# Lower bounds for some decision problems over $\mathbb{C}$

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**Abstract**

Lower bounds for some explicit decision problems over the complex numbers are given.

## 1 Introduction

This paper is about lower bounds for certain decision problems over $\mathbb{C}$. (See [3] for the model of computation and for background). In particular, we will provide lower bounds for the complexity of deciding, given $x$, if $p^d(x) = 0$ for some explicit polynomials $p^d$.

A related problem is to give lower bounds for the evaluation of explicit polynomials. This has been an active subject of research since [6]. See [4] for modern developments and for bibliographical remarks. More recent results appeared in [1] and [2].

Most of those bounds use the Ostrowsky model of computation ([4] page 6): sum and multiplication by an algebraic constant are free, and the complexity of a computation for polynomial $f(x)$ is the number of non-scalar multiplications, i.e., of multiplications of two polynomials in the variable $x$. For instance, Horner rule for a degree $d$ polynomial requires $d$ non-scalar multiplications.

All those bounds apply trivially to the complexity of evaluating polynomials by a 'machine over $\mathbb{C}$' as defined in [3], or to the (multiplicative-branching) complexity of a computation tree for evaluating the same polynomial.

Little is known, however, about the application of those bounds to decision problems (Over $\mathbb{C}$, in the sense of [3], or by a decision tree as in [4], Definition (4.19) page 115. In this definition, each node of a computation tree can perform one algebraic operation or comparison, and therefore a natural measure of complexity is the depth of the tree).

In this paper, only decision problems of the form below will be considered: let $X \subseteq \mathbb{N} \times \mathbb{C}$, and let $X_d = \{x \in \mathbb{C} : (d, x) \in X\}$. Typically, $d$ is the problem size and $\#X_d \leq d$. One can think of $X$ as the disjoint union of the zero-set of a

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family of polynomials of degree $\leq d$, where $d \in \mathbb{N}$. The two following forms of a decision problem are natural in this setting:

**Problem 1.** For any fixed $d$, decide wether $x \in X_d$.

**Problem 2.** Decide wether $(d, x) \in X$

Problem 1 is non-uniform, in the sense that we allow for a different machine over $\mathbb{C}$ or a different decision tree to be used for each value of $d$. However, we want a bound on the running time or on the multiplicative complexity of the tree, as a function of $d$.

Problem 2 is uniform. It is harder than Problem 1, in the sense that it cannot be solved by a decision tree, since $\# X_d$ can be arbitrarily large. It requires a machine over $\mathbb{C}$, that will eventually branch according to the value of $d$.

Lower bounds for Problem 1 are also lower bounds for Problem 2.

A trivial, topological lower bound for Problems 1 and 2 when $\# X_d = d$ is $\log_2 d$. Sharper known bounds come from the 'Canonical Path' argument, see [3] section 2.5: Let $f$ be a univariate polynomial. The complexity of deciding $f(x) = 0$ is bounded below by the minimum of the complexity of evaluating $g(x)$, where $g$ ranges over the non-zero multiples of $f$.

If one assumes some property of $f$ that propagates to its multiple $g$, then one eventually obtains a sharper, non-trivial lower bounds.

In Lemma 1 below, we will give conditions on the roots of $f$ that will provide lower bounds for the evaluation of $g$. Essentially, we will require a subset of the roots to be rapidly growing. This will imply a rapid growth property for the coefficients of $g$. Then, the results of [1, 2] imply a lower bound for the complexity of evaluating $g$. Thus we will be able to construct specific polynomials that are hard to decide in the *non-uniform* sense, viz.

**Lower bound 1.** *The set $X = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : x = 2^{2^{di}}, 0 \leq i \leq d$, cannot be solved in time polylog(d) in the setting of Problem 1.*

**Lower bound 2.** *The set $Y = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : p^d(x) = 0\}$, where $p^d(t) = \sum_{i=0}^{d} 2^{2^{d(d-i)}} t^i$, cannot be decided in time polylog(d) in the setting of Problem 1.*

In a more classical computer-science language, we can define the input size of some $(d, x)$ as $\log d$. This means that the integer $d$ is represented in binary notation, while variable $x$ can contain an arbitrary complex number. In that case, 'time polylog($d$) in the setting of Problem 1' can be refrased as $\mathcal{P}_{/\mathrm{poly}}$. The lower bounds above become now: $X \notin \mathcal{P}_{/\mathrm{poly}}$ and $Y \notin \mathcal{P}_{/\mathrm{poly}}$.

Non-uniform lower bounds 1 and 2 can be compared to the following easier, uniform lower bound:

**Lower bound 3.** *The set $Z = \{(d, x) \in \mathbb{Z} \times \mathbb{C} : q^d(x) = 0\}$, where $q^d(t) = \sum_{i=0}^{d} 2^{2^i} t^i$, cannot be decided in time polylog(d) in the setting of Problem 2.*

This means that the set $Z$, where $d$ is represented in binary notation and $x$ is a complex number, does not belong to $\mathcal{P}$ over $\mathbb{C}$.

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## 2 Background and notations

**Definition 1.** Let $K \subset L$ be finite algebraic extensions of $\mathbb{Q}$. Let $\nu$ be a valuation in $M_K$. Then we extend the notation $\nu$ to $L$ by:

$$\nu(x) = \frac{\sum_\mu n_\mu \mu(x)}{\deg[L:K]}$$

where the sum ranges over all the valuations $\mu$ of $L$ that are 'above' $\nu$, and where $n_\mu$ is the 'local degree' of $L : K$. The local degree is defined as $n_\mu = \deg[L_\mu : K_\nu]$, where $K_\nu$ is the completion of $K$ under the metric induced by the absolute value $|.|_\nu$.

Recall that for $x \in K$, $\deg[L:K]\nu(x) = \sum_\mu n_\mu \nu(x)$. The case $K = \mathbb{Q}$ is an immediate consequence of Corollary 2 of Theorem 1 in Chapter II, p. 39 of [5].

**Definition 2.** Let $g$ be a polynomial with algebraic coefficients in some extension $K$ of $\mathbb{Q}$. Let $\nu$ be a valuation in $M_K$. The *Newton diagram* of $g$ at $\nu$ is the (lower) convex hull of the set $\{(i, \nu(g_i)), i = 0 \cdots d\}$.

The basic property of Newton diagrams used here is the following.

**Proposition 1.** *Suppose that $\zeta_1, \cdots, \zeta_d$ are the roots of a univariate polynomial $g \in K[x]$. Let the roots of $g$ be ordered so that*

$$\nu(\zeta_1) \geq \cdots \geq \nu(\zeta_d)$$

*and let the increasing sequence $i_j$ assume the values $0$, $d$ and all the values of $i$ where:*

$$\nu(\zeta_i) > \nu(\zeta_{i+1})$$

*Then the sharp corners of the Newton diagram are precisely the points of the form $(i_j, \nu(g_{i_j}))$ for all $j$.*

*Moreover, the slope of the segment $[(i_{j-1}, \nu(g_{i_{j-1}})), (i_j, \nu(g_{i_j}))]$ is precisely $-\nu(\zeta_{i_j})$.*

*Proof of Proposition 1.* The proof uses the following property of valuations: $\nu(\sum x_i) \geq \min \nu(x_i)$. Furthermore, when that minimum is attained in only one $x_i$, we have equality.

Let $i_{j-1} < k < i_j$. Writing

$$\begin{aligned} g_{i_{j-1}} &= g_d \sigma_{d-i_{j-1}}(\zeta_1, \cdots, \zeta_d) \\ g_k &= g_d \sigma_{d-k}(\zeta_1, \cdots, \zeta_d) \\ g_{i_j} &= g_d \sigma_{d-i_j}(\zeta_1, \cdots, \zeta_d) \end{aligned}$$

one can pass to the valuation by:

$$\begin{aligned}
\nu(g_{i_{j-1}}) &= \nu(g_d) + \nu(\zeta_{i_{j-1}+1}) + \cdots + \nu(\zeta_d)\\
\nu(g_k) &\geq \nu(g_d) + \nu(\zeta_{k+1}) + \cdots + \nu(\zeta_d)\\
\nu(g_{i_j}) &= \nu(g_d) + \nu(\zeta_{i_j+1}) + \cdots + \nu(\zeta_d)
\end{aligned}$$

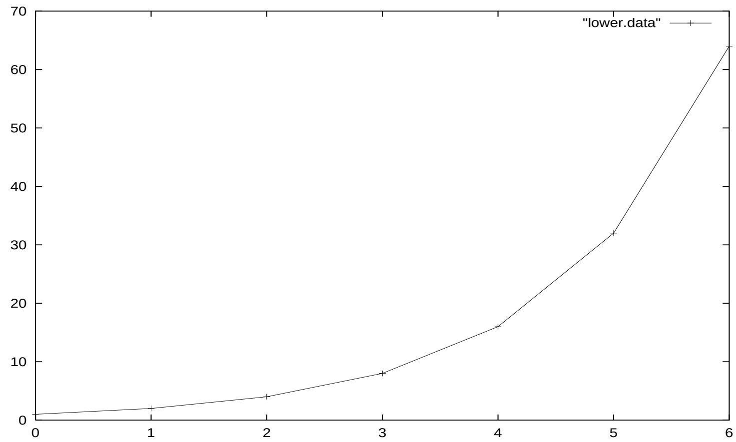
Subtracting, one obtains:

$$\begin{aligned}
\nu(g_{i_j}) - \nu(g_{i_{j-1}}) &= -\nu(\zeta_{i_{j-1}+1}) - \cdots - \nu(\zeta_{i_j})\\
&= -(i_j - i_{j-1})\nu(\zeta_{i_j})\\
\nu(g_{i_j}) - \nu(g_k) &\leq -\nu(\zeta_{k+1}) - \cdots - \nu(\zeta_{i_j})\\
&\leq -(i_j - k)\nu(\zeta_{i_j})
\end{aligned}$$

This concludes the proof. □

# 3 Uniform lower bounds

We can now prove Lower Bound 3.




*Proof of Lower bound 3.* The Newton diagram of $q^d$ at 2 is $\{(i, 2^i) : 0 \leq i \leq d\}$. (This latest set is convex, since the points lie on the curve $y = 2^x$ and this curve is convex). Therefore, there is a unique root $\zeta$ of $q^d$ that minimizes $\nu(\zeta)$.

Since $q^d_{d-1} = (-\sum \zeta_i) q^d_d$, where the sum ranges over all the roots, we have:

$$\nu_2(q^d_{d-1}) = \nu_2(q^d_d) + \min \nu_2(\zeta_i) = \nu_2(q^d_d) + \nu_2(\zeta)$$

Replacing by the actual values of the coefficients, one gets:

$$\nu_2(\zeta) = -2^{d-1} \tag{1}$$

Now, suppose that there is a machine $M$ that decides $q^d(t) = 0$ in time $\text{polylog}(d)$. One can assume without loss of generality that this machine has no constant but 0 and 1. Let its running time be bounded by $T = a(\log d)^b$.

Let us fix $d > 2 + T^2$. We will derive a contradiction.

Let $g$ be the polynomial defining the canonical path (recall that $d$ is fixed now, so this is the path followed by generic $t \in \mathbb{C}$). It can be computed in time $\leq T^2$, so we have the following bounds:

$$\begin{aligned} \deg g &\leq 2^{T^2} \\ 0 \leq \nu_2(g_p) &\leq 2^{T^2} \end{aligned}$$

Since $\zeta$ is also a root of $g$, there are coefficients $g_i$ and $g_j$, $i \neq j$, such that:

$$(j-i)\nu_2(\zeta) = \nu_2(g_i) - \nu_2(g_j) \tag{2}$$

Thus, $|\nu_2(\zeta)| \leq |\nu_2(g_i)| + |\nu_2(g_i)|$. This implies:

$$|\nu_2(\zeta)| \leq 2^{1+T^2} < 2^{d-1}$$

Replacing by equation 1, one obtains $2^{d-1} < 2^{d-1}$, a contradiction. □

# 4 Non-uniform lower bounds

**Lemma 1.** *Let $g = g(t)$ be a degree $D$ polynomial with algebraic coefficients. Let $\nu$ be a (non-archimedian) valuation of $K = \mathbb{Q}[g_0, \cdots, g_D]$. Let $\xi_1, \cdots \xi_D$ be the roots of $g$, and assume they are ordered in such way that:*

$$\nu(\xi_1) \geq \cdots \geq \nu(\xi_D)$$

*Suppose that there is a subsequence $\zeta_j = \xi_{i_j+1}$, $j = 1 \cdots d$, such that the following holds:*

1. $\nu(\zeta_d) \geq 1$
2. $\nu(\zeta_j) \geq 2(i_{j+1} - i_j)\ \nu(\zeta_{j+1})$, *for* $0 \leq j \leq d-1$.

*Then $g$ cannot be evaluated in less than*

$$L \geq \sqrt{\frac{d}{28 \log_2 D + 1}}$$

*multiplications.*

*Proof of Lemma 1.* We can assume without loss of generality that the ordering of the $\xi_i$ satisfies:

$$\cdots \xi_{i_j} < \xi_{i_j+1} = \zeta_j \leq \xi_{i_j+2} \cdots$$

For $j \in \{1, \cdots, d-1\}$ we have:

$$\nu(g_{i_j}) - \nu(g_{i_{j+1}}) = \nu(\xi_{i_j+1}) + \cdots + \nu(\xi_{i_{j+1}})$$

Hence, using $\nu(\xi_{i_{j+1}}) > \nu(\zeta_d) \geq 1$:

$$\nu(\zeta_j) \leq \nu(g_{i_j}) - \nu(g_{i_{j+1}}) \leq (i_{j+1} - i_j)\nu(\zeta_j)$$

By the same argument, for $j \in \{0, \cdots, d-2\}$:

$$\nu(\zeta_{j+1}) \leq \nu(g_{i_{j+1}}) - \nu(g_{i_{j+2}}) \leq (i_{j+2} - i_{j+1})\nu(\zeta_{j+1})$$

Hence,

$$\frac{\nu(g_{i_j}) - \nu(g_{i_{j+1}})}{\nu(g_{i_{j+1}}) - \nu(g_{i_{j+2}})} \geq \frac{\nu(\zeta_j)}{(i_{j+1} - i_j)\nu(\zeta_{j+1})} \geq 2$$

Set $G_j = \nu(g_{i_j})$ for $j = 0, \cdots, d-1$. We know that the $G_j$ are such that $|G_{j+1} - G_j| < \frac{1}{2}|G_j - G_{j-1}|$. Hence

$$\#\{\sum s_j G_j, s_j \in \{0; 1\}\} = 2^d$$

Hence:

$$\#\{\nu(\prod_{s \in S} g_s), S \subset \{0, \cdots, D\}\} \geq 2^d$$

and hence

$$\mu(g) = \#\{\sum_{S \subset \{0, \cdots, D\}} \theta_S \prod_{s \in S} g_s, \theta_S \in \{0; 1\}\} \geq 2^{2^d}$$

By Lemma 1 in [1] or by Lemma 4 in [2],

$$\mu(g) \leq 2^{(D+1)^{28L^2}}$$

and hence, taking logs:

$$(D+1)^{28L^2} \geq 2^d$$

Taking logs again:

$$28L^2 \geq \frac{d}{\log_2 D + 1}$$

and hence:

$$L \geq \sqrt{\frac{d}{28 \log_2 D + 1}}$$

$\square$

Note: Lemma 1 in [1] is slightly more general than Lemma 4 in [2]. However, using Lemma 4 in [2] it is possible to replace all the appearances of the number 28 in the statement and proof of Lemma 1 above by the number 21.

*Proof of Lower Bound 2.* We see from its Netwon diagram that the polynomial $p$ has distinct roots $\zeta_1, \cdots \zeta_d$ with:

$$\nu_2(\zeta_i) = 2^{d(d-i+1)} - 2^{d(d-i)} = 2^{d(d-i)}(2^d - 1)$$

So we have $\nu_2(\zeta_d) = 2^d - 1 > 1$, and

$$\nu_2(\zeta_i)/\nu_2(\zeta_{i+1}) = 2^d \tag{3}$$

Assume that there are $a, b$ such that for each $d$, there is a machine $M$ over $\mathbb{C}$ deciding $p(t) = 0$ in time $T = a(\log d)^b$. Its generic path is defined by a polynomial $g(t)$ of degree $\leq 2^T$.

Let us fix $d > 28(T+1)T^2$. In particular $d \geq T+1$. We are in the conditions of Lemma 1, where $D = 2^T$. From that Lemma, it follows that

$$T \geq \sqrt{\frac{d}{28 \log_2 2^T + 1}} \geq \sqrt{\frac{d}{28(T+1)}}$$

Hence,

$$28T^2(T+1) \geq d$$

contradicting our choice of $d$. □

Equation (3) holds trivially in the proof of Lower bound 1. The rest of the proof is verbatim the same.

# References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