COPRODUCTS OF BOUNDED DISTRIBUTIVE LATTICES: CANCELLATION
A Banff Conference Problem

Jonathan David Farley

Abstract. Let $L * M$ denote the coproduct of the bounded distributive lattices L and M. At the 1981 Banff Conference on Ordered Sets, the following question was posed: What is the largest class $\mathcal{L}$ of finite distributive lattices such that, for every non-trivial Boolean lattice B and every $L \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$?

In this note, the problem is solved.

1. Introduction.

A *Post algebra* is a bounded distributive lattice (with least element 0 and greatest element 1) with a non-trivial finite chain $0 = c_0 < \cdots < c_{n-1} = 1$ and a Boolean sublattice B such that every element has a unique representation $\bigvee_{i=1}^{n-1} (b_i \wedge c_i)$ where $b_1 \geqslant \ldots \geqslant b_{n-1}$ are in B. The chain is called the *chain of constants*. It is well known that Post algebras are exactly the coproducts $B * C$ of non-trivial Boolean lattices B and non-trivial finite chains C in the category of bounded distributive lattices ([13], Theorem 2). It is also well known that the chain of constants in a Post algebra P is unique in the following sense: If B is a Boolean sublattice of P and C and C' are finite subchains containing 0 and 1 such that $P = B * C = B * C'$, then $C = C'$. That is, C and C' are equal as subsets, not simply isomorphic ([7], Theorem 2.1). More precisely, if $\iota_B : B \hookrightarrow B * C$, $\iota_C : C \hookrightarrow B * C$, $\iota'_B : B \hookrightarrow B * C'$, and $\iota_{C'} : C' \hookrightarrow B * C'$ are

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the natural monomorphisms and $\Psi : B * C \cong B * C'$ an isomorphism, then the image of $\Psi \circ \iota_C$ is the image of $\iota_{C'}$.

Balbes and Dwinger proved that if B and B' are non-trivial Boolean lattices and C and C' non-trivial bounded chains, then $B * C = B' * C'$ implies $B = B'$ and $C \cong C'$ ([1], Theorem 1). Moreover, for a given non-trivial bounded chain C, $B * C = B * C'$ implies $C = C'$ for all non-trivial Boolean lattices B and all bounded chains C' if and only if C is *rigid*, that is, has just one automorphism ([1], Theorem 6). Comer and Dwinger investigated bounded distributive lattices K such that, for all bounded distributive lattices L and M in a given class, $K * L = K * M$ implies $L = M$ ([3]).

At the 1981 Banff Conference on Ordered Sets, F. Yaqub asked for a description of the largest class $\mathcal{L}$ of finite distributive lattices such that, for every non-trivial Boolean lattice B and every $L \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$ ([12], p. 849).

The problem as stated must be formulated more precisely, but we answer the question, as well as the corresponding one for isomorphisms (Theorems 2 and 3). Our tool is Priestley duality for distributive lattices.

2. Definitions, Notation, and Basic Results.

For basic notions, see [6]. A *component* of a poset is a non-empty maximal connected subset. A subset U of a poset P is an *up-set* if, for all $u \in U$ and $p \in P$, $u \leqslant p$ implies $p \in U$. A partially ordered topological space P is *totally order-disconnected* if, for all p, $q \in P$ such that $p \nleqslant q$, there exists a clopen up-set U such that $p \in U$ and $q \notin U$. A *Priestley space* is a compact totally order-disconnected space.

Given an ordered space P, let $D(P)$ denote the bounded distributive lattice of clopen up-sets. Given a bounded distributive lattice L, let $P(L)$ denote the Priestley space of prime filters, with topology generated by the subbasis

$$\big\{\{F \in P(L) \mid a \in F\}, \{F \in P(L) \mid a \notin F\} \;\big|\; a \in L\big\}.$$

The operators D and P extend to functors which yield a dual equivalence between the categories of bounded distributive lattices with $\{0,1\}$-preserving homomorphisms and Priestley spaces with continuous order-preserving maps. For some consequences of Priestley duality, see [9] and [10].

It is well known that $P(L)$ is an antichain if L is a Boolean lattice ([6], Theorem 9.8). Also, $P(L)$ is a finite poset with the discrete topology if L is a finite distributive lattice. Further, $P(L \times M)$ is *order-homeomorphic* (order-isomorphic via a map that is a homeomorphism) to $P(L) + P(M)$, the disjoint sum of the ordered spaces ([6], Exercise 10.3(iii)). Lastly, $P(L * M)$ is order-homeomorphic to $P(L) \times P(M)$. Hence, for bounded distributive lattices L and

M, $L * M$ is isomorphic to $L^{P(M)}$, the lattice of continuous order-preserving maps from $P(M)$ to L with the discrete topology, ordered pointwise; L is associated with the constant maps, and M with the maps $\mathbf{2}^{P(M)}$, where $\mathbf{2} = \{0, 1\}$ ([5], Theorem and Corollary, [4], Corollary 2.3, and [11], Theorem). In [14] and [15], this aspect of duality is used to understand *generalized Post algebras*, coproducts of Boolean lattices and bounded distributive lattices. (See Figures 1–4.)

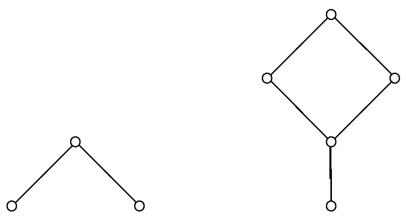

Figure 1. The poset P and the lattice $L = D(P)$.

Figure 2. The poset Q and the lattice $M = D(Q)$.

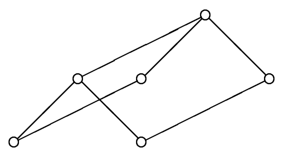

Figure 3. The poset $P \times Q$.

3. The Solution to the Problem for Isomorphisms and Equality.

We must emend Yaqub's problem because of the following example: Let B be a Boolean lattice such that $B^2 \cong B$. (For instance, let $B := \mathbf{2}^\omega$.) Then for any finite poset P, $B^P \cong B^{(2\times P)}$. Hence, for any finite distributive lattice L, $B * L \cong B * L^2$. Therefore, in the problem, we must restrict the class to which L' can belong. We insist that L and L' belong to the *same* class.

A class $\mathcal{L}$ of finite distributive lattices is *shabazz* if, for all lattices M and N, $M \times N^2 \in \mathcal{L}$ implies $M \times N \in \mathcal{L}$. A finite distributive lattice is *square-free* if it has no direct factor of the form N^2 (N a non-trivial lattice). Let $\mathcal{B}$ be the class of non-trivial Boolean lattices.

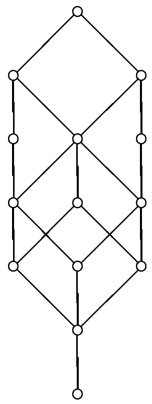

Figure 4. The lattice $L * M \cong D(P \times Q)$.

Lemma 1. *Let X be an antichain, P and P' posets, C a component of P, and $\Psi : X \times P \cong X \times P'$ an order-isomorphism. Let $\pi_X : X \times P' \to X$ be the projection.*

Then, for all $x \in X$, there exists $x' \in X$ such that

$$(\pi_X \circ \Psi)[\{x\} \times C] = \{x'\}.$$

Proof. Suppose that $\Psi(x, p_0) = (x', p_0')$ and $\Psi(x, p_1) = (x'', p_1')$. If $p_0 \leqslant p_1$, then $(x', p_0') \leqslant (x'', p_1')$, so that $x' = x''$. $\square$

Theorem 2. *Let $\mathcal{L}$ be a shabazz class of finite distributive lattices. The following are equivalent:*

1. *for all $B \in \mathcal{B}$ and $L, L' \in \mathcal{L}$, $B * L \cong B * L'$ implies $L \cong L'$;*
2. *every member of $\mathcal{L}$ is square-free.*

Proof. Assume $L \cong M \times N^2$ for some $L \in \mathcal{L}$ and lattices M and N. Let $L' := M \times N$ and let $B := \mathbf{2}^\omega$. As $B \cong B^2$, we have $B * L \cong B * L'$.

Now assume $B \in \mathcal{B}$ and let L and L' be square-free finite distributive lattices. Then $B * L \cong B * L'$ implies there exists an order-isomorphism $\Psi : X \times P \cong X \times P'$ where $X := P(B)$, $P := P(L)$, and $P' := P(L')$; P and P' are finite posets with pairwise non-isomorphic components. Let $\pi_{P'} : X \times P' \to P'$ be the projection and fix $x \in X$. By Lemma 1, the map

$$p \mapsto (\pi_{P'} \circ \Psi)(x, p) \qquad (p \in C)$$

is an order-isomorphism on each component C of P. $\square$

Theorem 3. *Let $\mathcal{L}$ be a class of finite distributive lattices. The following are equivalent:*

(1) *for all $B \in \mathcal{B}$ and $L,\, L' \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$;*
(2) *every member of $\mathcal{L}$ is rigid.*

Proof. Let L be a finite distributive lattice and $\phi : L \cong L$ a non-trivial automorphism. Define $\Phi : L^2 \to L^2$ by $\Phi(a, b) = \big(a, \phi(b)\big)$ $(a,\, b \in L)$. Then Φ is an automorphism of $\mathbf{2}^2 * L$ that does not map constants to constants.

Now let X be a non-empty Priestley space that is an antichain. Let P and P' be finite rigid posets. Let $\Psi : X \times P \cong X \times P'$ be an order-homeomorphism. We must show that, for all $U \in D(P)$, there exists $U' \in D(P')$ such that $\Psi[X \times U] = X \times U'$.

Without loss of generality, P and P' are connected. By Lemma 1, for all $x \in X$, there exists $x' \in X$ such that $\Psi[\{x\} \times P] = \{x'\} \times P'$. For all $x \in X$ and $U \in D(P)$, let $U'_x \in D(P')$ be such that $\Psi[\{x\} \times U] = \{x'\} \times U'_x$.

It suffices to show that, for all $x_0,\, x_1 \in X$ and $p \in P$,

$$(\pi_{P'} \circ \Psi)(x_0, p) = (\pi_{P'} \circ \Psi)(x_1, p).$$

By Lemma 1, the maps $p \mapsto (\pi_{P'} \circ \Psi)(x_0, p)$ and $p \mapsto (\pi_{P'} \circ \Psi)(x_1, p)$ $(p \in P)$ are order-isomorphisms, so, by rigidity, they are equal. $\square$

4. Related Results.

In [3], a research announcement of results that apparently were never published*, it is stated that if K, L, and M are bounded distributive lattices with L and M rigid, then $K * L = K * M$ implies $L = M$. (The author discovered Theorem 3 independently of [3].) The interpretation of this statement is *not* the same as in our paper, as is evidenced by the case where K, L, and M are each the three-element chain (cf. [8], Fig. 4).

In [2], Theorem 2.4(i), it is shown that there exist countable Boolean algebras L and L' such that $\mathbf{2}^2 * L \cong \mathbf{2}^2 * L'$ but $L \not\cong L'$. It also follows from (1.4) in [2] that, for non-trivial finite distributive lattices L, L', and M, $L * M \cong L' * M$ implies $L \cong L'$.

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*The author was unsuccessful in his attempt to obtain a copy of the (unpublished) paper announced in [3].

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, United States of America

Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, California 94720, United States of America

E-mail address: `farley@math.vanderbilt.edu`