# Weak Convergence and Deterministic Approach to Turbulent Diffusion.

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*Dedicated to Walter Strauss*

**Abstract.**

The purpose of this contribution is to show that some of the basic ideas of turbulence can be addressed in a deterministic setting instead of introducing random realizations of the fluid. Weak limits of oscillating sequences of solutions are considered and along the same line the Wigner transform replaces the Kolmogorov definition of the spectra of turbulence. One of the main issue is to show that, at least in some cases, this weak limit is the solution of an equation with an extra diffusion (the name turbulent diffusion appears naturally). In particular for a weak limit of solutions of the incompressible Euler equation (which is time reversible) such process would lead to the appearance of irreversibility. In the absence of proofs, following a program initiated by P. Lax [L], the diffusive property of the limit is analyzed, with the tools of Lax and Levermore [LL] or Jin Levermore and Mc Laughlin [JLM], on the zero dispersion limit of the Korteweg-deVries equation and of the Non Linear Schrodinger equation. The three authors are extremely happy to have the opportunity to publish this contribution in a volume dedicated to Walter Strauss as a mark of friendship and admiration for his achievement. They hope that this paper concerned with non linear fluid mechanics, non linear instabilities and inverse scattering, will find its place in the different domains that have interested Walter.

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## 1. Introduction.

Two types of objects appear frequently in the theory of turbulence for a fluid defined in an open set $\Omega \subset \mathbf{R}^d$, $d = 1, 2$ or $3$ : Models of Turbulence and Turbulent spectra, the first ones is used in most of practical numerical simulations and the second proposed as a tool for the understanding of the phenomena.
One of the most classical models of turbulence is the so called $k - \epsilon$ model which reads :

$$\partial_t U + \nabla_x (U \otimes U) + \nabla_x P - \nu \Delta_x U - c_\nu \nabla_x [\frac{k^2}{\epsilon} (\nabla_x U +^t \nabla_x U)] = 0, \qquad \nabla_x \cdot U = 0 \tag{1}$$

and

$$\begin{aligned} \partial_t k + U \nabla_x k - \frac{c_\nu k^2}{2\epsilon} |\nabla_x U + (\nabla_x U)^t|^2 - \nabla_x \cdot [c_\nu \frac{k^2}{\epsilon} \nabla_x k] + \epsilon = 0\,, \\ \partial_t \epsilon + U \nabla_x \epsilon - \frac{c_1 k}{2} |\nabla_x U + (\nabla_x U)^t|^2 - \nabla_x \cdot [c_3 \frac{k^2}{\epsilon} \nabla_x \epsilon] + c_2 \frac{\epsilon^2}{k} = 0\,. \end{aligned} \tag{2}$$

In the above system $c_i$ denote several constants usually given by experimentation. Equation (1) is the standard incompressible Navier Stokes equation modified by the introduction of a "turbulent viscous" term :

$$\nu \nabla_x u + c_\nu [\frac{k^2}{\epsilon} (\nabla_x U +^t \nabla_x U)]\,;\ \nu_T = \nu + c_\nu \frac{k^2}{\epsilon} \tag{3}$$

where the space-time dependent scalar quantities $k$ and $\epsilon$ are defined by the coupled system (2) and interpreted in term of local energy fluctuation and local enstrophy.
Intuitively the formula encompasses the idea that the fluid by its self interaction produces some averaging effect and therefore that the average $U$ is a smooth quantity solution of an equation with a viscosity greater than the initial one. Therefore one conjectures and for some variant of the system (1) (2) (cf. [MP] and [LM]) proves the relation :

$$\nu_T \geq \nu\,. \tag{4}$$

This relation becomes important when $\nu$ is very small. This situation corresponds to large Reynolds numbers. In some sense the idea is to have for the average an equation which would become valid when, due to the complexity, the initial equation cannot be computed. Such effect becomes crucial when $\nu = 0$. All what remains is the turbulent aspect of the viscosity. It has to be positive otherwise the turbulent model is an ill posed problem (in the sense of Hadamard) and the solution cannot be computed. Furthermore it gives one of the many examples where the limit of a reversible system (the incompressible Euler equation) becomes irreversible.
There is up to now no mathematical analysis of the range of validity of such formulas. The "phenomenological" proof usually given introduces some randomness in the description of the motion of the fluid. The randomness plays an even more crucial role in the definition

of the spectra of turbulence as given below. A random family of realizations $u(x,t,\omega)$, is said to be homogeneous whenever the tensor

$$\langle u(x+r,t)\otimes u(x,t)\rangle$$

is independent of $x$ and $t$ and it is said to be isotropic whenever it depends only on $|r|$. The homogeneity hypothesis implies that the tensor

$$\widehat{R}(k)=\int u(x+r,t,\omega)\otimes u(x,t,\omega)e^{-ir\cdot k}d\omega dr \tag{5}$$

is independent of $x$ and $t$. The isotropy hypothesis implies that it is given by the formula :

$$\widehat{R}(k,t)=\frac{E(|k|)}{4\pi|k|^2}(I-\frac{k\otimes k}{|k|^2})\,. \tag{6}$$

The expressions (5) and (6) are starting points for most of the phenomenological theory of turbulence according to Kolmogorov [Ko] and Kraichnan [Kr].

## 2 Weak convergence and Wigner Measures.

As said in the introduction one of the purpose of this contribution is to show that the recourse to randomness is not compulsory for the above construction. At variance, as already observed by Lax [L], one could consider the weak limit of a deterministic sequence of oscillatory solutions of the fluid mechanics equations and use recent results concerning defect measures and Wigner transform (cf.[Ge], [LP] and [Ta]).
The first observation is that with the homogeneity hypothesis, the formula (5) can also be written as :

$$\widehat{R}(k,t)=\int u(x+\frac{r}{2},t,\omega)\otimes u(x-\frac{r}{2},t,\omega)e^{-ir\cdot k}d\omega dr \qquad (7)\,,$$

and therefore the right hand side turns out to be the Wigner transform of $u$. As above it is a symmetric positive tensor. However the progress is that this tensor is can be written in term of a local quantity $\widehat{R}(x,t,k,\omega)$ :

$$\begin{aligned}\widehat{R}(x,t,k)&=<\widehat{R}(x,t,k,\omega)>\\ \widehat{R}(x,t,k,\omega)&=\int u(x+\frac{r}{2},t,\omega)\otimes u(x-\frac{r}{2},t,\omega)e^{-ir\cdot k}dr\end{aligned} \tag{8}$$

and homogeneity hypothesis can be relaxed.
In fact this idea already appeared as a basic ingredient of the contribution of D.C. Besnard, F.H. Harlow, R.M. Rauenzahn and C. Zemach [BHRZ]. In spite of the fact that it is a very natural approach, to the best of our knowledge, it has not been used elsewhere in turbulence theory. Furthermore the Wigner transform, in space, or (after time localization) in time

is well defined for any solution $u$ or sequence of solutions $u_n$ and one has the standard formulas :

$$\begin{aligned}
&\widehat{R}_n(x,t,k)=\int u_n(x+\frac{r}{2},t)\otimes u_n(x-\frac{r}{2},t,\omega)e^{-ir\cdot k}dr\,,\\
&u_n(x,t)\otimes u_n(x,t)=(\frac{1}{2\pi})^d\int \widehat{R}_n(x,t,k)dk\,,\\
&\widehat{R}^\theta_n(x,t,\tau)=\int \theta(s)u_n(x,t+\frac{s}{2})\otimes u_n(x,t-\frac{s}{2},\omega)e^{-is\tau}ds,\ \ \theta\in\mathcal{D}(\mathbf{R}_+),\ \theta(0)=1\,,\\
&u_n(x,t)\otimes u_n(x,t)=(\frac{1}{2\pi})\int \widehat{R}^\theta_n(x,t,\tau)d\tau\,.
\end{aligned} \tag{9}$$

The two last formulas of (9) are important because they involve instead of space correlations, time correlations which are the quantities more commonly involved in practical experiments.
The only physical *a priori* estimate, uniform with respect to the Reynolds number, is the energy estimate.

$$\tfrac{1}{2}\int_\Omega |u_n(x,t)|^2dx\le \tfrac{1}{2}\int_\Omega |u_n(0,t)|^2dx\le C\,. \tag{10}$$

This observation is valid in particular for the $3d$ Euler or Navier Stokes equation as proven by Di Perna and Lions (cf [Li] section 4.3). It is also valid even in $2d$ for the Navier Stokes equation with viscosity going to zero, when the natural viscous boundary condition $u=0$ on $\partial\Omega$ is assumed. In particular Grenier has constructed solutions to the Navier Stokes equation in the half plane $\Omega=\mathbf{R}_{y+}\times\mathbf{R}_x$ with vorticity blowing up in the $L^\infty$ norm when the viscosity goes to zero([Gre] theorem 2.1.). As observed by Grenier this is an instability phenomenon for the Prandtl layer which is of the same type as the non linear instability of the Euler equation proven by Friedlander Strauss and Vishik [FSV].
With the estimate (10) one concludes that, up to the extraction of a subsequence, $u_n$ converges in $weak^*L^\infty(\mathbf{R}_{t_+},L^2(\Omega))$ to a limit $u$. However due to the above considerations, in many cases, one will have :

$$\lim_{n\to\infty} u_n(x,t)\otimes u_n(x,t)=u(x,t)\otimes u(x,t)+R_{turb}(x,t) \tag{11}$$

with $R_{turb}(x,t)\neq 0$. In a follow up of ideas of Peter Lax [L] one could consider that the appearance of the tensor $R_{turb}$ plays the role of the Reynolds stress tensor as the emergence of turbulence in a deterministic approach. In fact it is the Defect Measure of the sequence $u_n$. The tensor $R_{turb}$ is symmetric positive definite and one has :

$$R_{turb}(x,t)=\lim_{n\to\infty}(u_n-u)\otimes(u_n-u)\,, \tag{12}$$

or, with the Wigner transform of $u_n-u$,

$$\widehat{R}_{turb}(x,t,k)=\lim_{n\to\infty}\int_{\mathbf{R}^d}(u_n-u)(x+\frac{r}{2},t)\otimes(u_n-u)(x-\frac{r}{2},t)e^{-ir\cdot k}dk \tag{13}$$

and

$$\lim_{n\to\infty} u_n(x,t)\otimes u_n(x,t) = u(x,t)\otimes u(x,t) + (\frac{1}{2\pi})^d \int_{\mathbf{R}^d} \widehat{R}_{turb}(x,t,k)dk\,. \tag{14}$$

The formula (14) displays the natural link, for our purpose, between the defect measure and the Wigner transform. Observe that the left hand side of (13) is the natural local and deterministic "avatar" of the Kolmogorv spectra for random turbulence and it is natural to conjecture that it will inherit the basic properties of isotropy and scale law for the dependence in $k$. The isotropy hypothesis is made plausible $3d$ by the following remarks : A necessary condition for $R_{turb}(x,t)$ to be non zero is that curl $u_n$ becomes unbounded in the neighborhood of $(x,t)$ then :
(i) for non zero viscosity it has been shown by Constantine and Fefferman [CF] that it is much more the oscillations in direction of the vorticity than its size that are responsible for instabilities in the fluid,
(ii) exterior constant Coriolis force stabilizes the fluid when the Rossby number goes to infinity as shown by Babin Nicolaenko and Mahalov [BNM] and others. Therefore a decomposition of the vorticity according to the formula :

$$\mathrm{curl}\ u_n = \Omega_n + \tilde{\omega}_n \tag{15}$$

with $\Omega_n$ having a constant direction and a modulus going to $\infty$ while $\omega_n$ remains bounded should not be possible.

In the sequel of this section we consider in the two dimensional case, sequences of solutions to the Euler equation with the impermeability condition :

$$\begin{aligned} \partial_t u_n + \nabla_x(u_n\otimes u_n) + \nabla_x p_n &= 0\,,\\ \vec{n}\cdot u = 0 \text{ on } \partial\Omega,\ \nabla_x\cdot u_n &= 0\,, \end{aligned} \tag{16}$$

or of the Navier Stokes equation in a domain $\Omega$ of $\mathbf{R}^2$ with the viscous boundary condition :

$$\begin{aligned} \partial_t u_n + \nabla_x(u_n\otimes u_n) + \nabla_x p_n - \nu_n\Delta_x u_n &= 0,\\ u_n = 0 \text{ on } \partial\Omega,\ \nabla_x\cdot u_n &= 0\,. \end{aligned} \tag{17}$$

In both cases existence and uniqueness of such solutions are well established facts with the hypothesis

$$u_n(x,0) = u_n^0(x) \in L^2(\Omega), \nabla_x\cdot u_n^0 = 0, \omega_n^0 = \nabla\times u_n^0 \in L^\infty(\Omega)\,, \tag{18}$$

in the first case and with the assumption

$$u_n(x,0) = u_0(x) \in L^2(\Omega), \nabla_x\cdot u_0 = 0\,,$$

in the second case ($\nu_n > 0$).

However in these two cases the turbulent Reynolds tensor may be present in the limit :

(i) if one considers a sequence $u_n$ of solutions of the Euler equation with initial data uniformly bounded in $L^2(\Omega)$ but with initial $\omega_n^0$ vorticity unbounded in $L^\infty$,
(ii) in the second case if, as already mentioned above, one keeps the initial data fixed but let the viscosity $\nu_n$ goes to zero.
In both cases the limit satisfies the equations

$$\partial_t u + \nabla_x(u \otimes u) + \nabla_x R_{turb} + \nabla_x p = 0,\ \nabla_x \cdot u = 0\,, \tag{19}$$

with

$$R_{turb}(x,t) = \lim_{n\to\infty} (u_n(x,t) - u(x,t)) \otimes (u_n(x,t) - u(x,t))\,. \tag{20}$$

Introducing the trace :

$$T = \frac{R_{turb}^{11} + R_{turb}^{22}}{2}$$

(19) is changed into

$$\partial_t u + \nabla_x(u \otimes u) + \nabla_x S_{turb} + \nabla_x P = 0,\ \nabla_x \cdot u = 0\,. \tag{21}$$

with $S_{turb}$ denoting a tracefree tensor and $P = p + T$. The space of tracefree tensors is of dimension 2 and assuming that that the limit $u$ is a smooth function, this space has a natural basis given by the matrix :

$$\frac{1}{2}\big(\nabla u + \nabla^t u\big) = \begin{pmatrix} \partial_{x_1} u_1, & \frac{1}{2}(\partial_{x_2} u_1 + \partial_{x_1} u_2) \\ \frac{1}{2}(\partial_{x_2} u_1 + \partial_{x_1} u_2) & \partial_{x_2} u_2 \end{pmatrix} \tag{22}$$

and an orthogonal complement

$$\Phi(u) = \begin{pmatrix} \frac{1}{2}(\partial_{x_2} u_1 + \partial_{x_1} u_2) & \partial_{x_2} u_2 \\ \partial_{x_2} u_2 & -\frac{1}{2}(\partial_{x_2} u_1 + \partial_{x_1} u_2) \end{pmatrix}\,. \tag{23}$$

Therefore there exist two space-time depending functions $\nu_{turb}(x,t)$ and $\delta(x,t)$ such that one has :

$$S_{turb} = \nu_{turb}\big(\nabla u + \nabla^t u\big) + \delta\Phi(u) \tag{24}$$

and the equation (21) becomes the equation :

$$\partial_t u + \nabla_x(u \otimes u) + \nabla_x\big(\nu_{turb}\big(\nabla u + \nabla^t u\big)\big) + \nabla_x(\delta\Phi(u)) + \nabla_x P = 0,\ \nabla_x \cdot u = 0\,. \tag{25}$$

As discussed above, the mechanism of "creation of turbulence" should be isotropic and this would imply that the tensor $S_{turb}$ is invariant under Galilean transformations and therefore proportional to $\big(\nabla u + \nabla^t u\big)$ reducing (25) to a diffusive type equation :

$$\partial_t u + \nabla_x(u \otimes u) + \nabla_x\big(\nu_{turb}\big(\nabla u + \nabla^t u\big)\big) + \nabla_x P = 0,\ \nabla_x \cdot u = 0\,. \tag{26}$$

Observe however that in any case, assuming that on the boundary $u$ is zero whenever the $\nu_{turb}$ is positive one has the energy estimate :

$$\frac{1}{2}\partial_t \int_\Omega |u(x,t|^2 dx + \int_\Omega \nu_{turb}(x,t)|\nabla_x u|^2 dx = 0\,. \tag{27}$$

A turbulent model would be obtained by coupling equations (25) with a system of equations which would determine the function $\nu_{turb}$. The necessary condition to obtain a such well posed system is that the function $\nu_{turb}(x,t)$ is non negative, a property which does not follow for the fact that the tensor $R_{turb}$ is itself non negative.
In the case of the Euler equation, with a sequence of initial data $u_n^0$ having unbounded vorticity one would derive an irreversible problem as the limit of reversible equations. Eventually from the formula (27) one deduces the

**Proposition 1** In the above configurations, assume that the sequence of initial data $u_n^0(x)$ converges strongly to $u_0(x)$ in $L^2(\Omega)$, then for $T>0$, the following assertions are equivalent :
(i) the sequence $u_n(x,t)$ converges strongly in $L^2(\Omega\times[0,T])$,
(ii) $\nu_{turb}(x,t)$ is identically zero on $\Omega\times[0,T]$,
(iii) one has :

$$\int_0^T\int_\Omega \nu_{turb}(x,t)|\nabla_x u|^2 dxdt \le 0\,. \tag{28}$$

**Proof** The only non classical point is the fact that (iii) implies the strong convergence or equivalently that one has :

$$\liminf_{n\to\infty}\tfrac{1}{2}\int_0^T\int_\Omega |u_n(x,t)|^2 dxdt = \tfrac{1}{2}\int_0^T\int_\Omega |u(x,t)|^2 dxdt\,. \tag{29}$$

With the classical energy estimate (for any given $n$) and the relation (27) one deduces the inequalities :

$$\begin{aligned}\frac{T}{2}\int_\Omega |u_0(x)|^2dx \ge \liminf_{n\to\infty}\tfrac{1}{2}\int_0^T\int_\Omega |u_n(x,t)|^2dxdt &= \tfrac{1}{2}\int_0^T\int_\Omega |u(x,t)|^2dxdt\,,\\ &\ge \frac{T}{2}\int_\Omega |u_0(x)|^2dx - \int_0^T\int_\Omega \nu_{turb}(x,t)|\nabla_x u|^2dxdt\,,\end{aligned} \tag{30}$$

and (29) follows from (28).

## 3 Positivity versus non positivity of the Diffusion Coefficient for the small Dispersion limit of KDV and NLS flows.

In the absence of a systematic theory it seems worth while to study, following the program of Lax, the issue of the positivity of the turbulent coefficient on the dispersive limit of the KDV and NLS equations using explicit formulas given by the inverse scattering theory.
For the KDV flow one considers the problem :

$$u_t^\epsilon - 6u^\epsilon u_x^\epsilon + \epsilon^2 u_{xxx}^\epsilon = 0,\ \text{ with initial data } u^\epsilon(x,0)=u_0(x) \tag{31}$$

and the for the NLS flow the problem :

$$\begin{aligned} i\epsilon u_t^\epsilon + \frac{\epsilon^2}{2} u_{xx}^\epsilon + (1 - |u^\epsilon|^2) u^\epsilon = 0, \\ \text{with initial data } u^\epsilon(x,0) = A(x) exp(i \frac{S(x)}{\epsilon}) . \end{aligned} \tag{32}$$

With the introduction of the functions :

$$\rho^\epsilon = |u^\epsilon|^2 - 1, \text{ and } \mu^\epsilon = \frac{-i\epsilon}{2}(u^\epsilon \bar{u_x^\epsilon} - u_x^\epsilon \bar{u^\epsilon}) , \tag{33}$$

the NLS equation is equivalent to the system :

$$\begin{aligned} {\rho^\epsilon}_t + {\mu^\epsilon}_x =& 0, \\ {\mu^\epsilon}_t + (\frac{\mu^{\epsilon 2}}{\rho^\epsilon} + \frac{\rho^{\epsilon 2}}{2})_x =& \frac{\epsilon^2}{4} (\rho^\epsilon (\log \rho^\epsilon)_{xx})_x, \end{aligned} \tag{34}$$

The equations (31) and (32) are time reversible and (34) is a reversible perturbation of the the usual isentropic compressible Euler equation. For $\epsilon$ going to zero the functions $u^\epsilon$, $\rho^\epsilon$ and $\mu^\epsilon$ converge weakly and the following notations are introduced :

$$\begin{aligned} \bar{u} &= weak \lim_{\epsilon \to 0} u^\epsilon , \\ \bar{u^2} &= weak \lim_{\epsilon \to 0} (u^\epsilon)^2 , \\ \bar{\rho} &= weak \lim_{\epsilon \to 0} \rho^\epsilon , \\ \bar{\mu} &= weak \lim_{\epsilon \to 0} \mu^\epsilon , \\ Q(\rho^\epsilon, \mu^\epsilon) &= (\frac{\mu^{\epsilon 2}}{\rho^\epsilon} + \frac{\rho^{\epsilon 2}}{2}) , \\ \bar{Q} &= weak \lim_{\epsilon \to 0} Q(\rho^\epsilon, \mu^\epsilon) . \end{aligned} \tag{35}$$

and one obtains the equation :

$$\partial_t \bar{u} - 6\partial_x (\frac{\bar{u}^2}{2}) - 6\partial_x (\frac{\bar{u^2}}{2} - \frac{\bar{u}^2}{2}) = 0 \tag{36}$$

and the system

$$\begin{aligned} \bar{\rho}_t + \bar{\mu}_x =& 0 , \\ \partial_t \bar{\mu} + \partial_x (\frac{\bar{\mu}^2}{\bar{\rho}} + \frac{\bar{\rho}^2}{2}) + \partial_x (\bar{Q} - Q(\bar{\rho}, \bar{\mu})) =& 0 . \end{aligned} \tag{37}$$

In the region where strong convergence occurs one has

$$(\frac{\bar{u^2}}{2} - \frac{\bar{u}^2}{2}) = 0 \tag{38}$$

or

$$\bar{Q} - Q(\bar{\rho}, \bar{\mu})) = 0\,. \tag{39}$$

As expected in these regions (36) is (up to a simple change in the $x$ variable ) the Burgers equation and (37) is the compressible Euler equation for isentropic fluids.
On the other hand it is known (cf [LL] and [JLM]) that the strong convergence does not hold everywhere. The regions where strong convergence fails are called the *Whitham regions.* By a simple convexity argument, (observe that both the functions

$$u \to u^2 \text{ and } (\rho,\, \mu) \to Q(\rho, \mu)$$

are convex), one has in the Whitham region :

$$\frac{\bar{u^2}}{2} - \frac{\bar{u}^2}{2} > 0 \text{ and } \bar{Q} - Q(\bar{\rho}, \bar{\mu})) > 0\,. \tag{40}$$

To analyze the possibility of the appearance of "turbulent viscosity." one writes (36) and (37) in the following form :

$$\bar{u}_t - 3((\bar{u})^2)_x - \partial_x(\nu_{turb}\partial_x\bar{u}) = 0,\ \text{with } \nu_{turb}(x,t) = \frac{\bar{u^2} - (\bar{u})^2}{\partial_x\bar{u}} \tag{41}$$

and

$$\begin{gathered}\bar{\rho}_t + \bar{\mu}_x = 0\,,\\ \bar{\mu}_t + (\frac{\bar{\mu}^2}{\bar{\rho}} + \frac{\bar{\rho}^2}{2})_x - \partial_x(\nu_{turb}\bar{\mu}_x) = 0\,,\\ \text{with } \nu_{turb}(x,t) = \frac{\bar{Q} - Q(\bar{\rho}, \bar{\mu}))}{\partial_x\bar{\mu}}\,.\end{gathered} \tag{42}$$

The existence of any kind of turbulent model requires that $\nu_{turb}$ be non negative which is equivalent here to the property :

$$\partial_x\bar{u}(x,t) > 0\,, \tag{43}$$

in the Witham region for the KDV dispersive limit and :

$$\frac{\bar{Q} - Q(\bar{\rho}, \bar{\mu}))}{\partial_x\bar{\mu}} \geq 0\,, \tag{44}$$

in the Witham region for the dispersive NLS limit. This is also equivalent to

$$\partial_x\bar{\mu} > 0\,. \tag{45}$$

Since diffusion properties may appear on a larger time scale it is natural to explore the properties (43) and (45) for large time.
Such program is done below using the tools of the inverse scattering following [LL] and the conclusion will be the fact that such properties are satisfied depends on the initial data. The starting point are the following theorems :

**Theorem 2.** ([LL]) Let $u(x,t;\epsilon)$ solve

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \tag{46}$$

with initial data $u(x,0;\epsilon) = u_0(x)$ belonging to the Schwartz class, strictly negative, and with only one minimum point $x_0$, at which $u_0(x_0) = -1$. Let $x_\pm(\eta)$ be defined for $0 < \eta < 1$ by

$$u_0(x_\pm(\eta)) = -\eta^2 \text{ and } x_- < x_0 < x_+ \,. \tag{47}$$

Define the function

$$\phi(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} \frac{\eta}{(-u_0(x) - \eta^2)^{1/2}} dx \,, \tag{48}$$

for $0 < \eta < 1$. Then,
(i) the weak limit

$$\bar{u}(x,t) = \lim_{\epsilon \to 0} u(x,t;\epsilon)$$

exists.
(ii) As $t$ goes to infinity, for $x$ such $\delta < x/t < 4 - \delta$, with $\delta$ is any given small positive constant one has

$$\bar{u}(x,t) = -\frac{1}{4\pi t}\phi((\frac{x}{4t})^{1/2}) + o(1/t) \,. \tag{49}$$

(iii) As $t$ goes to infinity, for $x/t < 0$ or $x/t > 4$ one has

$$\bar{u} = O(t^{-2}) \,. \tag{50}$$

This theorem is stated and proved in the third paper of Lax and Levermore [LL] pages 810-815. Furthermore, with some conjecture on the uniform effect of the remote part of the initial data on the solution of KDV equation the authors adapt their asymptotic analysis to the initial data :

$$u_0(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x > 0. \end{cases} \tag{51}$$

which correspond to the shock profile for the Burgers equation. They obtain for the weak limit the formula :

$$\begin{aligned} \bar{u}(x,t) &= -\,1 \text{ for } x < -6t \,, \\ \bar{u}(x,t) &= s(\frac{x}{t}) \text{ for } -6t < x < 4t \,, \\ \bar{u}(x,t) &= \text{ for } 4t < x \,, \end{aligned} \tag{52}$$

where the function $\xi \mapsto s(\xi)$ can be computed in term of complete elliptic integrals. Explicit numerical computations done in [LL] on the formula for $s(\xi)$ indicate that this is an increasing function on the interval $[-6,4]$.
The theorem 2 has a counterpart for the NLS dispersive limit.
**Theorem 3.** Let $u(x,t;\epsilon)$ solve the NLS flow :

$$i\epsilon u_t(x,t;\epsilon) + \frac{\epsilon^2}{2}u_{xx}(x,t;\epsilon) + (1 - |u(x,t;\epsilon)|^2)u(x,t;\epsilon) = 0 \,, \tag{53}$$

$$u^\epsilon(x,0) = |u_0(x)|^2 exp\big(i\frac{S(x)}{\epsilon}\big)\,, \tag{54}$$

with

$$\rho_0(x) = 1 + |u_0(x)|^2\,, \tag{55}$$

and

$$\mu_0(x) = \partial_x S(x)\,, \tag{56}$$

belonging to the Schwartz class. Let also assume, for simplicity, that the initial data are "single well" in the following sense (cf [JLM]). Introduce the functions

$$r_\pm(x) = \tfrac{1}{2}\partial_x S \pm A(x)\,, \tag{57}$$

and assume that $r_-$ has only one maximum $\lambda_{max}$ while $r_+$ has only one minimum $\lambda_{min}$ with the relation :

$$-1 \le r_-(x) \le \lambda_{max} < \lambda_{min} \le r_+(x) \le 1\,. \tag{58}$$

Define the numbers $x^\pm(\lambda)$ according to the formula :

$$\begin{aligned} \text{for } -1 \le \lambda \le \lambda_{max}, r_-(x_\pm(\lambda)) = \lambda, x_- < x_+\,,\\ \text{for } \lambda_{min} \le \lambda \le 1, r_+(x_\pm(\lambda)) = \lambda, x_- < x_+\,. \end{aligned}$$

Then, the weak limit

$$\bar\rho(x,t) = lim_{\epsilon\to 0}|u(x,t;\epsilon)|^2 + 1$$

exists. Furthermore in the Whitham region

$$\frac{x}{t} \in (-1, \lambda_{max}) \cup (\lambda_{min}, 1)\,, \tag{59}$$

as $t \to \infty$, one has :

$$\begin{aligned} \bar\rho(x,t) = 1 - \frac{4}{\pi t}\phi(x/t)(1-(x/t)^2)^{1/2}, \text{ where}\\ \phi(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{\lambda - 1/2(r_+(s)+r_-(s))}{(\lambda - r_+(s))^{1/2}(\lambda - r_-(s))^{1/2}}ds, \end{aligned} \tag{60}$$

and elsewhere one has, $\bar\rho \simeq 1$.
**Proof** : The proof is given in [K] and for sake of completeness the main steps are recalled here. The existence of the weak limit is proved in [JLM]. Following a method suggested in [LL] one begins with the multisolitons formula for fixed $\epsilon$ and then let $\epsilon$ go to zero.
For fixed $\epsilon$, the long time behavior of $|u|^2$ is as follows [FT, pp.168-176]. In the solitonless region $|x/t| > 1$ or $\lambda_{max} < x/t < \lambda_{min}$, one has

$$|u|^2 = 1 + O(t^{-1/2})\,. \tag{61}$$

As $t \to \infty$. In the Whitham region, the solution is a multisolitons solution :

$$|u(x,t;\epsilon)|^2 \sim 1 - \Sigma_{n=1}^{N(\epsilon)} s(x - \eta_n t - x_n, \eta_n), \text{ where}$$
$$s(x,\eta) = \frac{1-\eta^2}{cosh^2((1-\eta^2)^{1/2}\frac{x}{2\epsilon})}, \tag{62}$$

with exponentially small error. The $\eta_n$'s are the associated eigenvalues (of the underlying Dirac operator) and the $x_n$'s are some phase constants of no importance.
The width of each soliton is

$$O(\frac{\epsilon}{(1-\eta^2)^{1/2}}). \tag{63}$$

By Weyl's law for the distribution of eigenvalues in $(-1, \lambda_{max}) \cup (\lambda_{min}, 1)$ as $\epsilon \to 0$,

$$\eta_{n+1} - \eta_n = \frac{\pi\epsilon}{\phi(\bar{\eta}_n)}, \tag{64}$$

where $\bar{\eta}_n \in (\eta_n, \eta_{n+1})$.
Peaks of solitons are located at $\eta_n t$. As $t \to \infty$, they are separated by

$$\frac{\pi\epsilon t}{\phi(\eta_n)} \tag{65}$$

so for large $t$ they are well separated.
The wave number $\eta$ of the soliton that peaks at $x$ at time $t$ is $\eta = x/t$, if $t$ is large and either $-1 < x/t < \lambda_{max}$ or $\lambda_{min} < x/t < 1$.
Therefore the density of the solitons is

$$\frac{\phi(x/t)}{\pi\epsilon t}. \tag{66}$$

The area between a soliton and the line $u = 1$ is

$$4\epsilon(1-\eta^2)^{1/2} \sim 4\epsilon(1-(x/t)^2)^{1/2}, \tag{67}$$

so the asymptotic area density which is given by $1 - \bar{\rho}$ is the product of (66) and (67) :

$$\frac{4\phi(x/t)}{\pi}(1-(x/t)^2)^{1/2}. \tag{68}$$

Hence, the asymptotic formula for the weak limit $\bar{\rho}$ follows.

¿From the above statement several observations can be made concerning the appearance of a positive turbulent viscosity in the limit equation satisfied by $\bar{u}$ for the KDV dispersive limit and by $(\bar{\rho}, \bar{\mu})$ for the NLS dispersive limit. Such positivity would be related to the appearance of irreversibility in a weak limit of reversible models.

As said above the construction of [LL] (section 7 page 817) shows that a shock profile as initial data produces in the limit a smooth solution with a "turbulent viscosity".
Following Theorem 2, one considers initial data $u(x,0;\epsilon) = u_0(x)$ belonging to the Schwartz class, strictly negative, and with only one minimum point $x_0$, at which $u_0(x_0) = -1$. For large $t$ the solution is asymptotic to

$$-\frac{1}{4\pi t}\phi((\frac{x}{4t})^{1/2}) \text{ with } \phi(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} \frac{\eta}{(-u_0(x)-\eta^2)^{1/2}}dx\,. \tag{69}$$

Therefore one has :

$$\partial_x \bar{u} \simeq -\frac{1}{\pi(4t)^{\frac{3}{2}}}\phi'(\frac{x}{(4t)^{\frac{1}{2}}})\,. \tag{70}$$

And the "turbulent diffusion" hypothesis requires that

$$\phi'(\eta) < 0\,, \forall \eta \in [0,1]\,. \tag{71}$$

With $u_0(x) = -e^{-|x|^\beta}$ one has :

$$\phi(\eta) = 2\int_0^{\left(2\log\frac{1}{\eta}\right)^{\frac{1}{\beta}}} \frac{\eta}{(-u_0(x)-\eta^2)^{1/2}}dx\,. \tag{72}$$

On the table 1 the values of $\phi(\eta)$ and for the following $\beta$ exponents : $\beta = 1,\ 3/2, 2,$ and 4 and with 9 steps $\eta = k10^{-1}, 1 \le k \le 9$
In the first case $\phi$ is decreasing, in the second case its variation changes, and then $\phi$ is increasing in the two last cases.
For the NLS flow one considers initial data satisfying the hypothesis of Theorem 3 and observes that the existence for large time of a diffusive regime would be given by $\partial_x \bar{\mu} \ge 0$ in the Witham region and with the conservation law :

$$\bar{\rho}_t + \bar{\mu}_x = 0\,,$$

this is equivalent to $\bar{\rho}_t \le 0$ in the same region. With Theorem 3 this condition is equivalent, for large $t$ to the relation

$$\partial_\lambda\big(\lambda\phi(\lambda)(1-\lambda^2)^{\frac{1}{2}}\big) \ge 0, \forall \lambda \in [-1,+1]\,. \tag{73}$$

with $\phi(\lambda)$ given by

$$\phi(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{\lambda - 1/2(r_+(s)+r_-(s))}{(\lambda - r_+(s))^{1/2}(\lambda - r_-(s))^{1/2}}ds\,. \tag{74}$$

Solution with initial data having zero momentum and a symmetric density with an fractional exponential rate of convergence at infinity are analyzed :

$$\rho_0(x) = 1 - \tfrac{1}{2}e^{-|x|^\beta},\ \ S(x) = 0\,. \tag{75}$$

The fact that the momentum is zero gives $r_+(\lambda) + r_-(\lambda) = 0$. The $x$ symmetry of the density remains true for all time and all $\epsilon$ therefore it is enough to consider the behavior of

$$\big(\lambda\phi(\lambda)(1-\lambda^2)^{\frac{1}{2}}\big)\phi(\lambda), \lambda = \frac{x}{t}\,,$$

for

$$0 < \lambda_{min} \leq \lambda = \frac{x}{t} \leq 1\,. \tag{76}$$

One has :

$$\lambda\phi(\lambda)(1-\lambda^2)^{\frac{1}{2}}\big)\phi(\lambda) = 2\int_0^{\left(-\log 2(1-\lambda)\right)^{\frac{1}{\beta}}} \frac{\lambda^2\sqrt{(1-\lambda^2)}}{\lambda^2-(1-\frac{1}{2}e^{-|x|^\beta})^2}dx\,. \tag{77}$$

The numerical computation given on the table 2 are done for $\beta = 1.5,\ 2\,,3$ and $3.5$ with $\lambda = x/t$ varying from $\lambda_{min} = 0.5$ to $0.9$ with step $0.1$. They indicate that the "turbulent regime" appears for $\beta = 1.5$, $\beta = 2$, $\beta = 3$ but does not hold for $\beta = 3.5$.

| $\beta = 1$ | $\beta = 3/2$ | $\beta = 2$ | $\beta = 4$ |
|---|---|---|---|
| 5.88251 | 2.19345 | 1.75226 | .78565 |
| 5.47775 | 3.65183 | 2.07024 | 1.09422 |
| 5.06441 | 3.37626 | 2.29727 | 1.36578 |
| 4.63711 | 3.09141 | 2.47717 | 1.62689 |
| 4.18879 | 2.79252 | 2.62703 | 1.89375 |
| 3.70918 | 2.47278 | 2.75579 | 2.18372 |
| 3.18159 | 2.12106 | 2.86860 | 2.52387 |
| 2.57400 | 1.71600 | 2.96898 | 2.97357 |
| 1.80404 | 1.20273 | 3.05934 | 3.73515 |

**Table 1 Numerical computation for the dispersive KDV limit.**
Values of $\phi(\eta)$ are computed for $\beta = 1$, $\beta = 3/2$, which appears as a critical case, $\beta = 2$ and $\beta = 4$ with $\eta = 10^{-1}k$, $1 \leq k \leq 9$

| $\beta = 1.5$ | $\beta = 2$ | $\beta = 3$ | $\beta = 3.5$ |
|---|---|---|---|
| 0 | 0 | 0 | .0 |
| 1.10957 | 1.29635 | 1.48400 | 1.53471 |
| 1.62180 | 1.64627 | 1.63429 | 1.62240 |
| 2.16222 | 1.97843 | 1.76762 | 1.70143 |
| 2.69501 | 2.21615 | 1.76986 | 1.64733 |

**Table 2 Numerical computation for the dispersive NLS limit.**
Values of $\lambda\phi(\lambda)$ are computed for $\beta = 1.5$ $\beta = 2$, $\beta = 3$ and $\beta = 3.5$ which appears as a critical case with $\lambda = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$.

**Conclusion.**

In this contribution it has been shown that some of the basic questions of the statistical theory of turbulence could be formulated in a deterministic setting with the introduction of sequence of weakly converging solutions. The counterpart of the turbulent spectra being the Wigner transform and the turbulent diffusion being related to defect measures. Explicit computations done on integrable classical integrable system indicate that for these models it is not always possible to construct a "formal" turbulent equation. At this point of our analysis it depends on the behavior of the initial data and in particular on the fact that they should not be too much concentrated (their decay for $|x|$ going to infinity has to be not too small). It is worth while to notice that with the convenient conjectures of [LL] the shock profile leads always to a diffusive regime.

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