FREE FISHER INFORMATION WITH RESPECT TO A COMPLETELY POSITIVE MAP AND COST OF EQUIVALENCE RELATIONS

DIMITRI SHLYAKHTENKO

ABSTRACT. Given a family of isometries $v_1, \ldots, v_n$ in a tracial von Neumann algebra M, a unital subalgebra $B \subset M$ and a completely-positive map $\eta : B \to B$ we define the free Fisher information $F^*(v_1, \ldots, v_n : B, \eta)$ of $v_1, \ldots, v_n$ relative to B and η. Using this notion, we define the free dimension $\delta^*(v_1, \ldots, v_n \between B)$ of $v_1, \ldots, v_n$ relative to B, id.

Let R be a measurable equivalence relation on a finite measure space X. Let M be the von Neumann algebra associated to R, and let $B \cong L^\infty(X)$ be the canonical diffuse subalgebra. If $v_1, \ldots, v_n, \cdots \in M$ are partial isometries arising from a treeing of this equivalence relation, then $\lim_n \delta^*(v_1, \ldots, v_n, \ldots \between B)$ is equal to the cost of the equivalence relation in the sense of Gaboriau.

1. INTRODUCTION.

Recently [4], [3] Gaboriau introduced the notion of cost of a measurable equivalence relation R on a finite measure space X. The cost $C(R)$ measures "how many local morphisms from X to X are required to generate R", and is an invariant of R. If α is a free measure-preserving action of a discrete group G on a finite measure space X, he associated to it the cost $C(R_\alpha)$ of the equivalence relation R_α induced by the action. The cost of the group $C(G)$ was then defined as the infimum of $C(R_\alpha)$ over all possible actions α of G. Gaboriau showed that certain groups (e.g., cyclic groups and their free products) have a fixed price — $C(R_\alpha)$ is independent of α. He also showed that the cost of the group is well-behaved under the free product operation: $C(G * H) = C(G) + C(H)$.

Let G be a (finitely generated) discrete group, and let $g_1, g_2, \ldots, g_n \in G$. Then to each g_i one can associate a unitary $u(g_i)$ in the left regular representation $\ell^2(G)$. The weak closure of the $*$-algebra generated by all such $u(g_i)$ is called the von Neumann algebra of G, denoted by $L(G)$. $L(G)$ has a canonical trace (given by $\tau(u) = \langle u\delta_e, \delta_e \rangle$, where $\delta_e \in \ell^2(G)$ is the delta-function supported on the neutral element of G). To the family of unitaries $u(g_1), \ldots, u(g_n)$ Voiculescu [6], [9], [7] has associated two numbers, $\chi(u(g_1), \ldots, u(g_n))$ and $\chi^*(u(g_1), \ldots, u(g_n))$, called the "microstates" and "microstates-free" free entropy. These numbers arise naturally from Voiculescu's free probability theory. Using microstates free entropy χ, he defined the free entropy dimension (via microstates) $\delta(u(g_1), \ldots, u(g_n))$; replacing in his definition χ by χ^* leads to the microstates-free free dimension δ^*, which we use in this paper. The two notions (microstates-free and involving microstates) of free entropy dimension make sense for an arbitrary collection $u_1, \ldots, u_n$ of unitaries in a tracial von Neumann algebra. It is not known if the two quantities δ and δ^* are different; they coincide in every example in which their values have been computed.

Research supported by MSRI.

Voiculescu has proved that $\delta(u(g_1),\dots,u(g_n))$ is independent of the choice of generators $g_1,\dots,g_n$ of the group G, and therefore gives a non-trivial invariant of G, called the free dimension of G, $\delta(G)$. The corresponding invariance property for δ^* is suspected, but so far has not been proved.

If $G = K * H$, and $k_1,\dots,k_n \in K$, $h_1,\dots,h_m \in H$, then both δ and δ^* are additive:

$$\begin{aligned}\delta^*(u(k_1),\dots,u(k_n),u(h_1),\dots,u(h_n)) &= \delta^*(u(k_1),\dots,u(k_n))\\ &\quad+\delta^*(u(h_1),\dots,u(h_m))\end{aligned}$$

and similarly for δ. It is remarkable that if g is a generator of a cyclic group, then $\delta(u(g)) = \delta^*(u(g)) = C(G) = 1 - \frac{1}{|G|}$. Additivity of both quantities under free products imply that a (finite) free product of cyclic groups has a sequence of generators $g_1, g_2,\dots,g_n$ for which $\delta(u(g_1),\dots,u(g_n)) = \delta^*(u(g_1),\dots,u(g_n)) = C(G)$. There are indications that the three quantities: $C(G)$, $\delta(G)$ and $\delta^*(u(\text{set of generators of } G))$ are actually the same.

It is tempting to suggest that there is a more general notion of "cost" (and "free dimension") which unifies the notion of cost (free dimension) of a group, and the notion of cost of an equivalence relation. Indeed, groups and equivalence relations are both special cases of groupoids. In this paper:

1. If M is a tracial von Neumann algebra, $B \subset M$ a unital subalgebra, and $v_1,\dots,v_n \in M$ is an n-tuple of partial isometries, we define the free entropy dimension of the n-tuple $v_1,\dots,v_n$ relative to $(B, \mathrm{id} : B \to B)$, $\delta^*(v_1,\dots,v_n \between B)$. For R an equivalence relation on a measure space X, a graphing ("sequence of generators") $\phi_1,\dots,\phi_n,\dots$ gives rise to a family $v_1,\dots,v_n,\dots$ of partial isometries in $W^*(X,G)$, and hence to a number, $\delta^*(\phi_1,\dots,\phi_n,\dots) = \limsup_n \delta^*(v_1,\dots,v_n \between B)$.
2. In certain cases (when an equivalence relation R is treeable), R has a graphing $\phi_1,\dots,\phi_n,\dots$, giving rise to partial isometries $v_1,\dots,v_n,\dots \in W^*(X,R)$ for which $\lim_{n\to\infty} \delta^*(v_1,\dots,v_n \between B) = C(R)$.
3. Given a sequence of generators $\phi_1,\dots,\phi_n$ one can extend the resulting isometries $v_1,\dots,v_n \in W^*(X,G)$ to unitaries $u_1,\dots,u_n \in W^*(X,G)$ in a certain way. For a treeing $\phi_1,\dots,\phi_n,\dots$, $\delta^*(v_1,\dots,v_n \between B) = \delta^*(u_1,\dots,u_n \between B)$.
4. If α is a free action of a discrete group on a measure space X, each element $g \in G$ gives rise to a unitary $w(g)$ in the crossed product algebra $L^\infty(X) \rtimes_\alpha G \cong W^*(X,R_\alpha)$. It turns out that $\delta^*(w(g_1),\dots,w(g_n) \between B) = \delta^*(u(g_1),\dots,u(g_n))$ (here $u(g_j) \in B(\ell^2(G))$ are in the group von Neumann algebra of G), and therefore depends only on the generators $g_1,\dots,g_n$ and not on the action α.

The connections between δ^* and δ lead us to suspect that if $\phi_1,\dots,\phi_n,\dots$ is a graphing of an equivalence relation R on X, then the number $\delta^*(\phi_1,\phi_2,\dots)$ depends only on R. The fourth statement above would then imply that if R comes from a group action, δ^* of any graphing of R would have the same microstates-free free dimension as any sequence of generators of the group.

The remainder of the paper is organized as follows. We start with the definition of free Fisher information F^* for a family of isometries $v_1,\dots,v_n \in M$ relative to a subalgebra $B \subset M$ and a completely-positive map $\eta : B \to B$. The definition involves the notion of free Fisher information for unitaries with respect to a completely positive map, and is a straightforward extension of the definitions in [8], [7] and [5]. Some proofs are direct adaptations of the arguments from these papers; we include them for the convenience of the reader. Specializing to the case $\eta = \mathrm{id}$ we get as in [8] an entropy quantity Σ^*. Passing from free entropy to free dimension like in [6] gives a free dimension quantity, which is denoted by $\delta^*(v_1, \dots, v_n \between B)$. Next , we study the properties of F^*, Σ^* and δ^*. We conclude by exhibiting equality between cost and free dimension in certain cases.

Acknowledgements. This work was completed while the author was participating in the Random Matrix program at MSRI. We would like to thank the organizers for the friendly atmosphere. We would also like to acknowledge fruitful conversations with Prof. J. Feldman, Prof. S. Popa and Prof. D.-V. Voiculescu.

2. Free microstates-free Fisher information for unitaries.

The reader is referred to [10] for the basic notions from free probability theory.

2.1. **Notations.** In the remainder of the paper, (M, τ) will be a tracial non-commutative probability space, i.e., M is a W^*-algebra, and $\tau : M \to \mathbb{C}$ is a linear functional, so that

1. $\tau(xy) = \tau(yx)$ for all $x, y \in M$ (traciality)
2. $\tau(x^*x) \geq 0$ (positivity)
3. $\tau(x^*x) = 0$ iff $x = 0$ (faithfulness)

M is a bimodule over itself under right and left multiplication. The norm $\|x\|_2 = \tau(x^*x)$ defines a pre-Hilbert space structure on M.

We shall fix $B \subset M$ a unital subalgebra, and denote by $\mathrm{CP}(B)$ the set of all completely-positive maps from B to B. Recall that $\eta \in \mathrm{CP}(B)$ if η is a linear map from B to B and its natural extension $\eta \otimes 1 : B \otimes M_{n\times n} \to B \otimes M_{n\times n}$ to $n \times n$ matrices over B is positive: $\tau \otimes \mathrm{Tr}(\eta(m^*m)) \geq 0$ for all $m \in B \otimes M_{n\times n}$. We shall be frequently concerned with the case that B is the commutative algebra $B = L^\infty(X, \mu)$, where (X, μ) is a finite measure space, and $\tau(f) = \int f(t) d\mu(t)$ for $f \in B$. In this case, $\eta \in \mathrm{CP}(B)$ if and only if $\eta(f)$ is a.e. positive whenever $f \in B$ is an a.e. positive function.

If $\eta : B \to B$ is a completely-positive map, then $\langle a \otimes b, a' \otimes b' \rangle = \tau(b^* \eta(a^* b))$ defines a (possibly semi-definite) pre-Hilbert space structure on $B \otimes B$. The Hilbert space obtained from $B \otimes B$ after separation and completion with respect to this inner product is denoted $H^2(B, \eta)$. It carries the obvious structure of a B, B-bimodule.

If $B \subset M$, there exists a linear completely-positive map $E_B : M \to B$, called the conditional expectation from M onto B. Sometimes we write E for E_B when B is understood. E satisfies $E(bmb') = bE(m)b'$, $b, b' \in B$, $m \in M$. E is determined by $\tau(bE(m)) = \tau(bm)$ for all $b \in B$, $m \in M$. If $\eta \in \mathrm{CP}(B)$, it can be extended to all of M by composing it with E_B.

If $\eta \in \mathrm{CP}(B)$, we write $H(B, \eta)$ for $H^2(M, \eta \circ E_B)$.

We call an element $u \in M$ a partial isometry, i.e., uu^* and u^*u are projections, i.e., $u^*uu^*u = u^*u$, $uu^*uu^* = uu^*$. We denote by $B[u] = B[u, u^*]$ the $*$-algebra generated by B and u, u^*.

2.2. **Freeness with amalgamation.** If A_1, A_2 are subalgebras of M, we say that they are *free with amalgamation over* B, if whenever $a_j \in W^*(A_{i(j)}, B)$, $i(1) \neq i(2), \dots, i(n -$

$1) \neq i(n)$ and $E_B(a_j) = 0$, one has

$$E_B(a_1 \dots a_n) = 0.$$

We further say that the families $(a_1^{(1)}, \dots, a_{n_1}^{(1)}), (a_1^{(2)}, \dots, a_{n_2}^{(2)}), \dots$ of elements of M are free with amalgamation over B, if the algebras $A_j = W^*(B, \{a_i^{(j)}\}_{i=1}^{n_j})$ are free with amalgamation over B. If M and N are two tracial probability spaces, each containing B, one can form their amalgamated free product $M *_B N$, which is again a tracial probability space. $M *_B N$ is generated by N and M, with the copies of B in M and N identified. M and N as subalgebras of $M *_B N$ are free with amalgamation over B.

2.3. **Free Fisher information for an isometry.** Let v be a partial isometry and $B \subset M$ be a subalgebra. Consider the derivation $d_{u:B} : B[u, u^*] \to H(B[u, u^*], \eta)$, determined by:

1. $d_{u:B}(u) = uu^* \otimes u = (uu^*)1 \otimes u(u^*u)$, $d_{u:B}(u^*) = -u^* \otimes uu^*$
2. $d_{u:B}(wv) = d_{u:B}(w) \cdot v + w \cdot d_{u:B}(v)$
3. $d_{u:B}$ is B-bilinear.

Definition 2.1. We call a vector $\xi(u : B, \eta) \in L^2(B[u, u^*])$ *conjugate to* u *relative to* B *with respect to* η, if

$$\langle \xi(u : B, \eta), w \rangle = \langle 1 \otimes 1, d_{u:B}(w) \rangle \tag{2.1}$$

for all $w \in B[u, u^*]$.

Definition 2.2. If $\xi(u : B, \eta)$ exists, we let the *relative free Fisher information of* u *with respect to* B *and* η to be $F^*(u : B, \eta) = \|\xi(u : B, \eta)\|_2^2$. If the conjugate vector does not exist, we set $F^*(u : B, \eta) = +\infty$. If $B = \mathbb{C}$ and $\eta = \mathrm{id}$, we write just $F^*(u)$.

Example 2.3. Let $u = 0$. Then $d_{u:B} = 0$, since $d_{u:B}(b) = 0$ for all $b \in B$, and $d_{u:B}(u) = 0 \cdot 0 \otimes u = 0$. Hence $\xi(u : B, \eta) = 0$, and $F^*(u : B, \eta) = 0$.

Proposition 2.4. *Let $D \subset B$ be a unital subalgebra. Assume that $\eta|_D \in \mathrm{CP}(D)$. Assume that $\xi(u : B, \eta)$ exists. Then $\xi(u : D, \eta)$ exists and is given by $E_{L^2(D[u,u^*])}(\xi(u : B, \eta))$.*

Proof. Identical to that of [5, Porposition 3.4]. □

Proposition 2.5. *Let $D \subset B$ be a subalgebra. Assume that $\eta = \eta \circ E_D$ and u is $*$-free from B with amalgamation over D. Then $\xi(u : B, \eta) \in L^2(D[u, u^*])$, and $\xi(u : D, \eta) = \xi(u : B, \eta)$. Furthermore, $F^*(u : B, \eta) = F^*(u : D, \eta)$.*

Proof. If $\xi(u : B, \eta)$ exists, then also $\xi(u : D, \eta)$ exists. Hence we can assume that $\xi(u : D, \eta)$ exists. It is sufficient to verify that it satisfies (2.1) for all $w \in B[u, u^*]$. We may assume that $E_D(w) = 0$, and hence that $w = b_0 v_0 b_1 v_1 \dots b_m$, where $b_j \in B$, $v_j \in D[u, u^*]$, $E_D(b_j) = E_D(v_j) = 0$, except possibly b_0 and/or b_n is equal to 1.

Assume first that at least one of b_0, b_1 is not 1, or that $m > 1$. Then, since $\tau(\xi(u : D, \eta)^*) = \langle \xi(u : D, \eta), 1 \rangle = \langle 1 \otimes 1, 0 \rangle = 0$, we get by freeness that $\langle \xi(u : D, \eta), w \rangle = 0$.

If $m > 1$, then $d_{u:B}(w) \perp 1 \otimes 1$ in $H(B, \eta)$. If $m = 1$, so that $w = b_0 v_0 b_1$, assume that $d_{u:B}(v_0) = \sum z_i \otimes z_i'$, $z_j, z_j' \in D[u, u^*]$. Then $d_{u:B}(w) = \sum b_0 z_i \otimes z_i' b_1$, and

$$\begin{aligned}
\langle 1 \otimes 1, d_{u:B}(w) \rangle &= \sum \tau(\eta(b_0 z_i) z_i' b_1) \\
&= \sum \tau(\eta(E_D(b_0 z_i)) E_D(z_i' b_1)) \\
&= \sum \tau(\eta(E_D(b_0) E_D(z_i)) E_D(z_i') E_D(b_1)),
\end{aligned}$$

which is zero if (as we assumed) either $E_D(b_0)$ or $E_D(b_1) = 0$.

It remains to verify (2.1) in the case $b_0 = b_1 = 1$ and $m = 1$, i.e., that

$$\langle \xi(u : D, \eta), v_0 \rangle = \langle 1 \otimes 1, d_{u:B}(v_0) \rangle.$$

But this follows from the definition of $\xi(u : D, \eta)$ and the fact that $d_{u:B}(v_0) = d_{u:D}(v_0)$. $\square$

Proposition 2.6. *Let $v \in B$ be an isometry. Assume that $u = uvv^*$ and that $uu^*, u^*u \in B$. Then $F^*(u : B, \eta) = F^*(uv : B, \eta)$.*

Proof. Let $W = b_0 u^{g(1)} b_1 \dots b_n$, and $W' = b_0 (uvv^*)^{g(1)} b_1 \dots b_n$, where $b_i \in B$ and $g(j) \in \{\cdot, *\}$. Then $W = W'$.

We first claim that $d_{u:B}(W) = d_{u:B}(W') = d_{uv:B}(W')$. The proof proceeds by induction on n. If $n = 0$, $d_{u:B}(b_0) = d_{u:B}(b_0) = d_{uv:B}(b_0) = 0$. If the equality holds for $n = k$, then writing $W' = b_0 (uvv^*)^{g(1)} \overline{W}'$, where $\overline{W}'$ has length k, and similarly $W = b_0 u \bar{W}$, with $\bar{W}$ having length k, we have if $g(1) = \cdot$:

$$\begin{aligned}
d_{u:B}(W) &= b_0 uu^* \otimes u\bar{W} + b_0 u d_{u:B}(\bar{W}), \\
d_{u:B}(W') &= b_0 uu^* \otimes uvv^* \bar{W}' + b_0 uvv^* d_{u:B}(\bar{W}'), \\
d_{uv:B}(W') &= b_0 uvv^* u^* \otimes uvv^* \bar{W}' + b_0 uvv^* d_{uv:B}(\bar{W}'),
\end{aligned}$$

which are the same by the inductive assumption. The case that $g(1) = *$ is similar.

Assume now that $F^*(uv : B, \eta) < +\infty$, so that $\xi = \xi(uv : B, \eta) < +\infty$. Then

$$\begin{aligned}
\langle \xi, W \rangle &= \langle \xi, W' \rangle \\
&= \langle 1 \otimes 1, d_{uv:B}(W') \rangle \\
&= \langle 1 \otimes 1, d_{u:B}(W') \rangle \\
&= \langle 1 \otimes 1, d_{u:B}(W) \rangle.
\end{aligned}$$

It follows that $\xi(u : B, \eta) = \xi$.

Assuming now that $F^*(u : B, \eta) < +\infty$, we can apply the same reasoning as before, replacing u by (uv) and v by v^* to conclude that $\xi((uv)v^* : B, \eta) = \xi(uv : B, \eta)$. $\square$

Proposition 2.7. *Let v be an isometry, free with amalgamation over B from u, and assume that $uu^*, u^*u \in B$. Assume that $u = uvv^*$. Assume further that $F^*(v : B, \eta) < +\infty$. Then $F^*(uv : B, \eta) \le F^*(v : B, \eta) < +\infty$. In fact, $\xi(uv : B, \eta) = E_{B[u]}(\xi(u : B, \eta))$.*

Proof. Let $C = B[u]$. Then

$$F^*(v : B, \eta) = F^*(v : C, \eta \circ E_B) = F^*(vu : C, \eta \circ E_B),$$

and

$$\xi(v : B, \eta) = \xi(v : C, \eta \circ E_B) = \xi(vu : C, \eta \circ E_B)$$

Hence

$$F^*(vu : B, \eta) \leq F^*(vu : C, \eta \circ E_B) = F^*(v : B, \eta) < +\infty$$

and

$$\xi(vu : B, \eta) = E_{B[u]}\xi(vu : C, \eta \circ E_B) = E_{B[u]}\xi(v : B, \eta).$$

□

Corollary 2.8. *Let u be a unitary, and $p \in B$ be an isometry. Assume that $upu^* \in B$. Then*

$$F^*(up : B, \eta) \leq F^*(u : B, \eta).$$

3. Free Fisher information for several isometries.

Definition 3.1. Let $B \subset M$, $\eta \in \mathrm{CP}(B)$ be as before, and let $u_1, \ldots, u_n \in M$ be isometries. Then the mutual free Fisher information of $u_1, \ldots, u_n$ relative to B, η is defined to be

$$F^*(u_1, \ldots, u_n : B, \eta) = \sum_j F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_n], \eta \circ E_B).$$

Proposition 3.2. $F^*(u_1, \ldots, u_n) \leq F^*(u_1, \ldots, u_k) + F^*(u_{k+1}, \ldots, u_n)$.

Proof. If $1 \leq j \leq k$, then

$$F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_n]) \leq F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_k]),$$

by Proposition 2.4. Similarly, for $k < j \leq n$,

$$F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_n]) \leq F^*(u_j : B[u_{k+1}, \ldots, \hat{u}_j, \ldots, u_n]).$$

□

Proposition 3.3. *If $u_1, \ldots, u_k$ are free with amalgamation over B from $u_{k+1}, \ldots, u_n$, then*

$$F^*(u_1, \ldots, u_n) = F^*(u_1, \ldots, u_k) + F^*(u_{k+1}, \ldots, u_n).$$

Proof. If $1 \leq j \leq n$, then

$$F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_n]) = F^*(u_j : B[u_1, \ldots, \hat{u}_j, \ldots, u_k]),$$

by Proposition 2.5. Similarly, if $k < j \leq n$, we get that

$$F^*(: B[u_1, \ldots, \hat{u}_j, \ldots, u_n]) = F^*(u_j : B[u_{k+1}, \ldots, \hat{u}_j, \ldots, u_n]).$$

Applying the definition of $F^*(u_1, \ldots, u_n)$ proves the Proposition. □

Corollary 3.4. *If u_j are free with amalgamation over B, then $F^*(u_1, \ldots, u_n : B, \eta) = \sum F^*(u_j : B, \eta)$.*

Proposition 3.5. *Let $u_1, \ldots, u_n$ be isometries. Then:*

1. *If $D \subset B$ is such that $\eta(D) \subset D$, then*
$$F^*(u_1, \dots, u_n : D, \eta) \leq F^*(u_1, \dots, u_n : B, \eta)$$
2. *If $D \subset B$ is such that $\eta = E_D \circ \eta \circ E_D$, and $u_1, \dots, u_n$ are $*$-free with amalgamation over D from B, then*
$$F^*(u_1, \dots, u_n : D, \eta) = F^*(u_1, \dots, u_n : B, \eta).$$

Proof. Follows from 2.4 and 2.5. □

3.1. **F^* of a single unitary.** In the case that u is a unitary, $B = \mathbb{C}$ and $\eta = \mathrm{id}$, Voiculescu gave an explicit formula for $F^*(u) = F^*(u : \mathbb{C}, \mathrm{id})$:
$$3F^*(u) = -1 + \int p^3 d\lambda$$
where λ is the Haar measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and p is the density with respect to the Haar measure of the distribution of u. In particular, if $u(t)$ is unitary free Brownian motion starting at identity and $0 < \alpha < \frac{1}{2}$, then there exist constants C_1, C_2, independent of t, for which $F^*(u(t)) \leq C_1 t^{-1}$ for $0 < t \leq 1$, and $F^*(u(t)) \leq C_2 \exp(-t\alpha)$.

4. Free Fisher information for normalizing isometries.

Let $B \subset M$ be as before. Denote by $\mathcal{N}(B)$ the normalizer of B, consisting of those unitaries $w \in M$, for which $wBw^* = B$. If $w \in \mathcal{N}(B)$, then $\mathrm{Ad}_u(w) = uwu^*$ is an automorphism of B. We write $\mathcal{GN}(B)$ for the "full group" of the normalizer; $\mathcal{GN}(B)$ consists of all isometries $v \in M$, for which $v^*Bv \subset B$ and $vBv^* \subset B$. Notice that in particular vv^* and v^*v are assumed to be in B.

If $u \in \mathcal{N}(B)$, then frequently $F^*(u : B, \tau) = +\infty$. Indeed:

Proposition 4.1. *If the conjugate variable $\xi = \xi(u : B, \tau)$ exists, and $u \in \mathcal{N}(B)$, then $\tau(ubu^*c) = \tau(b)\tau(c)$ for all $b, c \in B$.*

Proof. If ξ were to exist, we would have, for $b, c \in B$
$$\begin{aligned} 0 = \langle \xi, ubu^*c \rangle &= \langle 1 \otimes 1, 1 \otimes ubu^*c - ubu^* \otimes c \rangle \\ &= \tau(ubu^*c) - \tau(ubu^*)\tau(c) \\ &= \tau(ubu^*c) - \tau(b)\tau(c), \end{aligned}$$
which would imply that $\tau(ubu^*c) = \tau(b)\tau(c)$. □

Definition 4.2. A unital subalgebra $A \subset M$ is said to be Aut-independent from the algebra B, if:

1. There exist generators $v_1, v_2, \dots$ of A, so that $v_j \in \mathcal{N}(B)$;
2. For all $a \in A$ and $b \in B$, one has $\tau(ab) = \tau(a)\tau(b)$. Equivalently, $E_B(A) = \mathbb{C}1$.

We say that unitaries $a_1, \dots, a_n$ are Aut-independent from B, if $W^*(a_1, \dots, a_n)$ is Aut-independent from B and $a_j \in \mathcal{N}(B)$.

Example 4.3. Let X be a finite measure space, and let α be a measure-preserving action of a discrete group G on X. Let $M = L^\infty(X) \rtimes_\alpha G$, $B = L^\infty(X)$. For $g \in G$, write $u_g = u(\alpha, g)$ for the unitary in M, implementing the action of g. Then for any $g_1, \dots, g_n \in G$, $u_{g_1}, \dots, u_{g_n}$ are Aut-independent from B. Furthermore, $L(G) = W^*(u_g : g \in G) \subset M$ is Aut-independent from B.

Proposition 4.4. *Let A_1, A_2 be two algebras, each Aut-independent from B. Assume that A_1 and A_2 are free with amalgamation over B. Then $A_1 \vee A_2$ is Aut-independent from B.*

Proof. It is clear that $A_1 \vee A_2$ have a sequence of generators from $\mathcal{N}(B)$. Thus we must prove that if $a \in A_1 \vee A_2$, then $E_B(a) \in \mathbb{C}$. Let $w = a_1 a_2 a_3 \cdots a_n \in A_1 \vee A_2$, so that $a_j \in A_{i(j)}$, $i(j) \neq i(j+1)$ for all j. Let $[w]$ be the number of j for which $E_B(a_j) \neq 0$. Then we have

$$E_B(w) = E(\underbrace{(a_1 - E_B(a_1))a_2 \dots a_n}_{w_1}) + E_B(a_1)E(\underbrace{a_2 \dots a_n}_{w_2}).$$

Since $E_B(a_1) \in \mathbb{C}$ by Aut-independence of $A_{j(1)}$ and B, it is sufficient to prove that $E_B(w_1)$ and $E_B(w_2) \in \mathbb{C}$. But $[w_1], [w_2] < [w]$. Hence proceeding inductively, we must prove that $E_B(w) \in \mathbb{C}$ if $[w] = 0$. But in that case $E_B(w) = 0$, by the freeness with amalgamation assumption. □

Proposition 4.5. *Let $D \subset B$ be a unital subalgebra. Assume that B is generated by D and another unital subalgebra C. Assume that $C[u_1, \dots, u_n]$ is Aut-independent from D, and $u_1, \dots, u_n \in \mathcal{N}(D)$. Assume that $\eta \in \mathrm{CP}(B)$ is such that $\eta|_D = \mathrm{id}$ and $\eta(C) \subset C$. Then*

$$F^*(u_1, \dots, u_n : B, \eta) = F^*(u_1, \dots, u_n : C, \eta|_C).$$

Proof. Using the definition of F^* and repeatedly replacing B by $B[u_1, \dots, \hat{u}_j, \dots, u_n]$, C by $C[u_1, \dots, \hat{u}_j, \dots, u_n]$, it is sufficient to prove the proposition in the case $n = 1$. We write $u = u_1$.

Since $\eta(C) \subset C$, by Proposition 2.4, we in general have

$$F^*(u : B, \eta) \geq F^*(u : C, \eta|_C).$$

Thus we may assume that $\xi = \xi(u : C, \eta|_C)$ exists. We'll show that $\xi(u : C, \eta|_C) = \xi(u : B, \eta)$, which would establish the Proposition.

Let $w \in B[u, u^*]$. We must show that

$$\langle \xi, w \rangle = \langle 1 \otimes 1, d_{u:B}(w) \rangle_{H(B,\eta)}.$$

By assumption, $u^* D u \subset D$, $u D u^* \subset D$. We may therefore assume that w has the form

$$w = c_0 d_0 u^{g(1)} c_1 d_1 u^{g(2)} \cdots c_n d_n,$$

where $c_j \in C$, $d_j \in D$, $g(j) \in \{\cdot, *\}$, and also $c_j \in \mathcal{N}(D)$. Then

$$\langle \xi, w \rangle = \langle \xi, c_0 u^{g(1)} c_1 \cdots c_n d_0' d_1' \cdots d_n' \rangle,$$

where $d_j' = \mathrm{Ad}^{-1}_{u^{g(n)} c_n} \cdots \mathrm{Ad}^{-1}_{u^{g(j+1)} c_{j+1}} \mathrm{Ad}^{-1}_{u^{g(j)} c_j}(d_j)$. Since $\xi \in L^2(C[u, u^*])$ and $C[u, u^*]$ is Aut-independent from D, it follows that

$$\langle \xi, w \rangle = \langle \xi, c_0 u^{g(1)} c_1 \cdots c_n \rangle \cdot \tau(d_0' d_1' \cdots d_n').$$

On the other hand,

$$\begin{aligned} d_{u:B}(w) &= \sum_k c_0 d_0 u^{g(1)} \cdots c_k d_k \left\{ \begin{array}{cl} uu^* \otimes u & \text{if } g(k)=1 \\ -u^* \otimes uu^* & \text{if } g(k)=-1 \end{array} \right\} \times \\ &\quad \times c_{k+1} d_{k+1} u^{g(k+1)} \cdots c_n d_n \\ &= d_{u:C}(w) \cdot d_0' d_1' \cdots d_n', \end{aligned}$$

since $a \cdot d \otimes b = a \otimes d \cdot b$ for all $a, b \in B[u]$ and $d \in D$. Because of independence again, we find that

$$\langle 1 \otimes 1, d_{u:B}(w) \rangle_{H(B,\eta)} = \langle 1 \otimes 1, d_{u:C}(w) \rangle_{H(B,\eta)} \cdot \tau(d_0' d_1' \cdots d_n') = \langle \xi, w \rangle,$$

because $\xi = \xi(u : C, \eta|_C)$. □

Corollary 4.6. *Let G be a finitely-generated group and $g_1, \ldots, g_n$ be its generators. Let α be a measure-preserving action of G on a finite measure space X. Let $u(g_1), \ldots, u(g_n)$ be the unitaries in $L(G) \subset L^\infty(X) \rtimes_\alpha G$, implementing the action. Then*

$$F^*(u(g_1), \ldots, u(g_n) : L^\infty(X), \mathrm{id}) = F^*(u(g_1), \ldots, u(g_n)).$$

In particular, let G be a free group, and let $g_1, \ldots, g_n$ be its standard generators. Then

$$\begin{aligned} F^*(u(g_1), \ldots, u(g_n) : L^\infty(X), \mathrm{id}) &= F^*(u(g_1), \ldots, u(g_n)) \\ &= \sum_{i=1}^n F^*(u(g_i)) = 0. \end{aligned}$$

Proof. It is easily seen 4.3 that $L(G)$ is Aut-independent from $L^\infty(X)$ inside $L^\infty(X) \rtimes_\alpha G$. This implies the first claim.

In the case G is a free group, $u(g_i)$ are free, hence

$$F^*(u(g_1), \ldots, u(g_n)) = \sum F^*(u(g_i)).$$

Since each $u(g_i)$ is a Haar unitary, it follows that $F^*(u(g_i)) = 0$. □

Proposition 4.7. *Let $D \subset B$ be a unital subalgebra of B, $u_1, \ldots, u_n \in \mathcal{GN}(B)$. Let $p_i \in D \subset B$, $i = 1, \ldots, N$, $N \in \mathbb{N} \cup \{\infty\}$ be orthogonal projections, so that $\sum_{i=1}^N p_i = 1$.Assume that $[p_j, B] = \{0\}$ and $[u_i, p_j] = 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, N$. Then*

$$F^*(u_1, \ldots, u_n : B, E_D) = \sum \tau(p_j) F^*(p_j u_1, \ldots, p_j u_n : p_j B, E_{p_j D}).$$

Proof. It is sufficient to prove the Proposition for $n = 1$. We write $u = u_1$. Let $H = H(B[u, u^*], E_D)$ be the bimodule associated to E_D. Then by the assumptions made, we have the direct sum decomposition (as a Hilbert space)

$$H = \oplus H_i,$$

where

$$H_i = p_i H p_i = p_i H$$

as bimodules. Each H_i can be identified with $H(p_i B p_i [p_i u p_i], E_{p_i D p_i})$ by multiplying its inner product with $\sqrt{\tau(p_i)^{-1}}$.

Similarly, $K = L^2(B[u]) = \bigoplus K_i$, where

$$K_i = p_i L^2(B[u]).$$

The space K_i can be identified with $L^2(p_iBp_i[p_iup_i])$ as bimodules over p_iBp_i by multiplying its inner product by $\sqrt{\tau(p_i)^{-1}}$. It is easily seen that $d_{u:B}$ preserves these direct sum decompositions, so that $d_{u:B} = \oplus d_i$, where $d_i : K_i \to H_i$. It is also clear that each $d_i = d_{p_iup_i:p_iBp_i}$. Hence we get that $\xi(u : B, E_D) = d^*_{u:B}(1 \otimes 1)$ exists if and only if all $\xi(p_iup_i : p_iBp_i, E_{p_iDp_i}) = d^*_{p_iup_i:p_iBp_i}(p_i 1 \otimes 1)$ exist, and moreover

$$\xi(u : B, E_D) = \sum \xi(p_iup_i : p_iBp_i, E_{p_iDp_i}).$$

It follows that

$$\begin{aligned} F^*(u : B, E_D) = \|\xi(u : B, E_D)\|_2^2 &= \sum \|\xi(p_iup_i : p_iBp_i, E_{p_iDp_i})\|^2_{K_i} \\ &= \sum \tau(p_i)\|\xi(p_iup_i : p_iBp_i, E_{p_iDp_i})\|^2_{L^2(p_iBp_i[p_iup_i])} \\ &= \sum \tau(p_i) F^*(p_iup_i : p_iBp_i, E_{p_iDp_i}). \end{aligned}$$

□

Remark 4.8. The same proof shows that Proposition 4.7 still holds, if rather than dealing with a discrete family of projections p_i, we are dealing with a projection-valued measure μ on X, $\mu(X) = 1$. In that case, we get that $F^*(u : B, E_D) = \int_X F^*(u(t) : B(t), E_{d\mu(t)D})\tau(d\mu(t))$, where $u(t)$ and $B(t)$ are the disintegrations of u and B over the commutative algebra $W^*(\mu(A) : A \subset X)$.

The following proposition is essentially the basis of all computations of F^* in this paper. Indeed, it reduced the computation of F^* of several unitaries relative to a subalgebra to the computation of F^* of a single unitaries relative to $\mathbb{C}$, for which there is an explicit formula (see Section 3.1).

We mention that the situation as in the hypothesis of the proposition occurs in the case of a treeing of an equivalence relation (see later in the paper).

Proposition 4.9. *Let $u_1, u_2, \dots$ be unitaries in M, $B \subset M$ unital subalgebra, so that:*

1. $u_j \in \mathcal{N}(B)$ *for all* j
2. *For each j, there exist central projections $p_1^{(j)}, \dots, p_{n(j)}^{(j)} \in B$, so that $[p_k^{(j)}, u_j] = 0$ for all k, $\sum p_k^{(j)} = 1$, and for all k, $p_k^{(j)} u_j p_k^{(j)}$ is* Aut*-independent from $p_k^{(j)} B p_k^{(j)}$.*
3. *$u_1, u_2, \dots$ are free with amalgamation over B.*

Then

$$F^*(u_1, \dots, u_n : B, \mathrm{id}) = \sum_{j=1}^{n} \sum_{i=1}^{n(j)} \tau(p_i^{(j)}) F^*(p_i^{(j)} u_j p_i^{(j)})$$

and

$$\lim_{n\to\infty} F^*(u_1, \dots, u_n : B, \mathrm{id}) = \sum_{j=1}^{\infty} \sum_{i=1}^{n(j)} F^*(p_i^{(j)} u_j p_i^{(j)}).$$

Proof. Since $u_1, \ldots, u_n$ are free with amalgamation over B, it follows that

$$F^*(u_1, \ldots, u_n : B, \mathrm{id}) = \sum_{j=1}^{n} F^*(u_j : B, \mathrm{id}).$$

Next, we get that

$$F^*(u_j : B, \mathrm{id}) = \sum \tau(p_i^{(j)}) F^*(p_i^{(j)} u_j p_i^{(j)} : p_i^{(j)} B, \mathrm{id}).$$

Since $p_i^{(j)} B$ is independent from $p_i^{(j)} u_j p_i^{(j)}$, it follows that

$$F^*(p_i^{(j)} u_j p_i^{(j)} : p_i^{(j)} B, \mathrm{id}) = F^*(p_i^{(j)} u p_i^{(j)}).$$

This implies the statement. □

5. Free dimension for normalizing isometries.

5.1. **Free Brownian motion.** Let $u_1, \ldots, u_n \in M$, $B \subset M$ be as before. Consider multiplicative free Brownian motion $w_1(t), \ldots, w_n(t)$, such that:

1. $w_1(t), \ldots, w_n(t)$ are free among each other with amalgamation over B
2. $w_1(t), \ldots, w_n(t)$ are free from $u_1, \ldots, u_n$ with amalgamation over B
3. $[w_j(t), B] = \{0\}$, and $\{w_j(t)\}_j$ is Aut-independent from B.

To construct such a family, take $w(t)$ to be free multiplicative Brownian motion [1] in an algebra C. Let $C_1, \ldots, C_n$ be n isomorphic copies of C. Consider $*_B(C_i \otimes B)$, and let $w_i(t)$ be $w(t) \otimes 1$ in the i-th copy C_i of C.

Proposition 5.1. *Let $u_1, \ldots, u_n$, $w_1(t), \ldots, w_n(t)$ be as in Section 5.1. Then, given $0 < \alpha < \frac{1}{2}$, there exist constants C_1, C_2, independent of t, so that :*

(a) $F^*(w_j(t) : B, \mathrm{id}) < +\infty$; more precisely,

$$F^*(w_j(t) : B, \mathrm{id}) \leq \frac{C_1}{t}, \qquad 0 < t \leq 1,$$

$$F^*(w_j(t) : B, \mathrm{id}) \leq C_2 e^{-t\alpha}, \qquad 1 \leq t.$$

(b)$F^*(uw_1(t), \ldots, uw_n(t)) < +\infty$; more precisely,

$$F^*(uw_1(t), \ldots, uw_n(t)) \leq nF^*(w_1(t)) \leq \begin{cases} n\frac{C_1}{t}, & 0 < t \leq 1 \\ nC_2 e^{-t\alpha}, & 1 \leq t \end{cases}.$$

Proof. By Proposition 3.2,

$$F^*(uw_1(t), \ldots, uw_n(t) : B, \mathrm{id}) \leq \sum F^*(uw_j(t) : B, \mathrm{id}).$$

By Proposition 2.7, we get that

$$F^*(uw_j(t) : B, \mathrm{id}) \leq F^*(w_j(t) : B, \mathrm{id}).$$

Since $w_j(t)$ is Aut-independent from B, we get by Proposition 4.5 that $F^*(w_j(t) : B, \mathrm{id}) = F^*(w_j(t))$. The estimates now follow from Voiculescu's results for a single unitary (see Section 3.1). □

Definition 5.2. Let $v_1,\ldots,v_n\in M$ be partial isometries. The free entropy of $v_1,\ldots,v_n\in M$ relative to B,id is defined to be

$$\Sigma^*(v_1,\ldots,v_n \between B)=\frac{1}{2}\int_0^\infty F^*(v_1w_1(t),\ldots,v_nw_n(t))dt.$$

By [8, Corollary 10.9], we get that if u is a single unitary, then

$$\Sigma^*(u\between\mathbb{C})=\Sigma(u)=\iint\log|z_1-z_2|d\mu(z_1)d\mu(z_2),$$

where μ is the distribution of u, supported on the unit circle.

Definition 5.3. Fix a an element ψ of the β-compactification of $I=(0,1]$, which is not in I. The free dimension of $u_1,\ldots,u_n$ relative to B,id is defined to be

$$\delta^*(u_1,\ldots,u_n\between B)=\sum\tau(u_j^*u_j)-\lim_{t\to\psi}\frac{\Sigma^*(u_1w_1(t),\ldots,u_nw_n(t)\between B)}{\log t^{1/2}}.$$

In the case that $B=\mathbb{C}$, we write simply $\delta^*(u_1,\ldots,u_n)$.

5.2. **Properties of δ^*.** The properties of F^*, together with Proposition 4.4 immediately give rise to the following properties of $\delta^*(\cdots\between B)$:

Proposition 5.4. *Let $u_1,\ldots,u_n$, B be as above. Then we have:*

1. $$0\leq\delta^*(u_1,\ldots,u_n)\leq\sum_{i=1}^n\tau(u_iu_i^*)$$
2. *If $u_1,\ldots,u_k$ are free from $u_{k+1},\ldots,u_n$ with amalgamation over B, then*
$$\delta^*(u_1,\ldots,u_n)=\delta^*(u_1,\ldots,u_k)+\delta^*(u_{k+1},\ldots,u_n)$$
3. $$\delta^*(u_1,\ldots,u_n)\leq\delta^*(u_1,\ldots,u_k)+\delta^*(u_{k+1},\ldots,u_n)$$
4. *If $D\subset B$ is a unital subalgebra, then*
$$\delta^*(u_1,\ldots,u_n\between B)\leq\delta^*(u_1,\ldots,u_n\between D)\leq\delta^*(u_1,\ldots,u_n).$$
5. *If $D\subset B$ is a unital subalgebra, so that $B=D\vee C$, and $C[u_1,\ldots,u_n]$ is* Aut-*independent from B, and $u_1,\ldots,u_n\in\mathcal{GN}(D)$, then*
$$\delta^*(u_1,\ldots,u_n\between B)=\delta^*(u_1,\ldots,u_n\between C).$$
In particular, if $u_1,\ldots,u_n$ are Aut-*independent from B, then*
$$\delta^*(u_1,\ldots,u_n\between B)=\delta^*(u_1,\ldots,u_n).$$
6. *If p_i, $i=1,\ldots,N$, $N\in\mathbb{N}\cup\{\infty\}$ are orthogonal central projections in B, so that $\sum p_i=1$, and so that $[p_i,u_j]=0$ for all i,j, then*
$$\delta^*(u_1,\ldots,u_n\between B)=\sum_i\tau(p_i)\delta^*(p_iu_1p_i,\ldots,p_iu_np_i\between p_iBp_i).$$

The same conclusion holds if μ is a projection-valued measure on X, $\mu(X) = 1$, so that $\mu(Y) \in B$ is in the center of $B[u_1, \ldots, u_n]$ for all Borel subsets $Y \subset X$:

$$\delta^*(u_1, \ldots, u_n) = \int_X \delta^*(u_1(t), \ldots, u_n(t) \between B(t))\tau(d\mu(t)),$$

where $u_j(t), B(t)$ is the disintegration of u_j, B over $W^*(\mu(A) : A \subset X)$.

6. Free dimension for one variable.

Proposition 6.1. *Let $u \in M$ be a unitary, and let μ be its distribution, supported on the circle $T = \{z : |z| = 1\}$. Let $w(t)$ be multiplicative free Brownian motion, free from u. Let $u_t = uw(t)$, and let μ_t be the distribution of u_t. Then*

$$2 \lim_{t \to 0} \frac{\iint \log|z - w| d\mu_t(z) d\mu_t(w)}{|\log t|} = -\sum_{t \in T} (\mu\{t\})^2.$$

Proof. We reduce the problem to the form dealt with in Voiculescu's proof of [6, Proposition 6.3]. We know by [8, Corollary 1.7] that μ_t is absolutely continuous with respect to Haar measure on T. Denote by p_t its density; then we also know that $\|p_t\|_\infty \leq 2/\sqrt{t}$ for t sufficiently small. We have that $\|w(t) - 1\|_\infty \leq Kt^{1/2}$ for some constant K. We also have that

$$\mu_{t^2}(-\pi, b - Kt) \leq \mu(-\pi, b) \leq \mu_{t^2}(-\pi, b + Kt); \tag{6.1}$$

this follows from the fact that μ_t is the distribution of $uw(t)$, and $\|w(t^2) - 1\| \leq Kt$, so that the spectrum of $w(t^2)$ is contained in the arc $-Kt, Kt$.

The quantity $\iint \log|z - w| d\mu_t(z) d\mu_t(w)$ does not change if we replace $\mu_t(w)$ by $\mu_t(rw)$ for some fixed $r \in T$. Hence we may assume that there are no atom s of μ in some arc around -1. Let $-\pi \leq \theta \leq \pi$ be the polar coordinate on T. We'll from now on write $d\mu_t(\theta)$ to denote the density in $[-\pi, \pi]$ of the push-forward via θ of the measure μ_t.

Notice that

$$\log|e^{i\theta} - e^{i\psi}| = \frac{1}{2}\log(2 - 2\cos(\theta - \psi)) = \frac{1}{2}\log(1 - \cos(\theta - \psi)) + \frac{1}{2}\log 2.$$

It will thus suffice to prove that

$$\lim_{t \to 0} \frac{\iint \log(1 - \cos(\theta - \psi)) d\mu_t(\theta) d\mu_t(\psi)}{|\log t|} = -\sum_{t \in T} (\mu\{t\})^2.$$

There is a $\delta > 0$ so that for $|\alpha| < \delta$, $\frac{\alpha^2}{4} \leq 1 - \cos\alpha \leq \frac{\alpha^2}{2}$. It follows that

$$\begin{aligned}
\frac{2}{|\log t|} \iint_{|\theta - \psi| < \delta} \log|\theta - \psi| d\mu_t(\theta) d\mu_t(\psi) - \frac{\log 4}{|\log t|} \iint_{|\theta - \psi| < \delta} 1 d\mu_t(\theta) d\mu_t(\psi) \\
\leq \frac{1}{|\log t|} \iint_{|\theta - \psi| < \delta} \log(1 - \cos(\theta - \psi)) d\mu_t(\theta) d\mu_t(\psi) \\
\leq \frac{2}{|\log t|} \iint_{|\theta - \psi| < \delta} \log|\theta - \psi| d\mu_t(\theta) d\mu_t(\psi) - \frac{\log 2}{|\log t|} \iint_{|\theta - \psi| < \delta} 1 d\mu_t(\theta) d\mu_t(\psi).
\end{aligned}$$

When $2\pi-\delta \geq |\theta-\phi| \geq \delta$, $0 \geq \log(1-\cos(\theta-\psi)) \geq \log(1-\cos\delta)$ and when $|\theta-\phi| \geq \delta$, $0 \geq \log|\theta-\psi| \geq \log\delta$. Hence as $t \to 0$, the following expressions

$$|\log t|^{-1} \iint \log(1-\cos(\theta-\psi))d\mu_t(\theta)d\mu_t(\psi),$$

$$|\log t|^{-1} \iint_{|\theta-\psi|<\delta} \log(1-\cos(\theta-\psi))d\mu_t(\theta)d\mu_t(\psi)+$$

$$|\log t|^{-1} \iint_{|\theta-\psi|\geq 2\pi-\delta} \log(1-\cos(\theta-\psi))d\mu_t(\theta)d\mu_t(\psi),$$

$$|\log t|^{-1} \iint_{|\theta-\psi|<\delta} 2\log|\theta-\psi|d\mu_t(\theta)d\mu_t(\psi)+$$

$$|\log t|^{-1} \iint_{|\theta-\psi|\geq 2\pi-\delta} 2\log|2\pi-(\theta-\psi)|d\mu_t(\theta)d\mu_t(\psi) \qquad \text{and}$$

$$|\log t|^{-1} \iint 2\log|\theta-\psi|d\mu_t(\theta)d\mu_t(\psi)+$$

$$|\log t|^{-1} \iint 2\log|2\pi-(\theta-\psi)|d\mu_t(\theta)d\mu_t(\psi)$$

all have the same limit, if any.

It is therefore sufficient to show that

$$\lim_{t\to 0} 2\frac{\iint \log|\theta-\psi|d\mu_t(z)d\mu_t(w)}{|\log t|} = -\sum_{t\in T}(\mu\{t\})^2,$$

$$2\lim_{t\to 0} \frac{\iint \log|2\pi-(\theta-\psi)|d\mu_t(\theta)d\mu_t(\psi)}{|\log t|} = 0, \qquad \text{i.e.,}$$

$$\lim_{t\to 0} \frac{\iint \log|\theta-\psi|d\mu_t(z)d\mu_t(w)}{|\log t^{1/2}|} = -\sum_{r\in T}(\mu\{r\})^2,$$

$$\lim_{t\to 0} \frac{\iint \log|2\pi-(\theta-\psi)|d\mu_t(\theta)d\mu_t(\psi)}{|\log t^{1/2}|} = 0.$$

Let $t = s^2/K^2$, and write ν_t for μ_{t^2/K^2}. Then we must prove that

$$\lim_{s\to 0} \frac{\iint \log|\theta-\psi|d\nu_s(z)d\nu_s(w)}{|\log s|} = -\sum_{t\in T}(\mu\{t\})^2,$$

$$\lim_{t\to 0} \frac{\iint \log|(\theta-\psi)|d\nu_s(\theta-\pi)d\nu_s(\psi+\pi)}{|\log s|} = 0.$$

Let v_s be the density of ν_s. We now proceed exactly as in the proof of [6, Proposition 6.3], noticing that the only estimates on ν_s needed in that proof are that: (a) $\|v_s\|_2 \leq Cs^{-1/2}$

and (b) $\nu_s(-\pi, b-s) \leq \mu(-\pi, b) \leq \nu_s(-\pi, b+s)$. Voiculescu's proof then shows that

$$\lim_{s\to 0} \frac{\iint \log|\theta-\psi| d\nu_s(z) d\nu_s(w)}{|\log s|} - \mu\otimes\mu(\Delta) = 0,$$

$$\lim_{t\to 0} \frac{\iint \log|(\theta-\psi)| d\nu_s(\theta-\pi) d\nu_s(\psi+\pi)}{|\log s|} - \mu^{-\pi}\otimes\mu^{\pi}(\Delta) = 0,$$

where $\Delta = \{(t,t) : t\in[-\pi,\pi]\}$ and μ^r denotes the translation of μ by r. Since μ has no atoms in around π and $-\pi$, $\mu^{-\pi} = \mu^{-\pi} = 0$, while $\mu\otimes\mu(\Delta) = \sum_{t\in T}(\mu\{t\})^2$.

It remains to prove estimates (a) and (b). To prove estimate (a), it is sufficient to prove that $\|p_t\|_2 \leq \hat{C}t^{-\frac{1}{4}}$. Since $\|p_t\|_\infty \leq Kt^{-1/2}$ and p_t is supported on an arc of length $Ct^{1/2}$, it follows that

$$\|p_t\|_r^r = \int_{\operatorname{supp} p_t} |p_t(\theta)|^r d\theta \leq Ct^{1/2}\cdot K^r t^{-r/2} = CK^r t^{(1-r)/2}.$$

Hence $\|p_t\|_r \leq \hat{C}t^{\frac{1}{2}(-1+1/r)}$. Setting $r=2$ gives the estimate $\|p\|_2 \leq \hat{C}t^{-\frac{1}{4}}$. Estimate (b) follows by substituting the definition of ν_s and s into (6.1). □

Corollary 6.2. *Let $u\in M$ be a unitary, and let μ be its distribution, supported on the circle $T = \{z : |z| = 1\}$. Then $\delta^*(u) = 1 - \sum_{t\in T}\mu(\{t\})^2$.*

Corollary 6.3. *Let G be a cyclic singly-generated group, and $g\in G$ be its generator. Let $u\in L(G)$ be the unitary implementing g. Then $\delta^*(u) = 1 - \frac{1}{|G|}$.*

Proof. If $|G| = \infty$, the distribution of u is a Haar measure. It follows that $\delta^*(u) = 1$.

If $|G| = n$, then the distribution of u is equal to $\frac{1}{n}\sum\delta_{\omega^n}$, where ω^n is a primitive n-th root of unity. We have that $\mu(\{\omega^n\})^2 = \frac{1}{n^2}$, so that $\sum_{t\in T}\mu(\{t\})^2 = n\frac{1}{n^2} = \frac{1}{n}$. □

Proposition 6.4. *Let v be an isometry, $v\in\mathcal{GN}(B)$. Let w be a Haar unitary, independent from B and free from v with amalgamation over B. Assume that the B-valued distributions of v and vw are the same. Then $\delta^*(v \between B) = \tau(vv^*)$.*

Proof. Since $v\in\mathcal{GN}(B)$, one has in particular $vv^*, v^*v\in B$. Then $\delta^*(v)\leq\tau(v^*v) = \tau(vv^*)$. Let w be a Haar unitary, independent from B and free from v with amalgamation over B. Assume that vw has the same B-valued distribution as v. Then $F^*(w : B, \mathrm{id}) = F^*(w) = 0$ because w is independent from B; and by Proposition 2.4, we get that $0\leq F^*(vw : B, \mathrm{id}) \leq F^*(w : B, \mathrm{id}) = 0$. Hence $F^*(vw : B, \mathrm{id}) = 0$. Let $w(t)$ be multiplicative unitary free Brownian motion, which is independent from B and free from w, v with amalgamation over B. Then the B-valued distributions of $vw(t)$ and $vww(t)$ coincide. Since $ww(t)$ is free from v with amalgamation over B and has the same B-valued distribution as w, it follows that the B-valued distributions of $vw(t)$ and vw are the same. Hence we get that

$$\delta^*(v \between B) = \tau(v^*v) - \lim_{t\to\psi}\frac{F^*(vw(t) : B, \mathrm{id})}{F^*(w(t))} = \tau(v^*v) - 0.$$

□

Problem 6.5. Let $u_1,\dots,u_n\in\mathcal{GN}(B)$. Is $\delta^*(u_1,\dots,u_n \between B)$ is an invariant for the pair $(W^*(u_1,\dots,u_n \between B), B)$?

7. Free dimension and cost.

7.1. **Local morphisms.** Let (X,μ) be a finite measure space. If $A \subset X$ is a subset, a *local morphism* $\phi : A \to \phi(A)$ is a measure-preserving measurable monomorphism from its domain A to its range $\phi(A)$. Let R be a measurable equivalence relation on X. We say that ϕ is consistent with R, if $\phi(x) \sim_R x$ for almost all $x \in A$.

If ϕ, ψ are local morphisms, then so are $\phi \circ \psi$ and ϕ^{-1}; these are defined on the subsets $\{x \in \operatorname{dom} \psi : \psi(x) \in \operatorname{dom} \phi\}$ and $\operatorname{ran} \phi$, respectively.

7.2. **Graphings and treeings.** Following Gaboriau [3], we say that a set $\{\phi_i\}_{i\in I}$ of local morphisms is a graphing of an equivalence relation R, if a.e. in X, $x \sim_R y$ iff $x = \phi_{i_1}^{g(1)}(\phi_{i_2}^{g(2)}(\dots\phi_{i_n}^{g(n)}(y)\dots))$ for some $i_1,\dots,i_n \in I$ and $g(1),\dots,g(n) \in \{\pm 1\}$. A graphing is a treeing, if for any $i_1 \neq i_2$, $i_2 \neq i_3$,... and $g(1),\dots,g(n) \in \pm\mathbb{N}$, the set $\{x : \phi_{i_1}^{g(1)}(\phi_{i_2}^{g(2)}(\dots\phi_{i_n}^{g(n)}(x)\dots)) = x\}$ has measure zero.

7.3. **Cost of an equivalence relation.** The cost $C(R)$ of R was defined by Gaboriau to be the infimum of $\sum_{i\in I}\mu(\operatorname{dom}\phi_i)$ over all graphings $\{\phi_i\}_{i\in I}$ of R. If $\{\phi_i\}_{i\in I}$ is a treeing, then $C(R)$ is given by $\sum_{i\in I}\mu(\operatorname{dom}\phi_i)$.

7.4. **Free equivalence relations.** Two equivalence relations R_1 and R_2 are called free, if for each $n \geq 2$, the set of all $x \in X$ for which $x \sim_{R_{i_1}} x_1 \sim_{R_{i_2}} \sim_{R_{i_3}} \cdots \sim_{R_{i_n}} x$ with $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$ has measure zero. Gaboriau proved that $C(R_1 \vee R_2) = C(R_1) + C(R_2)$, if $R_1 \vee R_2$ is the equivalence relation generated by R_1 and R_2, and R_1, R_2 are free.

7.5. **The von Neumann algebra of an equivalence relation.** Feldman and Moore [2] associated to a measurable equivalence relation R on X a von Neumann algebra $W^*(X,R)$. We summarize some properties of $W^*(X,R)$:

1. $W^*(X,R)$ contains $L^\infty(X)$ as a unital subalgebra;
2. Every local morphism ϕ consistent with R defines a partial isometry $v_\phi \in M$, so that $v_\phi^* v_\phi = \chi_{\operatorname{dom}\phi}$, $v_\phi v_\phi^* = \chi_{\operatorname{ran}\phi}$, and $v_\phi f v_\phi^* = f \circ \phi$ for all $f \in L^\infty(\operatorname{dom}\phi) \subset L^\infty(X)$ (χ denotes the characteristic function of a set). Furthermore, $v_\phi v_\psi = v_{\phi\psi}$, $v_\phi^* = v_{\phi^{-1}}$.
3. If ϕ is a local morphism, and $C \subset \operatorname{dom}\phi$, then $v_{\phi|_C} = v_\phi \chi_C$.
4. M is generated by all v_ϕ, where ϕ ranges over all possible local morphisms, consistent with R.
5. There is a conditional expectation E from M onto $L^\infty(X)$, so that $\tau = \mu \circ E$ is a trace on M. Moreover, if ϕ is a local morphism, then $E(v_\phi) = \chi_{\{x\in X:\phi(x)=x\}}$

7.6. **Extensions of local morphisms.** Let $\phi : A \to G$ be a local morphism. Let $A_\infty = \{x \in X : x \in \operatorname{dom}\phi^{-n}, \forall n\}$. Then $\phi(A) \cdot A_\infty = \phi^{-1}(A) \cdot A_\infty = A_\infty$. Let $A^\infty = \{x \in X : x \in \operatorname{dom}\phi^n, \forall n\}$. Let $C = \operatorname{dom}\phi \cup \operatorname{ran}\phi$. Then $\phi(A) \cdot C = \phi^{-1}(A) \cdot C = C$. Let $D = C \setminus A_\infty \setminus A^\infty$. Then for all $x \in D$, there exists a number $n(x)$, so that $x \in \operatorname{dom}\phi^{-n}$, but $x \notin \operatorname{dom}\phi^{-(n+1)}$. Let $A_n = \{x : x(n) = n\}$. Now define $\Theta_\phi(x) = \phi(x)$ if $x \in A_\infty$, $\Theta_\phi(x) = \phi^{-1}(x)$ if $x \in A^\infty \setminus A_\infty$, $\Theta_\phi(x) = x$ if $x \notin C$, and $\Theta_\phi(x) = \phi^{-1}(x)$ if $x \in A_n$, $n > 0$, $\Theta_\phi(x) = \phi^{m(x)}(x)$ if $x \in A_0$, where $m(x) = \max\{p : x \in \operatorname{dom}\phi^p\}$.

Proposition 7.1. *Let $\phi : A \to G$ be a local morphism, compatible with an equivalence relation R. Then Θ_ϕ is an automorphism of X, compatible with R.*

Proof. We must check that $\Theta_\phi(x) \neq \Theta_\phi(y)$ if $x \neq y$. This is clear if at least one of (x, y) is not in C, or if $x \in A_n$, $y \in A_m$ for $n = m$. Assume that $x \in A_n$, $y \in A_m$, but $n \neq m$. For definiteness, assume that $n > 0$. Then $\Theta_\phi(x) \in A_{n-1}$. If $m > 0$, then $\Theta_\phi(y) \in A_{m-1}$, so that $\Theta_\phi(x) \neq \Theta_\phi(y)$. If $m = 0$, $\Theta_\phi(y) = \Theta_\phi(x)$ would imply that $\phi^{m(y)}(y) = \Theta_\phi(x) \in \operatorname{dom}\phi$. □

Given a graphing $\{\phi_i\}$ of an equivalence relation R, we can associate to it unitaries $u_i \in W^*(X, R)$ by taking the unitaries u_{ϕ_i} associated to bisections Θ_{ϕ_i}.

Proposition 7.2. *Let $\{\phi_i\}_{i\in I}$ be a graphing of an equivalence relation R on a finite measure space X. Then $\{\Theta_{\phi_i}\}_{i\in I}$ is also a graphing of R. In particular, the von Neumann algebra $W^*(X, R)$ is generated by $L^\infty(X)$ and $\{v_{\phi_i}\}_{i\in I}$.*

Proof. Clearly, only the first statement needs to be proved. It is clearly sufficient to prove that if $x = \phi_i(y)$, then x and y are equivalent in the equivalence relation generated by Θ_{ϕ_i}. From the definition of Θ, we have that either $\Theta_{\phi_i}(x) = y$, or $\Theta_{\phi_i}(y) = x$, unless $y \notin \operatorname{dom}\phi_i^{-1}$, and $\phi_i^n(y) \notin \operatorname{dom}\phi_i$ for some $n \in \mathbb{N}$. In that case, we have that $\Theta_{\phi_i}(x) = y$. □

Lemma 7.3. *A treeing of R is a graphing $\{\phi_i\}$, such that $R = *R_i$, where R_i is the equivalence relation generated by ϕ_i, and so that $\phi_i^n(x) \neq x$ for all $x \in X$ and $n \in \pm\mathbb{N}$.*

Conjecture 7.4. *Let $\{\phi_i\}$ be a graphing of an equivalence relation R. Let u_i be the unitary in $W^*(X, R)$ associated to Θ_{ϕ_i}, and let $v_i \in W^*(X, R)$ be the partial isometry associated to ϕ_i. Then*

$$\delta^*(\{\phi_i\}) = \lim_{n\to\infty} \delta^*(u_1, u_2, \ldots, u_n \between L^\infty(X)) = \lim_{n\to\infty} \delta^*(v_1, v_2, \ldots, v_n \between L^\infty(X))$$

is independent of the graphing $\{\phi_i\}$.

Proposition 7.5. *Let R_1, R_2 be two equivalence relations, which are free. Then the algebras $W^*(X, R_1)$ and $W^*(X, R_2)$ are free in $W^*(X, R)$ with amalgamation over $L^\infty(X)$.*

Proof. Let $\phi_i^{(1)}$, $i \in I$ be local morphisms compatible with R_1, and $\phi_i^{(2)}$, $i \in I$ be local morphisms compatible with R_2. Then $W^*(X, R_1 \vee R_2) = W^*(X, R_1) \vee W^*(X, R_2)$ is densely linearly spanned by $L^\infty(X)$ and words of the form $w = v_{\phi_{i_1}^{j_1}} \ldots v_{\phi_{i_n}^{j_n}}$, where $j_1 \neq j_2$, $j_2 \neq j_3$, etc., and $E_{L^\infty(X)}(v_{\phi_{i_k}^{j_k}}) = 0$. We must prove that $E_{L^\infty(X)}(w) = 0$. But $E_{L^\infty(X)} = \chi_{\{x\in X:\phi_{i_1}^{j_1}\circ\ldots\phi_{i_n}^{j_n}(x)=x\}} = 0$ by the assumption that R_1 and R_2 are free. □

Proposition 7.6. *Let $\{\phi_i\}$ be a treeing of an equivalence relation R. Let u_{ϕ_i} be the unitary associated to the bisection Θ_{ϕ_i}. Then*

$$C(R) = \lim_{n\to\infty} \delta^*(u_{\phi_1}, u_{\phi_2}, \ldots, u_{\phi_n} \between L^\infty(X)).$$

Proof. In view of additivity of both $C(R)$ and δ^* under free products, it is sufficient to verify the equality for a single ϕ. Let $A = \operatorname{dom}\phi$, $B = \operatorname{ran}\phi$, $C = A \cup B$. R be the equivalence relation generated by ϕ, and let

$$A_n = \{x \in X : \#\{y : y \sim_R x\} = n\}.$$

Then $C(R) = \mu(A_\infty) + \sum \mu(A_n)(1 - \frac{1}{n})$, since A_n are disjoint, and R is cyclic of order n on A_n.

On the other hand, let p_n be the characteristic function of A_n. Then $[u_\phi, p_n] = 0$. Moreover, $u_n = p_n u_\phi p_n$ implements a free action of a cyclic group of order n on A_n. Therefore,

$$\delta^*(u_\phi \between L^\infty(X)) = \tau(p_\infty)\delta^*(u_\infty \between L^\infty(A_\infty)) + \sum \tau(p_n)\delta^*(u_n \between L^\infty(A_n)).$$

Since u_n is Aut-independent from $L^\infty(A_n)$, it follows that $\delta^*(u_n \between L^\infty(A_n)) = \delta^*(u_n) = 1 - \frac{1}{n}$. Thus $C(R) = \delta^*(u \between L^\infty(X))$. $\square$

If conjecture 7.4 is correct, then $\delta^*(\{\phi_i\}) \leq C(R)$. Indeed, assume that each ϕ_i by itself is a treeing. Let R_{ϕ_i} be the equivalence relation generated by ϕ_i.

$$\begin{aligned} \delta^*(\{\phi_i\}) &= \lim_{n\to\infty} \delta^*(u_{\phi_1}, u_{\phi_2}, \ldots, u_{\phi_n} \between L^\infty(X)) \\ &\leq \lim_{n\to\infty} \sum \delta^*(u_{\phi_i} \between L^\infty(X)) = \sum C(R_{\phi_i}) \leq C(R), \end{aligned}$$

the last inequality because given a graphing $\{\phi_i\}$ we can always assume that *each* ϕ_i is a treeing by itself by shrinking the domains of ϕ_i.

Proposition 7.7. *Let $\{\phi_i\}$ be a treeing of an equivalence relation R. Let v_{ϕ_i} be the partial isometry in $W^*(X,R)$ associated to ϕ_i. Then*

$$C(R) = \lim_{n\to\infty} \delta^*(v_{\phi_1}, \ldots, v_{\phi_n} \between L^\infty(X)) = \sum_{j=1}^{\infty} \delta^*(v_{\phi_i} \between L^\infty(X)).$$

Proof. Since $\{\phi_i\}$ is a treeing, v_{ϕ_j}, $j = 1, 2, \ldots$ are free with amalgamation over $L^\infty(X)$. Hence

$$\delta^*(v_{\phi_1}, \ldots, v_{\phi_n} \between L^\infty(X)) = \sum_{j=1}^{n} \delta^*(v_{\phi_i} \between L^\infty(X)).$$

Since each ϕ_i is a treeing, the $L^\infty(X)$-valued distribution of v_{ϕ_i} remains the same if v_{ϕ_i} is replaced by $v_{\phi_i} w$, where w is a Haar unitary, independent from $L^\infty(X)$ and free from v_{ϕ_i} with amalgamation over $L^\infty(X)$. From Proposition 6.4 we get that $\delta^*(v_{\phi_i}) = \tau(v_{\phi_i}^* v_{\phi_i}) = \mu(\operatorname{dom}\phi_i)$. Since $C(R) = \sum_{j=1}^\infty \mu(\operatorname{dom}\phi_i)$, $C(R) = \lim_n \delta^*(v_{\phi_1}, \ldots, v_{\phi_n})$. $\square$

Problem 7.8. Choose any graphing $\{\phi_i\}$ of R, let u_{ϕ_i} be the unitary in $W^*(X,R)$ associated to the bisection Θ_{ϕ_i}, and let v_{ϕ_i} be the partial isometry in $W^*(X,R)$ associated to ϕ_i. Is it true that $C(R) = \lim_{n\to\infty} \delta^*(u_{\phi_1}, \ldots, u_{\phi_n} \between L^\infty(X)) = \lim_{n\to\infty} \delta^*(v_{\phi_1}, \ldots, v_{\phi_n})$?

If this question is answered in the affirmative, then all groups have fixed cost. Indeed, if $\phi_i = \alpha_{g_i}$ comes from the free action of a group G with some fixed set of generators $g_1, g_2, \ldots$, then $\delta^*(u_{\phi_1}, \ldots, u_{\phi_n} \between L^\infty(X)) = \delta^*(u_{\phi_1}, \ldots, u_{\phi_n})$, the latter depending only on $g_1, g_2, \cdots \in G$, and independent of the action α.

REFERENCES

1. P. Biane, *Free Brownian motion, free stochastic calculus and random matrices*, Free Probability (D.-V. Voiculescu, ed.), Fields Institute Communications, vol. 12, American Mathematical Society, 1997, pp. 1–19.
2. J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras I, II*, Trans. AMS **234** (1977), 289–359.
3. D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Preprint, 1998.
4. ______, *Mercuriale de groupes et de relations*, C.R. Acad. Sci. Paris **t. 326** Série I (1998), 219–222.
5. D. Shlyakhtenko, *Free entropy with respect to a completely-positive map*, Preprint, 1998.
6. D.-V. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory II*, Invent. Math. **118** (1994), 411–440.
7. ______, *The analogues of entropy and of Fisher's information measure in free probabilility, V*, Invent. Math. **132** (1998), 189–227.
8. ______, *The analogues of entropy and of Fisher's information measure in free probability, VI*, Preprint, Berkeley, 1998.
9. ______, *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, IMRN **1** (1998), 41 – 64.
10. D.-V. Voiculescu, K. Dykema, and A. Nica, *Free random variables*, CRM monograph series, vol. 1, American Mathematical Society, 1992.

DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095
E-mail address: `shlyakht@member.ams.org`