s-CONVEXITY, MODEL SETS AND THEIR RELATION

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ABSTRACT. The relation of s-convexity and sets modeling physical quasicrystals is explained for quasicrystals related to quadratic unitary Pisot numbers. We show that 1-dimensional model sets may be characterized by s-convexity for finite set of parameters s. It is shown that the three Pisot numbers $\frac{1}{2}(1+\sqrt{5})$, $1+\sqrt{2}$, and $2+\sqrt{3}$ related to experimentally observed non-crystallographic symmetries are exceptional with respect to s-convexity.

1. INTRODUCTION

In this paper we study properties of a class of binary operations defined on a real vector space. As we show in this article such operations can be used for characterizations of point sets modeling physical quasicrystals. Such operation was first introduced in a purely mathematical article by I. Calvert in 1978. However, from his paper [3] and absence of references in it, one cannot deduce the motivation for the study of this operations. The problem was then elaborated by R. G. E. Pinch in 1985 in [7]. The connection of this operation with quasicrystals was first recognized by Berman and Moody in [1], where they work with a special case of such operation.

For arbitrary real parameter s, Calvert defined a binary operation

$$x \vdash_s y := sx + (1-s)y\,, \qquad x, y \in \mathbb{R}^n\,. \tag{1}$$

Pinch in [7] calls a set closed under such operation an **s-convex** set. Pinch shows that an s-convex set $\Lambda \subset \mathbb{R}$ is either dense in an interval of $\mathbb{R}$, or it is uniformly discrete. A set $\Lambda \subset \mathbb{R}$ is called **uniformly discrete**, if there exists an $\varepsilon > 0$, such that $|x-y| > \varepsilon$ for any $x, y \in \Lambda$, $x \neq y$. Pinch studies the question for which parameters s there exists a uniformly discrete s-convex set containing at least two points. We shall denote by $Cl_s A$ the closure of the set A under the operation (1). A most suitable candidate for a uniformly discrete s-convex set (s fixed) is $Cl_s\{0,1\}$. If s belongs to $(0,1)$, then $Cl_s\{0,1\}$ is a closure under a ordinary convex combination and therefore is dense in $[0,1]$. Pinch shows that for $s \notin (0,1)$, the closure $Cl_s\{0,1\}$ may be uniformly discrete only if s is an algebraic integer. On the other hand he proves that if s is a totally real algebraic integer, such that all its algebraic conjugates are in $(0,1)$, then $Cl_s\{0,1\}$ is uniformly discrete.

The connection to sets modeling quasicrystals was observed by Berman and Moody on the example of model sets based on the golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$, defined by

$$\Sigma_\tau(\Omega) := \{a + b\tau \mid a, b \in \mathbb{Z}\,, a + b\tau' \in \Omega\}\,, \tag{2}$$

where $\tau' = \frac{1}{2}(1-\sqrt{5})$ is the algebraic conjugate of τ and Ω is a bounded interval in $\mathbb{R}$. For such Ω, the set $\Sigma_\tau(\Omega)$ is not only uniformly discrete, but also **relatively dense**, which means that the distances of adjacent points in $\Sigma_\tau(\Omega)$ are bounded by a fixed constant. A set which is both uniformly discrete and relative dense is called a **Delone** set. It is easy to show that $\Sigma_\tau(\Omega)$ is closed under the operation (1) for $s = -\tau$. The operation $\vdash_{-\tau}$ has an important property. It has been shown in [1] that the quasicrystal $\Sigma_\tau[0,1]$ may be generated by this operation starting from initial seeds $\{0,1\}$, more precisely $Cl_{-\tau}\{0,1\} = \Sigma_\tau[0,1]$. This means that the operation is 'strong enough' so that any uniformly discrete set closed under it has to be a model set of the form (2).

It can be shown that all model sets (2) are s-convex for infinitely many $s \in \mathbb{Z}[\tau]$. A question arises whether there exists another s 'strong enough' to ensure $Cl_s\{0,1\} = \Sigma_\tau[0,1]$. In this article we provide an answer to a more general question.

The golden ratio τ is a quadratic unitary Pisot number related to 5-fold symmetry quasicrystals. Other experimentally observed non-crystallographic symmetries (8-fold and 12-fold) are related also to quadratic unitary Pisot numbers $1+\sqrt{2}$ and $2+\sqrt{3}$. We consider here two infinite families of quadratic unitary Pisot numbers $\beta > 1$, solutions of the equations

$$x^2 = mx + 1\,, \qquad m \in \mathbb{Z}\,, m \geq 1\,, \text{ or}$$
$$x^2 = mx - 1\,, \qquad m \in \mathbb{Z}\,, m \geq 3\,.$$

Let us denote by β' the algebraic conjugate of β, i.e. the second root of the equation. The number β is called unitary Pisot, since $|\beta'| < 1$, and $\beta\beta' = \mp 1$. A model set is then defined as a subset of the ring $\mathbb{Z}[\beta] := \mathbb{Z} + \mathbb{Z}\beta$, equipped with the Galois automorphism $x = a + b\beta \;\mapsto\; x' = a + b\beta'$. We let

$$\Sigma_\beta(\Omega) := \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\}\,, \tag{3}$$

where Ω is a bounded interval in $\mathbb{R}$. The set $\Sigma_\beta(\Omega)$ is called a model set in $\mathbb{Z}[\beta]$ with the acceptance window Ω.

Remark: Let us mention that the solution of equation $x^2 = 3x - 1$ gives the same ring $\mathbb{Z}[\beta]$ as the equation $x^2 = x + 1$, and thus we shall consider the equation $x^2 = mx - 1$ with $m > 3$ only.

Question 1. *Let β be quadratic unitary Pisot number. Which parameter s ensures that any uniformly discrete s-convex set is a model set in $\mathbb{Z}[\beta]$?*

We show in this paper that $s = -\frac{1}{2}(1+\sqrt{5})$ is exceptional for model sets in $\mathbb{Z}[\tau]$. The number $\tau^2 = 1 - s$ has the same property, since generally s-convexity is equivallent to $(1-s)$-convexity. For no other parameters s the operation (1) is 'strong enough'. The only other quadratic unitary Pisot numbers, for which a suitable parameter exists are $\beta = 1+\sqrt{2}$ and $2+\sqrt{3}$. It means that in the majority of cases the invariance with respect to one operation (1) does not imply being a model set. One may further ask the following question.

Question 2. *Let β be quadratic unitary Pisot number. Is it possible to find a finite set $\mathcal{N}$ of parameters, such that s-convexity of a uniformly discrete set $\Lambda \subset \mathbb{Z}[\beta]$ for all $s \in \mathcal{N}$ implies Λ being a model set in $\mathbb{Z}[\beta]$?*

The answer is YES, and we provide a constructive proof for each β.

The solutions to Questions 1 and 2 are formulated as Theorems 4.1 and 4.2 in Section 4. Their proofs require different approaches and different mathematical tools, therefore they are found in separate Sections 2 and 3.

2. Auxiliary lemmas I

Recall that we are working in a ring $\mathbb{Z}[\beta] \subset \mathbb{Q}[\beta]$, and that the mentioned conjugation $\;'\;$ is a Galois automorphism on the field $\mathbb{Q}[\beta]$. One has $(x+y)' = x' + y'$ and $(xy)' = x'y'$ for any two elements $x, y \in \mathbb{Z}[\beta]$. With this in mind, we may state that for $s \in \mathbb{Z}[\beta]$ and $\Lambda \subset \mathbb{Z}[\beta]$,

$$\Lambda \text{ is } s\text{-convex} \qquad \text{if and only if} \qquad \Lambda' = \{x' \mid x \in \Lambda\} \text{ is } s'\text{-convex.} \tag{4}$$

From the definition of model sets it follows that $\big(\Sigma_\beta(\Omega)\big)' = \mathbb{Z}[\beta] \cap \Omega$. Hence from (4) we obtain

$$\Sigma_\beta(\Omega) \text{ is } s\text{-convex} \qquad \text{if and only if} \qquad \mathbb{Z}[\beta] \cap \Omega \text{ is } s'\text{-convex.}$$

Since Ω is an interval, $s'x + (1-s')y$ has to be a convex combination in the ordinary sense, i.e. $s' \in [0,1]$, which is equivallent to $s \in \Sigma_\beta[0,1]$.

In this formalism, Question 1 may be rewritten as follows: Does there exist $s \in \Sigma_\beta[0,1]$, such that for any uniformly discrete s-convex set $\Lambda \subset \mathbb{Z}[\beta]$ it holds that

$$\Lambda' = \mathbb{Z}[\beta] \cap \Omega \tag{5}$$

for some Ω? If (5) should be valid for every uniformly discrete Λ, it must be true also for $\Lambda = Cl_s\{0,1\}$. Since $\Lambda' = Cl_{s'}\{0,1\}$ contains $0' = 0$ and $1' = 1$ and since $s'x + (1-s')y$ is a convex combination, the only suitable candidate for Ω in this case is the interval $\Omega = [0,1]$. It is the reason why in the sequel we focus our attention to investigation of equatlity

$$[0,1] \cap \mathbb{Z}[\beta] = Cl_{s'}\{0,1\}\,, \qquad \text{for } s' \in (0,1)\,.$$

Lemma 2.1. *Let β be quadratic unitary Pisot number. Assume that $Cl_{s'}\{0,1\} = \mathbb{Z}[\beta] \cap [0,1]$. Then for any $y \in \mathbb{Z}[\beta] \cap [0,1]$, the scaling factor s' divides either y or $y-1$ in the ring $\mathbb{Z}[\beta]$.*

Proof. Pinch in [7] has proved that any $y \in Cl_{s'}\{0,1\}$ may be written in the form

$$y = \sum_{i=0}^{n} b_i (s')^i (1-s')^{n-i}\,, \qquad b_i \in \mathbb{Z}\,, 0 \le b_i \le \binom{n}{i}\,, \tag{6}$$

for some non negative integer n. Therefore $b_0 \in \{0,1\}$. If b_0 is equal to 0, then y is divisible by s', otherwise s' divides $y - b_0 = y - 1$. □

Before the following corollary let us mention several number theoretical facts. On the field $\mathbb{Q}[\beta]$ on may define a 'norm' $N(x) := xx' \in \mathbb{Q}$ and a 'trace' $\mathrm{tr}(x) := x + x' \in \mathbb{Q}$. Since $|N(\beta)| = 1$ and $\mathrm{tr}(\beta) = m$ are both integers, it holds for every $x \in \mathbb{Z}[\beta]$ that $N(x) \in \mathbb{Z}$ and $\mathrm{tr}(x) \in \mathbb{Z}$. A divisor of unity in $\mathbb{Z}[\beta]$ is an element u such that $\frac{1}{u} \in \mathbb{Z}[\beta]$; u is a divisor of unity iff $|N(u)| = 1$.

Corollary 2.2. *If $Cl_{s'}\{0,1\} = \mathbb{Z}[\beta] \cap [0,1]$, then both s' and $1-s'$ divide 2 in the ring $\mathbb{Z}[\beta]$.*

Proof. Note that the roles of s' and $1-s'$ are symmetric. Thus it suffices to show that s' divides 2. If s' is a divisor of unity then the assertion is true. Assume the opposite, i.e. s' is not a divisor of unity and s' does not divide 2. Choose $y = 1/\beta^2$. Since y is a divisor of unity, the scaling factor s' does not divide neither y, nor $2y$. According to Lemma 2.1 this implies $s'|(y-1)$ and $s'|(2y-1)$. Therefore s' divides $(2y-1) - (y-1) = y$, thus a contradiction. □

Proposition 2.3. *Let β be a quadratic unitary Pisot number. Assume that $Cl_{s'}\{0,1\} = \mathbb{Z}[\beta] \cap [0,1]$ is satisfied for some s'. Then either of the possibilities below is true:*

- *$\beta = \tau$ (root of $x^2 = x + 1$) and $s' = -\tau'$ or $s' = 1 + \tau'$.*
- *$\beta = 1 + \sqrt{2}$ (root of $x^2 = 2x + 1$) and $s' = -\beta'$ or $s' = 1 + \beta'$.*
- *$\beta = 2 + \sqrt{3}$ (root of $x^2 = 4x - 1$) and $s' = \beta'$ or $s' = 1 - \beta'$.*

Proof. From the relation (6) it is clear that all elements of $Cl_{s'}\{0,1\}$ are polynomials in s'. Since $s' \in \mathbb{Z}[\beta]$, it satisfies a quadratic equation with integer coefficients, namely $x^2 = (s+s')x - ss'$, i.e. $x^2 = \mathrm{tr(s)}x - N(s)$. Using this quadratic equation any polynomial y from (6) may be reduced to the form $y = a + bs'$. Therefore clearly $Cl_{s'}\{0,1\} \subset \mathbb{Z}[s']$. The condition $Cl_{s'}\{0,1\} = \mathbb{Z}[\beta] \cap [0,1]$ implies that we need $\mathbb{Z}[\beta] = \mathbb{Z}[s']$. This can be satisfied only if $s' = \pm\beta + k$ for some integer k. The restriction $0 < s' < 1$ gives only two admissible values for the parameter s'.

(a) Let us consider $\beta^2 = m\beta + 1$, i.e. $[\beta] = m$. Here $s' = \beta - m = 1/\beta$ or $s' = m + 1 - \beta = 1 - 1/\beta$. Corollary 2.2 states that both s' and $1 - s'$ divide 2 in $\mathbb{Z}[\beta]$. Since $N(1/\beta) = -1$, $s' = 1/\beta$ is a divisor of unity and hence divides 2 automatically. Let us assume that $1 - 1/\beta$ divides 2. It means that there exists an element $c + d\beta \in \mathbb{Z}[\beta]$, such that

$$2 = (c + d\beta)\left(1 - \frac{1}{\beta}\right) = c - d + mc + \beta(d - c)\,.$$

It follows that $c = d$ and $mc = 2$. Since m is a positive integer, this may happen only for $m = 1$ and $m = 2$. These are the two cases given in the statement.

(b) Let us now consider $\beta^2 = m\beta - 1$, i.e. $[\beta] = m - 1$. Here $s' = m - \beta = 1/\beta$ or $s' = \beta - m + 1 = 1 - 1/\beta$. Again from Corollary 2.2, $1 - 1/\beta$ divides 2. It means that there exists an element $c + d\beta \in \mathbb{Z}[\beta]$, such that

$$2 = (c + d\beta)\left(1 - \frac{1}{\beta}\right) = c - d - mc + \beta(d + c)\,.$$

This implies $c = -d$ and $(2 - m)c = 2$. For the equation $\beta^2 = m\beta - 1$ we consider $m \ge 4$, so that the only solution is $m = 4$. □

3. Auxiliary lemmas II

In order to prove the results concerning the second question, we shall need the notion of β-expansions.

The ring $\mathbb{Z}[\beta] := \mathbb{Z} + \mathbb{Z}\beta$, can be characterized using β-expansions [8]. A β-expansion of a real number $x \geq 0$ is defined for any real $\beta > 1$ as an infinite sequence $(x_i)_{k \geq i > -\infty}$ given by the 'greedy' algorithm in the following way:

$$x_k := \left[\frac{x}{\beta^k}\right],$$

where k satisfies $\beta^k \leq x < \beta^{k+1}$. Denote $r_k = x/\beta^k - x_k$. Numbers x_{i-1} and r_{i-1} are computed from x_i and r_i by prescription:

$$x_{i-1} := [\beta r_i], \qquad r_{i-1} := \beta r_i - x_{i-1}.$$

Clearly $x_i \in \{0, 1, \ldots, [\beta]\}$ for each i. In [8] it is proven that

$$x = \sum_{i=-\infty}^{k} x_i \beta^i,$$

i.e. the sum converges for each positive real x and for each $\beta > 1$.

Parry in [6] answered the question for which sequences $(x_i)_{k \geq i > -\infty}$ there exists a positive real x, such that $(x_i)_{k \geq i > -\infty}$ is its β-expansion. Let the Rényi β-representation of 1 be

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \cdots, \qquad a_i \in \{0, 1, \ldots, [\beta]\},$$

and let

$$x = \sum_{i=-\infty}^{k} x_i \beta^i, \qquad x_i \in \{0, 1, \ldots, [\beta]\}.$$

Then the sequence $(x_i)_{k \geq i > -\infty}$ is a β-expansion of x if and only if for any integer $j \leq k$, the sequence $x_j x_{j-1} x_{j-2} \ldots$ is lexicographically strictly smaller than sequence $a_1 a_2 a_3 \ldots$. Let us apply the rule on β-expansions for quadratic unitary Pisot numbers β.

At first let us discuss the case $\beta^2 = m\beta + 1$, i.e. $[\beta] = m$. The Rényi representation of 1 is $1 = m/\beta + 1/\beta^2$, so that $a_1 = m$, $a_2 = 1$, $a_3 = 0$, $\ldots$. It means that $x = \sum x_i \beta^i$ is a β-expansion iff any $x_i = m$ occuring in the sequence $(x_i)_{k \geq i > -\infty}$ is followed by $x_{i-1} = 0$.

Let now $\beta^2 = m\beta - 1$, so that $[\beta] = m - 1$. The Rényi representation of 1 in this case is

$$1 = (m-1)/\beta + (m-2) \sum_{k=2}^{\infty} \frac{1}{\beta^k}.$$

Therefore $(x_i)_{k \geq i > -\infty}$ is a β-expansion of some $x > 0$ iff $x_i x_{i-1} x_{i-2} \ldots$ is strictly lexicographically smaller than $(m-1)(m-2)(m-2) \ldots$ for any $i \leq k$.

Let $x \in \mathbb{R}$. If x is negative we put $x_i = -|x|_i$, negatives of β-expansion coefficients for $|x|$. If the β-expansion of an $x \in \mathbb{R}$ ends in infinitely many zeros, it is said to be finite, and the zeros at the end are omitted. Denote the set

$$\mathrm{Fin}(\beta) := \{\varepsilon x \mid x \in \mathbb{R}_0^+, \varepsilon = \pm 1, x \text{ has a finite } \beta\text{-expansion}\}.$$

In the sequel, it will be useful to use the relation between the set $\mathrm{Fin}(\beta)$ of all numbers with finite β-expansion and the ring $\mathbb{Z}[\beta]$. In [4] it is proved that for β, which is a root of the equation $x^2 = mx + 1$, the set $\mathrm{Fin}(\beta)$ is a ring. It follows immediately that

$$\mathbb{Z}[\beta] = \mathrm{Fin}(\beta). \tag{7}$$

For β being a solution of $x^2 = mx - 1$, the situation is different. In this case, $\mathrm{Fin}(\beta)$ is not closed under addition, (for example $\beta - 1 \notin \mathrm{Fin}(\beta)$), while $\mathbb{Z}[\beta]$ is. Burdík et al. in [2] prove that for such β, the set

$\mathrm{Fin}(\beta)$ can be characterized as the set of those elements x of $\mathbb{Z}[\beta]$, for which $N(x) = xx'$ is non negative. Therefore

$$\mathbb{Z}[\beta] \supsetneqq \mathrm{Fin}(\beta) = \{x \in \mathbb{Z}[\beta] \mid xx' \geq 0\} \,. \tag{8}$$

The crucial proposition for the answer to Question 2 is the following one.

Proposition 3.1. *Let β be a quadratic unitary Pisot number, satisfying the equation $x^2 = mx \pm 1$. Let Λ be a uniformly discrete subset of $\mathbb{Z}[\beta]$, containing $0, 1$. Put*

$$\mathcal{M} = \left\{ \frac{i}{\beta} \;\middle|\; i = 1, 2, \dots, \left[\frac{m \pm 1}{2}\right] \right\} .$$

If Λ' is s-convex for every $s \in \mathcal{M}$, then there exists a bounded interval Ω, such that $\Lambda' = \Omega \cap \mathbb{Z}[\beta]$.

The proof of the above proposition is based on the following lemma. It is useful to introduce the notion of a closure under a set of operations $\vdash_s$. We denote by $Cl_{\mathcal{M}}A$ the smallest set containing A, which is s-convex for all s in a finite set $\mathcal{M}$.

Lemma 3.2. *Let β be a quadratic unitary Pisot number and $\mathcal{M}$ defined in Proposition 3.1. Then*

$$Cl_{\mathcal{M}}\{0,1\} = [0,1] \cap \mathbb{Z}[\beta]$$

Proof. Consider the operations $\vdash_s$ with $s = i/\beta$. For simplicity, denote

$$x \Vdash_i y := \frac{i}{\beta}x + \left(1 - \frac{i}{\beta}\right) y \,, \qquad i = 1, 2, \dots$$

The statement of the lemma is equivalent to the fact that any point $\mathbb{Z}[\beta]$ in the interval $[0,1]$ can be generated from 0, and 1 using operations $\Vdash_i$, $i = 1, 2, \dots, [(m \pm 1)/2]$. We first show that $[0,1] \cap \mathbb{Z}[\beta]$ can be generated using all operations $\Vdash_1, \dots, \Vdash_r$, with $r = [\beta]$. In the second step we find expression for $\Vdash_i$, $i = [(m \pm 1)/2] + 1, \dots, [\beta]$, as a composition of operations $\Vdash_i$, $i = 1, \dots, [(m \pm 1)/2]$.

(a) Consider β to be the solution of the equation $x^2 = mx + 1$, $m \geq 1$. According to (7), any $x \in \mathbb{Z}[\beta] \cap (0,1)$ has a finite β-expansion

$$x = \sum_{i=1}^{k} \frac{x_i}{\beta^i} \,, \qquad k \in \mathbb{Z} \,.$$

By induction on k we show that x belongs to $Cl_{\mathcal{M}}\{0,1\}$. Clearly, for the first step of induction we have

$$\frac{j}{\beta} = 1 \Vdash_j 0 \,, \tag{9}$$

for any $j = 1, \dots, m$. Suppose that the β-expansion of a point $x \in \mathbb{Z}[\beta] \cap (0,1)$ is of the form

$$x = \frac{j}{\beta} + \sum_{i=2}^{k} \frac{x_i}{\beta^i} \,, \qquad j = 0, \dots, m-1 \,.$$

Then x can be rewritten as the combination

$$x = \frac{j}{\beta} + \frac{z}{\beta} \,, \qquad z = \sum_{i=1}^{k-1} \frac{x_{i+1}}{\beta^i} \,.$$

By induction hypothesis, z could be generated from 0 and 1 using the given operations; i.e. $z \in Cl_{\mathcal{M}}\{0,1\}$. Since

$$(x \Vdash_i y) + c = (x + c) \Vdash_i (y + c) \,, \quad \text{and} \quad \frac{x}{\beta} \Vdash_i \frac{y}{\beta} = \frac{1}{\beta}(x \Vdash_i y) \,, \qquad \text{for any } i = 1, \dots, m \,, \tag{10}$$

one has the following relation,

$$Cl_{\mathcal{M}} \left\{ \frac{j}{\beta}, \frac{j+1}{\beta} \right\} = \frac{j}{\beta} + \frac{1}{\beta} Cl_{\mathcal{M}}\{0,1\} \,.$$

Therefore

$$x = \frac{j}{\beta} + \frac{z}{\beta} \in \frac{j}{\beta} + \frac{1}{\beta} Cl_{\mathcal{M}}\{0,1\} = Cl_{\mathcal{M}}\left\{\frac{j}{\beta}, \frac{j+1}{\beta}\right\} \subset Cl_{\mathcal{M}}\{0,1\}\,.$$

The later inclusion is valid due to the first step of induction (see (9)).

Assume that the coefficient x_1 of x in its β-expansion is equal to m. Necessarily, from the properties of β-expansions, x_2 is equal to 0. In this case we use

$$x = \frac{m}{\beta} + \frac{z}{\beta^2} \in \frac{m}{\beta} + \frac{1}{\beta^2} Cl_{\mathcal{M}}\{0,1\} = Cl_{\mathcal{M}}\left\{\frac{m}{\beta}, 1\right\} \subset Cl_{\mathcal{M}}\{0,1\}\,.$$

Thus we have shown that all points in $[0,1] \cap \mathbb{Z}[\beta]$ can be generated starting from 0, and 1, using the operations $\Vdash_i$, $i = 1, \ldots, m$. The following relation can be used to reduce the number of necessary operations from m to $[(m+1)/2]$. The relation is valid for any k, $k = 1, \ldots, m-1$, independently of x, y.

$$(x \Vdash_{k+1} y) \Vdash_1 (x \Vdash_k y) = \big(1 - \frac{m-k}{\beta}\big)x + \frac{m-k}{\beta} y = y \Vdash_{m-k} x\,.$$

(b) Let now β be the root of $x^2 = mx - 1$, $m \geq 4$. In this case, if we want to use the β-expansions of numbers, we have to encounter a complication: There exist points in the ring $\mathbb{Z}[\beta]$ which do not have a finite β-expansion.

Recall that an $x \in \mathbb{Z}[\beta]$ has a finite β-expansion if and only if $N(x) = xx' \geq 0$, (see (8)). For us, only $x \in (0,1)$ are of interest. The following statement is valid. Let $x \in \mathbb{Z}[\beta] \cap (0,1)$. Then either x or $(1-x)$ has a finite β-expansion. Indeed, since $N(x) = xx'$ is an integer and $x < 1$, necessarily $|x'| > 1$. Therefore $x' > 0$ if and only if $1 - x' < 0$.

Now we can proceed, similarly as for (a) by induction on the length of the β-expansion in order to show that any element of $\mathrm{Fin}(\beta) \cap [0,1]$ can be generated by corresponding operations $\Vdash_i$, $i = 1, \ldots, m-1$. For elements $x \in (0,1) \cap \big(\mathbb{Z}[\beta] \setminus \mathrm{Fin}(\beta)\big)$, we have $1 - x \in (0,1) \cap \mathrm{Fin}(\beta)$. Therefore $1 - x$ can be generated from 0 and 1 using the given operations. In the corresponding combination we replace all 0 by 1 and vice versa, which gives the desired combination for element x.

The coefficients in a β-expansion take one of the values $0, \ldots, m-1$. For an $x \in \mathrm{Fin}(\beta) \cap (0,1)$, whose $x_1 = j$, $j = 0, \ldots, m-2$, the procedure is the same as in the case (a). Consider an

$$x = \frac{m-1}{\beta} + \frac{x_2}{\beta^2} + \sum_{i=3}^{k} \frac{x_i}{\beta^i} = 1 - \frac{1}{\beta} + \frac{x_2+1}{\beta^2} + \sum_{i=3}^{k} \frac{x_i}{\beta^i} = \frac{1}{\beta} z + \left(1 - \frac{1}{\beta}\right) 1 = z \Vdash_1 1\,.$$

Since the segment $x_1 x_2 \ldots x_k$ of a β-expansion is strictly lexicographically smaller than the sequence $(m-1)(m-2)(m-2)\ldots$, the string $x_2 x_3 \ldots x_k$ is strictly smaller than $(m-2)(m-2)\ldots$. Therefore $(x_2+1)x_3 \ldots x_k$ is the β-expansion of z. By induction hypothesis $z \in Cl_{\mathcal{M}}\{0,1\}$, therefore $x = z \Vdash_1 1 \in Cl_{\mathcal{M}}\{0,1\}$.

By that we have shown that any $x \in [0,1] \cap \mathbb{Z}[\beta]$ can be generated by given operations $\Vdash_i$, $i = 1, \ldots, m-1$. The number of operations can be reduced from $m-1$ to $[(m-1)/2]$ by the following relation, which is valid for any k, $k = 1, \ldots, m-2$, for all x, y.

$$(x \Vdash_k y) \Vdash_1 (x \Vdash_{k+1} y) = \left(1 - \frac{m-k-1}{\beta}\right) x + \frac{m-k-1}{\beta} y = y \Vdash_{m-k-1} x\,.$$

□

Due to (10), the Lemma 3.2 can be generalized in the following way. If $c \in \mathbb{Z}[\beta]$, we have

$$Cl_{\mathcal{M}}\{c, c+1\} = [c, c+1] \cap \mathbb{Z}[\beta]\,. \tag{11}$$

Proof of Proposition 3.1. In order to prove the statement of the proposition, we have to find Ω, such that $\Lambda' = \Omega \cap \mathbb{Z}[\beta]$. We show that this property is satisfied by the convex hull of Λ', so we put $\Omega = \langle \Lambda' \rangle$. We have to justify that any element of $\mathbb{Z}[\beta] \cap \langle \Lambda' \rangle$ can be generated using the given operations starting from points of Λ'.

Consider $x \in \mathbb{Z}[\beta] \cap \langle \Lambda' \rangle$. Since $0, 1 \in \Lambda$, then also $0, 1 \in \Lambda'$, and hence $\Lambda' \supset ([0,1] \cap \mathbb{Z}[\beta])$, as a consequence of Lemma 3.2. Therefore, if $x \in [0,1]$, then x belongs to Λ'. Without loss of generality let $x > 1$. Observe that the points of Λ' cover densely the interval $\langle \Lambda' \rangle$. Therefore one can find a finite sequence of intervals $[y_1-1,y_1],\dots,[y_k-1,y_k]$, such that the elements y_i belong to Λ', and it holds that $0<y_1-1<1$, $y_{i+1}-1<y_i$ and $y_k-1<x\le y_k$. According to (11), one has

$$\Lambda' \supset Cl_{\mathcal{M}}\{y_i-1,y_i\}=[y_i-1,y_i]\cap\mathbb{Z}[\beta]\,.$$

Therefore x can be generated from points of Λ' by the operations $\Vdash_i$, hence it is contained in Λ'. Thus we have proven that $\Lambda'=\Omega\cap\mathbb{Z}[\beta]$.

It suffices now to justify that Ω is bounded. This is true, because otherwise $\Lambda=\Sigma_\beta(\Omega)$ would not be uniformly discrete. □

4. Main results, comments and open problems

Due to Proposition 2.3, for any quadratic Pisot number $\beta\neq\frac12(1+\sqrt5)$, $1+\sqrt2$, $2+\sqrt3$, s-convexity of a uniformly discrete set Λ with an arbitrary fixed parameter s does not mean that Λ is a model set in $\mathbb{Z}[\beta]$. The example of an s-convex set which is not a model set in $\mathbb{Z}[\beta]$ is the closure $Cl_s\{0,1\}$ for arbitrary $s\in\mathbb{Z}[\beta]$, $s'\in(0,1)$.

In the three exceptional cases $\beta=\frac12(1+\sqrt5)$, $(x^2=x+1)$, $\beta=1+\sqrt2$, $(x^2=2x+1)$, $\beta=2+\sqrt3$, $(x^2=4x-1)$, the set $\mathcal{M}$ of parameters in Proposition 3.1 has only one element $s'=1/\beta$, i.e. $s=1/\beta'$. The s-convexity with this specific parameter s implies being a model set in the corresponding $\mathbb{Z}[\beta]$. However, Proposition 2.3 states that $s=1/\beta'$ and $1-s$ are the only parameters in $\mathbb{Z}[\beta]$ with this property.

Theorem 4.1. *Let β be a quadratic unitary Pisot number and $s\in\mathbb{Z}[\beta]$. We say that a parameter s is model-set-forcing if any uniformly discrete s-convex set $\Lambda\subset\mathbb{Z}[\beta]$, containing 0 and 1, is a model set in $\mathbb{Z}[\beta]$.*

The parameter s is model-set-forcing if and only if s or $1-s$ takes one of the three values $-\frac12(1+\sqrt5)$, $-1-\sqrt2$, or $2+\sqrt3$.

In the majority of cases a single parameter $s\in\mathbb{Z}[\beta]$ does not suffice to characterize model sets in $\mathbb{Z}[\beta]$ by s-convexity. The following theorem, which is a direct consequence of Proposition 3.1, states that for such characterization one may use more, but a finite number of parameters s.

Theorem 4.2. *Let β be quadratic unitary Pisot number, satisfying the equation $\beta^2=m\beta\pm1$. Let Λ be a uniformly discrete subset of $\mathbb{Z}[\beta]$, containing 0,1. Set*

$$\mathcal{N}=\left\{s\;\middle|\; s'=\frac1\beta,\frac2\beta,\dots,\frac1\beta\left[\frac{m\pm1}{2}\right]\right\}.$$

Then Λ is a model set in $\mathbb{Z}[\beta]$ if and only if Λ is s-convex for any $s\in\mathcal{N}$.

The only cases, when the set $\mathcal{N}$ has one element, are the three exceptional quadratic unitary Pisot numbers singled out by Theorem 4.1. For other irrationalities β, the two theorems state that for a characterization of model sets in $\mathbb{Z}[\beta]$ one has to consider at most $[(m\pm1)/2]$ but at least two parameters. It would be interesting to determine in which cases one may reduce the number of necessary parameters from the suggested $[(m\pm1)/2]$.

Note that if instead of a model set $\Sigma_\beta(\Omega)$ we consider its multiple $\xi\Sigma_\beta(\Omega)$, for $\xi\in\mathbb{Z}[\beta]$, the resulting set is generally not a model set in $\mathbb{Z}[\beta]$ according to the the definition (3), but only its affine image. In order to avoid such trivial examples of a uniformly discrete s-convex set Λ, which is not a model set, we require for our statements $0,1\in\Lambda$. One may impose on Λ a weaker assumption using the notion of greatest common divisors in $\mathbb{Z}[\beta]$. However, the formulations and proofs would require complicated technicalities without really obtaining something new.

Non-trivial examples of uniformly discrete s-convex sets, which are not model sets in any $\mathbb{Z}[\beta]$, are the closures $Cl_s\{0,1\}$ for practically all parameters $s\in\mathbb{Z}[\beta]$. Such closures are proper subsets of model sets. However, it remains an open question, whether these subsets satisfy in addition to the uniform discreteness also the other part of Delone property, namely the relative density. If $Cl_s\{0,1\}$ is Delone, it is an interesting new aperiodic structure with abundant self-similarities. Investigation of even basic properties of such sets would be desirable.

The content of the article concerns the relation of s-convexity and 1-dimensional model sets. Probably a more interesting question is, whether the notion of s-convexity can be used for a characterization of model sets in any dimensions. Such question was solved in [5] for model sets based on the golden mean irrationality. We have shown there that any $(-\tau)$-convex uniformly discrete set $\Lambda \subset \mathbb{R}^n$ is an affine image of a model set. We expect similar fact to be true for other quadratic unitary Pisot numbers as well.

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