ASYMPTOTICS VIA STEEPEST DESCENT FOR AN OPERATOR RIEMANN-HILBERT PROBLEM

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ABSTRACT

In this paper, we take the first step towards an extension of the nonlinear steepest descent method of Deift, Its and Zhou to the case of operator Riemann-Hilbert problems. In particular, we provide long range asymptotics for a Fredholm determinant arising in the computation of the probability of finding a string of n adjacent parallel spins up in the antiferromagnetic ground state of the spin 1/2 XXX Heisenberg Chain. Such a determinant can be expressed in terms of the solution of an operator Riemann-Hilbert factorization problem.

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1. INTRODUCTION-MOTIVATION

It has been established by Korepin, Izergin and Bogoliubov [KIB] that many important problems arising in the theory of Quantum Inverse Scattering can be reduced to Classical Inverse Scattering Problems. In particular, the correlation functions for quantum solvable models can be described by solutions of classical integrable equations, or, equivalently, are reducible to the solution of Riemann-Hilbert problems. An important observation is that many such quantities are described by Fredholm determinants of particular ('integrable') integral operators. As shown by different authors (see [DIZ] for a concise proof) the computation of these determinants reduces to the solution of a Riemann-Hilbert problem. Long range or long time asymptotics for such a quantity can then be recovered via an asymptotic analysis of an oscillatory Riemann-Hilbert problem.[1] In general, such problems are not local; in fact they can be considered as operator Riemann-Hilbert problems, which makes their analysis harder. In this paper we present a solution of one of these problems, by reducing it to two distinct (and essentially different) matrix (2x2) problems and thus easily extracting asymptotics.

We are motivated by the following observation [KIEU].

FACT 1.1. Let $\Pi(n, \psi, \phi)$ be the probability of finding a string of n adjacent parallel spins up in the antiferromagnetic ground state of the spin 1/2 XXX Heisenberg chain. Here ψ is an angle related to the external magnetic field h,[2] and $\phi(z)$ is a dual quantum field, which can be considered as a holomorphic function in a neighborhood of the unit circle. Then, $\Pi(n, \psi, \phi) = \frac{(0|det(1+V)|0)}{det(1+K)}$, where 0 is the Fock vacuum, K is the integral operator on $L^2[0, \infty)$ with kernel $\frac{1}{\pi(1+(\lambda-\mu)^2)}$ and V is an integral operator defined below.

Our aim is the following.

THEOREM 1.2. Let ψ be as above and $\phi(z)$ be an entire function. Let $P(n, \psi, \phi) = det(I + V)$, where V is given by (2.1) below. As $n \to \infty$, we have

$$P(n, \psi, \phi) \sim const\ exp(n^2 log|sin(\psi/2)| + O(logn)). \tag{1.1}$$

Note that there is no ϕ dependence.

REMARK. Theorem 1.2 falls short of computing the asymptotics for the actual correlation function $\Pi(n, \psi, \phi)$. This may require the complete asymptotic expansion of $P(n, \psi, \phi)$. We plan to consider this problem in a later publication.

In the next section, we set up the operator Riemann-Hilbert problem that enables us to compute the Fredholm determinant $P(n, \psi, \phi)$. Using deformations inspired by [DZ2], [FIK] and [K2] we will reduce this problem to two standard matrix ones, which we can solve explicitly (as $n \to \infty$).

[1] See e.g.[DZ1] and [DIZ] for the Ising Chain and the XXO model. Such analysis has also been performed in the context of long time asymptotics for soliton equations by several authors; see e.g. [DZ2], [K1], [DVZ], [DKKZ], [K2].

[2] In fact ψ is defined by $e^{-i\psi} = -i\frac{e^{-2L}-i}{e^{-2L}+i}, -\pi < \psi < 0$, where $cosh2L = 2/h, L > 0$.

2. THE OPERATOR RIEMANN-HILBERT PROBLEM

We are interested in the long range asymptotics of the Fredholm determinant $det(I+V^{(n)})|_{\gamma=1}$ where the integral operator $V^{(n)}$ acts on a function f as follows:

$$(V^{(n)}f)(z_1) = \int_C V^{(n)}(z_1,z_2)f(z_2)dz_2, \tag{2.1}$$

where C is the contour $\theta \to z = exp(i\theta),\ -\psi < \theta < 2\pi + \psi$ and

(2.2)
$$V^{(n)}(z_1,z_2) = -\gamma\frac{i}{2\pi}\frac{1}{z_1-z_2}[e_+^{(n)}(z_1)e_-^{(n)}(z_2)r(z_1,z_2) - e_-^{(n)}(z_1)e_+^{(n)}(z_2)r(z_2,z_1)],$$
$$r(z_1,z_2) = \frac{2(z_2-1)(z_1-1)}{2(z_2-1)(z_1-1)+z_1-z_2},\ e_\pm^{(n)}(z) = z^{\mp n/2}e^{\pm\phi(z)/2}.$$

Here $\psi \in (-\pi,0)$ is defined as in Theorem 1.1 and is related to an external magnetic field.

It is established that for special ('integrable') operators where the kernel is of the form

$$V(z_1,z_2) = \Sigma_1^N \frac{a_j(z_1)b_j(z_2)}{z_1-z_2},\ with\ \Sigma_1^N a_j(z)b_j(z) = 0, \tag{2.3}$$

the computation of the related Fredholm determinant can be reduced to the solution of an NxN matrix Riemann-Hilbert problem (see [DIZ] for the proof). In our case, the kernel (2.2) is not of the desired type. However, using

$$r(z_1,z_2) = \int_0^\infty e^{-s+\frac{s}{2}\frac{z_1+1}{z_1-1}-\frac{s}{2}\frac{z_2+1}{z_2-1}}ds \tag{2.4}$$

we obtain ([FIK])

$$V^{(n)}(z_1,z_2) = -\gamma\frac{i}{2\pi}\int_0^\infty e_+^{(n)}(z_1|s)e_-^{(n)}(z_2|s) - e_-^{(n)}(z_1|s)e_+^{(n)}(z_2|s)ds, \tag{2.5}$$

where

$$e_\pm^{(n)}(z|s) = (z^{-n}exp(\phi(z)+s\frac{z+1}{z-1}))^{\pm 1/2}e^{-s/2}. \tag{2.6}$$

In other words, we end up with an operator Riemann-Hilbert problem as follows.

THEOREM 2.1. [FIK] Let $\Psi(z)$ (similarly $M(z)$) be a 2x2 matrix of integral operators depending on the complex parameter z:

$$(\Psi_{jk}(z)f)(s) = \int_0^\infty \Psi_{jk}(z|s,t)f(t)dy,\ f \in L^2[0,\infty), \tag{2.7}$$

such that

1. $\Psi(z)$ is analytic for $z \in \mathbb{C} \setminus \bar{C}$.
2. $\Psi(\infty) = I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $I(s,t) = \delta(s-t)$.
3. $\Psi_+(z) = \Psi_-(z) M(z),\ z \in C$, where Ψ_+ and Ψ_- are the normal limits of Ψ from inside and outside the unit circle respectively, and

(2.8a)
$$M(z|s,t) = \begin{pmatrix} \delta(s-t) + \gamma e_-^{(n)}(z|s) e_+^{(n)}(z|t) & -\gamma e_+^{(n)}(z|s) e_+^{(n)}(z|t) \\ \gamma e_-^{(n)}(z|s) e_-^{(n)}(z|t) & \delta(s-t) - \gamma e_+^{(n)}(z|s) e_-^{(n)}(z|t) \end{pmatrix},$$

or equivalently,

$$M(z) = \begin{pmatrix} I + \gamma P^T & -Q(z) z^{-n} e^{\phi(z)} \\ Q(1/z) z^n e^{-\phi(z)} & I - \gamma P(z) \end{pmatrix},\ z \in C, \tag{2.8b}$$

where P and Q are given by the kernels

$$\begin{aligned} P(z|s,t) &= exp[\frac{1}{2}(s-t)\frac{z+1}{z-1} - \frac{s+t}{2}], \\ Q(z|s,t) &= exp[\frac{1}{2}(s+t)\frac{z+1}{z-1} - \frac{s+t}{2}]. \end{aligned} \tag{2.8c}$$

Then, $P(n,\psi,\phi) = det(I + V^{(n)})$ satisfies the relation

$$\frac{P(n+1,\psi,\phi)}{P(n,\psi,\phi)} = det\Psi_{22}(0). \tag{2.9}$$

PROOF: The proof is analogous to the proof of the pure matrix equivalent (see [DIZ]); it is essentially given in [FIK].

REMARK: We note that P and $I-P$ are orthogonal projection operators. In fact, $P^* = P$ and $P^2(z) = P(z)$; also $P(z)Q(z) = Q(z)$, and $Q(z)Q(1/z) = P(z)$. By definition, $P^T(z|s,t) = P(z|t,s)$. Formulae (2.8c) define Q at least for $Rez < 1$. P is defined for at least the unit circle minus the point $z = 1$. An essential singularity exists at $z = 1$.

From now on, we focus our attention to the (physically interesting case) $\gamma = 1$.

We use the orthogonal decomposition into $Im(I-P)$ and ImP. Our Riemann-Hilbert problem defined by (2.8) splits into two distinct ones, of different nature. For convenience, we consider the jump contour as the whole unit circle, with jump

$$\begin{gathered} M^* = M,\ z \in C, \\ M^* = I,\ z \in (|z| = 1) \setminus \bar{C}. \end{gathered}$$

We then have

$$\begin{aligned} (\Psi(I-P))_+ &= \Psi_-(M^*(I-P)), \\ (\Psi P)_+ &= \Psi_- M^* P, \end{aligned}$$

across the unit circle, with identity asymptotics at infinity. In other words, we have two new problems. Let $\Phi = \Psi(I-P)$ inside the unit circle and $\Phi = \Psi$ outside the unit circle. Then

$$\begin{aligned} \Phi_+ &= \Phi_- M^*(I-P),\ z \in C, \\ lim_{z\to\infty}\Phi &= I. \end{aligned} \tag{2.10}$$

Furthermore, Φ_+ has an essential singularity at $z=1$.

Likewise, let $\chi = \Psi P$ inside the unit circle and $\chi = \Psi$ outside the unit circle. Then

$$\begin{aligned} \chi_+ &= \chi_- M^* P,\ z \in C, \\ lim_{z\to\infty}\chi &= I. \end{aligned} \tag{2.11}$$

Again, χ_+ is essentially singular at $z=1$.

Note that

$$\Psi(0) = \chi(0) + \Phi(0). \tag{2.12}$$

These new problems are degenerate since none of $P,\ I-P$ is invertible. They do become nondegenerate once we restrict $\Phi(z), \chi(z)\ |z|<1$ and $M^*(z),\ |z|=1$ to the two orthogonal subspaces $I-P$ and P respectively. Note that for each of the new problems the jump across $\{|z|=1\}\setminus\bar{C}$ is again the identity. Even more, the singularity at $z=1$ is removed.

On $Im(I-P)$ we have $(I-P)|_{Im(I-P)} = I$. Thus,[3]

$$\begin{aligned} \Phi|_{Im(I-P),+} &= \Phi|_{ImM(I-P),-} M|_{Im(I-P)},\ where \\ M|_{Im(I-P)}(z) &= \begin{pmatrix} I & -Q(z)z^{-n}e^{\phi(z)} \\ Q(1/z)z^n e^{-\phi(z)} & I \end{pmatrix},\ z \in C. \end{aligned} \tag{2.13}$$

This is essentially the problem appearing in the case $0<\gamma<1$, treated in [FIK]. We have the obvious factorization (note $Q(z)Q(1/z)_{|Im(I-P)} = 0$)

$$\begin{aligned} M|_{Im(I-P)}(z) &= M_U(z)M_L^{-1}(z),\ where \\ M_U(z) &= \begin{pmatrix} I & -Q(z)z^{-n}e^{\phi(z)} \\ 0 & I \end{pmatrix}, \\ M_L(z) &= \begin{pmatrix} I & 0 \\ -Q(1/z)z^n e^{-\phi(z)} & I \end{pmatrix}. \end{aligned}$$

By using the analyticity of $Q(z)$ for $|z|<1$ and $Q(1/z)$ for $|z|>1$, we deform the problem as follows.

Let Σ be the augmented contour consisting of the union of:

[3]Note here that $Im\ M(I-P)$ is independent of z by continuity.

1. The contour C.
2. A smooth curve C_{int} joining the endpoints of the contour C, lying entirely within the open disc $|z| < 1$ and close to $\{|z| = 1\} \setminus C$.
3. A smooth curve C_{ext} joining the endpoints of the contour C, lying entirely in $|z| < 1$.

All contours are meant to have a counterclockwise orientation. The complex plane is now divided into three regions.

1. The region containing 0, say R_1.
2. The unbounded region, say R_2.
3. The region between C_{int} and C_{ext}, say R_3.

Let

$$\begin{aligned} \tilde{\Phi} &= \Phi|_{Im(I-P)} M_L, \ z \in R_1, \\ &= \Phi|_{ImM(I-P)} M_U, \ z \in R_2, \\ = \Phi|_{Im(I-P)} &= \Phi|_{ImM(I-P)}, \ z \in R_3. \end{aligned}$$

Then, the jumps for $\tilde{\Phi}$ are equal to M_U on C_{ext} and M_L^{-1} on C_{int}. On C, there is no jump since $\Psi_+ M_{L+} = \Psi_- M_{U-}$. As $n \to \infty$, we trivially get the identity solution. So,

$$\chi_{|Im(I-P)}(0) \sim I, \ as \ n \to \infty. \tag{2.14}$$

Note that the essential singularity at $z = 1$ never plays a role since $z = 1$ is not on C.

Also, clearly $\chi_{|ImP}(0) = 0, \Phi_{|Im(I-P)}(0) = 0$. So, it remains to calculate $\Phi_{|ImP}(0)$; in fact,

$$det \Psi_{22}(0) \sim det \Phi|_{ImP;22}(0), \ as \ n \to \infty. \tag{2.15}$$

Hence, we can simply focus our attention on ImP.

On ImP, $Q(z) = \frac{1-z}{2} I$. In particular, no essential singularity exists and $Q(z)$ and $Q(1/z)$ can be defined meromorphically in the whole complex plane using $Q(z)Q(1/z) = I$. Our operator Riemann-Hilbert problem has jump matrix

$$M|_{ImP}(z) = I \begin{pmatrix} 2 & -Q(z) \ z^{-n} e^{\phi(z)} \\ Q(1/z) \ z^n e^{-\phi(z)} & 0 \end{pmatrix}, \ z \in C. \tag{2.16}$$

The conjugation used in the previous case is useless as the resulting scalar problem has jump with determinant 0. Recognizing that we have here an operator version of a 'shock'-type Riemann-Hilbert problem, we have to use a conjugation with a function defined appropriately as a radical (cf. [DVZ], [K2], [DIZ]).

Let $\alpha = -sin^2 \psi/2 < 0$. As in [DIZ], p.218, we define the function g such that

$$\begin{aligned} g(z) \ is \ analytic \ in \ \mathbb{C} \setminus \bar{C}; \\ g(z) \to 1, \ as \ z \ \to \infty; \\ g_+(z) g_-(z) = \frac{\alpha}{z}, \ z \ \in \ C; \\ |\frac{g_+}{g_-}| < 1, \ z \ \in \ C. \end{aligned}$$

It actually follows that

$$g(z) = \frac{((z-e^{-i\psi})(z-e^{i\psi}))^{1/2} + z - 1}{2z}, \; so \; g(0) = sin^2\frac{\psi}{2}. \tag{2.17}$$

We now define a new operator valued 2x2 matrix by

$$\begin{aligned} F_n(z) &= \alpha^{n\sigma_3/2}\Phi_{|ImP}(z)g(z)^{n\sigma_3}\alpha^{-n\sigma_3/2}, \; |z|<1, \\ F_n(z) &= \alpha^{n\sigma_3/2}\Phi|_{ImMP}(z)g(z)^{n\sigma_3}\alpha^{-n\sigma_3/2}, \; |z|>1, \end{aligned} \tag{2.18}$$

where $\sigma_3 = diag(1,-1)$ is a Pauli matrix. Then, $F_n(\infty) = I$ and the jump matrix for F_n is

$$M_{g,n}^{ImP}(z) = \begin{pmatrix} 2(\frac{g_+}{g_-})^n I & -Q(z)e^{\phi(z)} \\ Q(1/z)e^{-\phi(z)} & 0 \end{pmatrix}, \; z \in C. \tag{2.19}$$

As $n \to \infty$,

$$M_{g,n}^{ImP}(z) \to \tilde{M} = I \begin{pmatrix} 0 & -\frac{1-z}{2}e^{\phi(z)} \\ \frac{2}{1-z}e^{-\phi(z)} & 0 \end{pmatrix}, \; z \in C. \tag{2.20}$$

It is thus appropriate to consider the Riemann-Hilbert

$$\begin{aligned} \tilde{F}_+ &= \tilde{F}_-\tilde{M}, \; z \in C, \\ \tilde{F}(\infty) &= I. \end{aligned} \tag{2.21}$$

hoping that $\tilde{F} = F^\infty = lim_{n\to\infty}F_n$. It turns out that the solution of (2.21) has a pole at $z = 1$. This problem can be conjugated to one with jump equal to $I\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the solution can be derived easily. We diagonalize $\tilde{M}$ as follows.

$$\begin{aligned} \tilde{M} &= SDS^{-1}, \\ S &= \begin{pmatrix} 1 & 1 \\ \frac{2i}{z-1}e^{-\phi(z)} & \frac{-2i}{z-1}e^{-\phi(z)} \end{pmatrix}, \\ D &= diag(i,-i). \end{aligned} \tag{2.22}$$

Now D can be factorized as

$$\begin{aligned} D &= B_-B_+^{-1}, \\ B &= diag(\beta^{-1}, \beta), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \beta(z) &= (\frac{z-e^{-i\psi}}{z-e^{i\psi}})^{1/4}, \\ \beta(\infty) &= 1. \end{aligned} \tag{2.24}$$

Hence $(\tilde{F}SB)_+ = (\tilde{F}SB)_-$ across the jump contour; but $\tilde{F}SB \sim S$ near $z=\infty$, hence in fact $\tilde{F}SB = S$ and

$$\tilde{F} = SB^{-1}S^{-1} = \frac{I}{2}\begin{pmatrix} \beta+\beta^{-1} & \frac{i(z-1)}{4}e^{\phi}(\beta-\beta^{-1}) \\ \frac{i}{z-1}(\beta-\beta^{-1}) & \beta+\beta^{-1} \end{pmatrix}. \tag{2.25}$$

Even though $\tilde{F}$ has a pole at $z=1$, the (22)-entry which concerns us does not. It is thus possible to show that $F_{n,22} \to \tilde{F}_{22}$, as $n \to \infty$. The following formulae for the Fredholm determinants below follow immediately.

$$\begin{aligned} det\tilde{F}_{22}(0) &= sin\frac{\psi}{2}, \\ det\Psi_{22}(0) &= -sin^{2n+1}\frac{\psi}{2}. \end{aligned} \tag{2.26}$$

Using (2.9), (2.17) and (2.18) we get

$$\frac{P_{n+1}}{P_n} = sin^{2n+1}\frac{\psi}{2}\ (1+0(\frac{logn}{n})).$$

Theorem 1.2 now follows.

PROOF OF THEOREM 1.2. A rigorous proof that $F_{22}^{\infty} = \tilde{F}_{22}$ (as well as formula (2.14)) requires a Beals-Coifman type formula which reduces the operator Riemann-Hilbert problem to a singular integral equation. This can be easily done and the details of the proof are essentially contained in [DIZ], p.159. The Cauchy operator involved takes bounded operators in L^2 to bounded operators in L^2 and is itself bounded.

A small complication arises because the convergence of $|\frac{g_+}{g_-}|^n$ is not uniform near the endpoints of C. This is dealt with by constructing a parametrix near those points, as suggested in [DIZ] for the analogous matrix problem. g being scalar, there is no complication due to the operator nature of the underlying problem.

Finally, one may worry about the fact that $\tilde{F}$ has a pole at $z=1$ and the possible effect on the validity of the limiting procedure. This, however has been shown not to be the case, since any meromorphic problem can be made into a holomorphic problem with an extra jump on a circle around the pole (see e.g.[DKKZ]). The important fact is that the (22)-entry of $\tilde{F}$ (which is all we are interested in) has no singularity.

The limit F_{22}^{∞} is meant to be taken in the trace class norm. The result for determinants follows readily and the proof of Theorem 1.2 is complete.

REMARKS:

1. It may initially seem that an essential singularity at $z=1$ is introduced in the Riemann-Hilbert problem when one multiples the unknown with the operators P and $I-P$ (cf. remark after the proof of theorem 2.1). However, this singularity vanishes once we restrict our operators to smaller subspaces.

2. In Theorem 1.2, we have been assuming that $\phi(z)$ has an analytic extension on the complex plane. This is not necessarily true in the original physical problem (see Fact 1.1); however, we can always approximate ϕ by an analytic $\tilde{\phi}$ that enables our calculations to go through, and recover the same result (see [DZ2] for details).

3. In conclusion, we would like to point out that the importance of the above procedure lies in the fact that operator Riemann-Hilbert problems appear in several contexts, apart from the computation of correlation functions for exactly solvable models of statistical mechanics. For example, it is well known that the inverse problem for integrable equations in 2+1 dimensions (Davey-Stewartson, KP, etc.) can often be expressed as an operator Riemann-Hilbert problem, and hence deformed along the lines described here.

4. We also note that, even though operators do not generally commute, and hence even 'scalar' operator Riemann-Hilbert problems are not trivial to solve explicitly, scalar conjugating functions (like g of (2.10)) can still play the important 'deforming' role they play in the standard matrix factorization problems.

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