TORIC NEWTON METHOD FOR POLYNOMIAL HOMOTOPIES

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ABSTRACT. This paper defines a generalization of Newton's method to deal with solution paths defined by polynomial homotopies that lead to extremal values. Embedding the solutions in a toric variety leads to explicit scaling relations between coefficients and solutions. Toric Newton is a symbolic-numeric algorithm where the symbolic pre-processing exploits the polyhedral structures. The numerical stage uses the additional variables introduced by the homogenization to scale the components of the solution vectors to the complex unit circle. Toric Newton generates appropriate affine charts and enables to approximate the magnitude of large solutions of polynomial systems.

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1. INTRODUCTION

If a polynomial system has a large root, we first want to estimate its magnitude before the approximation of its actual value. In the context of polyhedral homotopies we can certify [15] diverging solution paths and separate those from paths leading to extremal values. The purpose of this paper is to show how polyhedral methods can approximate the magnitude of large roots. In [14] and [26] polyhedral homotopies were introduced to exploit sparsity. The number of solution paths equals the bound of Bernshteǐn's theorem. First we give some notations to state Bernshteǐn's theorems.

Denote $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. The *support* of a polynomial $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ is the set $A = \{ \mathbf{a} \in \mathbb{Z}^n \mid c_{\mathbf{a}} \neq 0 \}$. The *Newton polytope* P of f is the convex hull of its support A. The structure of a polynomial system $F(\mathbf{x}) = \mathbf{0}$ is represented by the Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ spanned by the supports $\mathcal{A} = (A_1, A_2, \ldots, A_n)$. Denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In [1] Bernshteǐn proved that the number of isolated solutions in $(\mathbb{C}^*)^n$ of a polynomial system with generic coefficients equals the mixed volume $V_n(\mathcal{P})$ of its Newton polytopes $\mathcal{P}$.

In the nongeneric case we examine faces of systems. A *face* $\partial_{\mathbf{v}} P$ of a polytope P is characterized by a direction $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and spanned by $\partial_{\mathbf{v}} A = \{ \mathbf{a} \in A \mid \forall \mathbf{b} \in A : \langle \mathbf{a}, \mathbf{v} \rangle \leq \langle \mathbf{b}, \mathbf{v} \rangle \}$. A *face polynomial of* f is $\partial_{\mathbf{v}} f(\mathbf{x}) = \sum_{\mathbf{a} \in \partial_{\mathbf{v}} A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$. A *face system of* F is $\partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$ with equations $\partial_{\mathbf{v}} f_i(\mathbf{x}) = 0$, $i = 1, 2, \ldots, n$. For $V_n(\mathcal{P}) > 0$, Bernshteǐn's second theorem [1] states that $F(\mathbf{x}) = \mathbf{0}$ has fewer than $V_n(\mathcal{P})$ isolated solutions in $(\mathbb{C}^*)^n$ if and only if there is some $\partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$ that has a solution in $(\mathbb{C}^*)^n$.

A toric variety X is a solution set of a polynomial system that contains the torus $(\mathbb{C}^*)^n$ as a dense open subset with $(\mathbb{C}^*)^N$ as a group acting on X. In section 2 we make this definition from [11] concrete in applying the homogenization procedure described in [5]. The group action is used in the third section to scale the solution components to the complex unit circle,

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narrowing the torus $(\mathbb{C}^*)^n$ to a numerically favorable region. See [6], [7], [11], [13], and [20] for more on toric varieties.

Representing diverging and badly-scaled solutions is a common struggle in the daily life of a numerical analysist. In [19, Chapter 5], [18], [27, 28] we find coefficient and equation scaling, combined with centering, an effective and useful tool. The classical homogenization [2], [17], [19] in projective space easily leads to clustering and fails to recognize the deficiency behavior when applied to sparse polynomial systems. In the third section of this paper we exemplify this situation.

In the fourth section we define Toric Newton method and analyze the cost of its execution. This algorithm is a symbolic-numeric approach. The Toric Newton method consists of a symbolic pre-processing stage, where the homogeneous forms are determined and the normal fans are refined to be simplicial. This stage is performed only once. The numerical stage is executed iteratively while following the solution paths and takes place in the original space of dimension n.

Toric Newton augments the robustness of polyhedral end games [15] as used in numerical homotopies [17]. Convergence properties of Toric Newton are addressed in the fifth section. The complexity of Newton in the dense case has been analyzed in [2] and has recently been extended to sparse complexity models [8], multi-homogeneous [9] and over-determined systems [10]. The Toric Newton method parallels similar developments in providing algorithms using torus embeddings to compute sparse resultants [21], residues [3], [4], and Bezoutians [22]. All these methods rely on polytopes [29]. Here we give practical evidence for the performance of the polyhedral constructions.

2. Newton Polytopes and Homogeneous Coordinates

If the Newton polytope lies in a hyperplane, then the coordinates of the polynomial are homogeneous. The exponents of the scaler λ are given by the components of the direction that is perpendicular to the hyperplane, as expressed formally in the following proposition.

Proposition 2.1. ([13, page 193]) *Consider* $f(\mathbf{x}) = \sum_{\mathbf{a}\in A} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}}$.
If $\exists \mathbf{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, $\exists d \in \mathbb{Z}$: $\forall \mathbf{a} \in A, \langle \mathbf{a}, \mathbf{v}\rangle = d$, *then* $f(\lambda^{v_1}x_1, \lambda^{v_2}x_2, \ldots, \lambda^{v_n}x_n) = \lambda^d f(\mathbf{x})$.

Proof. Denote $\lambda^{\mathbf{v}}\mathbf{x} = \lambda^{v_1}x_1\lambda^{v_2}x_2\cdots\lambda^{v_n}x_n$, then $(\lambda^{\mathbf{v}}\mathbf{x})^{\mathbf{a}} = (\lambda^{v_1}x_1)^{a_1}(\lambda^{v_2}x_2)^{a_2}\cdots(\lambda^{v_n}x_n)^{a_n} = \lambda^{\langle \mathbf{a},\mathbf{v}\rangle}$. So, $f(\lambda^{\mathbf{v}}\mathbf{x}) = \sum_{\mathbf{a}\in A} c_{\mathbf{a}}(\lambda^{\mathbf{v}}\mathbf{x})^{\mathbf{a}} = \sum_{\mathbf{a}\in A} c_{\mathbf{a}}\lambda^{\langle \mathbf{a},\mathbf{v}\rangle}\mathbf{x}^{\mathbf{a}} = \lambda^d f(\mathbf{x})$. □

The multilinearity of the inner product provides us with a whole range of homogeneous coordinates when the polytope lies in the intersection of many hyperplanes.

Example 2.1. *(projective space)* Let $(x, y, z) \in \mathbb{P}^2$ and consider $f(x, y, z) = x + y + z$. The support of f is the set $A = \{(1,0,0),(0,1,0),(0,0,1)\}$ and lies in the hyperplane $\langle \mathbf{a}, \mathbf{v}\rangle = d$ with $\mathbf{v} = (1,1,1)$, $d = 1$. The homogeneous coordinates are $(\lambda^1 x, \lambda^1 y, \lambda^1 z)$. ■

Example 2.2. *(multi-projective space)* Let $(x_0, x_1, y_0, y_1) \in \mathbb{P}^1 \times \mathbb{P}^1$ and consider $f(x_0, x_1, y_0, y_1) = x_1y_1 - x_0y_0$. The support of f is the set $A = \{(0,1,0,1),(1,0,1,0)\}$ and lies in the hyperplanes $\langle \mathbf{a}, \mathbf{v}^i\rangle = d$ with $\mathbf{v}^1 = (1,1,0,0)$, $\mathbf{v}^2 = (0,0,1,1)$, $\mathbf{v}^3 = (1,0,0,1)$, and $d = 1$. First we have the two usual bi-homogeneous coordinates $(\lambda^1 x_0, \lambda^1 x_1, y_0, y_1)$

and $(x_0, x_1, \lambda^1 y_0, \lambda^1 y_1)$, which allow to scale the x-variables independently from the y's. In addition, we have the third relation, because the Newton polytope is an edge and not of dimension two as one would expect. The corresponding homogeneous coordinates are $(\lambda^1 x_0, x_1, y_0, \lambda^1 y_1)$. ■

Examples 2.1 and 2.2 are particular cases of homogeneous coordinates. The general definition is given below. In applying the homogenization method in [5] we copy the original monomials $\mathbf{x}^{\mathbf{a}}$. By this adaption we recognize the structure of the polytope and have explicit scaling relations between the coefficients and solutions, as needed in the toric version of Newton's method.

Definition 2.1. Consider $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, with $P = \mathrm{conv}(A)$. Assume $\mathrm{vol}(P) > 0$. Let P have N facets given by N inward pointing normals $\mathbf{v}^i \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with all components relatively prime. For every $\mathbf{v}^i$, define $d_i = \min_{\mathbf{a} \in A} \langle \mathbf{a}, \mathbf{v}^i \rangle$. The facets are spanned by $\partial_{\mathbf{v}^i} A = \{\mathbf{a} \in A | \langle \mathbf{a}, \mathbf{v}^i \rangle = d_i\}$. The *homogenization map* ϕ_A introduces for every $\mathbf{v}^i$ a new variable z_i, as follows:

$$\phi_A(\mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathbf{a}} \prod_{i=1}^{N} z_i^{\langle \mathbf{a}, \mathbf{v}^i \rangle - d_i} \quad \text{and} \quad \phi_A \left(\sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \right) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \phi_A(\mathbf{x}^{\mathbf{a}}). \tag{1}$$

If $\mathrm{vol}(P) = 0$, then A lies in some hyperplanes $\langle \mathbf{x}, \mathbf{v}^i \rangle = c_i$, and we add $\pm \mathbf{v}^i$ to the normals of the k-faces of P, where $k = \dim(A) - 1$.

The result of the homogenization is the map $\phi_A(f) : (\mathbb{C}^*)^n \times \mathbb{C}^N \to \mathbb{C} : (\mathbf{x}, \mathbf{z}) \mapsto \phi_A(f)(\mathbf{x}, \mathbf{z})$. We may allow zero values for the z-variables, because of the following property.

Proposition 2.2. *Let* $\deg(f, x)$ *denote the maximal exponent of the variable* x *in the polynomial* f. *Then,*

$$\deg \left(\phi_A \left(\sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \right), z_i \right) \geq 0 \quad \text{and} \quad \deg(\phi_A(\mathbf{x}^{\mathbf{a}}), z_i) \left\{ \begin{array}{ll} = \ 0 & \text{if } \mathbf{a} \in \partial_{\mathbf{v}^i} A \\ > \ 0 & \text{if } \mathbf{a} \notin \partial_{\mathbf{v}^i} A \end{array} \right. . \tag{2}$$

Proof. The directions perpendicular to the facets are given by inward pointing normals, i.e.: for a facet normal $\mathbf{v}^i$: $d_i \leq \langle \mathbf{a}, \mathbf{v}^i \rangle$, $\forall \mathbf{a} \in A$. Therefore all exponents of z_i of ϕ_A are nonnegative. □

This homogenization corresponds geometrically to assigning as exponents of z_i the lattice distances relative to the ith facet of P. If we collect for every monomial the exponents of the new z-variables and store these vectors rowwise in a matrix, then we obtain the vertex-facet incident matrix [29] of the polytope P. The normal fan Σ_P collects all cones that are normal to the faces of P. The generators of those normal cones are the normals to the facets.

Example 2.3. *(running example)* Consider $f(x_1, x_2) = c_{10} x_1 + c_{01} x_2 + c_{20} x_1^2 + c_{12} x_1 x_2^2$. The Newton polytope has four facets, with inward pointing normals given by $\mathbf{v}^1 = (1, 1)$, $\mathbf{v}^2 = (0, 1)$, $\mathbf{v}^3 = (-2, -1)$, and $\mathbf{v}^4 = (1, -1)$. In Figure 1, the homogenizing along $\mathbf{v}^4$ is illustrated.

The homogenization is $\phi_A(f)(\mathbf{x}, \mathbf{z}) = c_{10} z_3^2 z_4^2 x_1 + c_{01} z_2 z_3^2 x_2 + c_{20} z_1 z_4^3 x_1^2 + c_{12} z_1^2 z_2^2 x_1 x_2^2$. ■

The homogeneous coordinates correspond to the inward pointing normals to the facets.

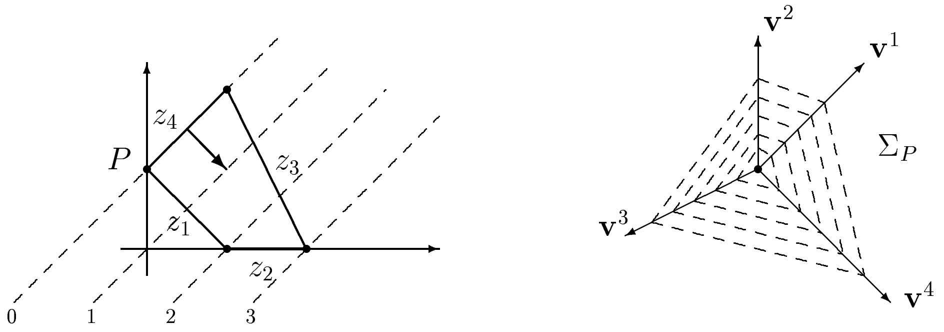


FIGURE 1. On the left we see that the exponents of z_4 are lattice distances relative to the fourth facet of P with normal $\mathbf{v}^4$. On the right the normal fan Σ_P of P is displayed.

Proposition 2.3. *Consider the facet normal* $\mathbf{v}^i$ *with corresponding* d_i*.*
Denote $\lambda^{\mathbf{v}^i}\mathbf{xz} = (\lambda^{v_1^i}x_1, \lambda^{v_2^i}x_2, \ldots, \lambda^{v_n^i}x_n, z_1, \ldots, z_{i-1}, \lambda^{-1}z_i, z_{i+1}, \ldots, z_n)$*.*
Then $\phi_A(f)(\lambda^{\mathbf{v}^i}\mathbf{xz}) = \lambda^{d_i}\phi_A(f)(\mathbf{x}, \mathbf{z})$*.*

Proof. First consider one monomial $\mathbf{x}^\mathbf{a}$:

$$\phi_A(\mathbf{x}^\mathbf{a}) \,|_{(\mathbf{x},\mathbf{z})=\lambda^{\mathbf{v}^i}\mathbf{xz}} = \mathbf{x}^\mathbf{a} \prod_{j=1}^{N} z_j^{\langle \mathbf{a}, \mathbf{v}^j \rangle - d_j} \,|_{(\mathbf{x},\mathbf{z})=\lambda^{\mathbf{v}^i}\mathbf{xz}} \tag{3}$$

$$= (\lambda^{v_1^i}x_1\lambda^{v_2^i}x_2 \cdots \lambda^{v_n^i}x_n)^\mathbf{a} \left(\prod_{j=1,\neq i}^{N} z_j^{\langle \mathbf{a}, \mathbf{v}^j \rangle - d_j} \right) (\lambda^{-1}z_i)^{\langle \mathbf{a}, \mathbf{v}^i \rangle - d_i} \tag{4}$$

$$= \lambda^{\langle \mathbf{a}, \mathbf{v}^i \rangle}\mathbf{x}^\mathbf{a}\lambda^{-\langle \mathbf{a}, \mathbf{v}^i \rangle + d_i} \prod_{j=1}^{N} z_j^{\langle \mathbf{a}, \mathbf{v}^j \rangle - d_j} \tag{5}$$

$$= \lambda^{d_i}\phi_A(\mathbf{x}^\mathbf{a}) \tag{6}$$

The multilinearity of ϕ_A completes the proof. □

The transition from toric to affine space is immediate by taking $\lambda = z_i$ in every $\lambda^{\mathbf{v}^i}\mathbf{xz}$.

Example 2.4. *(Example* 2.3 *continued.)* The homogeneous coordinates corresponding to the facets are given by

$$\begin{array}{ccccccccccccc}
(& \lambda^1 x_1 & , & \lambda^1 x_2 & , & \lambda^{-1} z_1 & , & z_2 & , & z_3 & , & z_4 &) \\
(& \lambda^0 x_1 & , & \lambda^1 x_2 & , & z_1 & , & \lambda^{-1} z_2 & , & z_3 & , & z_4 &) \\
(& \lambda^{-2} x_1 & , & \lambda^{-1} x_2 & , & z_1 & , & z_2 & , & \lambda^{-1} z_3 & , & z_4 &) \\
(& \lambda^1 x_1 & , & \lambda^{-1} x_2 & , & z_1 & , & z_2 & , & z_3 & , & \lambda^{-1} z_4 &)
\end{array} \tag{7}$$

for the respective normals $\mathbf{v}^i$, $i = 1, 2, 3, 4$. Observe that as $\lambda \to 0$, the x-values in affine space go either to zero or infinity. We have a multitude of homogeneous coordinates by taking combinations, see Σ_P in Figure 1. Positive combinations of any two vectors normal to adjacent facets correspond to constructing a vector inside the cone $\sigma \in \Sigma_P$ that has these vectors among its generators. The relations (7) illustrate how the group $(\mathbb{C}^*)^N$ acts on the variety. ■

For general polynomial systems not all supports are equal. Suppose there are r different supports which we group into $\mathcal{A} = (A_1, A_2, \dots, A_r)$ spanning the Newton polytopes $\mathcal{P} = (P_1, P_2, \dots, P_r)$. The idea is to embed the system using r distinct sets of homogenization variables, which provides a more cost-effective generalization than taking the Minkowski sum of the polytopes in $\mathcal{P}$. This idea works because any normal to the Minkowski sum is a sum of normals to the components in the sum, whence we can capture all faces in this way. Definition 2.1 is extended to systems with different supports as follows, taking $r = n$ for notational convenience.

Definition 2.2. Let $\mathcal{A}$ be the supports of $F(\mathbf{x}) = \mathbf{0}$. Suppose the polytope spanned by A_i has N_i facet normals. Abbreviate $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, \dots, \mathbf{z}^n)$. The *mixed homogenization map* $\phi_{\mathcal{A}}$ is

$$\phi_{\mathcal{A}}(F) : (\mathbb{C}^*)^n \times \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \cdots \times \mathbb{C}^{N_n} \to \mathbb{C}^n : (\mathbf{x}, \mathbf{z}) \mapsto \phi_{\mathcal{A}}(F)(\mathbf{x}, \mathbf{z}), \tag{8}$$

with $\phi_{\mathcal{A}}(F)(\mathbf{x}, \mathbf{z}) = (\phi_{A_1}(f_1)(\mathbf{x}, \mathbf{z}^1), \phi_{A_2}(f_2)(\mathbf{x}, \mathbf{z}^2), \dots, \phi_{A_n}(f_n)(\mathbf{x}, \mathbf{z}^n))$ and $\phi_{A_i}(f_i)$ as in (1). For a homotopy $H(\mathbf{x}, t) = \mathbf{0}$, the continuation parameter t is treated as a constant in setting up the homogenized homotopy $\phi_{\mathcal{A}}(H)(\mathbf{x}, \mathbf{z}, t) = \mathbf{0}$.

For several polytopes sharing the same facet normal, we can use the same corresponding z-variable. With this economical use of z-variables, Definition 2.2 yields the same homogenization map ϕ_A as in Definition 2.1 when applied to $\mathcal{A} = (A, A, \dots, A)$. Since the general homogenization is also determined by inward pointing normals $\mathbf{v}^i$, Propositions 2.2 and 2.3 have a direct generalization to the case of different supports and are omitted.

Proposition 2.4. *Suppose A is the support of $f(\mathbf{x})$.*
Let $(\mathbf{x}^*, \mathbf{z}^*) \in (\mathbb{C}^*)^n \times \mathbb{C}^N : \phi_A(f)(\mathbf{x}^*, \mathbf{z}^*) = 0$.

1. *If $z_i^* = 0$ and $z_j^* \neq 0$, $\forall j \neq i$, then $\partial_{\mathbf{v}^i} f(\mathbf{x}) = 0$ has a solution in $(\mathbb{C}^*)^n$.*
2. *Let $I = \{\ i \mid z_i^* = 0, \#(\cap \partial_{\mathbf{v}^i} A) \geq 2\ \}$, $z_j^* \neq 0$, $\forall j \notin I$, and $\mathbf{v} = \sum_{i \in I} \mathbf{v}^i$.*

   Then $\partial_{\mathbf{v}} f(\mathbf{x}) = 0$ has a solution in $(\mathbb{C}^*)^n$.

The solutions $\mathbf{x}^*$ of $\partial_{\mathbf{v}} f$ correspond to solutions of f with components equal to 0 and/or ∞.

Proof. Although the second statement generalizes the first, we distinguish two cases:

1. Suppose there is only one $z_i^* = 0$ with corresponding facet normal $\mathbf{v}^i$. Then all monomials in $\phi_A(f)$ whose exponent vector does not lie on the ith facet vanish when $z_i = z_i^* = 0$. For $z_j^* \neq 0$, $\forall j \neq i$, the homogeneous coordinates $\lambda^{\mathbf{v}^j} \mathbf{x}^* \mathbf{z}^*$ (as in Proposition 2.3) are used successively, each time setting $\lambda = z_j^*$, $\forall j \neq i$, to obtain a solution in $(\mathbb{C}^*)^n$ of $\partial_{\mathbf{v}^i} f(\mathbf{x}) = \mathbf{0}$.
2. When several z_i^*'s are zero, more monomials in the system vanish. The condition on I guarantees that there are still enough monomials left in $\partial_{\mathbf{v}} f$ to admit a solution in $(\mathbb{C}^*)^n$. Successive application of Proposition 2.3 setting $\lambda = z_j^*$, $\forall j \notin I$, yields a solution in $(\mathbb{C}^*)^n$ of $\partial_{\mathbf{v}} f(\mathbf{x}) = \mathbf{0}$.

The homogeneous coordinates contain as exponents for λ the components of the normals. Any normal $\mathbf{v}$ has at least one component different from zero, call it v_k. Setting $\lambda = z_i \to 0$ to transform to affine space will lead to $x_k \to \infty$, if $v_k < 0$, or $x_k \to 0$, in case $v_k > 0$. □

In [5], solutions that have no z-variables equal to zero, or only in cases covered by Proposition 2.4 are called *nontrivial*. Indeed, without the conditions on I in Proposition 2.4, all monomials could vanish. The generalization of Proposition 2.4 to polynomial systems is immediate.

Proposition 2.5. *Let $F(\mathbf{x}) = \mathbf{0}$ have supports $\mathcal{A}$. The nontrivial solutions of $\phi_{\mathcal{A}}(F)(\mathbf{x},\mathbf{z}) = \mathbf{0}$ with some z-variables equal to zero correspond to solutions of face systems.*

We close this section with some remarks on systems with different supports. In the homogenization we care for all coefficients in the system, therefore we distinguish between supports, rather than between polytopes. Secondly, for sparse polynomials it often happens that the supports lie in lower-dimensional planes. In that case we treat the normal vectors that define the hyperplanes that contain the supports as facet normals in addition to the normal to the faces that define the lower-dimensional polytopes. Finally, aside from this complication, working with different supports does not alter the homogenization method.

3. Scaling Coefficients and Solutions

The variables introduced by the homogenization allow to scale $\phi_A(f)(\mathbf{x},\mathbf{z}) = \mathbf{0}$, so that every component of $\mathbf{x}$ lies on the unit circle in $\mathbb{C}^*$, while keeping the z-variables bounded.

Proposition 3.1. *Suppose $\mathbf{x} \in (\mathbb{C}^*)^n$ satisfies $\phi_A(f)(\mathbf{x},\mathbf{z}) = \mathbf{0}$ for fixed $\mathbf{z}$ with $0 < z_j \leq 1$, $\forall j = 1,2,\ldots,N$. Let $\mathbf{v} = (-\log|x_1|, -\log|x_2|, \ldots, -\log|x_n|)$, with $\log$ of base 10. Then*

1. $\exists \mathbf{c} \in \mathbb{R}^n_+$, $\exists J = (j_1, j_2, \ldots, j_n)$, $j_i \in \{1,2,\ldots,N\} : \mathbf{v} = \displaystyle\sum_{i=1}^{n} c_i \mathbf{v}^{j_i}$.
2. *The homogeneous coordinates*
$$\left(\lambda^{\left(-\sum_{i=1}^n c_i \mathbf{v}^{j_i}\right)_k} x_k \Big|_{k=1}^{n}, \ldots, \lambda^{c_i} z_{j_i}\big|_{i=1}^{n}, \ldots \right) \tag{9}$$
with $\lambda = 0.1$ scale $(\mathbf{x},\mathbf{z})$ to $(\widetilde{\mathbf{x}},\widetilde{\mathbf{z}})$ with $|\widetilde{x}_k| = 1$ and $0 < \widetilde{z}_j \leq 1$.

Proof. The above statements are true because:

1. The normal fan Σ_P of P is complete. Whence every vector $\mathbf{v}$ can be written as a positive combination of the generators of a cone $\sigma \in \Sigma_P$.
2. $\left(-\sum_{i=1}^n c_i \mathbf{v}^{j_i}\right)_k = \log|x_k|$, so $\widetilde{x}_k = (0.1)^{\log|x_k|} x_k = \frac{x_k}{|x_k|}$, whence $|\widetilde{x}_k| = 1$.
   $0 < z_{j_i} \leq 1 \Rightarrow 0 < \widetilde{z}_{j_i} = (0.1)^{c_i} z_{j_i} \leq 1$, as $c_i \geq 0$. Thus $0 < \widetilde{z}_j \leq 1$. □

Note that any base b can be used in the logarithm, it suffices to replace 0.1 in Proposition 3.1 by b^{-1}.

Example 3.1. *(toric scaling analogue to projective space)* Suppose the Newton polytope is a simplex. The homogeneous coordinates are
$$\begin{array}{ccccccccccc}
(& \lambda^1 x_1 & , & \lambda^0 x_2 & , & \lambda^{-1} z_1 & , & z_2 & , & z_3 &) \\
(& \lambda^0 x_1 & , & \lambda^1 x_2 & , & z_1 & , & \lambda^{-1} z_2 & , & z_3 &) \\
(& \lambda^{-1} x_1 & , & \lambda^{-1} x_2 & , & z_1 & , & z_2 & , & \lambda^{-1} z_3 &)
\end{array} \tag{10}$$
for the respective facet normals $\mathbf{v}^1 = (1,0)$, $\mathbf{v}^2 = (0,1)$, and $\mathbf{v}^3 = (-1,-1)$. Since this is a *toric* analogue, we are concerned about solutions with zero components and have more z-variables than in the case of ordinary homogenization. We consider two situations:

1. $(x_1,x_2) = (2,2)$, $\mathbf{v} = (-\log(2), -\log(2))$, $J = (3,3)$, $\mathbf{c} = (\log(2), 0)$
   $\Rightarrow z_3 = (0.1)^{\log(2)} z_3 = \frac{1}{2} z_3$.

2. $(x_1, x_2) = (2, 1)$, $\mathbf{v} = (-\log(2), -\log(1))$, $J = (2, 3)$, $\mathbf{c} = (\log(2), \log(2))$
   $\Rightarrow z_2 = (0.1)^{\log(2)} z_2 = \frac{1}{2} z_2$ and $z_3 = (0.1)^{\log(2)} z_3 = \frac{1}{2} z_3$.

In the first case it suffices to use one of the homogeneous coordinates, while in the second case a combination is constructed. ■

Example 3.2. *(Example* 2.3 *continued.)* Here we also consider two situations:

1. $(x_1, x_2) = (\frac{1}{2}, 2)$, $\mathbf{v} = (-\log(\frac{1}{2}), -\log(2))$, $J = (4, 4)$, $\mathbf{c} = (\log(2), 0)$
   $\Rightarrow z_4 = (0.1)^{\log(2)} z_4 = \frac{1}{2} z_4$.
2. $(x_1, x_2) = (2, 2)$, $\mathbf{v} = (-\log(2), -\log(1))$, $J = (2, 3)$, $\mathbf{c} = (\log(2), \log(2))$
   $\Rightarrow z_3 = (0.1)^{2\log(2)} z_3 = \frac{1}{4} z_3$ and $z_4 = (0.1)^{3\log(2)} z_4 = \frac{1}{8} z_4$.

The scaling relation is constructed by decomposing $\mathbf{v}$ in Σ_P, see Figure 1. ■

In the general case when the system has an r-tuple of different supports $\mathcal{A}$, we decompose $\mathbf{v} = -\log|\mathbf{x}|$ in every component of $\Sigma_{\mathcal{P}} = (\Sigma_{P_1}, \Sigma_{P_2}, \ldots, \Sigma_{P_r})$ to compute new values of the z-variables which are distinct for every different support.

Propositions 2.5 and 3.1 allow us to give a concise proof of Bernshteĭn's second theorem formulated below as Theorem 3.1 using a homotopy argument.

Definition 3.1. We say that the homotopy $H(\mathbf{x}, t) = \mathbf{0}$ is *in generic position* with respect to the supports $\mathcal{A}$ that span the Newton polytopes $\mathcal{P}$, if $\forall t \in [0, 1)$: $H(\mathbf{x}, t) = \mathbf{0}$ has supports $\mathcal{A}$ and has exactly $V_n(\mathcal{P})$ regular solutions in $(\mathbb{C}^*)^n$.

Given a system $F(\mathbf{x}) = \mathbf{0}$, the canonical example of a homotopy in generic position is

$$H(\mathbf{x}, t) = (1-t) F^{(0)}(\mathbf{x}) + t F(\mathbf{x}) = \mathbf{0}, \quad t \in [0, 1], \tag{11}$$

where the system $F^{(0)}(\mathbf{x}) = \mathbf{0}$ has random complex coefficients and the same supports as F.

Theorem 3.1. *Consider $F(\mathbf{x}) = \mathbf{0}$, with supports $\mathcal{A}$, Newton polytopes $\mathcal{P}$ and $V_n(\mathcal{P}) \neq 0$.*

$$\#\{\ m_{\mathbf{x}} \mid F(\mathbf{x}) = \mathbf{0}, \mathbf{x} \text{ isolated } \} < V_n(\mathcal{P}) \ \Leftrightarrow\ \exists \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \exists \mathbf{x} \in (\mathbb{C}^*)^n : \partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$$

where $m_{\mathbf{x}}$ denotes the multiplicity of the isolated solution $\mathbf{x}$ of $F(\mathbf{x}) = \mathbf{0}$.

Proof. Construct any homotopy $H(\mathbf{x}, t) = \mathbf{0}$ that is in generic position with respect to $\mathcal{P}$ and for which $H(\mathbf{x}, t = 1) = F(\mathbf{x})$. Consider the homogenized homotopy $\phi_{\mathcal{A}}(H)(\mathbf{x}, \mathbf{z}, t) = \mathbf{0}$.

$\Rightarrow$ If $F(\mathbf{x}) = \mathbf{0}$ has fewer than $V_n(\mathcal{P})$ isolated solutions in $(\mathbb{C}^*)^n$, then, as $t \to 1$, either some solution paths must diverge outside $(\mathbb{C}^*)^n$ or some solutions paths must go to a component of solutions, because the homotopy is in generic position. In the first case, by Proposition 3.1, the only way to diverge is that some $z_i \to 0$. Then, according to Proposition 2.5, we have a solution in $(\mathbb{C}^*)^n$ of some face system. In the second case, the solution component allows at least one free variable to choose in moving on that component. So we can move a free variable outside $(\mathbb{C}^*)^n$ for which some z_i is zero. Then we can again apply Proposition 2.5.

$\Leftarrow$ If there is a face system that has a solution $\mathbf{x} \in (\mathbb{C}^*)^n$, then in the corresponding tuple of z-variables for which $\phi_{\mathcal{A}}(H)(\mathbf{x}, \mathbf{z}, 1) = \mathbf{0}$, $\exists i$: $z_i = 0$. Since the homotopy is in generic position, there is at least one solution path that ended at $(\mathbf{x}, \mathbf{z})$. By Proposition 2.5, the transition to affine coordinates shows that the solution path that ended at this couple $(\mathbf{x}, \mathbf{z})$ diverged to a solution that has components at infinity and/or zero components. So $F(\mathbf{x}) = \mathbf{0}$ must have fewer than $V_n(\mathcal{P})$ isolated solutions in $(\mathbb{C}^*)^n$. □

4. The Static Scaling in Projective Newton

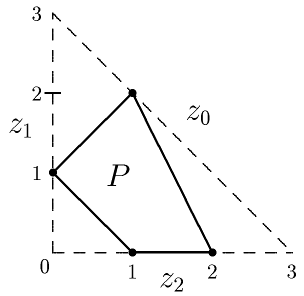
Let us recall Newton's method in projective space (see [17]). We assume we have a system $H = (h_1, h_2, \ldots, h_n)$ of homogeneous polynomials: $h_i(\lambda\mathbf{z}) = \lambda^{\deg(h_i)} h_i(\mathbf{z})$ for $i = 1, 2, \ldots, n$ with $\lambda\mathbf{z} = (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)$ the usual homogeneous coordinates. For a solution at infinity: $z_0 = 0$.

Algorithm 4.1. *(Newton's method in projective space)*

Input: $(H, \mathbf{z})$.	*(Homogeneous system, approximate root)*
Output: $\mathbf{z}$.	*refined approximation*
$i : \lvert z_i \rvert = \max_{j=1}^{n} \lvert z_j \rvert$;	*maximal component*
$\mathbf{z} := \left(\underbrace{\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}}, 1, \underbrace{\frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}} \right)$;	*scale the approximation*
$\mathbf{x} := \left(\overbrace{z_1, \ldots, z_{i-1}}, \overbrace{z_{i+1}, \ldots, z_n} \right)$;	*switch to an affine chart*
$F := H\rvert_{z_i=1}$;	*select evaluation map*
$\mathbf{x} := \mathbf{x} - D_{\mathbf{x}}^{-1} F(\mathbf{x}) F(\mathbf{x})$;	*affine Newton method*
$\mathbf{z} := (x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n)$.	*refined approximation*

In the context of path following, H is a parametrized family of systems.

Example 4.1. *(projective Newton on a sparse system)* Consider the application of projective Newton on a sparse system of polynomials. Suppose the supports of the polynomials are all equal. We see the Newton polytope displayed in Figure 2. In the homogenization the simplex is implicitly taken as Newton polytope and to every facet we associate a new variable z_i. Since in projective space one does not distinguish solutions with zero components, the original monomials are not copied.



$$f(\mathbf{x}) = c_{10}x_1 + c_{01}x_2 + c_{20}x_1^2 + c_{12}x_1x_2^2$$

$$\widetilde{f}(\mathbf{z}) = c_{10}z_0^2 z_1 + c_{01}z_0^2 z_2 + c_{20}z_0 z_1^2 + c_{12}z_1 z_2^2$$

Figure 2. On the left we see how the polytope P of f is approximated by a simplex which corresponds to the usual projective homogenization $\widetilde{f}$ given on the right.

In case of divergence, z_1 and/or z_2 go to infinity. The scaling step in Algorithm 4.1 selects $i = 1$ or $i = 2$ and z_0 strives to zero. In that case we see in Figure 2 that the monomial $z_1 z_2^2$ dominates. However, a diverging path cannot go to a monomial and goes to one of the other face systems. The multi-homogeneous version of Newton's method will make the same monomial dominant. ■

Example 4.2. *(projective Newton leads to clustering)* Consider the following simple system:

$$\begin{cases} x_1 - 10^8 = 0 \\ x_2^2 - 1 = 0 \end{cases} \tag{12}$$

In affine space, the solutions $(x_1, x_2) = (10^8, \pm 1)$ are well-separated. In projective space, they are represented as $(z_0, z_1, z_2) = (1, 10^8, \pm 1)$ and scaled to $(z_0, z_1, z_2) = (10^{-8}, 1, \pm 10^{-8})$, which is a pair of numerically clustered solutions. Although this example is cooked up for this special occasion, this phenomenon is observed as typical in many applications. As the sum of the two Newton polytopes of the two polynomials is a rectangle, the toric homogenization corresponds to the bi-homogenization, which allows to scale the variables independently: $(x_1, x_2) = (10^8, \pm 1) \mapsto (z_{10}, z_{11}, z_{20}, z_{21}) = (10^{-8}, 1, 1, \pm 1)$, avoiding the clustering of solutions. ■

The most important difference with multi-projective Newton is the dynamic use of the scaling procedure of Proposition 3.1 which guarantees distinct paths, as formulated below in Theorem 6.1.

5. Toric Newton scales in a Dynamic Way

The set up of Toric Newton enables an efficient numerical scaling while path following.

Algorithm 5.1. *(Toric Newton – symbolic set-up)*
Input: $(A, \triangle)$. *(support , triangulation)*
Output: H; *homogeneous coordinates matrix*
Σ_P. *simplicial normal fan*

1. $V :=$ Facet_Normals$(A, \triangle)$; *extract facet normals*
2. $H :=$ Homogenize(A, V); *relative lattice distances*
3. $\Sigma_P :=$ Cone_Subdivision(H, A, V). *refine normal fan*

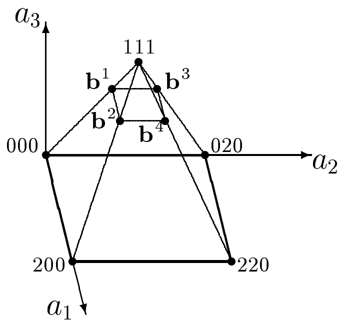
Next we analyze the cost of this algorithm. The dimension n and the cardinality $\#A$ measure the input size. The complexity of the output is bounded from below by the number of cells $\#\triangle$ in any triangulation $\triangle$ of A. Recall that we are interested in *all* isolated solutions and to count the number of roots we compute volumes. So $\#\triangle$ is a lower bound for the number of isolated solutions.

Proposition 5.1. *(Cost of Step 1 in Algorithm 5.1.) Consider a triangulation $\triangle$ of the polytope P spanned by A. Then all facets of P can be enumerated in $O(\#\triangle n(n^3 + \#A))$ operations.*

Proof. The candidate facets are the facets of the simplices in the triangulation. Every simplex has $n + 1$ facets. The computation of each candidate requires $O(n^3)$ arithmetical operations. To test whether this candidate normal constitutes a facet normal we have to compute inner products with the points in A and look whether the inner product is minimal for points on the facet of the simplex. This gives the bound $O(\#\triangle n(n^3 + \#A))$. □

The number of arithmetic operations of Step 2 of Algorithm 5.1 equals $O(n\#V)$. Step 3 is only needed when the polytope is not simple, i.e.: when some vertices are contained in more than n facets, see the top of the polytope in Figure 3. In Step 3 we make the fan simplicial subdividing the normal cones. One method to subdivide normal cones is to cut corners (see [13, page 190] and [29, pages 54–55]) at vertices that have more than n facets, as illustrated in Figure 3. We refer to [11, Section 2.6] for more on refinements of fans in connection with toric varieties.

For a normal cone $\sigma \in \Sigma_P$ at the vertex $\mathbf{a}$ we take the average $\overline{\mathbf{v}}$ of the facet normals that generate σ. Determine c such that the hyperplane $\langle \mathbf{x}, \overline{\mathbf{v}} \rangle = c$ separates $\mathbf{a}$ from the rest of the vertices that span the polytope. The normal cones τ to the intersection points of the hyperplane with the edges of the polytope refine the normal cone σ. The recursive structure of the normal fan Σ_P is illustrated at the right of Figure 3. In the decomposition of a vector $\mathbf{v}$ in a subdivision of σ, we replace the average vector $\overline{\mathbf{v}}$ in terms of the other generators of σ, so that we obtain a decomposition that uses only facet normals.



$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \quad V = \begin{array}{c} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{v}^1 \\ \mathbf{v}^2 \\ \mathbf{v}^3 \\ \mathbf{v}^4 \\ \mathbf{v}^5 \end{array} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\Sigma_P = \begin{bmatrix} 1\,1\,1 \to \begin{bmatrix} \mathbf{b}^1 \to (\overline{\mathbf{v}}, \mathbf{v}^2, \mathbf{v}^4) \\ \mathbf{b}^2 \to (\overline{\mathbf{v}}, \mathbf{v}^2, \mathbf{v}^3) \\ \mathbf{b}^3 \to (\overline{\mathbf{v}}, \mathbf{v}^4, \mathbf{v}^5) \\ \mathbf{b}^4 \to (\overline{\mathbf{v}}, \mathbf{v}^3, \mathbf{v}^5) \end{bmatrix} \\ 0\,0\,0 \to (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^4) \\ 2\,0\,0 \to (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \\ 0\,2\,0 \to (\mathbf{v}^1, \mathbf{v}^4, \mathbf{v}^5) \\ 2\,2\,0 \to (\mathbf{v}^1, \mathbf{v}^3, \mathbf{v}^5) \end{bmatrix}$$

FIGURE 3. Refinement of a normal cone of a nonsimple polytope by cutting the corner at $(1, 1, 1)$. The points $\mathbf{b}^i$ lie on the intersection of the hyperplane cut with the edges that contain $(1, 1, 1)$.

In three dimensions we use only one separating hyperplane, because every edge is enclosed in exactly two facets. This is no longer the case in higher dimensions, so we proceed recursively as the points in the intersection of the polytope with the separating hyperplane belong to a space of lower dimension. By Proposition 2.4 from [29, page 55], edges are in one-to-one correspondence with vertices of the polytope cut out by the separating hyperplane. Recursive application proves the following.

Proposition 5.2. *(Cost of Step 3 in Algorithm 5.1.) To triangulate an n-dimensional cone at most $n - 2$ separating hyperplanes are needed.*

Although the number of cones grows exponentially, one new refining cone requires only elementary operations to compute the intersection points of the edges with the separating hyperplane. Therefore we can perform a lazy refinement of Σ_P, decomposing only those normal cones when needed in the scaling procedure.

Just as projective Newton method, we switch to an appropriate affine chart of dimension n.

Algorithm 5.2. *(Toric Newton – number crunching)*

Input: $\phi = \phi_A(F)$;	*system in homogeneous coordinates*
$(\mathbf{x}, \mathbf{z})$;	*approximate root with* $\lvert x_i \rvert = 1$ *and* $\lvert z_j \rvert \leq 1$
Σ_P.	*simplicial normal fan*
Output: $(\mathbf{x}, \mathbf{z})$.	*refined approximation with* $\lvert x_i \rvert = 1$ *and* $\lvert z_j \rvert \leq 1$

$F := \phi\|_{Fix\ \mathbf{z}}$;	*evaluation map*
$\mathbf{x} := \mathbf{x} - D_{\mathbf{x}}^{-1}F(\mathbf{x})F(\mathbf{x})$;	*affine Newton method*
$\mathbf{v} := -\log\|\mathbf{x}\|$;	*direction of correction*
$J = (j_1, j_2, \dots, j_n)$	*find* $\sigma \in \Sigma_P : \mathbf{v} \in \sigma$
$:=$ Localize_Position$(\Sigma_P, \mathbf{v})$;	$\sigma = \mathrm{span}(\mathbf{v}^{j_1}, \mathbf{v}^{j_2}, \dots, \mathbf{v}^{j_n})$
$\mathbf{c} = (c_1, c_2, \dots, c_n) : \mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{v}^{j_i}$;	*decompose* $\mathbf{v}$
for all $i \in \{1, 2, \dots, n\}$ do	
$x_i := \frac{x_i}{\|x_i\|}$; $z_{j_i} := (0.1)^{c_i} z_{j_i}$;	*scale solution and coefficients*
end for.	

In analyzing the cost of localizing the position of the direction vector $\mathbf{v}$ we assume that the fan Σ_P has a recursive simplicial structure, as illustrated in Figure 3.

Proposition 5.3. *(Cost of* Localize_Position *in Algorithm 5.2.) For* $\mathbf{v} \in \mathbb{R}^n$ *and* Σ_P*,* $P = \mathrm{conv}(A)$*, finding* $\sigma \in \Sigma_P$ *so that* $\mathbf{v} \in \sigma$ *takes at least* $\#A$ *and at most* $\#A(n-1)$ *inner products.*

Proof. If P is simple, then σ is the normal cone at the point $\mathbf{a} \in A$ for which $\langle \mathbf{a}, \mathbf{v} \rangle = \min_{\mathbf{b} \in A} \langle \mathbf{b}, \mathbf{v} \rangle$. Computing this point requires $\#A$ inner products. If P is not simple, there are at most $n-2$ refinement levels, according to Proposition 5.2. At every level, there are at most $\#A$ points to consider. □

For r-tuples of supports $\mathcal{A}$, we construct $\Sigma_{\mathcal{P}} = (\Sigma_{P_1}, \Sigma_{P_2}, \dots, \Sigma_{P_r})$. The cost of the geometric constructions is scaled by a factor r. From our experiences with mixed-volume computations [25] we find linear upscaling in the cost rather remarkable.

6. Numerical Convergence Properties

In the absence of singularities, Newton's method converges quadratically, doubling the precision of the approximate root in each step. With a homotopy in generic position, we will not hit a solution component, except perhaps only at the very end of the paths. What may happen however, is that paths become clustered, hereby not only limiting the convergence but possibly also terminating the numerical path tracking with a failure to reach the end.

Affine path following methods deal well with paths diverging outside $(\mathbb{C}^*)^n$, for those paths exhibit a steady direction leading to the face systems that cause the deficiency. To obtain a certificate of divergence, polyhedral end games [15] can effectively trace the paths in affine space, because to compute the direction of divergence one does not need to follow the paths up to infinity. Reaching badly-scaled though probably still meaningful solutions by affine path following methods is often impossible due to the finiteness of numbers a computer can represent. Projective Newton brings infinity to our working space, but roots that are clearly separated in affine space can cluster by a projective transformation.

The dynamic of the scaling procedure of Proposition 3.1 provides the proof of the numerical convergence properties of Toric Newton.

Theorem 6.1. *The solution paths defined by a homotopy in generic position compactified torically remain separated, except possibly at the end when converging to multiple solutions.*

Proof. At the start of the paths the solutions $(\mathbf{x}, \mathbf{z})$ are all mutually distinct tuples. We will show that they stay separated, as the continuation parameter t progresses. By Proposition 3.1, the components of $\mathbf{x}$ remain on the unit circle and the components of $\mathbf{z}$ may possibly go from 1 to 0.

Either a path stays inside $(\mathbb{C}^*)^n$ or diverges, as $t \to 1$. Only in the latter case, according to Theorem 3.1, the solution paths have the same direction. So if the path stays inside $(\mathbb{C}^*)^n$, its direction $\mathbf{v}$ in Proposition 3.1 is different from the other paths yielding distinct values for $\mathbf{z}$. Thus all paths that stay inside $(\mathbb{C}^*)^n$ remain separated, unless they converge to a multiple solution.

Still according to Theorem 3.1, only paths that diverge to solutions of the same $\partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$ can have the same direction $\mathbf{v}$. However, this direction becomes only apparent at the end of the paths, when the monomials of $\partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$ are dominating the rest of the system. In that case, even when two paths would converge to the same $\mathbf{z}$-vector, they would converge to the same $\mathbf{x}$-vector only if that vector would be a multiple solution of $\partial_{\mathbf{v}} F(\mathbf{x}) = \mathbf{0}$. □

The following is an immediate reformulation of Theorem 6.1.

Corollary 6.1. *For a homotopy in generic position, Toric Newton method converges quadratically in correcting the solutions along the paths, except when approaching singularities at the end. Regular solutions are approximated by a quadratically converging Toric Newton method.*

We close this section with an important numerical observation. One may object that as z-variables can go to zero we could have to rely on very tiny values to distinguish the solutions. However, we must not look at the actual values of the z-variables, but at their exponents. These exponents have been obtained in a numerically stable way, by adding consecutive coefficients in the decomposition of the directions of the paths, see the scaling procedure of Proposition 3.1. To deal with extremal values of solutions, we usually work with a logarithmic scale, as the magnitude of the solutions is most relevant.

7. Computational Experiences

Here we report on the geometric computations showing the practical feasibility of the method.

7.1. **Implementation Issues.** The experiments are conducted with the polynomial homotopy continuation environment PHC [23] created, developed and maintained by the author. On the occasion of reported calculations with SAGBI homotopies [24], a new version of the package was recently released.

The construction of the triangulation is performed with the dynamic lifting algorithm [25]. This algorithm is efficient for sparse inputs. To overcome the space complexity, one should refine the fan in a lazy way subdividing only those parts of the cones that are strictly needed while path following. The implementation of Toric Newton produces values $(\theta, \mathbf{z}) \in [0, 2\pi)^n \times [0,1]^N$, where $x_i = e^{i\theta_i}$. To minimize numerical round off, the affine Newton's method must be executed in $[0, 2\pi)^n \times [0,1]^N$. This stage of implementation in PHC is still under construction.

Multi-precision arithmetic (as available in the latest release of PHC) is recommended for transforming from toric to affine coordinates at the end of the paths and for evaluation of the residuals. Only and only at that stage, we fine multi-precision arithmetic useful. Alternatively and numerically, one could treat the z_i's one after the other and apply Newton's method intermediately to correct the rounding errors. For well-conditioned affine roots, this method works well.

7.2. **Geometric Samples.** Here we give an impression on the complexity of the geometry involved on a 15-point configuration randomly generated for several dimensions. Table 1 summarizes the characteristics and timings. The computations were done on a 166 MHz Pentium II processor with 64 Mb internal memory running Linux.

	15 points in $\{0,1,2,3\}^n$ for $n = 3,4,5,6,7$							cpu min, sec, millisec	
n	$\#\triangle$	vol	$\#V$	max ω	$\deg(\phi_A, z_i)$	$\#\sigma$	$\#\Sigma_P$	construct	decompose
3	16	64	17	57	15	6	63	20 ms	380 ms
4	21	172	25	1,227	36	15	341	90 ms	500 ms
5	57	1,081	59	5,904	70	27	17,299	5 s 40 ms	850 ms
6	110	4,159	140	3,619,556	660	78	55,725	20 s 30 ms	1 s 450 ms
7	157	29,041	248	12,567,604	1,364	135	664,135	5 m 31 s 470 ms	3 s 30 ms

TABLE 1. Characteristics and timings for random samples. The first part lists the number of cells $\#\triangle$, the volume of the polytope and the number of facet normals $\#V$. In the second part we compare the maximal height (max ω) to construct $\triangle$ to the highest degree of a z-variable in the homogenization. The sixth column displays the maximum number of generating facet normals $\#\sigma$ of a cone and the total number of cones in the refinement of the fan Σ_P. The last part shows timings for constructing this refinement and for decomposing 1000 random vectors.

From Table 1 we observe the growth in complexity as the dimension increases. The maximal lifting value ω is generated by the dynamic lifting algorithm [25]. Although the integer lifting values generated by the implementation of the dynamic lifting algorithm can be improved by balanced floating-point liftings developed in [12], it is noteworthy that the degrees of the z-variables are of a much lesser magnitude.

Newton polytopes are in general not simple, as experienced from the $\#\sigma$. The exhaustive refinement of the normal fan is quite memory exhaustive. For larger dimensions we must use a lazy decomposition of normal cones. The fast timings in constructing and consulting this geometric data structure are promising and encouraging.

7.3. **A Numerical Example.** The following system is inspired on an example in [16].

$$\left\{\begin{array}{l} x_1 - 10 = 0 \\ x_2 - x_1^2 = 0 \\ x_3 - x_2^2 = 0 \\ x_4 - x_3^2 = 0 \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \gamma_{11}(x_1 - 10) + \gamma_{12}(x_2 - x_1^2) + \gamma_{13}(x_3 - x_2^2) + \gamma_{14}(x_4 - x_3^2) = 0 \\ \gamma_{21}(x_1 - 10) + \gamma_{22}(x_2 - x_1^2) + \gamma_{23}(x_3 - x_2^2) + \gamma_{24}(x_4 - x_3^2) = 0 \\ \gamma_{31}(x_1 - 10) + \gamma_{32}(x_2 - x_1^2) + \gamma_{33}(x_3 - x_2^2) + \gamma_{34}(x_4 - x_3^2) = 0 \\ \gamma_{41}(x_1 - 10) + \gamma_{42}(x_2 - x_1^2) + \gamma_{43}(x_3 - x_2^2) + \gamma_{44}(x_4 - x_3^2) = 0 \end{array}\right.$$

where the γ-coefficients are randomly generated complex numbers. Both systems are equivalent. The embedding at the right is only meant to illustrate how four nice quadratic polynomial equations can hide one root with large components. This problem is not recognized by conventional numerical scaling methods and is the kind of target application for Toric Newton.

8. Closing Remarks

To solve a wide class of polynomial systems in a reliable and efficient manner by homotopy continuation, a variety of different methods has been developed. Coefficient and equation scaling, often used in combination with centering, is a remedy to deal with systems that have extremal coefficients. Projective transformations compactify our working space. Polyhedral end games used in affine path following methods certify diverging paths. Toric Newton generates an optimal affine chart to approximate the magnitude of extremal roots.

Much is left undone. At a practical level, path trackers have to be developed and implemented that recognize the toric situation. Instead of complex condition numbers there is a need for structured condition numbers that measure the accuracy of the computations.

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