A note on the Lax pairs for Painlevé equations

A. A. Kapaev
St Petersburg Department of
Steklov Mathematical Institute
Fontanka 27
St Petersburg 191011, Russia

E. Hubert*
Mathematical Sciences Research
Institute
1000 Centennial Drive
Berkeley, CA 94720-5070, USA

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Abstract

For the classical Painlevé equations, besides the method of similarity reduction of Lax pairs for integrable PDEs, there are known two ways of Lax pair generation. The first of them is based on the confluence procedure in the Fuchs' linear ODE with four regular singularities isomonodromy deformation of which is governed by PVI equation. The second method treats the hypergeometric equation and confluent hypergeometric equations as the isomonodromy deformation equations for the triangular systems of ODEs which nontriangular extensions give rise to the Lax pairs for the Painlevé equations.

The theory of integrable integral operators suggests a new way of the Lax pair generation for the classical Painlevé equations. This method involves the gauge transformation of special kind applied to the linear systems exactly solvable in terms of the classical special functions. Some of the Lax pairs we introduce are known, other are new. The question of gauge equivalence of different Lax pairs for the Painlevé equations is considered.

1 Introduction

The Painlevé equations [1] originally found as the only irreducible second order ODEs $y_{xx} = R(x, y, y_x)$ rational in y_x, algebraic in y and analytic in x such that their general solutions have no movable branch points and essential singularities are known to govern the so-called isomonodromy deformations of certain linear differential equations with rational coefficients. In other words, the Painlevé equations can be written in the form of compatibility conditions for some over-determined linear systems of 2×2 matrix equations called now the Lax pairs.

The first of such system was found by Fuchs [2] who investigated the isomonodromy deformations of the linear Fuchsian second order scalar ODE with four singular points and revealed that preserving of the monodromy properties of such linear equation with necessity leads to deformations of its coefficients in accord with the sixth Painlevé equation PVI. Observation that other Painlevé equations can be obtained by use of certain scaling limits from PVI equation [1] gave rise to consideration of confluences of singular points in the Fuchs' linear equation for PVI [3] and finally to the pairs of linear systems for all the Painlevé equations [4, 5].

Another way of generating Lax pairs for the Painlevé equations was introduced in ref. [6]. In this paper, the authors have shown that the Airy, Bessel and parabolic cylinder equations [7] can be written as the compatibility conditions for triangular 2×2 matrix systems. Non-triangular

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generalizations of these Lax pairs yield the Lax pairs for the Painlevé equations of the second, third and fourth kind, respectively.

In the present paper, we explain a new scheme of generating Lax pairs for the classical Painlevé equations which allows us to elucidate another relationship between linear and nonlinear special functions, i.e. between (confluent) hypergeometric functions and Painlevé functions. The idea of this scheme comes from the fact [8] of close relationship between the integrable integral operators, Riemann-Hilbert problem and isomonodromy deformations of the linear equations with rational coefficients.

First, let us outline this relationship. Consider integral operator in the proper functional class on the set of finite intervals $\Gamma = \cup_{j=0}^{n}(a_j; a_{j+1})$,

$$(Ky)(\lambda) = \int_\Gamma K(\lambda,\mu) y(\mu)\, d\mu, \quad \lambda,\mu \in \Gamma \subset \mathbb{C}, \tag{1}$$

with the scalar kernel of the form introduced in ref. [9] (look for more general situations in refs. [10, 11]):

$$K(\lambda,\mu) = \frac{\varphi(\lambda)\psi(\mu) - \varphi(\mu)\psi(\lambda)}{\lambda-\mu}, \tag{2}$$

where the vectors

$$f(\lambda) = \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix}, \quad g(\mu) = \begin{pmatrix} -\varphi(\mu) \\ \psi(\mu) \end{pmatrix}, \tag{3}$$

are related to the linear differential equation

$$\frac{df}{d\lambda} = A_0(\lambda) f, \quad A_0 = \begin{pmatrix} a_3 & a_+ \\ a_- & -a_3 \end{pmatrix}, \tag{4}$$

with A_0 rationally dependent on λ.

The first example of such kernels introduced in [12] for description of quantum correlation functions and in [13] for description of the classical level spacing distribution function in "bulk" of spectrum of Gaussian Unitary Ensemble (GUE) is the famous sine kernel

$$K(\lambda,\mu) = \frac{\sin\pi(\lambda-\mu)}{\pi(\lambda-\mu)}, \tag{5}$$

with the exponent functions $\varphi(\lambda)$ and $\psi(\lambda)$,

$$\varphi(\lambda) = \frac{e^{i\pi\lambda}}{2\pi i}, \quad \psi(\lambda) = \frac{e^{-i\pi\lambda}}{2\pi i}.$$

The corresponding linear matrix equation (4) is a constant diagonal one,

$$A_0(\lambda) = \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix}.$$

Surprisingly, Fredholm determinant of the integral operator with the kernel (5) on the segment $(-x;x)$, $\det\big(I - K\chi_{(0;x)}\big)$, where we denote by χ_Γ the characteristic function of the set Γ, occurs to be related to the classical fifth Painlevé equation.

Next examples of such kind presented in refs. [14, 15] are the Airy kernel

$$K(\lambda,\mu) = \frac{Ai(\lambda)Ai'(\mu) - Ai'(\lambda)Ai(\mu)}{\lambda-\mu}, \tag{6}$$

where $Ai(\lambda)$ and $Ai'(\lambda)$ are the classical Airy function and its derivative [7], describing the level spacing distribution function for the GUE at the "edge" of spectrum, and the Bessel kernel

$$K(\lambda,\mu) = \frac{J_\alpha(\sqrt{\lambda})\sqrt{\mu}J'_\alpha(\sqrt{\mu}) - \sqrt{\lambda}J'_\alpha(\sqrt{\lambda})J_\alpha(\sqrt{\mu})}{2(\lambda-\mu)} \tag{7}$$

for "hard edge" of this spectrum at $\lambda = 0$. In the Airy kernel case, in our notations, the functions

$$\varphi(\lambda) = Ai(\lambda), \quad \psi(\lambda) = Ai'(\lambda),$$

with the Airy function $Ai(\lambda)$ form the vector solution of the equation (4) with the matrix $A_0(\lambda)$ given by

$$A_0(\lambda) = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}, \tag{8}$$

while in the Bessel kernel case, the functions

$$\varphi(\lambda) = J_\alpha(\sqrt{\lambda})/2, \quad \psi(\lambda) = \sqrt{\lambda} J'_\alpha(\sqrt{\lambda}),$$

with the Bessel function $J_\alpha(x)$ yield the vector solution of (4) with

$$A_0(\lambda) = \begin{pmatrix} 0 & \frac{1}{4\lambda} \\ -1 + \frac{\alpha^2}{\lambda} & 0 \end{pmatrix}, \tag{9}$$

As it is shown in [14, 15], Fredholm determinants of such operators on the half-infinite interval $(x, +\infty)$ are related to the second and third Painlevé equations, respectively.

Some kernels "beyond Airy" introduced in ref. [16] are expressed in terms of Hermite, Laguerre and Jacobi polynomials and related to the fourth, fifth and sixth Painlevé equations.

Following [9, 10], let us introduce the functions F and G as the solutions of the integral equations $F = \varphi + KF$, $G = \psi + KG$, i.e.

$$F = (I - K)^{-1}\varphi, \quad G = (I - K)^{-1}\psi. \tag{10}$$

and the matrix function $Y(\lambda)$ by use of the Cauchy integral

$$Y(\lambda) = I - \int_\Gamma \frac{F(\mu) g^T(\mu)}{\mu - \lambda}\, d\mu. \tag{11}$$

It follows from the properties of the Cauchy integrals (see, e.g., [17]) that this function is holomorphic outside Γ, normalized to the unit at infinity, may have logarithmic or integrable algebraic singularities at the points $a_j \in \partial\Gamma$ and is characterized by the jump on Γ

$$Y_+(\lambda) = Y_-(\lambda) H(\lambda), \quad H(\lambda) = I - 2\pi i \begin{pmatrix} -\psi(\lambda)\varphi(\lambda) & \psi^2(\lambda) \\ -\varphi^2(\lambda) & \psi(\lambda)\varphi(\lambda) \end{pmatrix}, \quad \lambda \in \Gamma. \tag{12}$$

The special form of the jump immediately leads to a very specific gauge transformation of the matrix 2×2 equation

$$\frac{d\Phi}{d\lambda} = A_0(\lambda)\Phi. \tag{13}$$

Indeed, let $\Phi(\lambda)$ be a fundamental solution of (13), i.e. $\det \Phi(\lambda) = 1$. Than the vector $f(\lambda) = \big(\psi(\lambda); \varphi(\lambda)\big)^T$ is given by the product

$$f(\lambda) = \Phi(\lambda) p, \quad p = \begin{pmatrix} q \\ p \end{pmatrix} = const. \tag{14}$$

At the same time, the vector $g(\lambda) = \big(-\varphi(\lambda); \psi(\lambda)\big)^T$ is given by the product

$$g(\lambda) = \big(\Phi(\lambda)^{-1}\big)^T q, \quad q = \begin{pmatrix} -p \\ q \end{pmatrix} = const. \tag{15}$$

Therefore, the product $\Phi^{-1}(\lambda)H(\lambda)\Phi(\lambda)$ does not depend on λ

$$\Phi^{-1}(\lambda)H(\lambda)\Phi(\lambda) = I - 2\pi i \begin{pmatrix} -pq & q^2 \\ -p^2 & pq \end{pmatrix} = const. \tag{16}$$

Thus the new 2×2 matrix function

$$\Psi(\lambda) = Y(\lambda)\Phi(\lambda) \tag{17}$$

has constant jump on Γ and satisfies a linear differential equation [11]

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi \tag{18}$$

with the rational in λ matrix A related to the "vacuum" matrix A_0 by the equation

$$A = YA_0Y^{-1} + Y_\lambda Y^{-1}. \tag{19}$$

The mentioned above Riemann-Hilbert problem for $\Psi(\lambda)$ (for details, see [11]) contains natural isomonodromy deformation parameters $a_j \in \partial\Gamma$, so that besides the equation in λ (18)

$$\frac{\partial\Psi}{\partial\lambda} = A(\lambda)\Psi \equiv \Big(\tilde{A}_0(\lambda) + \sum_j \frac{A_j}{\lambda - a_j}\Big)\Psi, \tag{20}$$

where A_j are constant matrices and $\tilde{A}_0(\lambda)$ is a rational in λ matrix with no singularity at a_j, the function $\Psi(\lambda)$ satisfies

$$d\Psi = U(\lambda)\Psi \equiv \Big(-\sum_j \frac{A_j}{\lambda - a_j}\, da_j\Big)\Psi. \tag{21}$$

Compatibility condition for the system (20), (21),

$$dA - \frac{\partial U}{\partial\lambda} + [A, U] = 0, \tag{22}$$

which holds identically in λ, gives rise to the system of nonlinear differential equations called the system of isomonodromy deformations [5].

Below we show that, in the simplest situations, such systems reduce to the second order ODE's equivalent to some of the classical Painlevé equations. The investigations described in this paper were essentially made with symbolic computations. In particular, the reductions of the compatibility condition systems (22) to Painlevé equations are computed according to an algorithm in differential algebra as presented in [18]. In practice we used the Maple V.5.1 package `diffalg` which is the implementation of such an algorithm. Information about this package can be found on the web page `http://daisy.uwaterloo.ca/~ehubert/Diffalg`.

Actually, the Lax pair for P34 equation related to the Airy kernel was found "by hand" in [11] and occurs to be gauge equivalent to the pair of Flaschka and Newell [19] for PII. The new Lax pair for PIII equation is found by use of symbolic computation. The Lax pair for PIV equation related to the integral operator with the Weber-Hermite kernel of the first kind (41), (42) coincides up to a diagonal gauge transformation with the pair of Garnier [4, 5]. The Weber-Hermite kernel of the second kind (54), (55) leads to another Lax pair for PIV equation evaluated by use of symbolic computations. We show that this Lax pair as well as the pair for PIV found in [21] are both gauge equivalent to the pair of Garnier. Finally, we observe that the Garnier systems [4, 5] for PV and PVI equations are related in the very same way to the Whittaker and hypergeometric kernels, respectively.

2 Airy-kernel and PII equation.

The simplest non-trivial example of the Lax pairs related to the integral operators (1) with the Tracy-Widom kernel (2), (3) is the pair coming from consideration of the operator with the Airy kernel (6) on the ray $(x;\infty)$. This example is found and investigated in full details in ref. [11]. Here, we give skim outlook of it.

"Vacuum" matrix equation (13) has the form (8), and because the gauge matrix $Y(\lambda)$ tends to I at infinity and contour Γ in this case consists of the ray $(x;\infty)$, the perturbed matrix equation is characterized by the matrix (19) of Jordan form in the main order at infinity with the only regular singularity at $\lambda = x$:

$$\Psi_\lambda \Psi^{-1} = A = \begin{pmatrix} a & \lambda + b \\ 1 & -a \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \tag{23}$$

$$\Psi_x \Psi^{-1} = U = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}. \tag{24}$$

The compatibility condition of (23), (24) is equivalent to the system of equations

$$\begin{aligned}
&b = -r - a^2,\\
&p = ar + \epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2}, \quad \epsilon^2 = 1,\\
&q = r^2 - a^2 r - xr - 2a\epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2}.\\
&a_x = r,\\
&r_x = 2\epsilon\sqrt{-r^3 + xr^2 - ar + \nu^2}, \quad \nu = const.
\end{aligned} \tag{25}$$

The last two equations of the system (25) yield the second-order ODE for the function r,

$$r_{xx} = \frac{(r_x)^2}{2r} - 4r^2 + 2xr - \frac{2\nu^2}{r}, \tag{26}$$

which up to scales coincides with the classical P34 equation from the list of Painlevé-Gambier [1].

The Lax pair (23), (24) is gauge equivalent to the Lax pair of Flaschka and Newell [19]

$$\frac{\partial \Psi^{FN}}{\partial \xi} = \mathcal{A}\Psi^{FN}, \quad \frac{\partial \Psi^{FN}}{\partial t} = \mathcal{U}\Psi^{FN}, \tag{27}$$

with the matrix coefficients $\mathcal{A}$ and $\mathcal{U}$ defined by

$$\mathcal{A} = \begin{pmatrix} -i\big(4\xi^2 + t + 2y^2\big) & i\big(4y\xi + \frac{\alpha}{\xi}\big) - 2z \\ -i\big(4y\xi + \frac{\alpha}{\xi}\big) - 2z & i\big(4\xi^2 + t + 2y^2\big) \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} -i\xi & iy \\ -iy & i\xi \end{pmatrix}. \tag{28}$$

The compatibility condition of (27), (28),

$$y_t = z\,, \quad z_t = 2y^3 + ty - \alpha\,, \quad \alpha = const, \tag{29}$$

is equivalent to the second Painlevé equation PII

$$y_{tt} = 2y^3 + ty - \alpha, \quad \alpha = const. \tag{30}$$

The gauge equivalence of the Lax pairs (23), (24) and (27), (28) is given by

$$\begin{aligned}
&\Psi(\lambda, x) = 2^{\frac{\sigma_3}{6}}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\xi^{\frac{\sigma_3}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\Psi^{FN}(\xi, t),\\
&\lambda = -2^{2/3}\xi^2 - 2^{-1/3}t, \quad x = -2^{-1/3}t,
\end{aligned} \tag{31}$$

where the function $u(t)$ satisfies the equation

$$u_t + \frac{1}{2}\big(z - y^2 - \frac{t}{2}\big) = 0. \tag{32}$$

The function $\Psi(\lambda, x)$ defined by (31) solves the system (23), (24) with the coefficients

$$\begin{aligned}
&a = 2^{1/3}(y+u), \quad b = -2^{2/3}(y+u)^2 + 2^{-1/3}\big(z + y^2 + \frac{t}{2}\big),\\
&p = -\frac{\alpha}{2} + \frac{1}{4} - u\big(z + y^2 + \frac{t}{2}\big), \quad q = 2^{1/3}u\Big(\alpha - \frac{1}{2} + u\big(z + y^2 + \frac{t}{2}\big)\Big),\\
&r = -2^{-1/3}\big(z + y^2 + \frac{t}{2}\big).
\end{aligned} \tag{33}$$

Inversion of the gauge transformation (31) is given by the formula

$$\begin{aligned}
&\Psi^{FN}(\xi, t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3}(\lambda - x)^{-\frac{\sigma_3}{4}}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\Psi(\lambda, x),\\
&\xi = e^{i\frac{\pi}{2}}2^{-1/3}(\lambda - x)^{1/2}, \quad t = -2^{1/3}x\,,
\end{aligned} \tag{34}$$

where the function $v(t)$ is defined by the equation

$$v = -\frac{p \pm \nu}{r} \equiv -2^{1/3}u. \tag{35}$$

The function Ψ^{FN} defined by (34) solves the system (27), (28) with the parameters

$$\begin{aligned}
&y = 2^{-1/3}(a+v), \quad \alpha = \frac{1}{2} \pm 2\nu,\\
&z = 2^{-2/3}x - 2^{1/3}r - 2^{-2/3}(a+v)^2
\end{aligned} \tag{36}$$

The equations (33) and (36) establish the known Bäcklund transformation between PII and P34 equations [1].

Note that the λ-equation in the Garnier system for PII equation found in [4, 5] has the only irregular singularity at infinity. This fact suggests there is no algebraic gauge transformation between the Lax pairs of Flaschka and Newell [19] and Garnier [4, 5].

3 Bessel kernel and PIII.

Let us consider the integral operator on $\Gamma = (x; \infty)$ with the kernel (7) related to the "vacuum" equation (13) with the matrix A_0 given by (9). Since A_0 is a Jordan matrix at infinity and the matrix of gauge transformation $Y(\lambda)$ behaves like I there, the matrix A coming from the matrix representation for the Bessel equation is also Jordan at infinity and has the only singularity at $\lambda = x$ (for our convenience, we have used before some constant transformation to make the Jordan block of the standard form):

$$\Psi_\lambda \Psi^{-1} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \tag{37}$$

$$\Psi_x \Psi^{-1} = U = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}. \tag{38}$$

The compatibility condition for the system (37), (38) is equivalent to the system of equations

$$\begin{aligned} & b=(a^2-\mu^2)\frac{y-1}{\rho}, \quad c=-\frac{\rho}{y-1}, \\ & p=\frac{x}{2}\frac{y_x}{y-1}-ay, \quad q=-(p^2-\nu^2)\frac{y-1}{\rho y}, \\ & r=\frac{\rho y}{y-1}, \\ & a_x+a\,\frac{y_x}{y-1}=\frac{x}{4}\,\frac{(y_x)^2}{y(y-1)^2}+\frac{1}{x}\big(\mu^2 y-\frac{\nu^2}{y}\big), \\ & \rho,\mu,\nu=const, \end{aligned} \tag{39}$$

where the function y satisfies the second order ODE

$$y_{xx}=\Big(\frac{1}{2y}+\frac{1}{y-1}\Big)y_x^2-\frac{y_x}{x}+\frac{2(y-1)^2}{x^2}\Big(\mu^2 y-\frac{\nu^2}{y}\Big)+\frac{2\rho}{x}\,y. \tag{40}$$

Equation (40) is the special case of the fifth Painlevé equation PV [1] with the parameters

$$\alpha=2\mu^2, \quad \beta=-2\nu^2, \quad \gamma=2\rho, \quad \delta=0.$$

It is well known [20], equation (40) is equivalent to the general third Painlevé equation PIII.

Since the known Lax pair for PIII equation [4, 5] has two irregular singularities at zero and infinity, there is no algebraic gauge transformation between the Garnier system for PIII and (37), (38).

4 Lax pairs for PIV equation.

In this section we describe several Lax pairs coming from applying the gauge transformations (17), (19) to the matrix equations exactly soluble in terms of parabolic cylinder functions and show that the known at presence Lax pairs for PIV equation are gauge equivalent.

4.1 Weber-Hermite kernel and PIV equation. Case 1.

Let us consider the integral operator (1) with the kernel (2) on the ray $\Gamma=(x;\infty)$ where $\varphi(\lambda)$, $\psi(\lambda)$ are given by the parabolic cylinder functions [7]

$$\varphi(\lambda)=w_\nu(\sqrt{2}\,\lambda), \quad \psi(\lambda)=w_{\nu-1}(\sqrt{2}\,\lambda), \quad \frac{d^2w_\nu(z)}{dz^2}=\Big(\frac{z^2}{4}-\nu-\frac{1}{2}\Big)w_\nu(z)\,. \tag{41}$$

For integer $\nu=N$ and polynomial $w_N(z)e^{z^2/2}$, such kernel was introduced in [16].

Because $w_{\nu-1}(z)=\frac{\sqrt{2}}{v}\big(w_\nu'(z)+\frac{z}{2}w_\nu(z)\big)$, where $v\neq 0$ does not depend on z, the vector $f(\lambda)=\big(\psi(\lambda);\varphi(\lambda)\big)^T$ solves the equation (4) with the matrix

$$A_0(\lambda)=\begin{pmatrix}\lambda & u\\ v & -\lambda\end{pmatrix}, \quad u=-\frac{2\nu}{v}. \tag{42}$$

The gauge transformation (17) yields the Lax pair of the following form:

$$\Psi_\lambda\Psi^{-1}=A(\lambda,x)=\begin{pmatrix}\lambda+a & b\\ c & -\lambda-a\end{pmatrix}+\frac{1}{\lambda-x}\begin{pmatrix}p & q\\ r & -p\end{pmatrix}, \tag{43}$$

$$\Psi_x\Psi^{-1}=U(\lambda,x)=-\frac{1}{\lambda-x}\begin{pmatrix}p & q\\ r & -p\end{pmatrix}, \tag{44}$$

compatibility condition of which is equivalent to the system of scalar equations

$$\begin{cases} a_x = 0, \\ b_x = 2q, \\ c_x = -2r, \\ p_x = br - cq, \\ q_x = 2(x+a)q - 2bp, \\ r_x = -2(x+a)r + 2cp. \end{cases} \tag{45}$$

This system allows the integrals

$$\begin{cases} a = a_0 = const, \\ a^2 + bc + 2p = \rho = const, \\ p^2 + qr = \gamma^2 = const, \end{cases} \tag{46}$$

moreover without loss of generality, we put

$$a = a_0 = 0, \tag{47}$$

that is equivalent to simultaneous shift of λ and x in the constant a_0. As to other integrals in (46), the parameter ρ describes the formal monodromy of the function Ψ at infinity, while γ has the same sense for the singular point $\lambda = x$. Using the integrals, we exclude p,

$$p = \frac{\rho - bc}{2}, \tag{48}$$

and one of the functions r or q,

$$r = \frac{1}{4q}\big(4\gamma^2 - (bc-\rho)^2\big), \quad \text{or} \quad q = \frac{1}{4r}\big(4\gamma^2 - (bc-\rho)^2\big), \tag{49}$$

to reduce the system (45) to one for the functions b, c and q (or r):

$$\begin{cases} b_x = 2q, \\ c_x = -\frac{1}{q}\big(2\gamma^2 - \frac{1}{2}(bc-\rho)^2\big), \\ q_x = 2xq + b(bc-\rho), \end{cases} \quad \text{or} \quad \begin{cases} b_x = \frac{1}{r}\big(2\gamma^2 - \frac{1}{2}(bc-\rho)^2\big), \\ c_x = -2r, \\ r_x = -2xr - c(bc-\rho). \end{cases} \tag{50}$$

Elementary computation shows that the functions $Q = -2q/b$ and $R = -2r/c$ satisfy the classical fourth Painlevé equation PIV [1]

$$Q = -\frac{2q}{b}: \quad Q_{xx} = \frac{Q_x^2}{2Q} + \frac{3}{2}Q^3 + 4xQ^2 + 2\big(x^2 + \rho + 1\big)Q - \frac{8\gamma^2}{Q}, \tag{51}$$

with the parameters

$$\alpha = -\rho - 1, \quad \beta = 16\gamma^2,$$

moreover b satisfies the first order linear ODE and becomes the so-called auxiliary function

$$\frac{b_x}{b} = -Q, \tag{52}$$

while all other coefficients are determined uniquely:

$$c = \frac{1}{b}\Big(-\frac{1}{2}Q_x + \frac{1}{2}Q^2 + xQ + \rho\Big), \quad p = -\frac{1}{2}(bc-\rho),$$

$$q = -\frac{1}{2}bQ, \quad r = \frac{1}{4q}\big(4\gamma^2 - (bc-\rho)^2\big).$$

Similarly,

$$R = -\frac{2r}{c}: \quad R_{xx} = \frac{R_x^2}{2R} + \frac{3}{2}R^3 + 4xR^2 + 2\big(x^2 + \rho - 1\big)R - \frac{8\gamma^2}{R}, \tag{53}$$

which coincides with PIV equation [1] with the parameters

$$\alpha = -\rho + 1, \quad \beta = 16\gamma^2,$$

while

$$\frac{c_x}{c} = R, \quad b = \frac{1}{c}\Big(\frac{1}{2}R_x + \frac{1}{2}R^2 + xR + \rho\Big), \quad p = -\frac{1}{2}(bc - \rho),$$
$$r = -\frac{1}{2}cR, \quad q = \frac{1}{4r}\big(4\gamma^2 - (bc - \rho)^2\big).$$

4.2 Weber-Hermite kernel and PIV equation. Case 2.

Another kind of the integral operator on the ray $\Gamma = (x;\infty)$ with the kernel (2) is determined by the functions $\varphi(\lambda)$, $\psi(\lambda)$ given by the parabolic cylinder function and its derivative:

$$\varphi(\lambda) = w_\nu(\sqrt{2}\,\lambda), \quad \psi(\lambda) = \sqrt{2}\,w_\nu'(\sqrt{2}\,\lambda), \quad \frac{d^2 w_\nu(z)}{dz^2} = \Big(\frac{z^2}{4} - \nu - \frac{1}{2}\Big)w_\nu(z)\,. \tag{54}$$

Now, the vector $f(\lambda) = \big(\psi(\lambda), \varphi(\lambda)\big)^T$ solves (4) with the matrix

$$A_0(\lambda) = \begin{pmatrix} 0 & \lambda^2 - 2\nu - 1 \\ 1 & 0 \end{pmatrix}. \tag{55}$$

The gauge transformation (17) yields the Lax pair of the following form:

$$\Psi_\lambda \Psi^{-1} = A(\lambda, x) = \begin{pmatrix} a\lambda + u & \lambda^2 + b\lambda + v \\ w & -a\lambda - u \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \tag{56}$$

$$\Psi_x \Psi^{-1} = U(\lambda, x) = -\frac{1}{\lambda - x}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \tag{57}$$

with the compatibility condition as being equivalent to the system of scalar equations

$$\begin{cases} a_x = r, \\ b_x = -2p, \\ u_x = (b + x)r, \\ v_x = -2p(b + x) + 2aq, \\ w_x = -2ar, \\ p_x = xr(b + x) + vr - wq, \\ q_x = -2xp(b + x) + 2xaq + 2uq - 2pv, \\ r_x = -2xar - 2ur + 2pw, \end{cases} \tag{58}$$

which allows the integrals

$$\begin{cases} a^2 + w = 1, \\ 2au + r + bw = 0, \\ 2ap + u^2 + (b + x)r + vw - a = \rho, \quad \rho = const, \\ p^2 + qr = \gamma^2, \quad \gamma = const. \end{cases} \tag{59}$$

As before, we choose the concrete values of the constants $a^2 + w$ and $2au + r + bw$ without loss of generality because these parameters are related to the affine transformations of $\mathbb{C}^2$ space of pairs (λ, x), so putting these constants to unit and zero, respectively, we do nothing but fix certain origin and scales.

In contrast, the parameters ρ and γ describe formal monodromy of Ψ-function at infinity and at the point $\lambda = x$, respectively.

Using the integrals (59), we reduce the system (58) to the system of four first order equations for the functions a, b, p and r, and observe that the ratio $R = -2r/w$ satisfies PIV equation [1]

$$R = -\frac{2r}{w}: \quad R_{xx} = \frac{R_x^2}{2R} + \frac{3}{2}R^3 + 4xR^2 + 2\big(x^2 + \rho\big)R - \frac{8\gamma^2}{R}, \tag{60}$$

with the parameters

$$\alpha = -\rho, \quad \beta = 16\gamma^2,$$

while the functions a and b satisfy the linear first order ODE and become auxiliary functions

$$\frac{a_x}{a^2-1} = \frac{1}{2}R, \quad b_x = \frac{a^2-1}{2a}Rb + \frac{1}{2}R_x + xaR + \frac{a^2+1}{4a}R^2.$$

The rest of the coefficients are determined uniquely by the expressions

$$u = \frac{1-a^2}{4a}(R-2b), \quad w = 1-a^2, \quad p = -\frac{1}{4}R_x - \frac{1}{2}(xa+u)R - \frac{1}{4}aR^2,$$

$$q = 2\frac{p^2-\gamma^2}{(1-a^2)R}, \quad v = \frac{1}{1-a^2}\big(a + \rho - 2ap - u^2\big) + \frac{1}{2}(b+x)R.$$

4.3 Gauge equivalence of the Lax pairs for PIV equation

Gauge equivalence of the Garnier system and (43), (44). Jimbo and Miwa [5] presented the Lax pair originated in the work of Garnier [4], which, after proper change of notations and the shift of λ in x, can be written in the following form:

$$\frac{\partial \Psi^{JM}}{\partial \lambda} = \mathcal{A}\Psi^{JM}, \quad \frac{\partial \Psi^{JM}}{\partial x} = \mathcal{U}\Psi^{JM}, \tag{61}$$

with the connection coefficients $\mathcal{A}$ and $\mathcal{U}$ defined as follows:

$$\mathcal{A} = \begin{pmatrix} \lambda & u \\ \frac{2}{u}(z-\tau) & -\lambda \end{pmatrix} + \frac{1}{\lambda - x}\begin{pmatrix} -z & -\frac{uy}{2} \\ \frac{2(z^2-\theta^2)}{uy} & z \end{pmatrix}, \tag{62}$$

$$U = -x\sigma_3 - \frac{1}{\lambda - x}\begin{pmatrix} -z & -\frac{uy}{2} \\ \frac{2(z^2-\theta^2)}{uy} & z \end{pmatrix}. \tag{63}$$

The matrix $\mathcal{A}$ (62) up to proper change of notations coincides with the matrix A (43), but the matrix $\mathcal{U}$ (63) contains an extra term $-x\sigma_3$ in compare with U (44). This term can be eliminated by the diagonal gauge transformation independent from λ, i.e.

$$\Psi(\lambda, x) = e^{\frac{x^2\sigma_3}{2}}\Psi^{JM}(\lambda, x). \tag{64}$$

Gauge equivalence of an alternative Lax pair and (43), (44). Another Lax pair for PIV equation found by Kitaev [21] and later by Milne, Clarkson, Bassom and Hicks [22]

$$\frac{\partial \Psi^{K}}{\partial \lambda} = \mathcal{A}\Psi^{K}, \quad \frac{\partial \Psi^{K}}{\partial x} = \mathcal{U}\Psi^{K}, \tag{65}$$

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2}\lambda^3 + \lambda(x+uv) + \frac{\alpha}{\lambda} & i\big(\lambda^2 u + 2xu + u'\big) \\ i\big(\lambda^2 v + 2xv - v'\big) & -\frac{1}{2}\lambda^3 - \lambda(x+uv) - \frac{\alpha}{\lambda} \end{pmatrix}, \tag{66}$$

$$\mathcal{U} = \begin{pmatrix} \frac{1}{2}\lambda^2 + uv & i\lambda u \\ i\lambda v & -\frac{1}{2}\lambda^2 - uv \end{pmatrix}, \tag{67}$$

comes from the Lax pair of Kaup and Newell [23] for the Modified Nonlinear Schrödinger Equation (MNS) by use of similarity reduction of MNS equation to PIV equation of Ablowitz, Ramani and Segur [24]. This pair is also gauge equivalent to the Lax pair (61)–(63). Indeed, if $\Psi^K(\lambda, x)$ satisfies (65)–(67), then

$$\Psi(\lambda,x) = e^{\frac{x^2\sigma_3}{2}} \begin{pmatrix} 1 & iu \\ 0 & 1 \end{pmatrix} \big(2(\lambda - x)\big)^{\frac{\sigma_3}{8}} \Psi^K\big(\sqrt{2(\lambda-x)}, x\big) \tag{68}$$

satisfies our first degree Lax pair (43)–(44) with

$$a = 0, \quad b = ie^{x^2}(u' - u^2 v), \quad c = ie^{-x^2} v, \quad p = \frac{\alpha}{2} + \frac{1}{4} - xuv + \frac{uv'}{2},$$

$$q = -iue^{x^2}\big(\alpha + \frac{1}{2} - xuv + \frac{uv'}{2}\big), \quad r = \frac{e^{-x^2}}{iu}\big(-xuv + \frac{uv'}{2}\big).$$

Gauge equivalence of (43), (44) and (56), (57). The Lax pairs (43), (44) and (56), (57) are also gauge equivalent to each other. Indeed, if $\Psi(\lambda)$ solves the system (56), (57), then the function

$$\tilde{\Psi}(\lambda) = \begin{pmatrix} 1 & \lambda s + t \\ 0 & 1 \end{pmatrix} \Psi(\lambda), \quad s = \frac{1}{a+1}, \quad t = 2\frac{ab(a+1)+u}{(a-2)(a+1)^2}, \tag{69}$$

satisfies the system (43), (44) with the parameters

$$\tilde{a} = u + sr + tw, \quad \tilde{b} = v + s - 2sp - 2tu - xs^2 r - 2str - t^2 w, \quad \tilde{c} = w,$$

$$\tilde{p} = p + sx + tr, \quad \tilde{q} = q - 2spx - 2tp - x^2 s^2 r - 2xstr - t^2 r, \quad \tilde{r} = r,$$

so that the Painlevé functions $R = -2r/w$ (60) and $\tilde{R} = -2\tilde{r}/\tilde{c}$ (53) coincide with each other. For the integral operator (2), the transformation (69) yields its regular perturbation $K \mapsto K + R$ with the degenerated integral operator R with the kernel $R(\lambda, \mu) = f(\lambda) g(\mu)$, where f and g are some parabolic cylinder functions.

5 Hypergeometric and confluent hypergeometric kernels

In this short section we note that the Garnier systems [4, 5] for PV and PVI equation, i.e.

$$\Psi_\lambda \Psi^{-1} = A_0(\lambda) + \frac{B}{\lambda - x}, \quad \Psi_x \Psi^{-1} = -\frac{B}{\lambda - x}, \tag{70}$$

with the matrix

$$A_0(\lambda) = \frac{1}{2}\sigma_3 + \frac{B_0}{\lambda}, \tag{71}$$

in the case of PV equation, and

$$A_0(\lambda) = \frac{B_0}{\lambda} + \frac{B_1}{\lambda - 1}, \tag{72}$$

in the case of PVI, have the structure described by (20), (21). Furthermore, the corresponding "vacuum" equations (13)

$$\Phi_\lambda \Phi^{-1} = A_0(\lambda)$$

are soluble in terms of the Whittaker functions for (71) and in terms of hypergeometric functions for (72). Thus there is no surprise that the Laguerre and Jacobi polynomial kernels introduced by Tracy and Widom in [16] are related to the fifth and the sixth Painlevé equations.

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References

[1] E. L. Ince, *Ordinary differential equations,* New York: Dover, 1959.

[2] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen. *Math. Annalen*, **63** (1907) 301-321.

[3] P. Boutroux, Recherches sur les transcendantes de M.Painlevé et l'étude asymptotique des équationes differéntielle du second ordre. *Ann. Sci. Ec. Norm. Super.*, (3) **30** (1913) 255-375; **31** (1914) 99-159.

[4] R. Garnier, Etude de l'intégrale générale de l'équation VI de M.Painlevé, *Ann. Sci. Ecole Norm. Super.*, 3^e série, **34** (1917) 239-353.

[5] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II. *Physica* **2** *D* (1981) 407-448.

[6] A. R. Its, A. A. Kapaev and A. V. Kitaev, The Isomonodromic deformation method and special functions. *in: Problems of theoretical physics. III.*, Yu. N. Demkov, Yu. V. Novozhilov and P. P. Pavinsky, eds., Leningrad St. Univ., 1988, 182-192.

[7] H. Bateman and A. Erdélyi, *Higher Transcendental Functions,* McGraw-Hill, NY, 1953.

[8] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, Differential equations for quantum correlation functions, *Int. J. Mod. Phys.*, **B4** (1990) 1003-1037; The quantum correlation function as the τ function of classical differential equations, 407-417, in: *Important developments in soliton theory*, eds. A. S. Fokas and V. E. Zakharov, Springer-Verlag, Berlin, 1993.

[9] C. A. Tracy and H. Widom, Systems of partial differential equations for a class of operator determinants, Partial differential operators and mathematical physics (Holzhau, 1994), 381-388, eds.: M. Denuth and B. Schulze, *Operator Theory: Advances and Applications*, **78** Birkhäuser, Basel, 1995; ——— Fredholm determinants, differential equations and matrix models, *Commun. Math. Phys.*, **163** (1994) 33-72.

[10] P. Deift, Integrable operators, in: Differential Operators and Spectral Theory, M. Sh. Birman's 70-th Anniversary Collection, V. Buslaev, M. Solomyak and D. Yafaev eds., *AMS Translations, Advances in the Mathematical Sciences–41*, ser. 2, Vol. 189 (1999) 69-84.

[11] P. Bleher, A. Bolibrukh, A. Its and A. Kapaev, Linearization of P34 equation of Painlevé-Gambier (to appear).

[12] M. Jimbo, T. Miwa, Y. Môri and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Physica* **1D** (1980) 80-158.

[13] M. L. Mehta, Random matrices. 2-nd edition, San Diego: Academic Press, 1991.

[14] C. A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel. *Commun. Math. Phys.* **159** (1994) 151-174.

[15] C. A. Tracy and H. Widom, Level-spacing distributions and the Bessel kernel. *Commun. Math. Phys.* **161** (1994) 289-309.

[16] C. A. Tracy and H. Widom, Fredholm Determinants, Differential Equations and Matrix Models. *PBB: xxx.lanl.gov*, hep-th/9306042 v3 2 Feb 1999.

[17] N. I. Muskhelishvili, *Singular Integral Equations*, 2-nd Edition, P. Noordhoff N. V., Groningen-Holland, 1953.

[18] E. Hubert, Factorization free decomposition algorithms in differential algebra. *Journal of Symbolic Computations*, to appear.

[19] H. Flaschka and A. C. Newell, Monodromy- and spectrum preserving deformations I. *Commun. Math. Phys.*, **76** (1980) 67-116.

[20] V. I. Gromak, Theory of Painlevé's equation. *Diff. Uravnen.* **11** (1975) 373-376.

[21] A. V. Kitaev, On the self-similar solutions of the modified nonlinear Schrödinger equation. *Teor. Mat. Fizika*, **64** (1985) 347-369.

[22] A. E. Milne, P. A. Clarkson and A. P. Bassom, Application of the isomonodromy deformation method to the fourth Painlevé equation. *Inverse Problems* **13** (1997) 421-439.

[23] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **19** (1978) 798-801.

[24] M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of p-type. *J. Math. Phys.* **21** (1980) 1006-1015.