POSETS THAT LOCALLY RESEMBLE DISTRIBUTIVE LATTICES
An Extension of Stanley's Theorem

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ABSTRACT. Let P be a graded poset with 0 and 1 and rank at least 3. Assume that every rank 3 interval is a distributive lattice and that, for every interval of rank at least 4, the interval minus its endpoints is connected. It is shown that P is a distributive lattice, thus resolving an issue raised by Stanley. Similar theorems are proven for semimodular, modular, and complemented modular lattices.

As a corollary, a theorem of Stanley for Boolean lattices is obtained, as well as a theorem of Grabiner (conjectured by Stanley) for products of chains. Applications to geometry and connections with the theory of buildings are discussed.

1. Introduction.

Can one determine if a given graded poset is a distributive lattice merely by looking at small intervals? Stanley has suggested investigating this question ([15]).

Stanley has proven that one may recognize *Boolean* lattices by looking at small intervals. Specifically, he has proven that a finite poset is a Boolean lattice if and only if (1) P is a graded poset with 0 and 1; (2) every interval of rank at most 3 is a Boolean lattice; and (3) for every interval of rank at least 4, the interval minus its endpoints is connected (see Corollary 5.6; [8], Lemma 8).

Grabiner has proven Stanley's conjecture that "Boolean lattice" may be replaced by "product of chains" (see Corollary 5.4; [8], Theorem 1).

The first author has proven that "Boolean lattice" may be replaced by "distributive lattice" (Theorem 5.2); the proof here is due to the second author. Indeed, the authors have independently shown that "Boolean" may be replaced by "modular," "complemented modular," or "semimodular" (Theorems 3.4, 4.2, and 4.4).

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The theorems of Stanley and Grabiner are fascinating because it is not even obvious why a poset satisfying (1)–(3) should even be a lattice, much less one with nice properties.

The motivation for the theorem comes from representation theory — precisely, actions of the symmetric group on the maximal chains of a graded poset with 0 and 1 ([14], §5). The theorems are reminiscent of results by Regonati, Hibi and Terai, who show that certain global properties of distributive and modular lattices may be deduced by looking at rank 3 intervals ([13], Part 3; [10], Theorem 0.1; [11], Theorem 3.3).

Grabiner takes a bare-hands approach, explicitly establishing the order relations that exist amongst the elements of the poset in order to prove it is a product of chains. Given the huge variety of distributive, modular, and semimodular lattices, such an approach is not feasible for the more general theorems. Instead, our proof is lattice-theoretic.

It turns out that these lattice-theoretic results intersect nicely with ideas from geometry and the theory of buildings.

2. Definitions and Notation.

A basic reference is [5].

Let P be a poset. Denote its least element by 0, if it exists, and its greatest element by 1, if it exists. For $x \in P$, let $\uparrow x := \{ p \in P \mid x \leqslant p \}$ and let $\downarrow x := \{ p \in P \mid p \leqslant x \}$. For $x, y \in P$ such that $x \leqslant y$, the set

$$\{ p \in P \mid x \leqslant p \leqslant y \}$$

is an *interval.* A *down-set* is a subset $D \subseteq P$ such that $\downarrow x \subseteq D$ for all $x \in D$. The poset is *connected* if, for all $x, y \in P$, there exists $k \in \mathbb{N}$ (which, without loss of generality, may be chosen to be odd) and $a_0, \dots, a_k \in P$ such that $x =: a_0 \leqslant a_1 \geqslant a_2 \leqslant \cdots \leqslant a_k := y$. The *product* of two posets P and Q is the poset $P \times Q := \{ (p, q) \mid p \in P \text{ and } q \in Q \}$ where $(p, q) \leqslant (p', q')$ if $p \leqslant p'$ and $q \leqslant q'$ (p, $p' \in P$; q, $q' \in Q$).

For x and y in a poset P, we write $x \lessdot y$ if $x < y$ and $x \leqslant p < y$ implies $x = p$ for all $p \in P$; we say x is a *lower cover* of y. An element is *join-irreducible* if it has a unique lower cover. Let $\mathcal{J}(P)$ be the set of join-irreducibles. In a poset with least element 0, a cover of 0 is called an *atom.*

A *chain* is a totally ordered set. A finite chain of cardinality $n + 1$ has *rank* n ($n \in \mathbb{N}_0$). A poset has *rank* (or *height*) n if the longest maximal chain is finite of rank n ($n \in \mathbb{N}_0$). The *rank* of an interval is its rank as a poset. A poset is *graded* if every maximal chain has the same finite rank.

A *lattice* is a non-empty poset L such that, for all $x, y \in L$, the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ exist. The lattice is *distributive* if, for all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. It is *modular* if, for all $x, y, z \in L$ such that $x \leqslant z$, $x \vee (y \wedge z) = (x \vee y) \wedge z$. If L has a 0 and a 1, it is *complemented* if, for all $x \in L$, there exists $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$. A lattice is *Boolean* if it is complemented and distributive.

It is well known that a lattice is modular if and only if it does not contain the lattice $N_5 := \{0, a, b, c, 1\}$ as a sublattice, where $0 < a < b < 1$; $0 < c < 1$; and no other non-trivial comparabilities hold. It is also well known that a modular lattice is distributive if and only if it does not contain the lattice $M_3 := \{0, x, y, z, 1\}$ as a sublattice, where $0 < x, y, z < 1$ and no other non-trivial comparabilities hold. (See [5], 6.10 and Figure 1.)

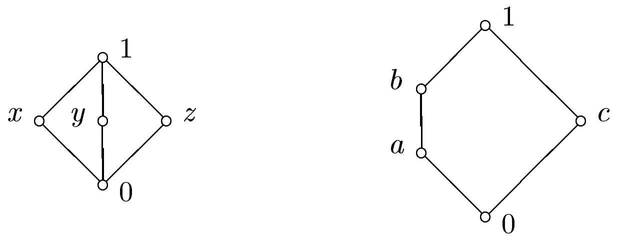


Figure 1. The lattices M_3 and N_5.

Let L be a lattice of finite rank. It is *(upper) semimodular* if, for all x, $y \in L$, $x \wedge y \lessdot x, y$ implies $x, y \lessdot x \vee y$. It is *lower semimodular* if, for all x, $y \in L$, $x, y \lessdot x \vee y$ implies $x \wedge y \lessdot x, y$. Every semimodular lattice of finite rank is a graded poset with 0 and 1. It is well known that L is modular if and only if it is upper and lower semimodular ([1], Chapter II, Theorems 15 and 16).

Every finite distributive lattice L is isomorphic to the poset of down-sets of $\mathcal{J}(L)$. It is a product of chains exactly when $\mathcal{J}(L)$ is a disjoint union of chains (that is, elements from distinct chains are incomparable), and a Boolean lattice exactly when $\mathcal{J}(L)$ is an antichain (that is, distinct elements are incomparable). These and other facts we shall use below are trivial consequences of Priestley duality (see [5], Chapter 8).

Additional definitions are given in §§6–8.

3. Posets That Locally Resemble Semimodular Lattices.

In this section, we prove that if a poset resembles a semimodular lattice "up close," then it *is* a semimodular lattice (under certain weak conditions). We do not assume P is finite. See Theorems 3 and 4.

Proposition 3.1. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a semimodular lattice.*

Then P is a lattice.

Remark. We do not need the connectivity assumption mentioned in §1.

We do need the assumption about rank, as the poset Q_2 in Figure 2 demonstrates: For $n \geqslant 2$, let $Q_n := \{0, a_1, \ldots, a_n, b_1, \ldots, b_n, 1\}$ where

$$0 < a_1 < a_2 < \cdots < a_n < 1;\ 0 < b_1 < b_2 < \cdots < b_n < 1;\ a_1 < b_2;\ b_1 < a_2;$$

and no other comparabilities hold but the necessary ones.

The poset Q_2 is a graded poset with 0 and 1 such that every proper interval is a distributive lattice and such that $Q_2 \setminus \{0, 1\}$ is connected; it is not a lattice,

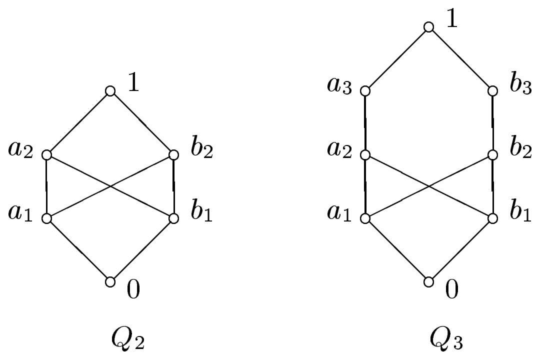


Figure 2. The posets Q_2 and Q_3.

however. Indeed, Q_2 is the only poset satisfying all of these conditions that is not a lattice.

The posets Q_n $(n \geqslant 3)$ show that we also need some assumption like semimodularity for the class of lattices to which our intervals belong.

Proof of proposition. Assume P is not a lattice for a contradiction. Then there exist distinct a, b, c, $d \in P \setminus \{0, 1\}$ such that a and b are maximal lower bounds of $\{c, d\}$.

Let $j \in P$ be such that $0 \lessdot j \leqslant a$ (so that $j \nleqslant b$). Let $c' := j \vee b$ in the lattice $\downarrow c$ and let $d' := j \vee b$ in the lattice $\downarrow d$. By semimodularity, $b \lessdot c', d'$. By maximality, $c' \neq d'$, so, by semimodularity again, $c', d' \lessdot c' \vee d' = 1$, $c = c'$ and $d = d'$. By symmetry, $a \lessdot c, d$.

Let $k \in P$ be such that $0 \lessdot k \leqslant b$. By semimodularity, there exist $c'' \leqslant c$ and $d'' \leqslant d$ such that $j, k \lessdot c'', d''$. By the preceding argument, $c'' \neq d''$ would imply that P has rank 3, a contradiction. Hence $c'' = d''$.

In the lattice $\uparrow j$, $c \wedge d$ exists and equals a, so that $a \geqslant d'' \geqslant k$, contradicting the fact that $a \wedge b = 0$. $\square$

Lemma 3.2. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a semimodular lattice. Also assume that $P \setminus \{0, 1\}$ is connected. Then for all distinct a, $b \gtrdot 0$ in P, $a \vee b \gtrdot a, b$.*

Remark. Some assumption about the rank is necessary: Let R be the poset $\{0, a, b, c, d, e, 1\}$ where $0 < a, b, c$; $a, b < d < 1$; $b, c < e < 1$; and no other comparabilities hold but the necessary ones (Figure 3).

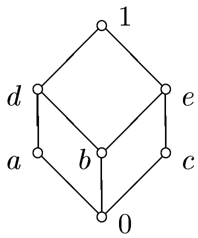


Figure 3. The poset R.

Proof of lemma. By Proposition 1, P is a lattice. For a, $b \gtrdot 0$ in P, write $a \triangle b$ if $a \lessdot a \vee b \gtrdot b$.

Assume we have distinct a, b, $c \gtrdot 0$ in P and that $a \triangle b \triangle c$. We claim that $a \triangle c$. Else $1 > a \vee b \vee c \gtrdot a \vee b, b \vee c$ so that $a \lessdot a \vee c \gtrdot c$.

By semimodularity and connectivity, if a and b are distinct atoms in P, there exist $k \in \mathbb{N}_0$ and $a_1, \ldots, a_k \gtrdot 0$ such that $a \triangle a_1 \triangle \cdots \triangle a_k \triangle b$. By the above claim, $a \triangle b$. $\square$

Theorem 3.3. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a semimodular lattice. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a semimodular lattice.

Remark. The poset R of Figure 3 shows that we need some assumption about the rank.

Proof of theorem. By Proposition 1, P is a lattice. By Lemma 2 and the hypothesis, P is semimodular. $\square$

The following theorem follows easily by induction on the rank.

Theorem 3.4. *Let P be a graded poset with 0 and 1. Assume that every rank 3 interval is a semimodular lattice. Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

Then P is a semimodular lattice. $\square$

4. Posets That Locally Resemble Modular Lattices.

In this section, we prove that, if a poset resembles a (complemented) modular lattice "up close," it *is* a (complemented) modular lattice, under certain weak assumptions.

Theorem 4.1. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a modular lattice. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a modular lattice.

Remark. The poset R of Figure 3 shows that we need some assumption about the rank.

Proof of theorem. The theorem follows from Theorem 3.3 and its dual. $\square$

Theorem 4.2. *Let P be a graded poset with 0 and 1. Assume that every rank 3 interval is a modular lattice. Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

Then P is a modular lattice. $\square$

Theorem 4.3. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a complemented modular lattice. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a complemented modular lattice.

Proof. By Theorem 1, P is a modular lattice. If P is not complemented, it has a join-irreducible j that does not cover 0 ([1], Chapter I, Theorem 14; and Chapter IV, Theorem 6). Hence P has a proper interval (which will be $\downarrow j$ if $j \neq 1$) that is not complemented, a contradiction. □

Theorem 4.4. *Let P be a graded poset with 0 and 1 of rank at least 3. Assume that every rank 3 interval is a complemented modular lattice. Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

Then P is a complemented modular lattice. □

5. Posets That Locally Resemble Distributive Lattices.

In this section, we prove that, if a poset resembles a distributive lattice "up close," it *is* a distributive lattice (under certain weak assumptions). We thus settle an issue raised by Stanley ([15]). See Theorems 1 and 2.

As a consequence, we obtain the theorem of Stanley, in which Boolean lattices replace distributive ones. We also obtain the theorem of Grabiner, in which products of chains replace distributive lattices in the statement of the result. (See Corollaries 4 and 6.)

The following theorem is due to the first author; the proof below is due to the second.

Theorem 5.1. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a distributive lattice. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a distributive lattice.

Remark. The poset R of Figure 3 shows that we need some assumption about the rank.

Proof of theorem. By Theorem 4.1, P is a modular lattice. It suffices to show that M_3 is not a sublattice.

Assume for a contradiction that it is, i.e., there exist distinct x, y, $z \in P$ such that $x \vee y = x \vee z = y \vee z$ and $x \wedge y = x \wedge z = y \wedge z$. Clearly $x \vee y = 1$ and $x \wedge y = 0$.

Let $a \in P$ be such that $0 \lessdot a \leqslant x$. Let $b := (a \vee z) \wedge y$ and let $c := (a \vee y) \wedge z$. By modularity, $0 \lessdot b, c$ and clearly a, b, and c are distinct.

By modularity,

$$a \vee b = (a \vee y) \wedge (a \vee z) = a \vee c$$

which clearly equals $b \vee c$, so that M_3 is a sublattice of a proper interval, a contradiction. □

Theorem 5.2. *Let P be a graded poset with 0 and 1 and rank at least 3. Assume that every rank 3 interval is a distributive lattice (Figure 4). Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

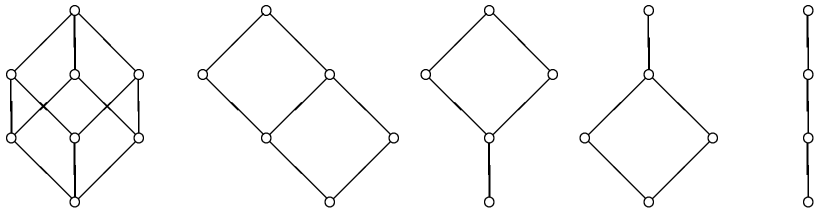


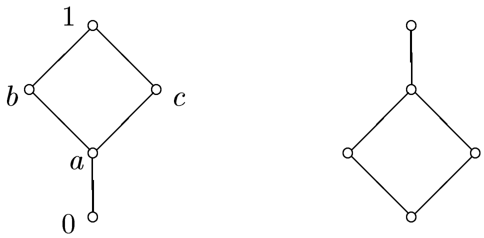
Figure 4. The rank 3 distributive lattices.

Then P is a distributive lattice. $\square$

Theorem 5.3. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a product of chains. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a product of chains.

Remark. The poset R of Figure 3 shows that some condition on the rank is necessary. Also, there are exactly two finite distributive lattices that are not products of chains with the additional property that every proper interval is a product of chains: S and its dual S^∂, where $S := \{0, a, b, c, 1\}$, $0 < a < b, c < 1$ and no other comparabilities hold but the necessary ones (Figure 5).



S $\quad$ S^∂

Figure 5. The posets S and S^∂.

Proof of theorem. By Theorem 1, P is a finite distributive lattice. Let $J := \mathcal{J}(P)$. For any minimal element $j \in J$, $J \setminus \{j\}$ is a disjoint union of chains. (For $a := \{j\}$ is a non-zero element of P, viewing P as the lattice of down-sets of J, and $\uparrow a$ is a product of chains.)

Similarly, for any maximal element $k \in J$, $J \setminus \{k\}$ is a disjoint union of chains.

Fix a minimal element $j \in J$ and let $J \setminus \{j\} = \bigcup_{i \in I} C_i$ be a disjoint union of chains. Assume for a contradiction that there exist distinct chains C and C' and elements $c \in C$ and $c' \in C'$ such that $j \leqslant c, c'$. (If j is only comparable to the elements of one chain C_i but J is not a disjoint union of chains, then we get the dual situation.)

If ${}^\# I \geqslant 3$, then we may choose a maximal element c'' of a third chain such that $J \setminus \{c''\}$ is *not* a disjoint union of chains, a contradiction.

If ${}^\# C \geqslant 2$ or ${}^\# C' \geqslant 2$, we again get a contradiction. Hence $J = \{j, c, c'\}$ and P only has rank 3, which is false. $\square$

Corollary 5.4 (Grabiner [8], Theorem 1)**.** *Let P be a graded poset with 0 and 1 and rank at least 3. Assume that every rank 3 interval is a product of chains (Figure 6). Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

Then P is a product of chains. □

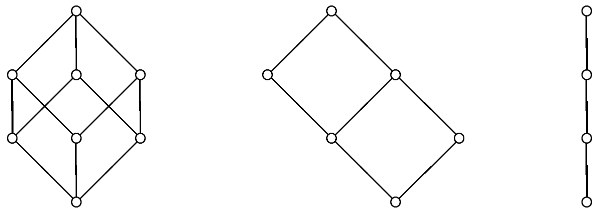


Figure 6. The rank 3 products of chains.

Theorem 5.5. *Let P be a graded poset with 0 and 1 of rank at least 4. Assume that every proper interval is a Boolean lattice. Also assume that $P \setminus \{0, 1\}$ is connected.*

Then P is a Boolean lattice.

Remark. The 3-element chain shows that some condition on the rank is needed. It is the only finite distributive lattice that is not a Boolean lattice with the property that every proper interval is a Boolean lattice.

Proof of theorem. The result follows from Theorem 4.3 and Theorem 1. □

Corollary 5.6 (Stanley [8], Lemma 8)**.** *Let P be a graded poset with 0 and 1 and rank at least 3. Assume that every rank 3 interval is a Boolean lattice (Figure 7). Also assume that, for every interval of rank at least 4, the interval minus its endpoints is connected.*

Then P is a Boolean lattice. □

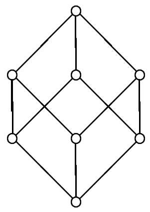


Figure 7. The rank 3 Boolean lattice.

6. Extensions to Posets of Infinite Rank.

It turns out that many of the theorems we have proven for graded posets of finite rank are true even for non-graded posets with infinite chains. (The proofs, however, are different.) The results of this section are mostly due to the second author, except for the results on semimodular posets, which are mostly due to the first.

Definitions. Given an order-theoretic property X, we say that a poset is *PI "X"* (or *Proper Interval "X"*) if all of its proper intervals have the property X. A poset with 0 is *atomic* if every non-zero element is above an atom. A poset P is *semimodular* if, for all x, y, $z \in P$ such that $z \lessdot x$ and $z \leqslant y$, there exists $w \in P$ such that $x \leqslant w$, and either $y = w$ or else $y \lessdot w$. A poset P is *SM* if, for all x, y, $z \in P$ such that $z \lessdot x, y$ and $x \neq y$, there exists $w \in P$ such that $y, z \lessdot w$.

First, we investigate the relation between semimodularity and the SM property for posets and lattices of finite or infinite rank.

Proposition 6.1. *Let P be a semimodular poset with 0 of rank $n \in \mathbb{N}_0$. Then*

1. *P is graded; and*
2. *P has the SM property.*

Proof (by induction on n). The result is trivial if $n = 0$, so suppose that the proposition holds for all semimodular posets with 0 of rank $k < n$ ($k \in \mathbb{N}_0$).

Assume that P does not satisfy the SM property; we shall derive a contradiction. Let x, y, $z \in P$ be such that $z \lessdot x, y$ and $x \neq y$, but x and y have no common upper cover. Clearly z equals 0.

By semimodularity, there exist x', x'', y', $y'' \in P$ such that $x \lessdot x'$ and $y \lessdot y'' < x'$ and $y \lessdot y'$ and $x \lessdot x'' < y'$ (Figure 8).

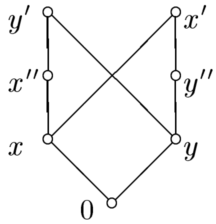


Figure 8. A subposet of P.

Let $\uparrow x$ have rank $k < n$. Then $\uparrow y'$ has rank at most $k - 2$, so $\uparrow y$ has rank at most $k - 1$; but then by symmetry $\uparrow x$ has rank at most $k - 2$, a contradiction. Thus P has the SM property.

It easily follows that P is graded. $\square$

Note. A semimodular *lattice* has the SM property. For posets of finite rank, semimodularity and the SM property are equivalent. The bounded semimodular poset of Figure 9 and the bounded SM poset of Figure 10 show, however, that neither property implies the other. (The poset of Figure 9 is the set $\{0, 1, x_1, y_1, x_2, y_2, \dots\}$ where $x_i < x_{i+1}, y_{i+2}$ and $y_i < y_{i+1}, x_{i+2}$ for $i \geqslant 1$. The poset of Figure 10 is the set $\{0, 1, a, b, c, z_1, z_2, \dots\}$ where $a, b < c, z_i$ and $z_i > z_{i+1}$ for $i \geqslant 1$.) Indeed, the poset of Figure 10 even has a finite maximal chain, but is not graded.

For the rest of the section, let L be an atomic bounded poset.

Proposition 6.2. *The following are equivalent:*

1. *L is a lattice;*
2. *L is a PI "lattice" and any two atoms have a least upper bound.*

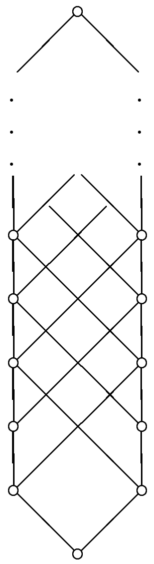


Figure 9. A semimodular poset that is not SM.

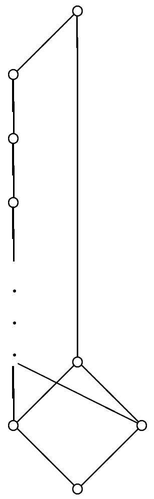


Figure 10. A non-semimodular poset with the SM property.

Proof. Clearly (1) implies (2).

Now assume (2) holds. Because L is bounded, it suffices to prove that, for all x, y, s, $t \in L \setminus \{0,1\}$ such that $x, y \leqslant s, t$, there exists $z \in L$ such that $x, y \leqslant z \leqslant s, t$.

We claim that, for all u, $v \in L$, if $0 < u \leqslant v \leqslant s, t$, then $u' := s \wedge t$ (in the proper interval $\uparrow u$) equals $v' := s \wedge t$ (in $\uparrow v$). (Clearly $v' \leqslant u'$; but since $v \leqslant u' \leqslant s, t$, we have $u' \leqslant v'$, hence $u' = v'$.)

Choose atoms a, $b \in L$ such that $a \leqslant x$ and $b \leqslant y$. Then $x, a, a \vee b, b, y \leqslant s, t$ and $x \geqslant a \leqslant a \vee b \geqslant b \leqslant y$, so that, by the claim, $z := s \wedge t$ (in the proper interval $\uparrow x$) equals $s \wedge t$ (in the interval $\uparrow y$).

Thus $x, y \leqslant z \leqslant s, t$. $\square$

For the rest of the section, assume that L has no *maximal chain of rank 3 or* *less.*

Proposition 6.3. *If L is a PI "semimodular lattice," then L is a lattice.*

Proof. By Proposition 2, we need only check that any two distinct atoms x and y have a least upper bound. If 1 is the only upper bound, we are done. Else, assume there exist s, $t \in L \setminus \{0, 1\}$ such that $x, y \leqslant s, t$. It suffices to find $z \in L$ such that $x, y \leqslant z \leqslant s, t$.

Let $u := x \vee y$ (in $\downarrow s$) and let $v := x \vee y$ (in $\downarrow t$). By semimodularity, $x, y \lessdot u, v$. If $u = v$, we are done. Else, by semimodularity, $u, v \lessdot w$, where $w := u \vee v$ in $\uparrow x$. Since L has no maximal chain of rank 3, $w \neq 1$, so that $u = v$ (since $\downarrow w$ is a lattice). $\square$

Note. If L is a PI "distributive lattice" such that $L \setminus \{0, 1\}$ is connected, but L *does* have a maximal chain of rank 3 or less, then either L is a lattice or else L is the poset Q_2 of §3.

Proposition 6.4. *Consider the following statements:*

(1) *L is a semimodular poset;*
(2) *L is an SM poset;*
(3) *L is a PI "semimodular poset" and $L \setminus \{0, 1\}$ is connected;*
(4) *L is a PI "SM poset" and $L \setminus \{0, 1\}$ is connected.*

Then (3) implies (1), but not conversely. Similarly, (4) implies (2), but not conversely.

Note. Figure 9 and Proposition 1 show that (1) does not imply (3) in general. Figure 10 shows that (2) does not imply (4) in general. Figure 11 shows that (1) does not imply (3) [and (2) does not imply (4)] even if L is finite and bounded. (The authors of [9] claim without proof that there are posets with the SM property that have intervals lacking the SM property.)

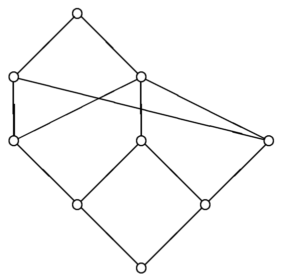


Figure 11. A semimodular poset that is not PI "semimodular."

Proof. We prove that (3) implies (1) [(4) implies (2)]. For $a, b \gtrdot 0$ in L, write $a \triangle b$ if there exists $x \in L$ such that $a \lessdot x \geqslant b$ ($a \lessdot x \gtrdot b$). Assume we have distinct atoms $a, b, c \gtrdot 0$ in L and that $a \triangle b \triangle c$. We claim that $a \triangle c$.

[For suppose $a \lessdot x \geqslant b \lessdot y \geqslant c$ where x, $y \in L$ ($a \lessdot x \gtrdot b \lessdot y \gtrdot c$). By semimodularity (the SM property) there exists $w \in L$ such that $y \leqslant w$, and either

$x = w$ or else $x \lessdot w$ ($x = y$ or else $x, y \lessdot w$). Because L has no maximal chains of rank 3, 0, a, and c are contained in a proper interval, so we can use semimodularity (the SM property).] □

The situation is better for semimodular *lattices*.

Proposition 6.5. *The following are equivalent:*

(1) *L is a semimodular lattice;*
(2) *L is a PI "semimodular lattice" and $L \setminus \{0, 1\}$ is connected.*

Proof. Assume (1). Since a semimodular lattice is SM and L contains no maximal chains of rank 2, $L \setminus \{0, 1\}$ is connected. Since L is a lattice, it is a PI "semimodular lattice." Hence (2) holds.

Now assume (2). By Proposition 3, L is a lattice. By Proposition 4, L is a semimodular poset. Hence (1) holds. □

For the next results, we apply the M_3–N_5 theorem.

Proposition 6.6. *If L is a semimodular lattice and a PI "modular lattice," then L is a modular lattice.*

Proof. Assume L is not modular, for a contradiction. Then let $\{s, x, y, z, t\}$ be a sublattice isomorphic to N_5, where $s < x < y < t$ and $s < z < t$. Clearly s equals 0 and t equals 1.

Let a, $b \in L$ be atoms such that $a \leqslant x$ and $b \leqslant z$. Because $\uparrow a$ is modular, we have $a < (a \vee z) \wedge y$ (since $x \vee [(a \vee z) \wedge y] = [x \vee (a \vee z)] \wedge y = y$). Because $\uparrow b$ is modular,

$$x \vee [(b \vee y) \wedge z] = (b \vee x) \vee [(b \vee y) \wedge z] = (b \vee x \vee z) \wedge (b \vee y) = b \vee y.$$

Hence $\{0, a, (a \vee z) \wedge y, z, a \vee z\}$ and $\{0, x, y, (b \vee y) \wedge z, b \vee y\}$ are both sublattices isomorphic to N_5. By PI "modularity," $a \vee z$ equals 1 and $b \vee y$ equals 1. By semimodularity, $y, z \lessdot 1$.

Since L has no maximal chains of rank 3, there exists $c \in L$ such that $b < c < z$, and so semimodularity implies that $a \vee b < a \vee c < 1$. As $\uparrow a$ is modular and $(a \vee b) \vee y = 1$, we know that $a < (a \vee c) \wedge y$.

Thus $\{0, a, (a \vee c) \wedge y, c, a \vee c\}$ is a sublattice isomorphic to N_5 in the proper interval $\downarrow a \vee c$, a contradiction. □

Proposition 6.7. *If L is a modular lattice and a PI "distributive lattice," then L is a distributive lattice.*

Proof. Assume for a contradiction that L is not distributive. Then there exist s, x, y, z, $t \in L$ such that $s < x, y, z < t$ and $\{s, x, y, z, t\}$ is a sublattice isomorphic to M_3. By PI "distributivity," s equals 0 and t equals 1.

Let $a \in L$ be an atom such that $a \leqslant x$. Let $b := (a \vee z) \wedge y$ and $c := (a \vee y) \wedge z$. Note that $a \vee b = (a \vee y) \wedge (a \vee z) = a \vee c \gtrdot b, c$ so that $a \vee b = b \vee c$. By modularity, $0 \lessdot b$.

Since L has no rank 2 maximal chains, $b \vee c < 1$, so that the proper interval $\downarrow b \vee c$ contains M_3 as a sublattice, a contradiction. $\square$

As a consequence, we get the next results:

Corollary 6.8. *The following are equivalent:*

(1) *L is a modular lattice;*
(2) *L is a PI "modular lattice" and $L \setminus \{0, 1\}$ is connected.* $\square$

Corollary 6.9. *The following are equivalent:*

(1) *L is a distributive lattice;*
(2) *L is a PI "distributive lattice" and $L \setminus \{0, 1\}$ is connected.* $\square$

A class $\mathcal{K}$ of bounded posets has the *interval extension property* with respect to a class of posets $\mathcal{L}$ if, for every K in $\mathcal{K}$, the following are equivalent:

(1) K is in $\mathcal{L}$.
(2) Every proper interval of K is in $\mathcal{L}$ and $K \setminus \{0, 1\}$ is connected.

The above results may be summarized as follows:

Theorem 6.10. *The class of bounded atomic posets with no maximal chains of rank 3 or less has the interval extension property for semimodular, modular, and distributive lattices, but not for semimodular posets.* $\square$

7. Applications to Geometry (in the Language of Geometry).

A lattice with 0 is *atomistic* if every element is a least upper bound of a set of atoms. A lattice of finite rank is *geometric* if it is semimodular and atomistic. Those geometric lattices that are modular and *irreducible* (in the sense that they cannot be expressed as direct products of non-trivial lattices) are known as *projective geometries.* For the definitions of *affine geometries* and *hyperbolic geometries*, we refer to [7]. A general construction of affine geometries from projective geometries is given by removing from any projective geometry a hyperplane (that is, the interval from 0 to a lower cover of 1, minus $\{0\}$). A semimodular lattice is said to be *locally projective* if $\uparrow a$ is a projective geometry for every atom a. (These lattices can often be embedded into projective geometries.)

A *weak linear space* is a two-sorted structure consisting of a set of *points* and a set of *lines* with an *incidence* relation between points and lines such that (1) any two points are incident with exactly one line, and (2) any line is incident with at least *one* point. If, moreover, every line is incident with at least *two* points, the structure is called a *linear space* (cf. [4]).

A linear space that possesses a quadrangle of lines is said to be a *projective plane* (or an *affine plane* or a *hyperbolic plane*) if, for every non-incident point-line pair, the number of lines incident with the point and not intersecting the line is 0 (or 1, or at least 2, respectively).

Obviously, every weak linear space with at least one line can be considered as a semimodular lattice of rank 3, and vice versa. [Here, points correspond to atoms, lines to coatoms (the lower covers of 1), and the incidence relation to the partial ordering.] The linear spaces with more than one line may be identified as the geometric lattices of rank 3.

Now we are prepared to give an interpretation of our previous considerations in the language of geometry (which indeed should also be viewed from the perspective of *diagram geometry* — cf. [4]; also see below).

Let L be a bounded graded poset of finite rank at least 4. Assume that, for any interval of rank at least 3, the interval minus its endpoints is connected.

1. *The following are equivalent:*

(1) *L is a semimodular lattice;*
(2) *every rank 3 interval of L is a weak linear space.*

2. *The following are equivalent:*

(1) *L is a geometric lattice;*
(2) *every rank 3 interval of L is a linear space.*

The poset L is a projective geometry (or affine geometry or hyperbolic geometry — cf. [7]) if and only if every rank 3 interval of L containing 0 is a projective plane (or affine plane or hyperbolic plane) and every other rank 3 interval is a projective plane.

3. *The following are equivalent:*

(1) *L is a semimodular locally projective lattice;*
(2) *every rank 3 interval of L that contains 0 is a weak linear space and every other rank 3 interval is a projective plane.*

8. Connections with Diagram Geometries and the Theory of Buildings.

To a graded poset P of finite rank, one may associate a geometry (in the sense of Buekenhout — cf. pp. 75–76 of [4]) using the comparability graph of P together with the rank function of P. (Buekenhout calls his concept of geometry "the graph theoretic approach.") An important notion for a geometry is that of being "residually connected." The geometry of a graded poset P is *residually connected* if and only if, for every interval of rank at least 3, the interval minus its endpoints is connected. We may call such a poset "residually connected."

Comment. The concept of residual connectivity goes back to J. Tits. Referring to P (usually the 0 and 1 of the poset are removed; that is, the interval from 0 to 1 minus the endpoints is considered), a chain is also called a "flag," and a maximal chain a "chamber."

What Tits was especially interested in is the "flag complex" (combinatorialists would say, "chain complex") of P. (Here a chamber is just a maximal simplex.)

Two chambers which differ by exactly one element are "adjacent"; a finite sequence of consecutive adjacent chambers is a "gallery." The complex is "connected" if any two chambers can be joined by a gallery; it is "strongly connected" if, for every simplex, its "star" (which is itself a complex) is connected — and this just means that P is residually connected. Actually, Tits was investigating strongly connected "numbered" complexes (for P, the rank function gives the numbering). Thus we can state that the chain complex of any graded poset is a numbered complex; it is strongly connected if and only if the poset is residually connected.

The "basic graph" of a graded poset is given by the Hasse diagram of its rank set (with its canonical ordering). It turns out (cf. pp. 486–487 of [4]) that there is a 1-1 correspondence between

(1) all strongly connected numbered complexes with a string as the basic graph; and
(2) all residually connected graded posets.

[Later Tits gave an intrinsic description in terms of "chamber systems." (The chamber system of a graded poset is its set of maximal chains.) For example, chamber systems with a group action are studied (possibly having some transitivity property). Here Coxeter groups appear too. A "building" is a certain connected numbered chamber complex; all of its galleries of reduced type are geodesic. If two simple galleries have the same origin, the same extremity, and the same reduced type, they coincide ... and so on. The posets that come from buildings are the ones with linear diagram; see [2], Proposition 4.18 and [16].]

To the geometry of a poset belongs its (basic) diagram, given by the Hasse diagram of its "type set," i.e., its rank set (which is canonically ordered). The geometry is "firm" if every rank 3 interval has at least two atoms.

The result of F. Buekenhout is now as follows:

Theorem. *The geometric lattices (as geometries with basic diagram) are exactly the firm residually connected geometries with string diagram — that is, those residually connected graded posets such that every rank 3 interval minus its endpoints is a linear space (or equivalently, every rank 3 interval is a geometric lattice).*

This theorem is proved in [3]. Pasini states it in a nice modern form in [12], Theorem 7.6 (p. 174). Also see [6], p. 198. (Clearly, this theorem is also a direct consequence of remark 2 in §7.)

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References

1. Garrett Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, Rhode Island, 1967.
2. Anders Björner, *Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings*, Advances in Mathematics **52** (1984), 173–212.
3. Francis Buekenhout, *Diagrams for geometries and groups*, Journal of Combinatorial Theory (A) **27** (1979), 121–151.
4. Francis Buekenhout (ed.), *Handbook of Incidence Geometry*, North–Holland, Amsterdam, 1995.
5. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990.
6. Anne Delandtsheer, *Dimensional linear spaces*, Handbook of Incidence Geometry, F. Buekenhout (ed.), North-Holland, Amsterdam, 1995, pp. 193–294.

7. Claude-Alain Faure and Alfred Frölicher, *The dimension theorem in axiomatic geometry*, Geometriae Dedicata **60** (1996), 207–218.
8. David J. Grabiner, *Posets in which every interval is a product of chains, and natural local actions of the symmetric group*, Discrete Mathematics, to appear.
9. L. Haskins and S. Gudder, *Semimodular posets and the Jordan–Dedekind chain condition*, Proceedings of the American Mathematical Society **28** (1971), 395–396.
10. Takayuki Hibi, *The comparability graph of a modular lattice*, preprint.
11. Takayuki Hibi and Naoki Terai, *Finite free resolutions and 1-skeletons of simplicial complexes*, Journal of Algebraic Combinatorics **6** (1997), 89–93.
12. Antonio Pasini, *Diagram Geometries*, The Clarendon Press, New York, 1994.
13. Francesco Regonati, *Whitney numbers of the second kind of finite modular lattices*, Journal of Combinatorial Theory (A) **60** (1992), 34–49.
14. Richard P. Stanley, *Flag-symmetric and locally rank-symmetric partially ordered sets*, The Electronic Journal of Combinatorics **3** (1996), #R6.
15. Richard P. Stanley, *lecture*, Workshop on Representation Theory and Symmetric Functions, Mathematical Sciences Research Institute, Berkeley, California, April 18, 1997.
16. Michelle L. Wachs, *Quotients of Coxeter complexes and buildings with linear diagram*, European Journal of Combinatorics **7** (1986), 75–92.

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