}$.

Definition 1 (Koiran). A problem L is said to be *definable without constants* if for each input size n there is a formula F_n in the first order theory of $\mathbb{C}$ such that 0 and 1 are the only constants occurring in F_n, and for any $x \in \mathbb{C}^n$, $x \in L$ if and only if $F_n(x)$ is true (there is no restriction on the size of F_n.

For future reference, we quote below Theorem 2 of [4]. The original statements of both Definition 1 and Theorem 3 are actually more general (for any algebraically closed field of characteristic 0).

Theorem 3 (Koiran). *Let $L \subseteq K^\infty$ be a problem which is definable without constants. If $L \in \mathcal{P}$, L can be recognized in polynomial time by a constant-free machine.*

The class of all the problems definable without constants will be denoted by $\mathcal{K}$.

We will need crucially in the sequel the notion of a *structured problem.* A structured problem is a pair (X, X^{yes}), $X^{\mathrm{yes}} \subseteq X \subseteq \mathbb{C}^\infty$. A non-structured problem X can always be written as the structured problem $(\mathbb{C}^\infty, X)$. The class $\mathcal{UP}$ will be meaningful only as a class of structured problems. But first of all, recall that

Definition 2. A structured problem (X, X^{yes}) belongs to the class $\mathcal{P}$ if and only if $X \in \mathcal{P}$ and $X^{\text{yes}} \in \mathcal{P}$.

Definition 3. A structured problem (X, X^{yes}) belongs to the class $\mathcal{K}$ if and only if $X \in \mathcal{K}$ and $X^{\text{yes}} \in \mathcal{K}$.

Definition 4. A structured problem (X, X^{yes}) belongs to the class $\mathcal{NP}$ if and only if:

(1) The problem X belongs to the class $\mathcal{P}$.

(2) There is a machine M with input x, g such that

$$x \in X \text{ and } \exists g \in \mathbb{C}^{\infty} \text{ s.t. } M(x,g) = 0 \Leftrightarrow x \in X^{\text{yes}}$$

(3) Furthermore, there is a polynomial p such that, for all $x \in X^{\text{yes}}$, there is $g \in \mathbb{C}^{\infty}$ such that $M(x,g) = 0$ and the running time of M with input x, g is no more than $p(\text{Size}(x))$.

Example 1. Let HN be the class of all lists $(m, n, f_1, \cdots, f_m)$ where $f_1, \cdots, f_m$ are polynomials in n variables. Each polynomial $f = \sum f_I x^I$ is represented sparsely by a list of monomials $(S, m_1, \cdots, m_S)$, where each monomial is a list $(f_I, I_1, \cdots, I_n)$.

An important convention to have in mind: integers appearing in the definition of a problem should be represented in bit representation. In this case, m, n, S, I_j are all lists of zeros and ones. Complex values are represented by one complex number. With this convention, HN is clearly in the class $\mathcal{P}$.

We also define HN^{yes} as the subset of polynomial systems in HN that have a common root over $\mathbb{C}$.

The definition above of the structured problem (HN, HN^{yes}) can be translated into first order constant-free formulae over $\mathbb{C}$. Therefore, $(HN, HN^{\text{yes}}) \in \mathcal{K}$. It is also $\mathcal{NP}$-complete over the complex numbers (Theorem 1 in Chapter 5 of [1]).

Example 2. Let

$$\begin{aligned} X &= \{(m, x) \in \mathbb{N} \times \mathbb{C}\} \\ X^{\text{yes}} &= \{(m, x) \in X \text{ such that } x \in \{1, 2, \cdots, m\}\} \end{aligned}$$

with the convention that m is in bit representation, while x is a complex number. Hence, $\text{Size}((m,x)) = O(1 + \lceil \log_2(m) \rceil)$. Then the problem (X, X^{yes}) is in $\mathcal{NP}$ over $\mathbb{C}$. The machine $M(x,g)$ can be constructing by guessing the bit decomposition g_i of x, and computing $x - \sum g_i 2^i$.

Again, (X, X^{yes}) is definable without constants.

3 Construction of the class $\mathcal{UP}$

In Chapter 7 of [1], it is proved that if the problem (X, X^{yes}) from Example 2 would happen to belong to the class $\mathcal{P}$, then condition (b) in Theorem 1 would be false. Therefore (b) implies $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$.

The class $\mathcal{UP}$ will be constructed by abstracting the same reasoning. The construction relies on some geometric properties of structured problems in $\mathcal{P}$. The notation that follows will be used in the sequel:

Let (X, X^{yes}) be a structured problem with $X \in \mathcal{P}$. We denote by $X \cap \mathbb{C}^i$ the set $\{x \in X : \text{Size}(x) = i\}$ of size i instances of the problem. Then we write $\overline{X \cap \mathbb{C}^i}$ for its Zariski closure over $\mathbb{C}$. We can define a new object associated to X as:

$$\overline{X} = \bigsqcup_{i=0,1,\cdots} \overline{X \cap \mathbb{C}^i}$$

We can think of $\overline{X}$ as the *closure* of X, indeed it is the smallest 'closed' problem containing X. Remark that in Examples 1 and 2, we have respectively $X = \overline{X}$ and $HN = \overline{HN}$.

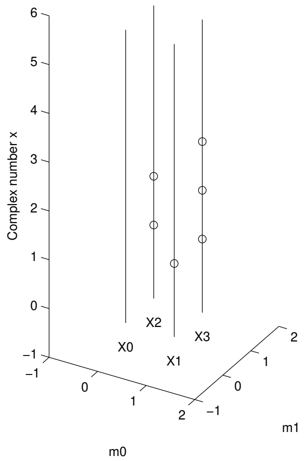
We can also decompose each Zariski-closed set $\overline{X \cap \mathbb{C}^i}$ into a finite union of irreducible components (affine varieties). Thus it makes sense to write $\overline{X}$ as the countable union:

$$\overline{X} = \bigcup X_j$$

where each X_j is an affine variety lying in some $\mathbb{C}^s$, where $s = \text{Size}(x), x \in X_j$. We can further define:

$$\begin{aligned} X_j^{\text{yes}} &= X_j \cap X^{\text{yes}} \\ X_j^{\text{no}} &= X_j \setminus X^{\text{yes}} \end{aligned}$$

(See Figure 1). Using this notation,



This is Problem (X, X^{yes}) from Example 2, restricted to the inputs (m_0, m_1, x) of size 3. X is represented by the four (complex !) lines and X^{yes} by the dots. Each of the complex lines is irreducible, and hence corresponds to a different X_i.

Figure 1: (X, X^{yes}) from Example 2

Definition 5. The class $\mathcal{UP}$ is the class of all structured problems (X, X^{yes}) such that $X \in \mathcal{P}$ and for all X_i, there is a non-zero polynomial $f_i \in \mathbb{Z}[x_1, \cdots, x_{s_i}]$, where $s_i = \text{Size}(x)$ for $x \in X_i$, with the following properties:

(1) $\tau(f_i)$ is polynomially bounded in S_i.

(2) $X_i^{\text{yes}} \subseteq Z(f)$ or $X_i^{\text{no}} \subseteq Z(f)$

Proposition 1. $\mathcal{P} \cap \mathcal{K} \subseteq \mathcal{UP}$

Proof of Proposition 1. Let (X, X^{yes}) be in $\mathcal{P} \cap \mathcal{K}$. Let $M = M(x)$ be the machine that recognizes $x \in X^{\text{yes}}$ in polynomial time, where the input x is assumed to be in $\overline{X}$. Although it is possible that an $x \in X_i$ is not in X, it is still possible to recognize $x \in X^{\text{yes}}$ in polynomial time. Indeed, X is also in $\mathcal{P}$. The machine $M(x)$ will check $x \in X$ and $x \in X^{\text{yes}}$.

Now we apply elimination of constants (Theorem 3), and choose M to be constant-free.

The nodes of the machine M are supposed to be numbered. Given an input x, the *path* followed by input x is the list of nodes traversed during the computation of $M(x)$.

When the input is restricted to one of the affine varieties X_i's, we can define the canonical path (associated to X_i as the path followed by the generic point of X_i. This corresponds to the following procedure:

At each decision node, at time T, branch depends upon an equality $F^T(x) = 0$, where x is the original input. The polynomial F can be computed within the machine running time. In case $F^T(x) = 0$ for all $x \in X_i$, we follow the Yes-path and say that this branching is trivial.

If not, we follow the no-path and say that this branching is non-trivial. The fact that X_i is a variety is essential here, since it guarantees that only a codimension ≥ 1 subset of inputs may eventually follow the Yes-path at this time.

The set of inputs that do NOT follow the canonical path can be described as the zero-set of

$$f_i = \prod F^T$$

where the product ranges over the non-trivial branches only. The polynomial f_i can be computed in at most twice the running time of the machine M restricted to X_i. By hypothesis, this is polynomial time in the size of $x \in X_i$.

Since we assumed that M returns only 0 or 1, the set of the inputs that follow the canonical path (i.e. $Z(f_i)$) is either all in X_i^{yes} or all in its complementary X_i^{no}.

There are now two possibilities. First possibility, X_i^{yes} has measure zero in X_i, and therefore it must be contained in $Z(f_i)$. Second possibility, X_i^{yes} has non-zero measure, hence it contains the complementary of $Z(f_i)$, and hence X_i^{no} is a subset of $Z(f_i)$. $\square$

4 Proof of the Theorems

Proof of Theorem 1.

(a) $\Rightarrow$ (b) is trivial, refer to [1] Chapter 7.

(b) $\Rightarrow$ (c): Let (X, X^{yes}) be the problem in Example 2. Since X_i^{no} is generic in X_i, all inputs in X_i^{yes} should escape the canonical path. Hence, if f_d is the polynomial that defines the canonical path, $f_d(i) = 0$ for $i = 1, 2, \cdots, d$. But then it cannot be evaluated in time polylog(d), by hypothesis (b). Hence, under the assumption (b), the problem (X, X^{yes}) is not in $\mathcal{UP}$. It does belong to $\mathcal{NP} \cap \mathcal{K}$, so $\mathcal{UP} \not\supseteq \mathcal{NP} \cap \mathcal{K}$.

(c) $\Rightarrow$ (d) : Using Theorem 2, Condition (c) implies that $(HN, HN^{\text{yes}}) \notin \mathcal{UP}$. However, since $(HN, HN^{\text{yes}}) \in \mathcal{K}$, Proposition 1 implies $(HN, HN^{\text{yes}}) \notin \mathcal{P}$. Hence $\mathcal{P} \neq \mathcal{NP}$ over $\mathbb{C}$. $\square$

Proof of Theorem 2. Let $(X, X^{\text{yes}}) \in \mathcal{NP} \cap \mathcal{K}$ and assume that $(HN, HN^{\text{yes}}) \in \mathcal{UP}$. We have to show that $(X, X^{\text{yes}}) \in \mathcal{UP}$.

For each X_i, one can embed (X_i, X_i^{yes}) into some $(HN_i, HN_i^{\text{yes}})$ as follows:

Let $M = M(x)$ be the deterministic polynomial time machine to recognize X, and let $N = N(x, g)$ be the non-deterministic polynomial time machine to recognize X^{yes}. We can assume without loss of generality that M and N are constant-free (Theorem 3).

Let T be the maximum running time of M and N when the input is restricted to X_i. Let $\phi(x)$ be the combined Register Equations of machines M and N for time T (Theorem 2 in Chapter 3 of [1]). Thus, $\phi(x)$ is a system of polynomial equations with integer coefficients and indeterminate coefficients $x_1, x_2, \cdots$. The polynomial system $\phi(x)$ can be constructed in polynomial time from x, and the size of $\phi(x)$ is polynomially bounded by the size of x.

We claim that $\phi(X_i)$ is contained in some HN_j, and that in that case $\phi(X_i^{\text{yes}}) \subseteq HN_j^{\text{yes}}$ and $\phi(X_i^{\text{no}}) \subseteq HN_j^{\text{no}}$.

Indeed, $X_i \subseteq \mathbb{C}^s$ for some s, and $\phi(\mathbb{C}^s) \subseteq HN_j$ for some j. Then $x \in X_i$ belongs to X^{yes} if and only if the corresponding $\phi(x)$ has a solution over $\mathbb{C}$.

We now distinguish two cases:

Case 1: HN_j^{yes} has measure zero in HN_j. Thus $HN_j^{\text{yes}} \subseteq Z(\hat{f}_j)$ for an easy-to-compute polynomial $\hat{f}_j$. In that case, since X_i^{yes} gets mapped into HN_j^{yes}, the composition $f_i = \hat{f}_j \circ \phi$ gives the polynomial associated to X_i.

Case 2: HN_j^{no} has measure zero in HN_j. Thus $HN_j^{\text{no}} \subseteq Z(\hat{f}_j)$ for an easy-to-compute polynomial $\hat{f}_j$. In that case, since X_i^{no} gets mapped into HN_j^{no}, $f_i = \hat{f}_j \circ \phi$ is the polynomial associated to X_i. $\square$

5 Ultimate Complexity

Let (Y, Y^{yes}) be a problem over $\mathbb{C}$, definable without constants and with Y semi-decidable (i.e. Y is the halting set of some machine). The closure $\overline{Y}$ is well-defined and can be written as a countable union of irreducible varieties Y_i.

For any machine M to solve (Y, Y^{yes}), one can produce a family of polynomials f_i, vanishing on the set of inputs that follow the canonical-path of M restricted to Y_i. As in item (2) of Definition 5, we have

$$Y_i^{\text{yes}} \subseteq Z(f_i) \text{ or } Y_i^{\text{no}} \subseteq Z(f_i)$$

Also, for each input size s, one has a finite number of indices i corresponding to components i $Y_i \subseteq \overline{Y}$ of size-s input. We can thus maximize over those indices i:

$$u_M(s) = \max_{i:Y_i \subseteq \mathbb{C}^i} \tau(f_i)$$

This invariant may be called 'ultimate running time', and is a lower bound (up to a constant) for the worst-case running time of M. As with ordinary complexity theory, one can define the 'ultimate complexity' class of a problem as the class of functions $u : \mathbb{N} \to \mathbb{R}$ such that $\exists M, c > 0 : \forall x u_M(x) \leq cu(x)$ and M recognizes (Y, Y^{yes}). This provides notions such as 'ultimate logarithmic time' or 'ultimate exponential time'.

In [6], a similar construction is used to obtain lower bounds for some specific decision problems. Those problems, however, had a very simple geometric structure (for each 'input size', X^{yes} was a finite set in $\mathbb{C}$). The motivation of this paper was to extend some of the ideas therein and in Chapter 7 of [1] to non-codimension-1 problems.

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