# COMMUTATIVE PARTIAL DIFFERENTIAL OPERATORS

ALEX KASMAN AND EMMA PREVIATO

Abstract. In one variable, there exists a satisfactory classification of commutative rings of differential operators. In several variables, even the simplest generalizations seem to be unknown and in this report we give examples and pose questions that may suggest a theory to be developed. In particular, we address the existence of a "spectral variety" generalizing the spectral curve of the one dimensional theory and the role of the differential resultant.

## 1. Introduction

In one variable, there exists a satisfactory classification of rings of differential operators that are (maximal) commutative. In several variables, even the simplest generalizations seem to be unknown and in this report we give examples and pose questions that may suggest a theory to be developed. To motivate what we do, we briefly recall the 1-variable case and selected results in several variables.

In the 1-variables case with analytic coefficients the classification was found by Burchnall and Chaundy in the 1920s [4, 5] by essentially formal methods of differential algebra and some algebraic function theory on the "spectral curve" $\operatorname{Spec}\mathcal{A}$, where $\mathcal{A}$ is the commutative ring, completed by a point "at infinity". The isospectral rings, roughlty speaking, form the Jacobi variety of this projective algebraic curve and the "Krichever map" [15] solves the inverse spectral problem explicitly. For a ring $\mathcal{A}$ generated by a pair $L = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0(x)$ (where $\partial = \partial/\partial x$) and $B = \partial^m + \cdots$ the determinant of the $(n+m)\times(n+m)$ resultant matrix of $L-\lambda$ and $B-\mu$ [21] is a non-zero, polynomial $p \in \mathbb{C}[\lambda, \mu]$ such that $p(L, B) = 0$. Moreover, in the "rank 1 case" $\gcd(n, m) = 1$ [21] one has that $p(\lambda, \mu) = 0$ *is* the affine equation of the spectral curve. A divisor of points on the curve corresponding to the given element of the Jacobi variety is the set of poles of the (normalized) gcd of $L - \lambda$ and $B - \mu$ at a reference point $x = x_0$; the flow in $x$ is linear on the Jacobian.

In view of this, it is natural to aks at least the following questions in several variables:

1. Is $\operatorname{Spec}\mathcal{A}$ an affine variety of dimension $N$ for any maximal commutative ring $\mathcal{A}$ of partial differential operators in $N$ variables?
2. For a ring $\mathcal{A}$ with generators $L_i$ $(1 \le i \le N+1)$, what is the relationship between the differential resultant of $L_i - \lambda_i$ and the equation of such a variety?

Research at MSRI is supported in part by NSF grant DMS-9701755. The second author was a member of the Symbolic Calculation program and wishes to express tremendous gratitude to organizer Michael Singer. The first author was a member of the Random Matrices program and wishes to thank all of the organizers of that program and also thank Michael Singer for his assistance and guidance.

More ambitiously, of course, one would ask for isospectral flows, a compete classification, and the inverse spectral problem, in increasing order of magnitude. A beautiful generalization of Burchnall-Chaundy theory was given by Nakayashiki [18, 19] using the Fourier-Mukai transform. He associates commutative rings in $N$ variables to a suitable $N$-dimensional abelian variety an some additional choices, one for each element of its Picard variety. But these are not scalar operators, rather they have $(N! \times N!)$ matrix coefficients.

In this report, we give a negative answer to question (1) by using techniques developed in [14] and we offer some observations, natural conjectures, and a strategy to treat (2). We deal with the scalar case only. We include as an appendix the *Mathematica* code that can be used to compute the differential resultant of any set of $N+1$ partial differential operators in $N$ variables, which is a handy tool for checking properties on the available class of examples such as [2, 7].

## 2. Geometric Structure of Maximal Commutative Rings

It is well known that the commutative rings of ordinary differential operators are finitely generated rings coordinate rings for algebraic curves. This forms the foundation of the Burchnall-Chaundy theory of such rings [4, 5, 21]. In contrast, very little is known about the algebro-geometric structure of commutative rings of partial differential operators. A question of interest is to address the problem of whether every commutative subring is contained in a (larger) commutative ring requiring only a finite number of generators over $\mathbb{C}$. Such a result is relevant, for instance, to the algebro-geometric investigations of quantum integrable systems [3, 12].

In this section we use techniques from soliton theory (namely Darboux transformation and Baker-Akhiezer functions) to study the structure of *certain* commutative rings of differential operators. We are able to show that these rings are *maximal* in the sense that they are not contained in any larger commutative subrings of the ring of differential operators. This in itself is a difficult task which is rarely achieved. Then, in one example, we study the structure of this ring more closely and note that it cannot be constructed with only a finite number of generators over $\mathbb{C}$ and hence is not the coordinate ring of an affine algebraic variety.

**Notation:** Let $\mathcal{D} = \mathbb{C}(x_1, \cdots, x_n)[\partial_1, \ldots, \partial_n]$ be the ring of rational coefficient differential operators in $n$ variables. It will be useful to be able to refer also to $\mathcal{D}_0 = \mathbb{C}[\partial_1, \ldots, \partial_n] \subset \mathcal{D}$ (the constant coefficient differential operators) as well as $\Psi\mathcal{D}$, a ring of microdifferential operators containing $\mathcal{D}$ as well as the inverse of the particular operator $K \in \mathcal{D}$ which will be important below [13] and $\Psi\mathcal{D}_0$, the contant coefficient microdifferential operators.

### 2.1. What Commutes with Many Constant Coefficient Operators?

In general, it is difficult to address the question of whether a commutative ring of partial differential operators is maximal. The key which allows us to do it here is the observation that although the centralizer of a single constant coefficient operator $p(\partial_1, \ldots, \partial_n)$ will contain non-constant coefficient operators, an operator commutes all constant coefficient multiples of $p(\partial_1,\dots,\partial_n)$ if and only if it also has constant coefficients.

**Lemma 2.1.** *Let $p \in \mathcal{D}_0$ be a non-zero constant coefficient differential operator. Then any operator $L \in \Psi\mathcal{D}$ which commutes with $p$ as well as all operators $P_i := \partial_i \circ p(\partial_1,\dots,\partial_n)$ ($1 \le i \le n$) is also constant coefficient (i.e. $L \in \Psi\mathcal{D}_0$).*

*Proof.* Let us suppose that $[p, L] = [P_i, L] = 0$. Then since $p \circ L = L \circ p$ and $p \circ \partial_i = \partial_i \circ p$ it follows that

$$\begin{aligned} 0 = [P_i, L] &= \partial_i \circ p \circ L - L \circ \partial_i \circ p \\ &= \partial_i \circ L \circ p - L \circ p \circ \partial_i \\ &= [L \circ p, \partial_i] \end{aligned}$$

Then we note that letting $L \circ p = \sum f_\alpha(x_1,\dots,x_n)\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ be a series representation for any microdifferential operator then

$$[\partial_i, L] = \sum f'_\alpha(x_1,\dots,x_n)\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$$

where prime denotes differentiation with respect to $x_i$. Hence, if $L \circ p$ commutes with each $\partial_i$ then $L \circ p$ has constant coefficients. However, this provides a linear relation between any coefficient of $L$ and certain higher ones. Since the coefficients of $L$ are bounded factorially with the order, this is only possible if the coefficients are all constant. □

2.2. **A Maximal Commutative Ring from Darboux Transformation.** Here we will consider a special class of commutative rings of differential operators for which we are able to demonstrate maximality using the results of the previous subsection. Suppose that the constant coefficient operator $p \in \mathcal{D}_0$ factors as

$$p(\partial_1,\dots,\partial_n) = L \circ K \qquad L, K \in \mathcal{D}. \tag{1}$$

(See [2] and [14] for a discussion of some methods for achieving such factorizations.) Then the method of Darboux transformation commonly used in the study of integrable systems [1, 8, 9, 17] is to consider the (more complicated) operator

$$P := K \circ L = K \circ p \circ K^{-1}$$

which shares many features with $p$ since the two operators are conjugate. For instance, one may try to conjugate other constant coefficient operators by $K$ to produce operators that commute with $P$. In fact, given any constant coefficient operator $r(\partial_1,\dots,\partial_n) \in \mathcal{D}_0$ it follows that $[K \circ r \circ K^{-1}, P] = 0$ . However, although $K \circ r \circ K^{-1} \in \Psi\mathcal{D}$ there is no reason to expect that it is in $\mathcal{D}$. The content of the next theorem is the statment that the ring of all *differential* operators commuting with $P$ which are produced in this way is a maximal commutative ring.

**Notation:** Let $K \in \mathcal{D}$ be a differential operator and define

$$R_0(K) := \{r \in \mathcal{D}_0 \mid K \circ r(\partial_1,\dots,\partial_n) \in \mathcal{D} \circ K\} \subset \mathcal{D}_0$$

to be the subring of elements $r \in \mathcal{D}_0$ such that $K \circ r$ has $K$ as a right factor. Then the ring

$$\begin{aligned} R(K) &:= \left(K \circ \mathcal{D}_0 \circ K^{-1}\right) \cap \mathcal{D} \\ &= K \circ R_0(K) \circ K^{-1} \end{aligned}$$

is a commutative subring of $\mathcal{D}$. In general, it will be the case that $R(K) = \mathbb{C}$ is trivial, but if $K$ is chosen to be a non-constant operator satisfying (1) then $R(K)$ will contain differential operators.

**Theorem 2.1.** *Let $K \in \mathcal{D}$ be a differential operator which is the right factor of some constant coefficient operator $p = L \circ K \in \mathcal{D}_0$. Then if $R'$ is a commutative ring such that*

$$R(K) \subset R' \subset \mathcal{D}$$

*it follows that $R(K) = R'$. In other words, $R(K)$ is a maximal commutative subring of $\mathcal{D}$.*

*Proof.* Let us suppose that $Q \in \mathcal{D}$ commutes with every element of $R(K)$. We must show that $Q$ is already in $R(K)$. Note that since $p = L \circ K$ one automatically has that $P_i := \partial_i \circ p$ is in $R_0(K)$. Thus, we know that

$$[K \circ P_i \circ K^{-1}, Q] = 0 \qquad \text{for} \qquad 1 \le i \le n$$

and hence conjugation in the ring of pseudo-differential operators gives

$$[P_i, K^{-1} \circ Q \circ K] = 0 \qquad 1 \le i \le n.$$

By Lemma 2.1, this implies that

$$K^{-1} \circ Q \circ K \in \Psi\mathcal{D}_0$$

and so it is clear that $Q$ is of the form $K \circ q(\partial_1, \ldots, \partial_n) \circ K^{-1}$ for some constant coefficient pseudo-differential operator $q \in \Psi\mathcal{D}_0$.

However, we may moreover note that $q \in \mathcal{D}_0 \subset \Psi\mathcal{D}_0$ is a constant coefficient *differential* operator. To show this we introduce a normalized common eigenfunction

$$\psi(x_1, \ldots, x_n, z_1, \ldots, z_n) = \frac{1}{g(z_1, \ldots, z_n)} K\left[e^{x_1 z_1 + \cdots + x_n z_n}\right]$$

where the polynomial $g$ is to be defined below. Note that regardless of the choice of $g$, we have by construction that

$$K \circ L[\psi] = p(z_1, \ldots, z_n)\psi \qquad Q[\psi] = q(z_1, \ldots, z_n)\psi.$$

One can write the function $K[\exp(\sum x_i z_i)]$ in the form

$$K\left[e^{x_1 z_1 + \cdots + x_n z_n}\right] = \left(\frac{\sum_{\alpha=1}^{N} \rho_\alpha(z_1, \ldots, z_n)\sigma_\alpha(x_1, \ldots, x_n)}{\sigma_0(x_1, \ldots, x_n)}\right) e^{x_1 z_1 + \cdots + x_n z_n}$$

where $\rho_\alpha$ are all non-zero polynomials in $z_1, \ldots, z_n$ and $\sigma_\alpha$ are *distinct, non-zero monomials* in $x_1, \ldots, x_n$. We choose $g(z_1, \ldots, z_n)$ to be the highest common factor of the polynomials $\rho_\alpha(z_1, \ldots, z_n)$. We have thus constructed $\psi$ so the product $f(z_1, \ldots, z_n)\psi$ is holomorphic in each $z_i$ for $f \in \mathbb{C}((z_1, \ldots, z_n)$ if and only if $f \in \mathbb{C}[z_1, \ldots, z_n]$ is actually a polynomial.

Then notice that for any $M \in \mathcal{D}$ one still has that $M[\psi]$ is holomorphic in $z_i$. In particular, $M[\psi]$ is always a polynomial in $\mathbb{C}(x_1, \dots, x_n)[z_1, \dots, z_n]$ multiplied by the exponential function $\exp \sum x_i z_i$. Putting this all together, since we have already seen that $Q[\psi] = q(z_1, \dots, z_n)\psi$, we conclude that $q(\partial_1, \dots, \partial_n) \in \mathcal{D}_0$ is a constant coefficient differential operator.

Finally, if $q \in \mathcal{D}_0$ has the property that $K \circ q \circ K^{-1} \in \mathcal{D}$ then this implies that $q \in R_0(K)$ and hence that $Q \in R(K)$ in the first place. So no operators outside of $R(K)$ have the property that they commute with every element of $R(K)$. □

2.3. **An Explicit Example.** It is still not clear from the theorem above whether the maximal commutative subrings of $\mathcal{D}$ constructed by Darboux transformation require an infinite number of generators over $\mathbb{C}$. Certainly in trivial cases (e.g. $K \in \mathcal{D}_0$) the resulting ring may require only a finite number of generators. But it would be nice to prove that this is always the case or alternatively to observe at least one example which does not. By considering a particular example in detail here, we achieve the latter.

2.3.1. *A Subring of* $\mathbb{C}[x, y]$. **Notation:** Let $\mathcal{R}_\lambda \subset \mathbb{C}[x, y]$ $(\lambda \in \mathbb{C})$ be the subset

$$\mathcal{R}_\lambda = \{q(x, y) \in \mathbb{C}[x, y] \mid q_x(z, \frac{\lambda}{z}) = q_y(z, \frac{\lambda}{z}) = q_{xy}(z, \frac{\lambda}{z}) \equiv 0\}.$$

In other words, $\mathcal{R}_\lambda$ is the set of polynomials $q \in \mathbb{C}[x, y]$ such that $q_x$, $q_y$ and $q_{xy}$ all have a factor of $xy - \lambda$. Note that $\mathbb{C} \subset \mathcal{R}_\lambda$ and more importantly that if $q_1, q_2 \in \mathcal{R}_\lambda$ are two such polynomials then $q_1 + q_2 \in \mathcal{R}_\lambda$ and $q_1 q_2 \in \mathcal{R}_\lambda$. This obviously gives us that

**Lemma 2.2.** *$\mathcal{R}_\lambda$ is a proper subring of $\mathbb{C}[x, y]$ containing $\mathbb{C}$ as well as every polynomial of the form $\rho(x, y)(xy - \lambda)^3$ for $\rho \in \mathbb{C}[x, y]$.*

It will be shown below that for a particular choice of $K \in \mathcal{D}$ the maximal commutative subring $R(K) \subset \mathcal{D}$ is isomorphic to $\mathcal{R}_\lambda$. Therefore it is interesting to note that this ring requires an infinite number of generators over $\mathbb{C}$.

**Lemma 2.3.** *The ring $\mathcal{R}_\lambda$ has the form $\mathbb{C}[\omega_1(xy - \lambda)^3, \omega_2(xy - \lambda)^3, \dots]$ where $\{\omega_i\}$ is any basis of $\mathbb{C}[x, y]$ as a vector space. In particular, $\mathcal{R}_\lambda$ is not finitely generated.*

*Proof.* We must show that a polynomial $q \in \mathbb{C}[x, y]$ is in $\mathcal{R}_\lambda$ if and only if it is of the form $g(x, y)(xy - \lambda)^3 + c$ for some $g \in \mathbb{C}[x, y]$ and $c \in \mathbb{C}$. Clearly, such a $q$ is an element of $\mathcal{R}_\lambda$. Alternatively, let us suppose that $q \in \mathcal{R}_\lambda$ and therefore $q_x = (xy - \lambda)r(x, y)$. Then, since

$$q_{xy} = xr(x, y) + (xy - \lambda)r_y(x, y)$$

also has a factor of $xy - \lambda$ one finds that $r$ has a factor of $xy - \lambda$ and hence $q_x$ actually has a factor of $(xy - \lambda)^2$. (Similarly for $q_y$.)

Now we have that $q_x = (xy - \lambda)^2 g(x, y)$ for some $g \in \mathbb{C}[x, y]$. Then integrating by parts with respect to $x$ one has

$$q(x, y) = \frac{1}{3y}\left[(xy - \lambda)^3 g(x, y) - \int (xy - \lambda)^3 g_x(x, y)\, dx\right].$$

Continuing to integrate by parts (choosing always to integrate $(xy-\lambda)^j$ so that one gets higher powers of $xy-\lambda$ and higher derivatives of $g$) one gets a finite sum (since a high enough derivative of $g$ will eventually vanish) of terms each having a factor of $(xy-\lambda)^3$, plus a constant of integration at the end.

Now we note that $\mathcal{R}_\lambda$ cannot be constructed by a finite number of generators over $\mathbb{C}$. We may, for instance, suppose that $R=\mathbb{C}[\nu_1,\nu_2,\dots]$ for some polynomials $\nu_i$ and w.l.o.g. we may take $\nu_i$ to have no constant term ($\nu_i(0,0)=0$). But then consider the polynomials $x^i(xy-\lambda)^3$ which are elements of $\mathcal{R}_\lambda$. These cannot involve any *products* of the generators $\nu_i$ else they would have a factor of $xy-\lambda$ to a higher degree. Thus, they must be a *linear* combination of some generators $\nu_i$. On the other hand, since the polynomials $x^i(xy-\lambda)^3$ are linearly independent for different $i$'s, it follows that you need infinitely many generators to construct $\mathcal{R}_\lambda$. □

2.3.2. *Isomorphism to a ring of Differential Operators.* Let us use the notation of the preceding subsection to describe a maximal commutative ring of differential operators. For this example, we will be working in two dimensions only, so $n=2$. The constant coefficient differential operator which we will factor is $p(\partial_1,\partial_2)=(\partial_1\partial_2-\lambda)^3$ ($\lambda\in\mathbb{C}$) which factors as $p=L\circ K$ with

$$K=x_1x_2(\partial_1\partial_2-\lambda)\circ\frac{1}{x_1x_2}.$$

and

$$\begin{aligned}L &= {\partial_1}^2{\partial_2}^2+\frac{1}{x_1}\partial_1{\partial_2}^2-{x_1}^{-2}{\partial_2}^2+\frac{1}{x_2}{\partial_1}^2\partial_2\\ &\quad+\frac{1-2\lambda x_1x_2}{x_1x_2}\partial_1\partial_2+\frac{-1-\lambda x_1x_2}{{x_1}^2x_2}\partial_2-{x_2}^{-2}{\partial_1}^2+\frac{-1-\lambda x_1x_2}{x_1{x_2}^2}\partial_1\\ &\quad+\lambda^2+\frac{1}{{x_1}^2{x_2}^2}+\frac{\lambda}{x_1x_2}\end{aligned}$$

**Lemma 2.4.** *A constant coefficient operator $q(\partial_1,\dots,\partial_n)\in\mathcal{D}_0$ is an element of $R_0(K)$ if and only if the function*

$$\psi(x_1,x_2,z):=x_1x_2e^{x_1z+x_2\frac{\lambda}{z}}$$

*is in the kernel of the operator $K\circ q$ for all values of $z\in\mathbb{C}$.*

*Proof.* One direction is especially simple. If $K\circ q=Q\circ K$ then

$$K\circ q[x_1x_2e^{x_1z+x_2\frac{\lambda}{z}}]=Q\circ x_1x_2(\partial_1\partial_2-\lambda)[e^{x_1z+x_2\frac{\lambda}{z}}]\equiv 0.$$

Conversely, let us suppose that $K\circ q$ annihilates this function. This means that $M:=K\circ q\circ x_1x_2$ applied to $exp(x_1z_1+x_2z_2)$ is zero for all $z_1z_2-\lambda=0$. But note that $M$ applied to this exponential results in a polynomial in $z_i$ with coefficients in $\mathbb{C}(x_1,x_2)$ multiplied by an exponential. This product vanishes on $z_1z_2-\lambda=0$ if and only if the polynomial has a factor of $z_1z_2-\lambda$ which implies that

$$M=Q\circ x_1x_2(\partial_1\partial_2-\lambda)$$

for *some* $Q\in\mathcal{D}$. Multiplying this equation on the right by $\frac{1}{x_1x_2}$ on the right proves the lemma. □

Using this lemma and the previous theorem, as well as the bispectrality [10, 11] of the constant coefficient operators, we demonstrate an isomorphism between $R(K)$ and $\mathcal{R}_\lambda$.

**Theorem 2.2.** *The ring $R(K)$, known to be maximal commutative by the preceding theorem, is isomorphic to the ring $\mathcal{R}_\lambda$ (cf. Lemma 2.2).*

*Proof.* Using Lemma 2.4 and Theorem 2.1, we know that $R(K)$ is isomorphic to the ring

$$R_0(K) = \{q \in \mathcal{D}_0 \mid K \circ q[x_1 x_2 e^{x_1 z + x_2 \lambda z}] \equiv 0\}.$$

However, since $\partial_{z_i} := \frac{\partial}{\partial z_i}$ commutes with differential operators in the variables $x_i$, this property is equivalent to saying that

$$\partial_{z_1}\partial_{z_2}\left[K \circ q[e^{x_1 z_2 + x_2 z_2}]\right] \equiv 0 \qquad \forall z_1 z_2 - \lambda = 0.$$

This can be written as differential equations for $q$ by applying all of these differential operators, clearing the denominator by multiplying by a polynomial in $x_1, x_2$ and looking at the coefficients of each monomial in $x_i$. These will be differential expressions for polynomials in $z_1$ and $z_2$ including $q$ which must vanish on $z_1 z_2 - \lambda$. For this to happen, it is necessary and sufficient that $q_x$, $q_y$ and $q_{xy}$ all have $z_1 z_2 - \lambda$ as a factor. □

## 3. Resultants of Commuting Differential Operators

In this section we give a definition of resultants for partial differential operators (cf. [6]) including "spectral parameters" and their significance in the commutative case.

3.1. **Definitions.** Fix $0 < n \in \mathbb{N}$ and denote by $\Omega^d$ the $\binom{n+d}{n}$-component vector

$$\Omega^d = (\omega_1^d, \omega_2^d, \dots)$$

where $\omega_i^d$ run over all monomial, monic differential operators in the variables $x_1, \dots, x_n$ of degree less than or equal to $d$. In other words,

$$\omega_i^d \in \{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \ : \ \alpha_i \in \mathbb{N}, \ \sum \alpha_i \leq d\}$$

By writing it as a vector, we are supposing that they have an ordering. Such an ordering is a choice, not determined canonically, but the particular choice is not important to the following. Then, for any differential operator $L$ of order $d$ or less, we denote by $\vec{\mathbf{v}}_d(L)$ the vector whose $i^{th}$ entry is the coefficient of $\omega_i^d$ in $L$. In particular, $L = \vec{\mathbf{v}}_d(L) \cdot \Omega^d$. Let $L_1, \dots, L_{n+1}$ be differential operators in the variables $x_1, \dots, x_n$ having orders $l_1, \dots, l_{n+1}$ respectively.

Let $N := -n + \sum l_i$ and construct the matrix $R_\mu = R_\mu(L_1, \dots, L_{n+1})$ whose rows are $\vec{\mathbf{v}}_N(\omega_j^{N-l_i} \circ (L_i - \mu_i))$ for all $1 \leq i \leq n+1$ and all $1 \leq j \leq \binom{n+N-l_i}{n}$. We call any maximal minor determinant of $R_\mu$ a *partial $\mu$-shifted differential resultant*. Note that each partial $\mu$-shifted differential resultant is a polynomial in the variables $\mu_i$ $(1 \leq i \leq n+1)$ with coefficients that may depend on $x_j$ $(1 \leq j \leq n)$. We define the *$\mu$-shifted differential resultant* of the operators $L_i$ to be the polynomial in the variables $\mu_i$ which is the greatest common divisor of all of these maximal minor determinants.

**Note:** In the case $L_i \in \mathbb{C}[\partial_1, \dots, \partial_n]$, this definition is a special case of the polynomial resultant [16] and that in one dimension with variable coefficients it reproduces the differential resultant of ordinary differential operators used to construct the spectral curve [20, 21]. The definition of the *differential resultant* of the operators $L_i$ given in [6] is, in our terminology, a particular *partial* $\mu$-shifted differential resultant of the operators $L_i$ with all $\mu_i = 0$.

3.2. **The Commutative Case.** As in the one dimensional case, we will here show that the differential resultant provides a polynomial equation satisfied by the operators $L_i$ in the case that they mutually commute. The remainder of the section will then be comprised of examples and counter-examples of what we would hope to have as a consequence. First, following the approach used in [6], we demonstrate the following essential lemma:

**Lemma 3.1.** *Any partial $\mu$-shifted differential resultant of the operators $L_1, \dots, L_{n+1}$ can be written as*

$$\sum_{i=1}^{n+1} D_i \circ (L_i - \mu_i) \tag{2}$$

*for some partial differential operators $D_i$ with coefficients depending on $\mu_j$ ($1 \le j \le n+1$) and $x_k$ ($1 \le k \le n$).*

*Proof.* Let $j$ be the integer $1 \le j \le \binom{n+N}{n}$ such that $\omega_j^N = 1$ is the differential operator of order zero in the vector $\Omega^N$. Construct the matrix $M$ of size $\binom{n+N}{n} \times \binom{n+N}{n}$ which is the identity matrix except for the fact that the $j^{th}$ column is replaced by the vector $\Omega^N$. Note that $\det M = 1$.

Let $\tilde{R}$ be a maximal square minor of the matrix $R_\mu(L_1, \dots, L_{n+1})$. Note that the elements in the $j^{th}$ column of the matrix $\tilde{R} \cdot M$ are all monic monomial differential operators composed with the operators $L_i - \mu_i$. Then, expanding down this column while taking determinants, one finds exactly something of the form (2) with the coefficients of $D_i$ coming from the other minor determinants of $\tilde{R}$.

On the other hand, it is an elementary fact of linear algebra that $\det \tilde{R} = \det \tilde{R} \cdot M$ and so (2) must actually be equal to the order zero operator which is the partial $\mu$-shifted differential resultant of the operators $L_i$. $\square$

Now suppose that the operators $L_i$ ($1 \le i \le n+1$) mutually commute. By definition, any partial $\mu$-shifted differential resultant of these operators is a polynomial in the variables $\mu_i$ with coefficients possibly depending on the variables $x_j$. As a consequence of Lemma 3.1 we then find that the operators $L_i$ *satisfy* this polynomial.

**Theorem 3.1.** *Let $p(\mu_1, \dots, m_{n+1})$ be any $\mu$-shifted differential resultant of the mutually commuting operators $L_i$, then*

$$p(L_1, \dots, L_{n+1}) = 0.$$

*Proof.* Only commutativity of the $\mu_i$'s with the $L_j$'s is required to rewrite $p(\mu_1, \dots, \mu_{n+1})$ in the form (2). So, since $[L_i, L_j] = 0$ we can write $p(L_1, \dots, L_{n+1})$ by substituting

$L_i$ for $\mu_i$ in (2). This, however, is clearly zero since every term has a factor of $L_i - \mu_i$ for some $i$. □

In the one dimensional case, we can moreover say that the $\mu$-shifted differential resultant is a polynomial in $\mu_1$ and $\mu_2$ with *constant* coefficients or a multiple of such a polynomial by a function of $x_1$. Here, the results proved thus far leave open the possibility that the differential resultant will only produce polynomial equations satisfied by the operators with *explicit* dependence on the variables $x_j$. We were not able to produce any such examples or exclude the possibility.

3.3. **The Zero Possibility.** As stated in the introduction, given two commuting ordinary differential operators $L_1$ and $L_2$ the determinant of $R_\mu(L_1, L_2)$ (which happens to always be square in the case $n = 1$) is a *non-zero* polynomial in $\mu_1$ and $\mu_2$ which is satisfied by the operators. Here we will see that the differential resultant does not always give such useful information in the higher dimensional case.

Consider the case $n = 2$ and

$$L_1 = \partial_1^2 - \partial_2^2 - 1 \qquad L_2 = \partial_1 \circ L_1 \qquad L_3 = \partial_2 \circ L_1.$$

Note that these operators satisfy the equation $L_2^2 - l_3^2 - L_1 - L_1^6 = 0$ and so one might hope, given Theorem 3.1, that the differential resultant of these operators is $\mu_2^2 - \mu_3^2 - \mu_1 - \mu_1^6$ (or at least is a non-zero multiple of this).

**Lemma 3.2.** *The differential resultant of the operators $L_i$ is the zero polynomial in the variables $\mu_i$ ($1 \le i \le 3$).*

*Proof.* One could, of course, merely compute the resultant according to the definition. However, there is a more direct and informative way to observe this fact. Since these operators are constant coefficient, the problem reduces to a problem of polynomial resultants. In particular, the resultant is the same as the resultant of the homogeneous polynomials

$$p_1(x_1, x_2, x_3) = x_1^2 - x_2^2 - (\mu_1 + 1)x_3^2 \qquad p_2(x_1, x_2, x_3) = x_1^3 - x_1x_2^2 - x_1x_3^2 - \mu_2x_3^3$$
$$p_3(x_1, x_2, x_3) = x_1^2x_2 - x_2^3 - x_2x_3^2 - \mu_3x_3^3$$

However, it is well known [16] that this resultant will be zero iff these polynomials have a common zero in projective space. Although it is true that no "finite" point (with $x_3 \neq 0$) is a common solution to these polynomials for all values of $\mu_i$, there are solutions at infinity. In particular, note that the point $(1, -1, 0)$ satisfies all three polynomials regardless of the values of $\mu_i$. □

It is interesting to note the geometry behind this situation. This problem of having a zero resultant never arises in the one dimensional case essentially because only one point is being added at infinity and that point is never a solution of the homogeneous polynomial. Whereas, in higher dimensions, there is "room" at infinity for many solutions.

Note that the same problem can also occur in a non-constant case (and so not simply an example of a polynomial resultant). In particular, the differential resultant of any three operators from the ring $R(K)$ described in Subsection 2.3.2 will be zero regardless of the values of the variables $\mu_i$. The mundane explanation of this fact here is merely that the $N^{th}$ powers of $\partial_1$ and $\partial_2$ never appear in $\omega_j^{N-l_i} \circ (L_i - \mu_i)$ and so there are columns of the resultant matrix with all zero entries.

3.4. **Positive Results.** A more encouraging example is to consider the operators

$$L_1 = \partial_1^2 - \partial_2^2 \qquad L_2 = x_2\partial_1 + x_1\partial_2 \qquad L_3 = L_1 \circ L_2 - \gamma L_1 \qquad \gamma \in \mathbb{C}.$$

It is a non-obvious fact that $[L_1, L_2] = 0$, but given this (which is easily checked) it is clear that $L_3$ also commute and that the three together satisfy a polynomial equation $p(L_1, L_2, L_3) = 0$ with

$$p(\mu_1, \mu_2, \mu_3) = \mu_3 - \mu_1\mu_2 + \gamma\mu_1.$$

Then the differential resultant (which can be most easily computed not by finding all maximal minor determinants but by the formula $D/A$ where $D$ and $A$ are the minor determinants specified in [16]) is exactly $p^3(\mu_1, \mu_2, \mu_3)$. This is very nearly what we would want (although there is presently no theory to explain the exponent "3" which arises).

It is intriguing and surprising that the resultant is independent of the variables $x_1$ and $x_2$ in this case. In the one dimensional case, the resultant of two monic differential operators is independent of $x$ if and only if the operators commute. Here, the situation involves one operator, $L_1$, with constant leading coefficients and others that are not, which cannot happen in the one dimensional case.

## 4. Mathematica Code

The following code (and assistance using it) is available by writing to the first author (`kasman@math.cofc.edu`). It is useful for performing many calculations with differential operators, and here we will only give a very brief description of how to use it. First set the value of the variable `dimen` equal to the number of variables you will be working with and then input the file containing the code. For instance:

```
In[1]:= dimen=3

Out[1]= 3

In[2]:= <<pdo-ak.m
String dimen already defined..using present value 3
READY: Partial Differential Operators in 3 dimensions
```

Then, a differential operator can be entered using `DX[1], DX[2],...` as the elementary differential operators and `X[1], X[2],...` as the corresponding variables. For example:

```
In[3]:= L=X[1] DX[2]+X[2] DX[1]

Out[3]= DX[2] X[1] + DX[1] X[2]

In[4]:= Q=DX[1]^2+DX[2]^2
```

```
               2        2
Out[4]= DX[1]  + DX[2]
```

Note that when entering or reading a differential operator in this notation, it is always assumed that all differentiation has been taken and functions are on the "left", even if it may not be written this way. (That is, `X[1] DX[1]=DX[1] X[1]`.) The non-commutativity is only apparent when multiplying two differential operators. Multiplication of differential operators is achieved using the command `pdomult`:

```
In[6]:= pdomult[DX[1],X[1]]

Out[6]= DX[1] X[1] + 1

In[7]:= pdomult[Q,L]
                              2
18:10 > multiplying by DX[1]
                              2
18:10 > multiplying by DX[2]
Simplifying...
.
.
.
18:10 > Simplifying {3, 0, 0} term.
.
18:10 > Simplifying {1, 1, 0} term.
18:10 > Simplifying {2, 1, 0} term.
.
.
18:10 > Simplifying {1, 2, 0} term.
.
.
18:10 > Simplifying {0, 3, 0} term.
.
.
.

                               2                3              3
Out[7]= 4 DX[1] DX[2] + DX[1]  DX[2] X[1] + DX[2]  X[1] + DX[1]  X[2] +

                2
>    DX[1] DX[2]  X[2]
```

Similarly, if you want to *apply* the operator `L` to the function `f` you simply say `pdoapply[L,f]`.

Finally, given a list of $N$ operators in $N-1$ variables, the matrix whose minor determinants give the differential resultant can be computed as `DiffResult[L1,L2,...,LN]`. (Note that this command automatically subtracts the indeterminate `mu[i]` from the

$i^{th}$ operator so that the differential resultant is a polynomial in these variables. The command `TakeRandomDeterminants[mat,n]` may come in handy as well, since it takes the determinant of `n` randomly chosen maximal minors of the matrix `mat`.

Here is the code:

```
(* PDO-AK.M  Mathematica Code for Differential Operators
by Alex Kasman, College of Charleston
kasman@math.cofc.edu *)

(* Set dimen to be the number of dimensions and then input this
file.  It will be 2 by default *)

(* Use X[1] ... X[dimen] as variables and DX[1] ... DX[dimen] as the
corresponding elementary differential operators. *)

(* Multiply two operators with pdomult[L,Q] *)

(* Conjugate one operator by another by pdoconj[L,Q], which yields the
operator Q L Q^1 if this is a differential operator. *)

(* DiffResult[{L1,....,LN}] gives the matrix whose minor determinants
have a gcd which is the differential resultant of L1-mu[1], L2-mu[2],
etc. *)

If[StringMatchQ[ToString[dimen],"dimen"],
dimen=2;
Print["String dimen undefined...setting to ",dimen," by default..."],
Print["String dimen already defined..using present value ",dimen]]


(* Call the variables X[1], X[2],...X[dimen] and the differential operators
DX[1],DX[2],...DX[dimen] *)

zerovect=Table[0,{i,1,dimen}]

DXpower[vect_]:=Module[{i},Product[DX[i]^(vect[[i]]),{i,1,Length[vect]}]]

DXv[alpha_]:=Module[{i},Table[DX[i],{i,1,Length[alpha]}] . alpha]

pdocoef[L_,zerovect]:=L/. DX[i_]->0

pdocoef[L_,vect_]:=Coefficient[Collect[((Expand[L]
/.DXpower[vect]->SPACE)/.DX[i_]->0),SPACE],SPACE]

pdotermmult[coef_,vect_,M_]:=Module[{i,sofar},
verbose["multiplying by ",DXpower[vect]];
For[i=1;sofar=M,i<Length[vect]+1,i=i+1,
sofar=pdopowmult[i,vect[[i]],sofar]];
```

```
coef sofar]

pdotermmult[0,vect_,M_]:=0

pdopowmult2[i_,1,M_]:=D[M,X[i]]+M DX[i]
pdopowmult[i_,0,M_]:=M

pdopowmult[i_,n_,M_]:=(pdopowmult2[i,n,M])
pdopowmult2[i_,n_,M_]:=pdopowmult2[i,n-1,pdopowmult2[i,1,M]]

maketablefor[M_]:=Module[{max,i,j,k,sofar},
For[i=1,i<=dimen+1,i=i+1,max[i]=Exponent[Collect[M,DX[i]],DX[i]]];
For[i=1;sofar=Table[j[k],{k,1,dimen}],i<dimen+1,i=i+1,
sofar=Table[sofar,{j[i],0,max[i]}]];
Flatten[sofar,dimen-1]]

maketablefor[M_,K_]:=Module[{max,i,j,k,sofar},
For[i=1,i<=dimen+1,i=i+1,max[i]=Exponent[Collect[M,DX[i]],DX[i]]
-Exponent[Collect[K,DX[i]],DX[i]]];
For[i=1;sofar=Table[j[k],{k,1,dimen}],i<dimen+1,i=i+1,
sofar=Table[sofar,{j[i],0,max[i]}]];
Flatten[sofar,dimen-1]]

pdoexp[ll_,n_]:=Module[{outL,j},(verbose["Raising operator to power
",n]; For[j=0;outL=1,j<n,j=j+1,verbose["pdoexp: power
",j];outL=pdomult[ll,outL]];outL)]

makeshortlist[L_]:=Module[{i,inlist,outlist}, verbose["making short
list"]; outlist={}; inlist=maketablefor[L];
For[i=1,i<Length[inlist]+1,i=i+1,
If[MatchQ[ToString[pdocoef[L,inlist[[i]]]],"0"],verbose["not in
list",inlist[[i]]],outlist=Union[outlist,{inlist[[i]]}];
verbose["makeshortlist: includes ",inlist[[i]]]]]; outlist]

pdomult[L_,M_]:=Module[{i,j,LM,liszt},
liszt=maketablefor[L];
For[i=1;LM=0,i<=Length[liszt],i=i+1,
LM=LM+pdotermmult[pdocoef[L,liszt[[i]]],liszt[[i]],M]];
pdosimp[LM]]

pdotermapply[coef_,vect_,M_]:=Module[{i,sofar},
verbose["Applying ",DXpower[vect]];
For[i=1;sofar=M,i<Length[vect]+1,i=i+1,
sofar=pdopowapply[i,vect[[i]],sofar]];
coef sofar]

pdotermapply[0,vect_,M_]:=0
```

```
pdopowapply[i_,n_,M_]:=(D[M,{X[i],n}])

pdoapply[L_,M_]:=Module[{i,j,LM,liszt},
liszt=maketablefor[L];
For[i=1;LM=0,i<=Length[liszt],i=i+1,
LM=LM+pdotermapply[pdocoef[L,liszt[[i]]],liszt[[i]],M]];
Simplify[LM]]

pdosimp[0]:=0

pdosimp[L_]:=Module[{liszt,i,sofar}, Print["Simplifying..."];
liszt=maketablefor[L]; For[i=1;sofar=0,i<Length[liszt]+1,i=i+1,
If[MatchQ[ToString[pdocoef[L,liszt[[i]]]],"0"],Print["."],
verbose["Simplifying ",liszt[[i]]," term."];
sofar=sofar+Simplify[pdocoef[L,liszt[[i]]]] DXpower[liszt[[i]]]]];
sofar]

pdodisplay[L_]:=Module[{liszt,i},
liszt=maketablefor[L];
For[i=Length[liszt],i>0,i=i-1,
pdotermdisplay[pdocoef[L,liszt[[i]]],liszt[[i]]]]]

pdotermdisplay[0,vect_]:=(bleh=0)

pdotermdisplay[coef_,vect_]:=Print["+ (",Simplify[coef],") ",DXpower[vect]]

pdoconj[P_,K_]:=Module[{L,lco,LK,KP,i,j,liszt,deg,bleh,slv,holdit},
liszt=maketablefor[P]; L=Sum[lco[liszt[[i]]]
DXpower[liszt[[i]]],{i,1,Length[liszt]}]; deg=pdodeg[P];
For[i=1,i<Length[liszt]+1,i=i+1,
If[Sum[liszt[[i]][[j]],{j,1,dimen}]>deg-1, bleh[liszt[[i]]]=0;
L=L/.lco[liszt[[i]]]->pdocoef[P,liszt[[i]]], bleh[liszt[[i]]]=100]];
Print["finding KP"]; KP=pdomult[K,P]; Print["finding LK"];
LK=pdomult[L,K]; brahms=maketablefor[LK];
For[i=Length[liszt],i>0,i=i-1, verbose["solving for
lco[",liszt[[i]],"]"]; If[bleh[liszt[[i]]]==0,verbose["...Already
set..."], For[j=Length[brahms],j>0,j=j-1,
holdit=pdocoef[LK,brahms[[j]]];
If[StringMatchQ[ToString[D[holdit,lco[liszt[[i]]]]],"0"],Print["."],
slv=Solve[pdocoef[LK,
brahms[[j]]]==pdocoef[KP,brahms[[j]]],lco[liszt[[i]]]][[1]];
Print["replacement ->",slv]; j=-5; L=L/.slv; LK=LK/.slv]]]];
(*Print["Okay if this is zero ",pdosimp[LK-KP]];*) L=pdosimp[L]]

pdodeg[L_]:=Module[{transL,i,j,deg,const},
transL=L/.DX[i_]->const[i] SPACE;
deg=Exponent[Collect[transL,SPACE],SPACE];
deg]
```

```
pdoTeX[L_]:=Module[{liszt,i},
liszt=maketablefor[L];
For[i=Length[liszt],i>0,i=i-1,
pdotermTeX[pdocoef[L,liszt[[i]]],liszt[[i]]]]]

pdotermTeX[0,vect_]:=(bleh=0)
pdotermTeX[1,vect_]:=Print["+ ",TeXForm[DXpower[vect]]]
pdotermTeX[coef_,zerovect]:=Print["+ ",TeXForm[coef]]

pdotermTeX[coef_,vect_]:=Print["+ ",TeXForm[Together[coef]]," **
",TeXForm[DXpower[vect]]]

Print["READY: Partial Differential Operators in ",dimen," dimensions"]

genminusone[deg_,liszt_]:=Module[{list2,i,j,n},
list2={};
For[i=1,i<=Length[liszt],i=i+1,
n=Sum[liszt[[i]][[j]],{j,1,Length[liszt[[i]]]}];
list2=Flatten[{list2,Table[Flatten[{liszt[[i]],j},1],
{j,0,deg-n}]},1]];
list2]

degcomplete[deg_,entry_]:=Module[{i},
Flatten[{entry,deg-Sum[entry[[i]],
{i,1,Length[entry]}]},1]]

AllMonomialsDegExactly[deg_]:=Module[{list2,list3,i,j,n},
If[dimen==1,DX[1]^deg,
list2=Table[{i},{i,0,deg}];
For[i=1,i<dimen-1,i=i+1,list2=genminusone[deg,list2]];
Table[degcomplete[deg,list2[[i]]],{i,1,Length[list2]}]]]

AllMonomialsDegAtMost[deg_]:=Module[{list2,i,j,n},
list2=Table[{i},{i,0,deg}];
For[i=1,i<dimen,i=i+1,list2=genminusone[deg,list2]];
list2];

multbynextdxupto[deg_,liszt_]:=Module[{list2,list3,i,j,k,n,n2,n3,n4,op,rem},
list2={}; For[i=1,i<=Length[liszt],i=i+1, op=liszt[[i]][[1]];
list3=liszt[[i]][[2]]; n=Length[list3]; k=Sum[list3[[j]],{j,1,n}];
rem=op; list2=Flatten[{list2,{{rem,Flatten[{list3,{0}},1]}}},1];
For[j=1,j<=deg-k,j=j+1, rem=Simplify[D[rem,X[n+1]]+rem*DX[n+1]];
list2=Flatten[{list2,{{rem,Flatten[{list3,{j}},1]}}},1]]]; list2]

MultByAllMonomialsUpto[deg_,L_]:=Module[{i,j,k,list2,rem},
rem={{L,{0}}}; verbose["MultByAllMonomials j=",1,"/",dimen];
For[i=1,i<deg+1,i=i+1, rem=Flatten[ {rem,
```

```
{{D[rem[[Length[rem]]][[1]],X[1]]+rem[[Length[rem]]][[1]]
DX[1],rem[[Length[rem]]][[2]]+1}}}, 1]]; For[j=2,j<=dimen,j=j+1,
verbose["MultByAllMonomials j=",j,"/",dimen];
rem=multbynextdxupto[deg,rem]];
Table[Expand[rem[[k]][[1]]],{k,1,Length[rem]}]]

pwr[liszt_]:=Product[DX[i]^liszt[[i]],{i,1,Length[liszt]}]

DiffResult[liszt_]:=Module[{i,j,k,maxdeg,n,n2,n3,mm,list2,list3,list4,rem},
n=Length[liszt];
maxdeg=Sum[pdodeg[liszt[[i]]],{i,1,n}]-dimen;
mm={};
list4={};
list2=AllMonomialsDegAtMost[maxdeg];
n2=Length[list2];
For[i=1,i<=n,i=i+1,
verbose["Diff result i=",i,"/",n];
list3=MultByAllMonomialsUpto[maxdeg-pdodeg[liszt[[i]]],liszt[[i]]-mu[i]];
mm=Flatten[{mm,list3},1]];
mm=makeamatrix[mm,list2];
verbose["The Matrix is:"];
If[Length[mm]+Length[mm[[1]]]>30,
verbose["TOO BIG TO PRINT"],
verbose[MatrixForm[mm]]];
(*TakeAllSquareMinors[mm,list4]*)
mm]

makeamatrix[liszta_,list2_]:=Module[{liszt,i,j,k,n,list3,list4,bleh},
wid=Length[list2]; list4=liszta/.DX[i_]->0; liszt=liszta-list4;
list3=Table[DXpower[list2[[i]]]->Table[Which[k==i,1,True,0],
{k,1,wid}],{i,Length[list2],2,-1}];
Simplify[(list4*bleh/.bleh->Table[Which[k==1,1,True,0],
{k,1,wid}])+(liszt/.list3)]]

TakeAllSquareMinors[mm_]:=Module[{liszt,
list2,i,j,k,n,ll,ww,deter,rem,minlist}, n=0; ll=Length[mm];
ww=Length[mm[[1]]]; If[ww==ll,{Det[mm]}, If[ww>ll,verbose["(Note:
Transposing the matrix for simplicity)"];
TakeAllSquareMinors[Transpose[mm]], For[k=0,k<=ll-ww,k=k+1,
verbose["Doing Young diagrams starting with ",k]; liszt={{k}};
For[i=1,i<ww,i=i+1, list2={}; For[j=1,j<=Length[liszt],j=j+1,
list2=Flatten[{list2,Table[Flatten[{liszt[[j]],
k},1],{k,0,liszt[[j]][[Length[liszt[[j]]]]]}]},1]]; liszt=list2];
verbose["Computing the corresponding determinants.  Listing nonzero
ones:"]; For[j=1,j<=Length[liszt],j=j+1,
deter=Simplify[Det[Table[mm[[ll-ww+i-liszt[[j]][[i]]]],{i,1,ww}]]];
If[StringMatchQ[ToString[deter],"0"],, n=n+1; rem[n]=deter; verbose["
******************* "]; verbose[" "]; verbose["Nonzero determinant
```

```
number ",n]; verbose[deter]]]]; Table[rem[j],{j,1,n}]]]]

time:=StringJoin[ToString[Date[][[4]]],":",ToString[Date[][[5]]]," > "]

verbose[f_]:=Print[time,f]
verbose[f_,g_]:=Print[time,f,g]
verbose[f_,g_,h_]:=Print[time,f,g,h]
verbose[f_,g_,h_,i_]:=Print[time,f,g,h,i]
verbose[f_,g_,h_,i_,j_]:=Print[time,f,g,h,i,j]
verbose[f_,g_,h_,i_,j_,k_]:=Print[time,f,g,h,i,j,k]

normp[a_,b_]:=Which[b<a,True,True,False]

TakeRandomDeterminants[mat_,n_]:=Module[{ll,ww,randomyoung,i,bleh},
ll=Length[mat];
ww=Length[mat[[1]]];
verbose[ll,"x",ww," matrix"];
randomyoung:=Sort[Table[Floor[Random[]*(ll-ww+1)],{i,1,ww}],normp];
For[i=1,i<n+1,i=i+1,
verbose["Try number ",i];
lst=randomyoung;
verbose["Young is ",lst];
bleh=Simplify[Det[Table[mat[[ll-ww+i-lst[[i]]]],{i,1,ww}]]];
verbose["Determinant is -> ",bleh]]]
```

## References


[1] A. Andrianov, N. Borisov and M. Ioffe, “The factorization method and quantum systems with equivalent energy spectra”, *Physics Letters A*, 105 (1984) no 1,2 pp. 19–22.

[2] Yu. Berest and A. Kasman “D-modules and Darboux Transformations”, *Lett. Math. Phys.* 43 (1998) 279–294

[3] A. Braverman, P. Etingof and D. Gaitsgory, “Quantum Integrable Systems and Differential Galois Theory”, *Transformation Groups* 2 (1997) pp. 31–56

[4] J.L. Burchnall and T.W. Chaundy, “Commutative ordinary differential operators”, *Proc. London Math. Soc.* **211** (1923), 420-440

[5] J.L. Burchnall and T.W. Chaundy, “Commutative ordinary differential operators”, *Proc. Roy. Soc. A* **118** (1928), 557-583

[6] G. Carrà Ferro, “A resultant theory for systems of linear partial differential equations”, Lie Groups Appl. **1** (1994), no. 1, 47–5

[7] O.A. Chalykh and A.P. Veselov, “Commutative rings of partial differential operators and Lie algebras”, Comm. Math. Phys. 125 (1990) 597-611.

[8] G. Darboux , “Sur une proposition relative aux equation lineaires” *Compt. Rend.* **94** (1882), 1456

[9] G. Darboux, “Leçons sur la théorie générale des surfaces.” 2éme partie, Paris, Gauthiers-Villars, 1889.

[10] J.J. Duistermaat and F.A. Grünbaum, “Differential equations in the spectral parameter”, *Comm. Math. Phys.* 103 (1986) 177-240

[11] F.A. Grünbaum, “Bispectral Musings” CRM Proceedings and Lecture Notes, volume 14, American Mathematical Society, Providence, RI, 1988 pp. 31–46.

[12] E. Horozov and A. Kasman, "Darboux Transformations for Bispectral Quantum Integrable Systems", to appear in *Letters in Mathematical Physics.*

[13] M. Kashiwara, T. Kawai, T. Kimura, "Foundations of algebraic analysis", Princeton University Press, 1986.

[14] A. Kasman, "Kernel Inspired Factorizations of Partial Differential Operators", *Journal of Mathematical Analysis and Applications*, 234/2 (1999) pp. 580-591.

[15] I.M. Krichever, "The integration of non-linear equations by methods of algebraic geometry", *Functional analy. Appl.* 12 (1977) 12-26.

[16] F.S. Macaulay, "The Algebraic Theory of Modular systems", *Cambridge Tracts in Mathematics and Mathematical Physics*, 19 (1916).

[17] V.B. Matveev and M.A. Salle, Darboux Transformations and Solitons, Springer Ser. Nonlinear Dynam., Springer-Verlag, Berlin, 1991.

[18] A. Nakayashiki "Structure of Baker-Akhiezer modules of principally polarized abelian varieties, commuting partial differential operators and associated integrable systems" *Duke Math. J.* 62 (1991), no. 2, 315–358.

[19] A. Nakayashiki "Commuting partial differential operators and vector bundles over abelian varieties" *Amer. J. Math.* 116 (1994), no. 1, 65–100.

[20] E. Previato, "Another algebraic proof of Weil's reciprocity" *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend.* (9) Mat. Appl. 2 (1991), no. 2, 167–171

[21] E. Previato, "Seventy years of spectral curves: 1923–1993", Integrable systems and quantum groups (Montecatini Terme, 1993), 419–481, *Lecture Notes in Math.*, 1620, Springer, Berlin, 1996

Department of Mathematics, College of Charleston, 66 George Street, Charleston, SC 29424-0001

*E-mail address*: `kasman@math.cofc.edu`

Department of Mathematics, Boston University, 111 Cummington Street, Boston, MA 02215

*E-mail address*: `ep@math.bu.edu`