Factorization of piecewise constant matrix functions and systems of linear differential equations

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Abstract

Let G be a piecewise constant $n \times n$ matrix function which is defined on a smooth closed curve Γ in the complex sphere and which has m jumps. We consider the problem of determining the partial indices of the factorization of the matrix function G in the space $L^p(\Gamma)$. We show that this problem can be reduced to a certain problem for systems of linear differential equations.

Studying this related problem, we obtain some results for the partial indices for general n and m. A complete answer is given for $n = 2$, $m = 4$ and for $n = m = 3$. One has to distinguish several cases. In some of these cases, the partial indices can be determined explicitly. In the remaining cases, one is led to two possibilities for the partial indices. The problem of deciding which is the correct possibility is equivalent to the description of the monodromy of n-th order linear Fuchsian differential equations with m singular points.

1 Introduction

Let Γ be a smooth closed positively oriented curve in the complex sphere $\dot{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which divides $\dot{\mathbb{C}}$ into two domains D_+ and D_-. Assume without loss of generality that $0 \in D_+$ and $\infty \in D_-$. Let $L^p(\Gamma)$ be the Banach space of all Lebesgue measurable functions on Γ for which

$$\|f\|_{L^p(\Gamma)} \;=\; \Big(\int_\Gamma |f(\tau)|^p \, |d\tau| \Big)^{1/p} \;<\; \infty. \tag{1.1}$$

For $1 < p < \infty$, consider the singular integral operator S_Γ defined on $L^p(\Gamma)$,

$$(S_\Gamma f)(t) \;=\; \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} \, d\tau, \qquad t \in \Gamma. \tag{1.2}$$

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The operator S_Γ is bounded on $L^p(\Gamma)$ and satisfies $S_\Gamma^2 = I$ (see [10]). Introduce the projections

$$P_\Gamma \quad = \quad (I + S_\Gamma)/2, \qquad Q_\Gamma \quad = \quad (I - S_\Gamma)/2, \tag{1.3}$$

and the Banach spaces

$$L^p_+(\Gamma) \quad = \quad P_\Gamma(L^p(\Gamma)), \qquad L^p_-(\Gamma) \quad = \quad Q_\Gamma(L^p(\Gamma)) \dotplus \mathbb{C}, \tag{1.4}$$

i.e., the image of P_Γ and the image of Q_Γ plus the set of constant functions.

It is well known ([11], see also [15]) that functions $f \in L^p_+(\Gamma)$ can be identified with functions $\hat{f}$ which are analytic in D_+ and for which there exists an expanding sequence of domains D_k with rectifiable boundaries Γ_k such that $D_k \cup \Gamma_k \subset D_+$, $\bigcup_k D_k = D_+$, and $\sup_k \int_{\Gamma_k} |f(t)|^p \, |dt| < \infty$. In fact, D_k can always be chosen as the image of $\{\, z\colon |z| < r_k \,\}$ under a conformal mapping of the unit disk $\mathbb{D} = \{\, z\colon |z| < 1 \,\}$ onto D_+, where r_k is an arbitrary increasing sequence of positive numbers converging to 1. The function f is given by the non-tangent limits of the function $\hat{f}$ at the boundary Γ of D_+. Conversely, the function $\hat{f}$ represents the analytic extension of f into D_+. A similar result holds for $L^p_-(\Gamma)$.

Let $L^\infty(\Gamma)$ denote the set of all Lebesgue measurable and essentially bounded functions on Γ. A *factorization* of an $n \times n$ matrix function $G \in L^\infty(\Gamma)^{n\times n}$ in the space $L^p(\Gamma)$ is a representation of the form

$$G(t) \quad = \quad G_+(t)\Lambda(t)G_-(t), \qquad t \in \Gamma, \tag{1.5}$$

where $\Lambda(t) = \operatorname{diag}(t^{\varkappa_1}, \ldots, t^{\varkappa_n})$ is a diagonal matrix with $\varkappa_1, \ldots, \varkappa_n \in \mathbb{Z}$, and the factors G_+ and G_- satisfy the following conditions:

(i) $G_+ \in L^p_+(\Gamma)^{n\times n}$, $G_+^{-1} \in L^q_+(\Gamma)^{n\times n}$,

(ii) $G_- \in L^q_-(\Gamma)^{n\times n}$, $G_-^{-1} \in L^p_-(\Gamma)^{n\times n}$.

Here $1/p + 1/q = 1$. The integers $\varkappa_1, \ldots, \varkappa_n$ are called the *partial indices* of the factorization.

Due to (i)-(ii), the operator $G_-^{-1} Q_\Gamma G_+^{-1}$ is defined on the (dense in $L^p(\Gamma)^n$) set of rational vector functions with poles off Γ and maps this set into $L^1(\Gamma)^n$. If this mapping is bounded in the L^p-norm, then it can be extended by continuity to a bounded operator on the whole space $L^p(\Gamma)^n$. In this case, the representation (1.5) is called a Φ-*factorization* in the space $L^p(\Gamma)$.

A detailed discussion of the Φ-factorization, its properties and the history of the subject can be found in [7, 15]; the latter monograph touches also on the (not necessarily Φ-) factorization in $L^p(\Gamma)$. The proofs of all the subsequent results stated in this section can be found in [7, 15]. The next theorem explains why the notion of the Φ-factorization is natural and important.

Theorem 1.1 *The matrix function $G \in L^\infty(\Gamma)^{n\times n}$ admits a Φ-factorization in the space $L^p(\Gamma)$ if and only if the singular integral operator $A = P_\Gamma + GQ_\Gamma$ is a Fredholm operator on the space $L^p(\Gamma)^n$. In this case, we have*

$$\dim\ker A \;=\; \sum_{\varkappa_j>0} \varkappa_j, \qquad \dim\ker A^* \;=\; -\sum_{\varkappa_j<0} \varkappa_j.$$

Hence the partial indices $\varkappa_1, \dots, \varkappa_n$ contain important information about the operator A. In particular, A is invertible if and only if the function G admits a so-called *canonical Φ-factorization*, i.e., a Φ-factorization where all partial indices are zero. If A is a Fredholm operator, then its index, $\operatorname{ind} A := \dim\ker A - \dim\ker A^*$, is equal to

$$\varkappa \;=\; \sum_{j=1}^n \varkappa_j, \tag{1.6}$$

the *total index* of the factorization.

The notion of Φ-factorization plays also an important role in the study of more general singular integral operators. One can show that the operator $G_1P_\Gamma + G_2Q_\Gamma$ with $G_1, G_2 \in L^\infty(\Gamma)^{n\times n}$ is a Fredholm operator if and only if the functions G_1 and G_2 are invertible in $L^\infty(\Gamma)^{n\times n}$ and the function $G_1^{-1}G_2$ admits a Φ-factorization.

The factorization of a matrix function is not unique. However, the partial indices are always uniquely determined up to the change of order. For this reason, we will assume that

$$\varkappa_1 \;\geq\; \varkappa_2 \;\geq\; \dots \;\geq\; \varkappa_n. \tag{1.7}$$

The relation between factors G_+ and G_- corresponding to different factorizations of a function G is described in the following theorem. The characterization involves certain rational block triangular matrix functions whose size is determined by the multiple occurrence of same values for the partial indices. Let I_l denote the identity matrix of size $l \times l$, and let $G\mathbb{C}^{l\times l}$ stand for the group of all invertible $l \times l$ matrices.

Theorem 1.2 *Assume that $G \in L^\infty(\Gamma)^{n\times n}$ admits two factorizations in $L^p(\Gamma)$,*

$$G(t) \;=\; G_+(t)\Lambda(t)G_-(t) \;=\; \widetilde{G}_+(t)\Lambda(t)\widetilde{G}_-(t), \qquad t \in \Gamma,$$

where $\Lambda(t) = \operatorname{diag}(t^{\overline{\varkappa}_1}I_{l_1}, t^{\overline{\varkappa}_2}I_{l_2}, \dots, t^{\overline{\varkappa}_k}I_{l_k})$ with $\overline{\varkappa}_1, \dots, \overline{\varkappa}_k \in \mathbb{Z}$, $\overline{\varkappa}_1 > \overline{\varkappa}_2 > \dots > \overline{\varkappa}_k$, $l_1, \dots, l_k \in \{1, 2, \dots\}$, $l_1 + \dots + l_k = n$. Then

$$G_+(t) \;=\; \widetilde{G}_+(t)V(t), \qquad \widetilde{G}_-(t) \;=\; U(t)G_-(t), \tag{1.8}$$

where $U(t)$ and $V(t)$ are matrix functions of the form

$$V(t) \;=\; \begin{pmatrix} A_{11} & V_{12}(t) & \dots & V_{1k}(t) \\ 0 & A_{22} & & V_{2k}(t) \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{pmatrix}, \quad U(t) \;=\; \begin{pmatrix} A_{11} & U_{12}(t) & \dots & U_{1k}(t) \\ 0 & A_{22} & & U_{2k}(t) \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{pmatrix},$$

with $A_{jj} \in G\mathbb{C}^{l_j\times l_j}$, $V_{ij}(t) = \sum_{r=0}^{\overline{\varkappa}_i-\overline{\varkappa}_j} A_{ij}^{(r)}t^r$, $U_{ij}(t) = t^{\overline{\varkappa}_j-\overline{\varkappa}_i}V_{ij}(t)$ and $A_{ij}^{(r)} \in \mathbb{C}^{l_i\times l_j}$.

In fact, also the converse of the above theorem is true. Suppose we are given a factorization (Φ-factorization) $G = G_+ \Lambda G_-$, and we determine $\widetilde{G}_+$ and $\widetilde{G}_-$ by (1.8), then also $G = \widetilde{G}_+ \Lambda \widetilde{G}_-$ is a factorization (Φ-factorization). Using this theorem and its converse, we can conclude the following statement.

Corollary 1.3 *Suppose $G \in L^\infty(\Gamma)^{n\times n}$ admits a Φ-factorization and $G = G_+ \Lambda G_-$ is a factorization. Then $G = G_+ \Lambda G_-$ is such a Φ-factorization.*

Let $PC(\Gamma)^{n\times n}$ stand for the set of all *piecewise continuous $n \times n$ matrix functions*, i.e., functions for which the one-sided limits $G(\tau+0)$ and $G(\tau-0)$ exist at each $\tau \in \Gamma$. Here $G(\gamma(\theta_0) \pm 0) := \lim_{\theta \to \theta_0 \pm 0} G(\gamma(\theta))$, where the periodic function $\gamma : \mathbb{R} \to \Gamma$ is a parameterization of the positively oriented curve Γ. The following result solves the problem of the existence of a Φ-factorization for functions in $PC(\Gamma)^{n\times n}$.

Theorem 1.4 *The matrix function $G \in PC(\Gamma)^{n\times n}$ admits a Φ-factorization in the space $L^p(\Gamma)$ if and only if the following is satisfied:*

(i) The matrices $G(\tau + 0)$ and $G(\tau - 0)$ are invertible for each $\tau \in \Gamma$.

(ii) For each $j = 1, \dots, n$ and each $\tau \in \Gamma$, we have

$$\frac{1}{2\pi} \arg \lambda_j(\tau) + \frac{1}{p} \;\notin\; \mathbb{Z}, \tag{1.9}$$

where $\lambda_1(\tau), \dots, \lambda_n(\tau)$ are the eigenvalues of the matrices $G(\tau - 0)G(\tau + 0)^{-1}$.

If the assumptions of this theorem are satisfied, then one can define the numbers $\zeta_j(\tau)$ as the (unique) values of $-(2\pi)^{-1} \arg \lambda_j(\tau)$ which lie in the interval $J_p := (1/p - 1, 1/p)$.

It is also possible to evaluate the total index $\varkappa$ related to the above factorization. For simplicity assume that G has only finitely many jumps at points $a_1, \dots, a_m \in \Gamma$, which lie in this order on the positively oriented curve Γ. Then

$$\varkappa \;=\; \sum_{k=1}^{m} \left[\frac{1}{2\pi} \arg \det G(\tau) \right]_{\tau = a_k + 0}^{a_{k+1} - 0} + \sum_{k=1}^{m} \sum_{j=1}^{n} \zeta_j(a_k). \tag{1.10}$$

Here $a_{m+1} = a_1$, and $[\dots]$ denotes the total increment along the subarcs on which G is continuous. Note that, in general, the value of $\varkappa$ depends on the space $L^p(\Gamma)$ because so do the numbers $\zeta_j(\tau)$. Only in the case where all $\lambda_j(\tau)$ are positive real numbers (hence $\zeta_j(\tau) = 0$), the value of $\varkappa$ does not depend on the underlying space $L^p(\Gamma)$.

The calculation of the partial indices $\varkappa_1, \dots, \varkappa_n$ in the case $n > 1$ is a more delicate problem and can be solved only in special situations. It is the purpose of this paper to consider the case of *piecewise constant matrix functions*, i.e., functions which have only a finite number of jump discontinuities and which are constant along the subarcs joining these points. We will show that the problem of determining the partial indices can be reduced to a certain problem for systems of linear differential equations. Elaborating on this related problem, we will obtain some information about the partial indices.

Preparing the following sections, we define certain matrices. Let $G \in PC(\Gamma)^{n\times n}$ be a piecewise constant matrix function with m jumps at the points $a_1, \dots, a_m \in \Gamma$, which are arranged on Γ in this order. Assume that G admits a Φ-factorization in the space $L^p(\Gamma)$. Introduce the matrices

$$M_k \;=\; G(a_k - 0)G(a_k + 0)^{-1}, \qquad k = 1, \dots, m. \tag{1.11}$$

Because G is constant along the subarcs joining a_k and a_{k+1} and takes the values $G(a_k + 0) = G(a_{k+1} - 0)$ there, these matrices satisfy the relation

$$M_1 M_2 \cdots M_m \;=\; I. \tag{1.12}$$

Moreover, let $E_1, \dots, E_m$ be matrices such that

$$M_k \;\sim\; \exp(-2\pi i E_k), \qquad k = 1, \dots, m, \tag{1.13}$$

and the real parts of the eigenvalues of E_k are contained in the interval J_p. Here "$\sim$" stands for similarity of matrices. Because G admits a Φ-factorization and J_p has length one, the matrices E_k exist and are uniquely determined up to similarity. In fact, the numbers $\zeta_1(a_k), \dots, \zeta_n(a_k)$ defined above are equal to the real parts of the eigenvalues of E_k. Formula (1.10) for the total index can also be simplified:

$$\varkappa \;=\; \sum_{k=1}^{m} \operatorname{trace} E_k. \tag{1.14}$$

Note that the matrices $E_1, \dots, E_m$ depend on the underlying space $L^p(\Gamma)$. Roughly speaking, they contain the information about the "proper" choice of the logarithm of the eigenvalues of $M_1, \dots, M_m$. For sake of further reference, the m-tuples of $n \times n$ matrices

$$[M_1, \dots, M_m] \quad \text{and} \quad [E_1, \dots, E_m] \tag{1.15}$$

will be called the "*data*" associated with the piecewise constant matrix function G with respect to the space $L^p(\Gamma)$.

Let us recall what is already known about the factorization of piecewise constant $n \times n$ matrix functions with m jump discontinuities. Any piecewise constant scalar function ($n = 1$) can be easily factored explicitly. Namely, $G(t) = G_+(t)t^{\varkappa}G_-(t)$, where

$$G_+(t) \;=\; c\prod_{k=1}^{m}(t - a_k)^{-\varepsilon_k}, \qquad G_-(t) \;=\; \prod_{k=1}^{m}\left(1 - \frac{a_k}{t}\right)^{\varepsilon_k}, \tag{1.16}$$

$\varepsilon_k = E_k$, and $\varkappa$ is given by (1.14). The branches of the analytic functions in (1.16) are chosen in such a way that $G_\pm$ is analytic on $D_\pm$, and c is a suitable nonzero constant.

Hence, the case $n = 1$ is solvable for any m. Another easy case is $m = 2$ for any n. Indeed, multiplying G by the constant matrix $G(a_1 - 0)^{-1}$ on the left, we may suppose without loss of generality that G assumes the values I and $M(= M_2)$ only. These values obviously commute with each other. In other words, G is a so-called *functionally commuting* matrix function, and its factorization can be constructed explicitly (see [15, Section 4.4]).

The simplest non-trivial case ($n = 2$, $m = 3$) was first studied by Zverovich and Khvoschinskaya [20] in a slightly different setting. In the setting considered here it was treated by Tashbaev and one of the authors in [18]. Using the appropriate modification of the results from [20], they gave explicit formulas for the factors and the partial indices. Therein they had to distinguish several cases. Apart from trivial cases (where G can be reduced to a functionally commuting matrix function), the factors were constructed by using the hypergeometric function. We will not restate here the complete results of [18], however, the results on the partial indices will be obtained again in a corollary in Section 6 in the setting of the related problem.

Let us describe the content of this paper in more detail. In Section 2, we recall some basic facts from the theory of systems of linear differential equations and from the theory of scalar linear differential equations (of higher order).

In Section 3, we introduce a class of $n \times n$ systems $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ of linear differential equations, which we call systems of standard form. Their solutions $Y(\tilde{z})$ are characterized by the monodromy, which is given by $[M_1, \ldots, M_m]$, and by the behavior at the singularities $a_1, \ldots, a_m \in \mathbb{C}$, which is given by $[E_1, \ldots, E_m]$. In addition, we allow an apparent singularity at infinity, where the behavior is described by certain integers $\varkappa_1, \ldots, \varkappa_n$, called the indices. Then we prove an "existence" and a "uniqueness" theorem for such systems. Moreover, we show that these systems of linear differential equations provide us with a solution to our factorization problem. Regarding the calculation of the partial indices, we are led to the following (well-posed) question: What are the indices of the systems which are characterized by given data $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ and singularities $a_1, \ldots, a_m$? In fact, the indices for the systems correspond to the partial indices of the factorization.

In Section 4, we reformulate the conditions regarding the behavior at the singularities at $a_1, \ldots, a_m$ and at infinity in terms of the matrix function $A(z)$ rather than in terms of the solution $Y(\tilde{z})$. This is a simple step, but all further considerations are based on it. We note that the above question can now be expressed as follows: What is the monodromy of a system given by $A(z)$, where $A(z)$ has a certain form?

In Section 5, we obtain two "general" results. Firstly, if the monodromy is irreducible, then the indices (which are supposed to be ordered decreasingly) satisfy $\varkappa_k - \varkappa_{k+1} \leq m - 2$ for each $k = 1, \ldots, n-1$. Secondly, we show that this condition is sharp. For this we produce a subclass of systems for which $\varkappa_k - \varkappa_{k+1} = m - 2$ for each $k = 1, \ldots, n-1$. It turns out that the monodromy of such systems coincides with the monodromy of certain scalar Fuchsian linear differential equations of n-th order and with m singularities. We remark that Bolibruch [5] already obtained the first result in the setting of vector bundles (and by using a different method) and that there is also a strong connection of the second result with his work.

In Section 6, we consider the case $n = 2$. One is led naturally to distinguishing three cases relating to the "reducibility type" of $[M_1, \ldots, M_m]$, i.e., the structure of the invariant subspaces of these matrices. For general m, we can determine the indices in some cases explicitly. In the remaining cases, we obtain the estimate $\varkappa_1 - \varkappa_2 \leq m - 2$. These remaining cases include irreducible ones (for which this result is already clear from Section 5) as well as certain reducible ones. The essential point in the proof is a theorem which characterizes the data (hence, in particular, the monodromy) of a certain class of 2×2 triangular systems.

With these results we are immediately in a position to give an explicit answer for the case $n = 2$, $m = 3$, hence restating the results of [18]. However, the results obtained here are slightly more general because the assumptions on the matrices $E_1, \ldots, E_m$ are less restrictive.

The complete answer is given for the case $n = 2$ and $m = 4$ by resorting in addition to the second result of Section 5. However, the answer is explicit only in some cases. In fact, in the other cases, we are led to two possibilities for the indices. The problem of deciding which possibility is the correct one is equivalent to the description of the monodromy of second order linear Fuchsian differential equations with four singular points.

In Section 7, we treat the case $n = m = 3$. It is technically much more complicated than the previous case as now nine reducibility types for $[M_1, M_2, M_3]$ occur. We also have to prove theorems which characterize the data (monodromy) of triangular and block-triangular systems. Finally, in the main theorem we arrive at several cases, where, similar as before, in some cases the indices can be given explicitly whereas in the other cases they depend on the description of the monodromy of third order linear Fuchsian differential equations with three singular points.

2 Systems of linear differential equations

A system of linear differential equations in the complex domain can be written as

$$\frac{dy}{dz} = A(z)y, \tag{2.1}$$

where $A(z)$ is a given $n \times n$ matrix function. We assume that $A(z)$ is analytic on the complex plane $\mathbb{C}$ with the exception of a finite number of points, which are called the *singularities* of the system. The point $a = \infty$ is by definition a singularity if the function $z^{-2}A(z^{-1})$ is not analytic at $z = 0$, which is justified by a change of variables $z \mapsto 1/z$.

Let $a_1, \ldots, a_m \in \dot{\mathbb{C}}$ be the singularities of the system, and let $\widetilde{S}$ be the *universal covering surface* of $S := \dot{\mathbb{C}} \setminus \{a_1, \ldots, a_m\}$. Then the solutions y of (2.1) are analytic functions defined on $\widetilde{S}$ and taking values in $\mathbb{C}^n$. Associated with $\widetilde{S}$ there is a *covering map* $\rho : \widetilde{S} \to S$. Points on $\widetilde{S}$ will usually be denoted by $\tilde{z}$ and points on S by $z = \rho(\tilde{z})$. The set of all solutions of (2.1) forms an n-dimensional linear space.

Taking n linearly independent solutions $y_1(\tilde{z}), \ldots, y_n(\tilde{z})$, one can consider the analytic $n \times n$ matrix function $Y(\tilde{z}) = (y_1(\tilde{z}), \ldots, y_n(\tilde{z}))$. It satisfies the matrix equation

$$Y'(\tilde{z}) \;=\; A(z)Y(\tilde{z}), \qquad \tilde{z} \in \widetilde{S}, \tag{2.2}$$

and we have $\det Y(\tilde{z}) \neq 0$ for all $\tilde{z} \in \widetilde{S}$. In fact, for any solution of (2.2) with $\det Y(\tilde{z}_0) \neq 0$ for some $\tilde{z}_0 \in \widetilde{S}$ we have $\det Y(\tilde{z}) \neq 0$ for all $\tilde{z} \in \widetilde{S}$. We will consider only such solutions (i.e., analytic functions $Y : \widetilde{S} \to G\mathbb{C}^{n\times n}$) because only they contain the full information. Any two solutions $Y_1(\tilde{z})$ and $Y_2(\tilde{z})$ are related to each other by $Y_2(\tilde{z}) = Y_1(\tilde{z})C$ with $C \in G\mathbb{C}^{n\times n}$.

Associated with the universal covering surface $\widetilde{S}$, there is a group Δ of deck transformations. A deck transformation is an analytic bijection $\sigma : \widetilde{S} \to \widetilde{S}$ which satisfies $\rho \circ \sigma = \rho$.

Given a solution $Y(\tilde{z})$ of (2.2) and $\sigma \in \Delta$, then $Y(\sigma(\tilde{z}))$ is also a solution of (2.2). Hence there exists a unique matrix $\chi(\sigma) \in G\mathbb{C}^{n\times n}$ such that

$$Y(\tilde{z}) \;=\; Y(\sigma(\tilde{z}))\chi(\sigma). \tag{2.3}$$

In fact, the mapping $\chi : \Delta \to G\mathbb{C}^{n\times n}$ is a representation (group homomorphism), i.e., $\chi(\sigma\tau) = \chi(\sigma)\chi(\tau)$. This mapping is called the *monodromy representation* of $Y(\tilde{z})$.

Now let $\widehat{Y}(\tilde{z})$ be another solution of (2.2), and let $\widehat{\chi} : \Delta \to G\mathbb{C}^{n\times n}$ be the monodromy representation of $\widehat{Y}(\tilde{z})$. If Y and $\widehat{Y}$ are related by $\widehat{Y}(\tilde{z}) = Y(\tilde{z})C$ with $C \in G\mathbb{C}^{n\times n}$, then

$$\widehat{\chi}(\sigma) \;=\; C^{-1}\chi(\sigma)C \tag{2.4}$$

for all $\sigma \in \Delta$. This means that to the system (2.2) there corresponds a class of mutually conjugate representations. This class is called the *monodromy* of the system (2.2).

It is interesting to note that the matrix function $A(z)$ can be reconstructed from a solution $Y(\tilde{z})$. Assume that $Y : \widetilde{S} \to G\mathbb{C}^{n\times n}$ is an analytic function which satisfies (2.3) with some representation $\chi : \Delta \to G\mathbb{C}^{n\times n}$. Then one can define

$$A(z) \;:=\; Y'(\tilde{z})Y(\tilde{z})^{-1}, \qquad \tilde{z} \in \widetilde{S}. \tag{2.5}$$

This definition is correct since the right hand side turns out to be a single-valued analytic function (i.e., it is invariant under any deck transformation).

We will occasionally have to modify a solution of a system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ with a (single-valued) analytic matrix function $V : S \to G\mathbb{C}^{n\times n}$ as follows: $\widehat{Y}(\tilde{z}) = V(z)Y(\tilde{z})$. Then formula (2.5) tells us that $\widehat{Y}(\tilde{z})$ is a solution of the system $\widehat{Y}'(\tilde{z}) = \widehat{A}(z)\widehat{Y}(\tilde{z})$ with

$$\widehat{A}(z) \;=\; V(z)A(z)V^{-1}(z) + V'(z)V^{-1}(z), \qquad z \in S. \tag{2.6}$$

Moreover, the modified system has the same monodromy as the original one. Conversely, suppose that $A(z)$ and $\widehat{A}(z)$ are related to each other by (2.6) with some single-valued matrix function $V(z)$. Then the solutions of the corresponding systems are connected with each other by $\widehat{Y}(\tilde{z}) = V(z)Y(\tilde{z})C$ with $C \in G\mathbb{C}^{n\times n}$.

Because the group Δ is finitely generated, by elements $\sigma_1, \dots, \sigma_m$ say, a complete characterization of the monodromy is given by the matrices $\chi(\sigma_1), \dots, \chi(\sigma_m)$. Although there is no distinguished choice for the generators, we will choose them in a specific way.

Assume that $a_1, \dots, a_m \in \mathbb{C}$ lie in this order on a smooth closed positively oriented curve $\Gamma \subset \mathbb{C}$. Let $\tilde{z}_0 \in \widetilde{S}$ be a point such that $\rho(\tilde{z}_0)$ lies inside of Γ. For $k = 1, \dots, m$, determine the (unique) point $\tilde{z}_k \in \widetilde{S}$ satisfying $\rho(\tilde{z}_k) = \rho(\tilde{z}_0)$ and the following condition: there exists a simple path $\widetilde{\gamma}_k : [0,1] \to \widetilde{S}$ with $\widetilde{\gamma}_k(0) = \tilde{z}_0$ and $\widetilde{\gamma}_k(1) = \tilde{z}_k$ such that the curve $\rho(\widetilde{\gamma}_k)$ intersects Γ exactly twice, first on the subarc joining a_{k-1} and a_k, and second on the subarc joining a_k and a_{k+1}. (That means, the curve $\rho(\widetilde{\gamma}_k)$ goes once around a_k in positive direction.) Having determined $\tilde{z}_k$, there exists a unique deck transformation $\sigma_k \in \Delta$ for which $\sigma_k(\tilde{z}_0) = \tilde{z}_k$. These deck transformations $\sigma_1, \dots, \sigma_m \in \Delta$ satisfy the condition

$$\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_m \;=\; \mathrm{id}. \tag{2.7}$$

The construction of $\sigma_1, \dots, \sigma_m$ depends primarily on the points $a_1, \dots, a_m$ and the curve Γ. It also depends on $\tilde{z}_0$, but for a different point $\tilde{z}_0$, we get deck transformations $\sigma'_k = \tau^{-1} \circ \sigma_k \circ \tau$ with some $\tau \in \Delta$.

The monodromy representation of a solution of (2.2) can now be described by the m-tuple $[\chi(\sigma_1), \dots, \chi(\sigma_m)]$, which (because of (2.7)) satisfies the relation

$$\chi(\sigma_1)\chi(\sigma_2)\cdots\chi(\sigma_m) \;=\; I. \tag{2.8}$$

As we are interested in monodromy, let us introduce the following equivalence relation:

$$[M_1, \dots, M_m] \;\sim\; [M'_1, \dots, M'_m] \tag{2.9}$$

if and only if there exists a $C \in G\mathbb{C}^{n\times n}$ such that $M'_k = C^{-1}M_kC$ for all $k = 1, \dots, m$. The equivalence classes are denoted by $[M_1, \dots, M_m]_\sim$ and will be called the *simultaneous similarity class.* Hence the monodromy of the system (2.2) is given by class $[\chi(\sigma_1), \dots, \chi(\sigma_m)]_\sim$.

In order to study the behavior of the solutions of the system (2.2) in a neighborhood of the singularities, we introduce the punctured neighborhoods $D_a = \{z \in \mathbb{C} : 0 < |z-a| < \varepsilon\}$ for $a \in \mathbb{C}$ and $D_\infty = \{z \in \mathbb{C} : |z| > 1/\varepsilon\}$ for $a = \infty$, where ε is sufficiently small. Let $\widetilde{D}_a$ be the universal covering surface of D_a. The associated group of deck transformations is singly generated by a transformation σ_a, which sends a point $\tilde{z} \in \widetilde{D}_a$ into its correspondent after going once around the point a in positive direction.

Let us restrict the solutions of the system (2.2) onto $\widetilde{D}_{a_k}$ and assume for simplicity that $a_k \neq \infty$. Then the function $Y : \widetilde{D}_{a_k} \to G\mathbb{C}^{n\times n}$ has the property $Y(\tilde{z}) = Y(\sigma_{a_k}(\tilde{z}))M_k$, where M_k is similar to $\chi(\sigma_k)$. Let E_k be *any* matrix satisfying $M_k = \exp(-2\pi i E_k)$, and introduce

$$(\tilde{z} - a_k)^{E_k} \;\; := \;\; \exp(E_k \ln(\tilde{z} - a_k)), \qquad \tilde{z} \in \widetilde{D}_{a_k}, \tag{2.10}$$

noting that the logarithm is well defined on $\widetilde{D}_{a_k}$. Then $Y(\tilde{z})$ can be written as

$$Y(\tilde{z}) \;\; = \;\; Z_k(z)(\tilde{z} - a_k)^{E_k}, \qquad \tilde{z} \in \widetilde{D}_{a_k}, \tag{2.11}$$

where $Z_k : D_{a_k} \to G\mathbb{C}^{n\times n}$ is a (single-valued) analytic function, which may have (along with its inverse) an isolated singularity at a_k. The description of the local behavior of the solution is now reduced to characterizing the matrix E_k and the behavior of $Z_k(z)$ and its inverse.

When speaking of a restriction of a solution, which is defined on $\widetilde{S}$, onto $\widetilde{D}_{a_k}$, we have to be a little bit careful. Namely, if $\rho : \widetilde{S} \to S$ is the covering map, then the preimage $\rho^{-1}(D_{a_k})$ can be identified with $\widetilde{D}_{a_k} \times \Omega$ where Ω is an index set, which is countably infinite for $m \geq 3$. For a solution $Y : \widetilde{S} \to G\mathbb{C}^{n\times n}$, the relation (2.11) should therefore correctly be written as

$$Y(\tilde{z}_*) \;\; = \;\; Z_k(z)(\tilde{z} - a_k)^{E_k} C_\omega, \qquad \tilde{z}_* = (\tilde{z}, \omega) \in \widetilde{D}_{a_k} \times \Omega, \tag{2.12}$$

where $C_\omega \in G\mathbb{C}^{n\times n}$ depends on the connected component Ω of the preimage $\rho^{-1}(D_{a_k})$.

A singularity a_k of a system is called *apparent* if the solutions have no branching behavior at a_k, i.e., if $\chi(\sigma_k) = I$. In this case it is not necessary to take this singularity into account when defining the universal covering surface.

The following fundamental theorem says that there exist systems with prescribed singularities and prescribed monodromy. A first proof was given by Plemelj [16], who employed the theory of singular integral equations. Another proof [17, 1] can be given by using the theory of vector bundles [9, 19].

Theorem 2.1 *For any distinct points $a_1, \ldots, a_m \in \dot{\mathbb{C}}$ and matrices $M_1, \ldots, M_m \in \mathbb{C}^{n\times n}$ satisfying $M_1 \cdots M_m = I$, there exists a system with singularities only at $a_1, \ldots, a_m$ and with monodromy given by $[M_1, \ldots, M_m]_\sim$.*

This theorem (as it is stated) does not say anything about the type of the singularities. A singularity a_k of a system is called *Fuchsian* if $A(z)$ has only a simple pole at $z = a_k$. A singularity a_k is called *regular* if the function $Z_k(z)$ in (2.12) has not an essential singularity at $z = a_k$. A Fuchsian singularity is always regular.

The result of Plemelj is actually stronger than the statement of the previous theorem. One can construct a system which has only Fuchsian singularities with the possible exception of one singularity, which is (in general) merely regular.

The question whether there exists a system with prescribed singularities and monodromy such that all singularities are Fuchsian is known as the Riemann–Hilbert problem or Hilbert's 21st problem. Surprisingly, Bolibruch [3, 1, 5] showed that the answer may be negative. Notice, however, that Plemelj's results still implies that the answer is always positive if one allows one additional apparent singularity.

A generalized version of the Riemann–Hilbert problem, where one asks for systems with prescribed Fuchsian singularities, prescribed monodromy and prescribed local behavior (in a certain specified sense) was also studied by Bolibruch [6].

The Riemann–Hilbert problem and its modifications have an intimate relation to the problem which will be proposed in Section 3. In fact, all these problems can be formulated by using the language of vector bundles. We will not elaborate on this direction, but merely confine ourselves to some comments later on.

Finally, let us recall some facts about (scalar) linear differential equations of higher order [13, 14, 8]. A linear differential equation of n-th order can be written as

$$y^{(n)} + q_1(z)y^{(n-1)} + \ldots + q_n(z)y \;=\; 0, \tag{2.13}$$

where $y^{(k)}$ stands for the k-th derivative of the complex valued function y. We assume that $q_1(z), \ldots, q(z)$ are analytic functions on $\mathbb{C}$ with the exception of a finite number of points, which represent the *singularities* of the equation. The point $a = \infty$ is a singularity if so is the point $z = 0$ for the equation resulting from a change of variables $z \mapsto 1/(z-a)$. (This definition does not depend on the choice of $a \in \mathbb{C}$.)

The solutions of the above equation with singularities $a_1, \ldots, a_m \in \dot{\mathbb{C}}$ are analytic functions $y : \widetilde{S} \to \mathbb{C}$ defined on the corresponding universal covering surface. The set of all solutions forms an n-dimensional linear space. Given n linearly independent solutions $y_1(\tilde{z}), \ldots, y_n(\tilde{z})$, the monodromy representation $\chi : \Delta \to G\mathbb{C}^{n\times n}$ is defined by

$$(y_1(\tilde{z}), \ldots, y_n(\tilde{z})) \;=\; (y_1(\sigma(\tilde{z})), \ldots, y_n(\sigma(\tilde{z})))\chi(\sigma), \tag{2.14}$$

where $\chi(\sigma) \in G\mathbb{C}^{n\times n}$ is a uniquely determined matrix and $\sigma \in \Delta$.

A singularity $a \in \mathbb{C}$ of equation (2.13) is called *Fuchsian* if one can write $q_k(z) = r_k(z)/(z-a)^k$ where $r_k(z)$ is analytic at $z = a$ for each $k = 1, \ldots, n$. The *local exponents* $\rho_1, \ldots, \rho_n$ of this singularity are by definition the roots of the *indical equation*

$$\rho(\rho-1)\cdots(\rho-n+1) + \sum_{k=1}^{n} r_k(a)\rho(\rho-1)\cdots(\rho-n+1+k) \;=\; 0. \tag{2.15}$$

The singularity at infinity is Fuchsian if $q_k(z) = O(1/z^k)$ as $z \to \infty$ for each $k = 1, \ldots, n$, and the local exponents are defined similarly.

Now suppose that equation (2.13) has exactly m Fuchsian singularities $a_1, \ldots, a_m$ with local exponents $\rho_1^{(k)}, \ldots, \rho_n^{(k)}$ for a_k. Then the well known Fuchs' relation says that

$$\sum_{k=1}^{m} (\rho_1^{(k)} + \ldots + \rho_n^{(k)}) \;=\; (m-2)\frac{n(n-1)}{2}. \tag{2.16}$$

A couple of more facts about linear differential equations will be stated in Section 5 when they are needed.

3 Systems of standard form and their relation to the factorization problem

In this section, we introduce a class of systems which are characterized by their singularities, their monodromy and their local behavior near the singularities. In addition, we allow an apparent singularity at infinity the behavior of which is described by certain integers.

We prove that such systems exist for given singularities, monodromy and local data. We also discuss the "uniqueness", i.e., we examine the relation between two systems having the same singularities and the same data. Finally, we show that the factors appearing in the factorization of a piecewise constant matrix function can be constructed by using the solutions of such systems. The problem of determining the partial indices reduces to a corresponding problem for systems of linear differential equations.

Let $a_1, \ldots, a_m \in \mathbb{C}$ be distinct points which lie in this order on a smooth closed positively oriented curve $\Gamma \subset \mathbb{C}$. Denote by $\widetilde{S}$ the universal covering surface of $S := \dot{\mathbb{C}} \setminus \{a_1, \ldots, a_m\}$. Define deck transformations $\sigma_1, \ldots, \sigma_m$ for $\widetilde{S}$ in the way explained in the previous section.

A matrix E is called *non-resonant* if the difference of any two eigenvalues of E is not a nonzero integer. A collection $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ of m-tuples of $n \times n$ matrices will be called *admissible data* if the following conditions are satisfied:

(a) $M_1 M_2 \cdots M_m = I$.

(b) $M_k \sim \exp(-2\pi i E_k)$ for each $k = 1, \ldots, m$.

(c) The matrices $E_1, \ldots, E_m$ are non-resonant.

Definition 3.1 *We say that the system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is of standard form with respect to admissible data $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$, with singularities $a_1, \ldots, a_m \in \mathbb{C}$ and with indices $\varkappa_1, \ldots, \varkappa_n \in \mathbb{Z}$, where $\varkappa_1 \geq \varkappa_2 \geq \ldots \geq \varkappa_n$, if the following conditions are satisfied:*

(i) The system has singularities only at $a_1, \ldots, a_m$ and at infinity, where the singularity at infinity is apparent.

(ii) The monodromy of the system is given by $[\chi(\sigma_1), \ldots, \chi(\sigma_m)]_\sim = [M_1, \ldots, M_m]_\sim$.

(iii) For each $k = 1, \ldots, m$, the solutions of the system defined on $\widetilde{D}_{a_k}$ can be written as

$$Y(\tilde{z}) = Z_k(z)(\tilde{z} - a_k)^{E_k} C, \qquad \tilde{z} \in \widetilde{D}_{a_k},$$

where the function $Z_k : D_{a_k} \cup \{a_k\} \to G\mathbb{C}^{n\times n}$ is analytic and $C \in G\mathbb{C}^{n\times n}$.

(iv) The solutions of the system defined on D_∞ can be written as

$$Y(z) \quad = \quad \operatorname{diag}(z^{\varkappa_1}, \ldots, z^{\varkappa_n}) Z_\infty(z) C, \qquad z \in D_\infty,$$

where the function $Z_\infty : D_\infty \cup \{\infty\} \to G\mathbb{C}^{n\times n}$ is analytic and $C \in G\mathbb{C}^{n\times n}$.

The necessity of the assumption (a) and (b) on the data is obvious. Assumption (c) is not necessary for this definition, but it will be important later on. Note that the solutions $Y(\tilde{z})$ are analytic and invertible functions on $\widetilde{S}^* = \widetilde{S} \setminus \rho^{-1}(\infty)$.

A simple observation is in order. The data $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ of a given system of standard form can be replaced by data $[CM_1C^{-1}, \ldots, CM_mC^{-1}]$ and $[C_1E_1C_1^{-1}, \ldots, C_mE_mC_m^{-1}]$, where $C, C_1, \ldots, C_m \in G\mathbb{C}^{n\times n}$. Conversely, to a given system of standard form one can uniquely associate the monodromy data $[M_1, \ldots, M_m]_\sim$ and the local data $[(E_1)_\sim, \ldots, (E_m)_\sim]$. Indeed, in order to see that the matrices E_k are unique up to similarity, one can use (iii) and consider the residue of $A(z) = Y'(\tilde{z})Y^{-1}(\tilde{z})$ at $z = a_k$.

Finally, also the indices of a system of standard form are unique. Here one can use (iv) and consider the residue of $Y^{-1}(z)Y'(z)$ of any solution defined on D_∞.

Before discussing the existence and uniqueness of systems of standard form for given singularities and data, we will illustrate how these systems provide us with the solution of the original factorization problem for piecewise constant matrix functions.

In the following theorem we assume that the curve Γ divides the complex sphere $\dot{\mathbb{C}}$ into two domains D_+ and D_- such that $0 \in D_+$ and $\infty \in D_-$. As usual, we identify functions which are analytic on the domains D_+ or D_- with their boundary values, considered as functions on Γ.

Theorem 3.2 *Let $G \in PC(\Gamma)^{n\times n}$ be a piecewise constant function with jumps only at $a_1, \ldots, a_m$. Assume that G admits a Φ-factorization in the space $L^p(\Gamma)$, $1 < p < \infty$, and let $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ be the data associated with G with respect to $L^p(\Gamma)$.*

Assume that there exists a system of standard form with these data, with singularities $a_1, \ldots, a_m$ and with indices $\varkappa_1, \ldots, \varkappa_n$. Let Y_1 and Y_2 be solutions of this system defined on D_+ and $D_- \setminus \{\infty\}$, respectively. Then there exist $C_1, C_2 \in G\mathbb{C}^{n\times n}$ such that the representation

$$G(t) \quad = \quad G_+(t)\Lambda(t)G_-(t), \qquad t \in \Gamma,$$

is a Φ-factorization of G in the space $L^p(\Gamma)$, where $\Lambda(t) = \operatorname{diag}(t^{\varkappa_1}, \ldots, t^{\varkappa_n})$ and

$$G_+(z) \;=\; C_1^{-1}Y_1^{-1}(z), \quad z \in D_+, \qquad G_-(z) \;=\; \Lambda^{-1}(z)Y_2(z)C_2, \quad z \in D_- \setminus \{\infty\}.$$

Proof. Note first that Y_1 and Y_2 exist because D_+ and D_- are simply connected and infinity is an apparent singularity of the system. Let $\widetilde{G}_+(z) = Y_1^{-1}(z)$, $z \in D_+$, and let $\widetilde{G}_-(z) = \Lambda^{-1}(z)Y_2(z)$, $z \in D_- \setminus \{\infty\}$. From the condition (iv) of Definition

3.1, it follows that $\widetilde{G}_-$ is also analytic and invertible at ∞. Hence G_+ and G_- are analytic and invertible on all of D_+ and D_-, respectively. It follows from (iii) that in a neighborhood of the singularity a_k, the entries of the functions $\widetilde{G}_+^{-1}$ and $\widetilde{G}_-$ behave as $(z-a_k)^\lambda(\ln(z-a_k))^\sigma$ and the entries of the functions $\widetilde{G}_+$ and $\widetilde{G}_-^{-1}$ behave as $(z-a_k)^{-\lambda}(\ln(z-a_k))^\sigma$, where λ is taken from the eigenvalues of E_k and $\sigma \in \{0,1,\ldots\}$ refers to their multiplicity. By assumption we have $-1/q = 1/p - 1 < \operatorname{Re}\lambda < 1/p$. This implies that $\widetilde{G}_+ \in L^p_+(\Gamma)^{n\times n}$, $\widetilde{G}_-^{-1} \in L^q_-(\Gamma)^{n\times n}$, $\widetilde{G}_+^{-1} \in L^p_+(\Gamma)^{n\times n}$ and $\widetilde{G}_- \in L^q_-(\Gamma)^{n\times n}$.

The functions Y_1 and Y_2 can be extended analytically onto the boundary $\Gamma \setminus \{a_1,\ldots,a_m\}$. The values of the functions defined by $\widetilde{G}_+(t) = Y_1^{-1}(t)$ and $\widetilde{G}_-(t) = \Lambda^{-1}(t)Y_2(t)$ with $t \in \Gamma\setminus\{a_1,\ldots,a_m\}$ coincide with the boundary values of the above functions $\widetilde{G}_+$ and $\widetilde{G}_-$, which are defined on D_+ and D_-.

We introduce the function $\widetilde{G}(t) = \widetilde{G}_+(t)\Lambda(t)\widetilde{G}_-(t)$, $t \in \Gamma$, and observe that this representation is a factorization of $\widetilde{G}$ in the space $L^p(\Gamma)$. On the other hand, we have $\widetilde{G}(t) = Y_1^{-1}(t)Y_2(t)$, and because Y_1 and Y_2 are solutions of the system, it follows that $\widetilde{G}$ is a piecewise constant function on Γ with jumps only at $a_1,\ldots,a_m$.

Now let us express the monodromy of the system in terms of $\widetilde{G}$. We start from the solution $Y_1(z)$ on D_+ and continue analytically by going once around the point a_k (see the construction of the deck transformations σ_k). When we first cross Γ, the proper analytic continuation is the function $Y_2(z)\widetilde{G}(a_k-0)^{-1}$ on D_-. After the second crossing of Γ, we obtain the function $Y_1(z)\widetilde{G}(a_k+0)\widetilde{G}(a_k-0)^{-1}$ on D_+. Hence, up to simultaneous similarity, we have $\chi(\sigma_k) = \widetilde{G}(a_k-0)\widetilde{G}(a_k+0)^{-1}$. On the other hand, by condition (ii), the monodromy is given by $[M_1,\ldots,M_m]$. Hence $\chi(\sigma_k) = C_1M_kC_1^{-1} = C_1G(a_k-0)G(a_k+0)^{-1}C_1^{-1}$ with some $C_1 \in G\mathbb{C}^{n\times n}$. This implies that $\widetilde{G}(a_k-0)\widetilde{G}(a_k+0)^{-1} = C_1G(a_k-0)G(a_k+0)^{-1}C_1^{-1}$, hence $\widetilde{G}(a_k+0)^{-1}C_1G(a_k+0) = \widetilde{G}(a_k-0)^{-1}C_1G(a_k-0)$ for each $k=1,\ldots,m$. Because both G and $\widetilde{G}$ are piecewise constant, there is a $C_2 \in G\mathbb{C}^{n\times n}$ such that $C_2 = \widetilde{G}(t)^{-1}C_1G(t)$. Hence $G(t) = C_1\widetilde{G}(t)C_2$, which implies that $G(t) = G_+(t)\Lambda(t)G_-(t)$ is a factorization in the space $L^p(\Gamma)$. By Corollary 1.3, this is even a Φ-factorization. □

Note that the data associated with a piecewise constant matrix function is necessarily admissible data. Actually, assumption (c) is even weaker than the condition that the real parts of the eigenvalues of the matrices E_k lie in the open interval J_p.

We need the following factorization-type result about analytic matrix functions defined on a punctured disk D_a. It is the analytic version of the Birkhoff-Grothendieck theorem [2]. A quite simple proof has also been given by Leiterer [12] (see also [1]).

Lemma 3.3 *For $a \in \mathbb{C}$, let $\Phi : D_a \to G\mathbb{C}^{n\times n}$ be an analytic function. Then there exist $\varkappa_1,\ldots,\varkappa_n \in \mathbb{Z}$ such that Φ can be written in the form*

$$\Phi(z) \;=\; U(z)\operatorname{diag}\left((z-a)^{\varkappa_1},\ldots,(z-a)^{\varkappa_n}\right)V(z), \qquad z \in D_a,$$

where $U : \dot{\mathbb{C}}\setminus\{a\} \to G\mathbb{C}^{n\times n}$ and $V : D_a \cup \{a\} \to G\mathbb{C}^{n\times n}$ are analytic functions.

Using a well known modification idea [1], we prove the existence of systems of standard form with prescribed data and singularities, which have certain (a priori not known) indices.

Theorem 3.4 *Let the $n \times n$ matrices $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ be admissible data, and let $a_1, \ldots, a_m \in \mathbb{C}$ be distinct points. Then there exists a system of standard form with these data and singularities and with certain indices $\varkappa_1, \ldots, \varkappa_n \in \mathbb{Z}$.*

Proof. From Theorem 2.1 it follows that there exists a system with singularities only at $a_1, \ldots, a_m$ whose monodromy is given by $[M_1, \ldots, M_m]_{\sim}$. Let Y_0 be a solution of this system, i.e., $Y_0 : \widetilde{S}^* \to G\mathbb{C}^{n\times n}$ is analytic and has the given monodromy representation.

We are going to modify Y_0 step by step in order to obtain an analytic matrix function $Y : \widetilde{S}^* \to G\mathbb{C}^{n\times n}$ which also satisfies (iii) and (iv) of Definition 3.1. Let us consider Y_0 near the point a_1. Because the monodromy is given as above, the solution changes by a matrix similar to M_1 after going once around a_1 in positive direction. Hence, restricted to $\widetilde{D}_{a_1}$, the function Y_0 can be written in the form $Y_0(\tilde{z}) = \Phi_1(z)(\tilde{z} - a_1)^{E_1} C$, where $\Phi_1 : D_{a_1} \to G\mathbb{C}^{n\times n}$ is a certain analytic function. We apply Lemma 3.3 with $\Phi = \Phi_1$ and $a = a_1$. We obtain that $\Phi_1(z) = \widetilde{U}_1(z)V_1(z)$, $z \in D_{a_1}$, where $\widetilde{U}_1$ is analytic and invertible on $\mathbb{C} \setminus \{a_1\}$ and V_1 is analytic and invertible on $D_{a_1} \cup \{a_1\}$. Hence $\widetilde{U}_1^{-1}(z)Y_0(\tilde{z}) = V_1(z)(\tilde{z} - a_1)^{E_1} C$ for $\tilde{z} \in \widetilde{D}_{a_1}$. We define the function $Y_1(\tilde{z}) := \widetilde{U}_1^{-1}(z)Y_0(\tilde{z})$ for $\tilde{z} \in \widetilde{S}^*$. This function satisfies the conditions (i), (ii), and also the condition (iii) at the point a_1.

The next steps consist in doing the same for $a_2, \ldots, a_m$. The essential point is that, at the k-step, we define the function $Y_k(\tilde{z}) = \widetilde{U}_k^{-1}(z)Y_{k-1}(\tilde{z})$, where $\widetilde{U}_k(z)$ is analytic and invertible on $\mathbb{C}\setminus\{a_k\}$. This modification does not produce additional singularities (except at infinity), it does not change the monodromy, and it changes the local behavior only at the a_k (in the desired way), but not at the other singularities. So we obtain a function $Y_m(\tilde{z})$ which satisfies the conditions (i)–(iii).

Finally, we consider the point infinity. Restricting $Y_m(\tilde{z})$ to D_∞ we can write $Y_m(z) = \Phi(1/z)C$, where $\Phi(1/z)$ is analytic and invertible on D_0. We apply Lemma 3.3 with $a = 0$ and factor $\Phi(z) = V(z)\Lambda(z)U(z)$, $z \in D_0$, where $V(z)$ is analytic and invertible on $\dot{\mathbb{C}} \setminus \{0\}$, $\Lambda(z)$ is the diagonal matrix, and $U(z)$ is analytic and invertible on $D_0 \cup \{0\}$. It follows that $V^{-1}(1/z)Y_m(z) = \Lambda(1/z)U(1/z)C$. We define $Y(\tilde{z}) = V^{-1}(1/z)Y_m(\tilde{z})$, $\tilde{z} \in \widetilde{S}^*$. Because $V(1/z)$ is analytic and invertible on $\mathbb{C}$, the function $Y(\tilde{z})$ satisfies also the required properties (i)–(ii). Since $Y(z) = \Lambda(1/z)U(1/z)C$ on D_∞, this function also satisfies condition (iv) after an appropriate permutation of the entries of $\Lambda(1/z)$.

Finally, we define the corresponding matrix function $A(z)$ by $A(z) = Y'(\tilde{z})Y^{-1}(\tilde{z})$. $\square$

Next we consider the question about the uniqueness of systems of standard form with prescribed data and singularities. First of all, the indices are uniquely determined. The systems themselves are not unique, but they are related to each other in a simple way. Because of the relation to factorization (described in Theorem 3.2) it should not be surprising that the description is analogous to that given in Theorem 1.2.

Theorem 3.5 *Let $Y_1'(\tilde{z}) = A_1(z)Y_1(\tilde{z})$ and $Y_2'(\tilde{z}) = A_2(z)Y_2(\tilde{z})$ be two systems of standard form with the same admissible data and singularities $a_1, \ldots, a_m \in \mathbb{C}$. Then the indices corresponding to both systems coincide. Moreover, if the indices are given by*

$$\overline{\varkappa}_1, \ldots, \overline{\varkappa}_1, \overline{\varkappa}_2, \ldots \overline{\varkappa}_2, \overline{\varkappa}_3, \ldots, \overline{\varkappa}_{k-1}, \overline{\varkappa}_k, \ldots, \overline{\varkappa}_k,$$

where $\overline{\varkappa}_j$ occurs exactly l_j times, $\overline{\varkappa}_1 > \overline{\varkappa}_2 > \ldots > \overline{\varkappa}_k$, $l_1, \ldots, l_k \in \{1, 2, \ldots\}$, $l_1 + \ldots + l_k = n$, then both systems are related to each other by

$$A_2(z) \;=\; V'(z)V^{-1}(z) + V(z)A_1(z)V^{-1}(z), \qquad z \in S \setminus \{\infty\}, \tag{3.1}$$

or, equivalently, by

$$Y_2(\tilde{z}) \;=\; V(z)Y_1(\tilde{z})C, \qquad \tilde{z} \in \widetilde{S}^*, \tag{3.2}$$

where $C \in G\mathbb{C}^{n\times n}$ and $V : \mathbb{C} \to G\mathbb{C}^{n\times n}$ is a rational matrix function of the form

$$V(z) \;=\; \begin{pmatrix} V_{11} & V_{12}(z) & \ldots & V_{1k}(z) \\ 0 & V_{22} & & V_{2k}(z) \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & V_{kk} \end{pmatrix} \tag{3.3}$$

with $V_{jj} \in G\mathbb{C}^{l_j\times l_j}$, $V_{ij}(z) = \sum_{r=0}^{\overline{\varkappa}_i-\overline{\varkappa}_j} V_{ij}^{(r)} z^r$, where $V_{ij}^{(r)} \in \mathbb{C}^{l_i\times l_j}$ for $i < j$.

Proof. Because the systems have the same singularities and the same monodromy, there exist an analytic function $V : S \setminus \{\infty\} \to G\mathbb{C}^{n\times n}$ and a $C \in G\mathbb{C}^{n\times n}$ such that $Y_2(\tilde{z}) = V(z)Y_1(\tilde{z})C$. We are first going to show that $V(z)$ is analytic and invertible on all of $\mathbb{C}$.

Let us restrict the solutions Y_1 and Y_2 onto a neighborhood $\widetilde{D}_{a_k}$ of the point a_k. From (iii) of Definition 3.1 it follows that there are $C_1, C_2 \in G\mathbb{C}^{n\times n}$ such that

$$Y_1(\tilde{z}) \;=\; Z_k^{(1)}(z)(\tilde{z}-a)^{E_k}C_1, \qquad Y_2(\tilde{z}) \;=\; Z_k^{(2)}(z)(\tilde{z}-a)^{E_k}C_2, \qquad \tilde{z} \in \widetilde{D}_{a_k},$$

where $Z_k^{(1)}$ and $Z_k^{(2)}$ are analytic and invertible at $D_{a_k} \cup \{a_k\}$. Hence

$$Z_k^{(2)}(z)(\tilde{z}-a)^{E_k}\widetilde{C} \;=\; V(z)Z_k^{(1)}(z)(\tilde{z}-a)^{E_k}, \tag{3.4}$$

where $\widetilde{C} = C_2C^{-1}C_1^{-1}$. By going once around the point a_k in positive direction it follows that

$$Z_k^{(2)}(z)(\tilde{z}-a)^{E_k}N_k\widetilde{C} \;=\; V(z)Z_k^{(1)}(z)(\tilde{z}-a)^{E_k}N_k,$$

where $N_k := \exp(2\pi i E_k)$. Combining both equations it follows that $N_k = \widetilde{C}^{-1} N_k \widetilde{C}$, i.e., N_k and $\widetilde{C}$ commute with each other. Now we use the fact that the matrix E_k is non-resonant. Because $N_k = \exp(2\pi i E_k)$, this implies that E_k can be written as a polynomial in the matrix M_k. Hence also E_k and $\widetilde{C}$ commute. Using this in connection with (3.4) we obtain

$$Z_k^{(2)}(z)\widetilde{C}(\tilde{z}-a)^{E_k} \;=\; V(z) Z_k^{(1)}(z)(\tilde{z}-a)^{E_k}.$$

Hence $V(z) = Z_k^{(2)}(z)\widetilde{C}(Z_k^{(1)}(z))^{-1}$, which implies that $V(z)$ is analytic and invertible at a_k.

So far we have shown that $V(z)$ and $V^{-1}(z)$ are entire analytic functions. We analyze their behavior at infinity by restricting the solutions Y_1 and Y_2 onto D_∞. From (iv) of Definition 3.1, it follows that

$$Y_1(z) \;=\; \Lambda_1(z) Z_\infty^{(1)} C_1, \qquad Y_2(z) \;=\; \Lambda_2(z) Z_\infty^{(2)} C_2, \qquad z \in D_\infty,$$

where $C_1, C_2 \in G\mathbb{C}^{n\times n}$ and $Z_\infty^{(1)}$ and $Z_\infty^{(2)}$ are analytic and invertible on a neighborhood of infinity. Here we assume that Λ_1 and Λ_2 are of the form

$$\Lambda_s(z) \;=\; \operatorname{diag}\left(z^{\overline{\varkappa}_1} I_{l_1^{(s)}}, z^{\overline{\varkappa}_2} I_{l_2^{(s)}}, \ldots, z^{\overline{\varkappa}_k} I_{l_k^{(s)}}\right), \qquad s = 1, 2,$$

where $\overline{\varkappa}_1, \ldots, \overline{\varkappa}_k \in \mathbb{Z}$, $\overline{\varkappa}_1 > \overline{\varkappa}_2 > \ldots > \overline{\varkappa}_k$, $l_1^{(s)}, \ldots, l_k^{(s)} \in \{0, 1, \ldots\}$, $l_1^{(s)} + \ldots + l_k^{(s)} = n$. Here I_l stands for the identity matrix of size $l \times l$. Now the above equations give

$$\Lambda_2(z) Z_\infty^{(2)} C_2 \;=\; V(z)\Lambda_1(z) Z_\infty^{(1)} C_1 C, \qquad z \in D_\infty.$$

This means that

$$V(z) \;=\; \Lambda_2(z) Z_\infty(z) \Lambda_1^{-1}(z), \qquad V^{-1}(z) \;=\; \Lambda_1(z) Z_\infty^{-1}(z) \Lambda_2^{-1}(z),$$

where $Z_\infty := Z_\infty^{(2)} C_2 C^{-1} C_1^{-1} (Z_\infty^{(1)})^{-1}$ is analytic and invertible in a neighborhood of infinity. Now we introduce block partitions of the matrices $V(z)$ and $V^{-1}(z)$:

$$V(z) \;=\; (V_{ij}(z))_{i,j=1}^k, \qquad V^{-1}(z) \;=\; (U_{ij}(z))_{i,j=1}^k,$$

where $V_{ij}(z)$ is of size $l_i^{(2)} \times l_j^{(1)}$ and $U_{ij}(z)$ is of size $l_i^{(1)} \times l_j^{(2)}$. Using the structure of $\Lambda_1(z)$ and $\Lambda_2(z)$, it follows that $\|V_{ij}(z)\| = O(z^{\overline{\varkappa}_i - \overline{\varkappa}_j})$ and $\|U_{ij}(z)\| = O(z^{\overline{\varkappa}_i - \overline{\varkappa}_j})$ as $z \to \infty$. Because $V_{ij}(z)$ and $U_{ij}(z)$ are entire analytic functions, we obtain that $V_{ij} = 0$ and $U_{ij} = 0$ for $i > j$, and that $V_{ij}(z)$ and $U_{ij}(z)$ are matrix polynomials of degree at most $\overline{\varkappa}_i - \overline{\varkappa}_j$ if $i \le j$. In particular, the diagonal blocks are constant matrices. The block triangular structure of $V(z)$ and $V^{-1}(z)$ implies that $V_{ii} U_{ii} = I_{l_i^{(2)}}$ and $U_{ii} V_{ii} = I_{l_i^{(1)}}$. Hence V_{ii} and U_{ii} are invertible square matrices, i.e., $l_i^{(1)} = l_i^{(2)}$. Hence the indices corresponding to both systems are the same. Thereby, we have also proved that $V(z)$ is of the desired form. $\square$

It is easy to see that also the converse of the above theorem holds. This suggests the following definition of an equivalence relation for systems of standard form. Two systems $Y_1'(\tilde z) = A_1(z)Y_1(\tilde z)$ and $Y_2'(\tilde z) = A_2(z)Y_2(\tilde z)$ of standard form will be called *equivalent* if their indices are the same and if there exists a matrix $V(z)$ of the form (3.3) and the properties stated there such that (3.1) holds (or, equivalently, (3.2) holds).

The next corollary resumes the results about existence a uniqueness in a very brief way. In fact, the uniqueness results will be employed implicitly in Section 6.

Corollary 3.6 *Let $a_1, \dots, a_m \in \mathbb{C}$ be distinct points. Then Definition 3.1 establishes a one-to-one correspondence between the equivalence classes of admissible data,*

$$[M_1, \dots, M_m]_\sim \quad \textit{and} \quad [(E_1)_\sim, \dots, (E_m)_\sim],$$

and the equivalence classes of systems of standard form with above singularities.

The previous considerations show that to each class of admissible data and given singularities one can associate a class of systems of standard form, and hence certain (uniquely determined) indices. For the problem of determining the partial indices for a piecewise constant matrix function, one has now to solve the following question: what are the indices associated to given admissible data and singularities? As we will see, this question can be answered explicitly only in some cases. In general, it leads to monodromy problems for which no explicit solution is known.

In order to indicate the importance of the non-resonance assumption for the matrices $E_1, \dots, E_m$, we show that Theorem 3.5 breaks down if this assumption is dropped. Consider the following matrix functions, which are the solutions of certain 2×2 systems with three singularities at a_1, a_2, a_3,

$$Y_1(z) = \begin{pmatrix} (z-a_1)(z-a_2)(z-a_3) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.5}$$

$$Y_2(z) = \begin{pmatrix} (z-a_1)(z-a_2) & 0 \\ 0 & (z-a_3) \end{pmatrix}, \tag{3.6}$$

$$Y_3(z) = \begin{pmatrix} (z-a_1)(z-a_2) & 0 \\ 1 & (z-a_3) \end{pmatrix}. \tag{3.7}$$

The corresponding systems have trivial monodromy ($M_1 = M_2 = M_3 = I$), the same local behavior (with $E_1 = E_2 = E_3 = \operatorname{diag}(1, 0)$), but their indices are $(3, 0)$ for Y_1 and $(2, 1)$ for Y_2 and Y_3. Moreover, one can show (which requires a little effort) that Y_2 and Y_3 are not equivalent in the above sense, although they have the same indices. Obviously, Corollary 3.6 also breaks down, and the question "what are the indices for given data?" does not make sense. One might think of replacing this question by asking for all possible indices for given data, but we will not pursue this direction in this paper.

Finally, let us mention the connection to vector bundles (we refer to [1, Sect. 5.1] for details). For given data and singularities one constructs in a certain way a vector bundle over $\dot{\mathbb{C}}$. Due to the fact that the matrices E_k are non-resonant, this construction is essentially unique. Any such vector bundle is characterized by certain integers $\varkappa_1, \dots, \varkappa_n$, which are called the *splitting type* of the vector bundle. Using the construction of the vector bundle and the splitting type, one can easily construct the corresponding system of standard form, which has indices $\varkappa_1, \dots, \varkappa_n$. So the above question takes the following form: what is the splitting type of the vector bundles constructed for given data and singularities?

4 Equivalent characterizations of systems of standard form

The definition of systems of standard form has been given entirely in terms of the solutions $Y(\tilde{z})$ of the system. In this section, we show by using standard arguments that the conditions (i), (iii) and (iv) of Definition 3.1 (i.e., except the monodromy condition) can be replaced by equivalent characterizations given entirely in terms of $A(z)$.

We prepare with the following well known lemma. Note that the assumption that E is non-resonant is essential for the validity of the lemma.

Lemma 4.1 *Let $A(z)$ be an analytic matrix function on D_a, $a \in \mathbb{C}$, and assume that $A(z)$ has only a simple pole at $z = a$ whose residue is a non-resonant matrix E,*

$$A(z) \;=\; \frac{E}{z-a} + \text{“analytic term at } z = a\text{”}, \qquad z \in D_a.$$

Then the solutions $Y : \widetilde{D}_a \to G\mathbb{C}^{n\times n}$ of $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ can be written as

$$Y(\tilde{z}) \;=\; Z(z)(\tilde{z}-a)^E C, \qquad \tilde{z} \in \widetilde{D}_a,$$

where the function $Z : D_a \cup \{a\} \to G\mathbb{C}^{n\times n}$ is analytic and $C \in G\mathbb{C}^{n\times n}$.

Proposition 4.2 *The system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is of standard form with (admissible) data $[M_1, \dots, M_m]$ and $[E_1, \dots, E_m]$, with singularities $a_1, \dots, a_m \in \mathbb{C}$, and with indices $\varkappa_1, \dots, \varkappa_n \in \mathbb{Z}$ if and only if the following conditions are satisfied:*

(i) The matrix function $A(z)$ is analytic on $\mathbb{C} \setminus \{a_1, \dots, a_m\}$.

(ii) The monodromy of the system is given by $[\chi(\sigma_1), \dots, \chi(\sigma_m)]_\sim = [M_1, \dots, M_m]_\sim$.

(iii) For each $k = 1, \dots, m$, the function $A(z)$ has at most a simple pole at $z = a_k$ whose residue is a matrix similar to E_k:

$$A(z) \;=\; \frac{S_k E_k S_k^{-1}}{z-a_k} + \text{“analytic term at } z = a_k\text{”}, \qquad z \in D_{a_k}.$$

(iv) At infinity, the function $A(z)$ is of the form

$$A(z) \;=\; \frac{\operatorname{diag}(\varkappa_1,\dots,\varkappa_n)}{z} \;+\; \Lambda(z)R(z)\Lambda^{-1}(z), \qquad z \in D_\infty,$$

where $\Lambda(z) = \operatorname{diag}(z^{\varkappa_1},\dots,z^{\varkappa_n})$ and $R : D_\infty \to \mathbb{C}^{n\times n}$ is an analytic function satisfying $\|R(z)\| = O(1/z^2)$ as $z \to \infty$.

Proof. Assume (i)–(iv) of Definition 3.1 holds. Then also conditions (i) and (ii) of this proposition are fulfilled. As to (iii), remark that $Y(\tilde z) = Z_k(z)(\tilde z - a_k)^{E_k}C$ implies

$$\begin{aligned} A(z) \;&=\; Y'(\tilde z)Y^{-1}(\tilde z) \;=\; Z_k'(z)Z_k^{-1}(z) + Z_k(z)\frac{E_k}{z-a_k}Z_k^{-1}(z) \\ &=\; \frac{Z_k(a_k)E_kZ_k^{-1}(a_k)}{z-a_k} \;+\; \text{“analytic term at } z = a_k\text{”} \end{aligned}$$

because $Z_k(z)$ is analytic and invertible at $z = a_k$. At infinity we have $Y(z) = \Lambda(z)Z_\infty(z)C$. Hence, similarly,

$$\begin{aligned} A(z) \;&=\; Y'(\tilde z)Y^{-1}(\tilde z) \;=\; \Lambda'(z)\Lambda^{-1}(z) + \Lambda(z)Z_\infty'(z)Z_\infty^{-1}(z)\Lambda^{-1}(z) \\ &=\; \frac{\operatorname{diag}(\varkappa_1,\dots,\varkappa_n)}{z} + \Lambda(z)R(z)\Lambda^{-1}(z), \end{aligned}$$

where $R(z) = Z_\infty'(z)Z_\infty^{-1}(z)$. Since $Z_\infty(z)$ is analytic and invertible, $\|R(z)\| = O(1/z^2)$ as $z \to \infty$. This implies (iv).

In order to prove the converse, we remark first that (i) and (ii) of this proposition imply (i) and (ii) of Definition 3.1 with the exception of the apparentness of the singularity at infinity, which will follow from (iv).

Condition (iii) of this proposition in connection with Lemma 4.1 immediately implies (iii) of Definition 3.1. Note that the mere similarity of the residue to E_k does not cause problems because $(\tilde z - a_k)^{S_kE_kS_k^{-1}} = S_k(\tilde z - a_k)^{E_k}S_k^{-1}$.

Finally, let us consider the singularity at infinity. Because of the asymptotics of $R(z)$ at infinity, there exists an analytic solution $Z_\infty : D_\infty \cup \{\infty\} \to G\mathbb{C}^{n\times n}$ of the system $Z_\infty'(z) = R(z)Z_\infty(z)$ near infinity. Now define $Y_\infty(z) = \Lambda(z)Z_\infty(z)$, $z \in D_\infty$. Then,

$$\begin{aligned} Y_\infty'(z)Y_\infty^{-1}(z) \;&=\; \Lambda'(z)\Lambda^{-1}(z) + \Lambda(z)Z_\infty'(z)Z_\infty^{-1}(z)\Lambda^{-1}(z) \\ &=\; \frac{\operatorname{diag}(\varkappa_1,\dots,\varkappa_n)}{z} + \Lambda(z)R(z)\Lambda^{-1}(z) \;=\; A(z). \end{aligned}$$

Hence $Y_\infty(z)$ is a solution of our original system considered on the domain D_∞. However, the relation of such a “restricted” solution to any solution of the system is given by $Y(z) = Y_\infty(z)C$ with some C. This is exactly condition (iv) of Definition 3.1. Obviously, this also shows that infinity is an apparent singularity. □

Now we present the desired description in terms of the matrix function $A(z)$. For $k \in \mathbb{Z}$, $k \geq 0$, let $\mathcal{P}_k$ stand for the set of all polynomials of degree less than or equal to

k. For $k \in \mathbb{Z}$, $k < 0$, let $\mathcal{P}_k$ stand for the set containing only the function identically equal to zero.

Theorem 4.3 *The system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is of standard form with (admissible) data $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$, with singularities $a_1, \ldots, a_m \in \mathbb{C}$ and indices $\varkappa_1, \ldots, \varkappa_n \in \mathbb{Z}$ if and only if the following conditions are satisfied:*

(i) $A(z) = [A_{ij}(z)]_{i,j=1}^n$, *where*

$$A_{ij}(z) \quad = \quad \frac{\varkappa_i \delta_{ij} z^{m-1} + p_{ij}(z)}{(z-a_1)(z-a_2)\cdots(z-a_m)}, \tag{4.1}$$

where $p_{ij} \in \mathcal{P}_{m-2+\varkappa_i-\varkappa_j}$ *and* δ_{ij} *is the Kronecker symbol.*

(ii) For each $k = 1, \ldots, m$, the residue of $A(z)$ at $z = a_k$ is similar to the matrix E_k.

(iii) The monodromy of the system is given by $[\chi(\sigma_1), \ldots, \chi(\sigma_m)]_\sim = [M_1, \ldots, M_m]_\sim$.

Proof. In regard to the previous proposition, we have to show that the characterization given by (4.1) is equivalent to the property that $A(z)$ is analytic on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$, has simple poles at $a_1, \ldots, a_m$ and behaves at infinity as described in (iv) of Proposition 4.2.

The fact that $A(z)$ is analytic on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$ and has simple poles at $a_1, \ldots, a_m$ is equivalent to writing the entries of $A(z)$ as $A_{ij}(z) = q_{ij}(z)(z-a_1)^{-1}(z-a_2)^{-1}\cdots(z-a_m)^{-1}$, where q_{ij} is an entire analytic function. The characterization of q_{ij} as a certain polynomial is now equivalent to (iv) of Proposition 4.2. $\square$

Considering the residues of $A(z)$ at $a_1, \ldots, a_m$ and at infinity and taking traces, we obtain the following simple relation,

$$\varkappa_1 + \ldots + \varkappa_n \quad = \quad \operatorname{trace} E_1 + \ldots + \operatorname{trace} E_m. \tag{4.2}$$

Hence the sum of the indices is determined by the the eigenvalues of E_k. This relation is the counterpart to formula (1.14) for the total index of the factorization of a piecewise constant matrix function.

5 Some general results in the irreducible case

In this section we present some necessary conditions for the indices of systems of standard form with irreducible monodromy data $[M_1, \ldots, M_m]$. By establishing a relationship to scalar linear Fuchsian differential equations of n-th order, we show that, in general, these conditions cannot be improved.

A subspace $X \subseteq \mathbb{C}^n$ is called an *invariant subspace* of a collection of $n \times n$ matrices $\{M_\omega\}_{\omega\in\Omega}$ if the image $M_\omega X$ is contained in X for each $\omega \in \Omega$. We say that a matrix function $F(z)$ has the invariant subspace X if $F(z)X \subseteq X$ for each z.

A collection of matrices $\{M_\omega\}_{\omega\in\Omega}$ is called *reducible* if there exists a non-trivial invariant subspace X (i.e., $X \neq \{0\}$ and $X \neq \mathbb{C}^n$). Otherwise, it is called *irreducible*. The monodromy of a system of linear differential equations is called reducible (resp., irreducible) if the collection $\{\chi(\sigma)\}_{\sigma\in\Delta}$ (or, equivalently, the collection $[\chi(\sigma_1),\dots,\chi(\sigma_m)]$) is reducible (resp., irreducible) for some (hence each) monodromy representation χ of the system.

The following auxiliary result relates invariant subspaces of a matrix function $A(z)$ with invariant subspaces for the solution and the monodromy of the corresponding system.

Lemma 5.1 *Let $A(z)$ be an analytic matrix function which has invariant subspaces $\{X_r\}_{r\in R}$. Then there exists a solution of the system $Y'(\tilde z) = A(z)Y(\tilde z)$ such that this solution and its corresponding monodromy representation also have the invariant subspaces $\{X_r\}_{r\in R}$.*

Proof. Let $P_r \in \mathbb{C}^{n\times n}$ be projections such that the image of P_r is the subspace X_r. Then $(I - P_r)A(z)P_r = 0$, and we obtain

$$(I - P_r)Y'(\tilde z)P_r \;=\; (I - P_r)A(z)(I - P_r)Y(\tilde z)P_r.$$

Choose any point $\tilde z_0 \in \widetilde S$ and consider the solution $Y(\tilde z)$ for which $Y(\tilde z_0) = I$. Hence $(I - P_r)Y(\tilde z_0)P_r = 0$. Because the matrix function $(I - P_r)Y(\tilde z)P_r$ is a solution of the above first order system, it follows that $(I - P_r)Y(\tilde z)P_r = 0$ for all $\tilde z \in \widetilde S$. This means that X_r is an invariant subspace of $Y(\tilde z)$. Hence X_r is also an invariant subspace of the inverse $Y^{-1}(\tilde z)$. Let χ be the monodromy representation of $Y(\tilde z)$ and $\sigma \in \Delta$. Then $\chi(\sigma) = Y^{-1}(\sigma(\tilde z))Y(\tilde z)$. Hence X_r is an invariant subspace of $\chi(\sigma)$. □

If we choose another solution of the system, $\widehat Y(\tilde z) = Y(\tilde z)C$, then the corresponding monodromy representation has the invariant subspaces $\{C^{-1}X_r\}_{r\in R}$ (see formula (2.4)).

Now we state the conditions on the indices in the case of irreducible monodromy. These conditions (formulated in the setting of vectors bundles and with a different proof) have already been obtained by Bolibruch [4]. Recall that the indices are ordered decreasingly, $\varkappa_1 \geq \varkappa_2 \geq \ldots \geq \varkappa_n$.

Theorem 5.2 *Let $Y'(\tilde z) = A(z)Y(\tilde z)$ be a system of standard form with m singularities $a_1,\dots,a_m \in \mathbb{C}$. If the monodromy of this system is irreducible, then the indices $\varkappa_1,\dots,\varkappa_n \in \mathbb{Z}$ satisfy, for each $k = 1,\dots,n-1$, the condition*

$$\varkappa_k - \varkappa_{k+1} \;\leq\; m - 2. \tag{5.1}$$

Proof. Suppose that this condition is not fulfilled for some k, i.e., $\varkappa_k - \varkappa_{k+1} > m-2$. Because the indices are ordered decreasingly, it follows that $\varkappa_j - \varkappa_i > m-2$ for each $j = 1,\dots,k$ and each $i = k+1,\dots,n$. Theorem 4.3 implies that for those i and j, the entries $A_{ij}(z)$ of the matrix $A(z)$ vanish identically. This means that the matrix

$A(z)$ is of block triangular form, i.e., it possesses the non-trivial invariant subspace $X = \mathbb{C}^k \oplus \{0\}^{n-k}$. By the previous lemma, the monodromy representation of some solution has also this invariant subspace. Hence the monodromy of the system is reducible, which contradicts the assumption. $\Box$

We want to emphasize that the above condition holds regardless of the location of the singularities and the values of $[E_1, \ldots, E_m]$. Reinterpreted in terms of the factorization problem, this implies the independence of this condition from the underlying space $L^p(\Gamma)$.

It is also easy to see that this condition (even combined with the knowledge of the total index $\varkappa$) is in general not sufficient to determine the values of the partial indices uniquely. In this connection, we are going to show that (5.1) cannot be improved.

For this purpose we introduce systems of standard form for which the above relation holds with equality, i.e., $\varkappa_k - \varkappa_{k+1} = m-2$ for each $k = 1, \ldots, n-1$. We single out a subclass of such systems which have – generically – irreducible monodromy.

Suppose we are given indices $\varkappa_1, \ldots, \varkappa_n \in \mathbb{Z}$ satisfying $\varkappa_k - \varkappa_{k+1} = m-2$. It follows from Theorem 4.3 that systems of standard form $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ with singularities $a_1, \ldots, a_m$ and above indices have a matrix function $A(z)$ which is of the form

$$\frac{1}{p}\begin{pmatrix} \varkappa_1 z^{m-1} + p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ \alpha_1 & \varkappa_2 z^{m-1} + p_{22} & p_{23} & & p_{2n} \\ 0 & \alpha_2 & \varkappa_3 z^{m-1} + p_{33} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p_{n-1,n} \\ 0 & 0 & \cdots & \alpha_{n-1} & \varkappa_n z^{m-1} + p_{nn} \end{pmatrix}, \qquad (5.2)$$

where $p(z) = (z-a_1)\cdots(z-a_m)$, $\alpha_k \in \mathbb{C}$ and $p_{ij} \in \mathcal{P}_{(m-2)(1+j-i)}$. The condition on the polynomials p_{ij} follows from $\varkappa_i - \varkappa_j = (m-2)(j-i)$. Conversely, any system with matrix $A(z)$ given by (5.2) such that the residues of $A(z)$ at $a_1, \ldots, a_m$ are non-resonant matrices is a system of standard form with above indices and singularities and with certain admissible data.

If $\alpha_k = 0$ for some $k = 1, \ldots, n-1$, then the monodromy of the above system is reducible because $A(z)$ is of block triangular form. Hence, as we are interested in irreducible monodromy, we will focus on systems with $\alpha_1 \cdots \alpha_{n-1} \neq 0$. A preliminary characterization of the data of such systems is given next.

For $S \in \mathbb{C}^{n\times n}$, let $\min(S)$ stand for the degree of the minimal polynomial of S, i.e., the polynomial p of smallest degree for which $p(S) = 0$.

Proposition 5.3 *Assume that $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is a system of standard form with admissible data $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ where $A(z)$ is given by (5.2) with $\alpha_1 \cdots \alpha_{n-1} \neq 0$. Then $\min(E_k) = \min(M_k) = n$ for each $k = 1, \ldots, m$.*

Proof. Because $\alpha_1 \cdots \alpha_{n-1} \neq 0$ the residue $\widehat{E}_k$ of $A(z)$ at $z = a_k$ has nonzero entries on the diagonal below the main diagonal. All entries below this diagonal are zero. By considering the powers of E_k, it follows that $\min(\widehat{E}_K) = n$. By Proposition 4.2(iii) we have $\widehat{E}_k \sim E_k$, which implies $\min(E_k) = n$. Because $M_k \sim \exp(-2\pi i E_k)$ and E_k is non-resonant, we obtain $\min(M_k) = n$ by considering the Jordan normal forms. $\square$

The next result relates the systems singled out above to scalar linear Fuchsian differential equations. Therein, the local exponents of the linear Fuchsian differential equations at the points $a_1, \ldots, a_m$ correspond to the eigenvalues of the matrices $E_1, \ldots, E_m$ of the given data.

This result has a strong relationship to a result obtained by Bolibruch [4]. He examined the minimal number of additional apparent singularities which a scalar Fuchsian differential equation must have in order to realize given irreducible monodromy. The answer he obtained involves the notion of the *maximal Fuchsian weight*, which is defined in terms of the splitting type for a class of vector bundles.

Theorem 5.4 *Let $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ be admissible data of $n \times n$ matrices, and let $a_1, \ldots, a_m \in \mathbb{C}$ be distinct points. For $k = 1, \ldots, m$, denote by $\varepsilon_k^{(1)}, \ldots, \varepsilon_k^{(n)}$ the eigenvalues of the matrices E_k. Then the following two statements are equivalent:*

(i) $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ is the data associated to some system with $A(z)$ given by (5.2) and $\alpha_1 \cdots \alpha_{n-1} \neq 0$.

(ii) $[M_1, \ldots, M_m]$ is the monodromy of some n-th order linear differential equation with Fuchsian singularities at $a_1, \ldots, a_m$ and local exponents given by $\{\varepsilon_k^{(1)}, \ldots, \varepsilon_k^{(n)}\}$ for $z = a_k$, $k = 1, \ldots, m-1$, and $\{\varepsilon_m^{(1)} - \varkappa_n, \ldots, \varepsilon_m^{(n)} - \varkappa_n\}$ for $z = a_m$.

Proof. (ii)$\Rightarrow$(i): Suppose we are given an n-th order linear differential equation (2.13) with the above properties. Let $y_1(\tilde{z}), \ldots, y_n(\tilde{z})$ be n linear independent solutions. We introduce

$$W(\tilde{z}) = \begin{pmatrix} y_1^{(n-1)}(\tilde{z}) & y_2^{(n-1)}(\tilde{z}) & \ldots & y_n^{(n-1)}(\tilde{z}) \\ \vdots & \vdots & & \vdots \\ y_1'(\tilde{z}) & y_2'(\tilde{z}) & \ldots & y_n'(\tilde{z}) \\ y_1(\tilde{z}) & y_2(\tilde{z}) & \ldots & y_n(\tilde{z}) \end{pmatrix}, \qquad \tilde{z} \in \widetilde{S}^*, \tag{5.3}$$

Then $W(\tilde{z})$ is analytic on $\widetilde{S}^*$, and it is well known that the Wronskian $\det W(\tilde{z})$ does not vanish on all of $\widetilde{S}^*$. In particular, $W(\tilde{z})$ is a solution of the $n \times n$ system $W'(\tilde{z}) = A_0(z)W(\tilde{z})$, where

$$A_0(z) = \begin{pmatrix} -q_1(z) & -q_2(z) & \ldots & -q_n(z) \\ 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}, \qquad z \in S \setminus \{\infty\}, \tag{5.4}$$

and $q_1(z), \ldots, q_n(z)$ are the coefficients of the scalar differential equation. Obviously, the system for $A_0(z)$ has the same singularities $a_1, \ldots, a_m$ (and possibly an apparent singularity at infinity) and the same monodromy as the scalar differential equation.

Below we will replace the system $W'(\tilde{z}) = A_0(z)W(\tilde{z})$ by a modified system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ which will have the desired properties. However, in order to analyze the behavior at the singular points a_k, we will first consider differently modified systems $Y_k(\tilde{z}) = A_k(z)Y_k(\tilde{z})$ near these singularities.

For $k = 1, \ldots, m$, we introduce

$$P_k(z) \quad = \quad \mathrm{diag}\Big((z-a_k)^{n-1}, (z-a_k)^{n-2}, \ldots, (z-a_k), 1\Big), \tag{5.5}$$

and write $q_j(z) = r_{jk}(z)/(z-a_k)^j$ for each $j = 1, \ldots, n$. Because the singularities of the scalar equation are assumed to be Fuchsian, the functions $r_{jk}(z)$ are analytic at $z = a_k$.

For $k = 1, \ldots, m-1$, we define $Y_k(\tilde{z}) = P_k(z)W(\tilde{z})$. Then $Y_k(\tilde{z})$ is a solution of the system $Y_k'(\tilde{z}) = A_k(z)Y_k(\tilde{z})$ with $A_k(z) = P_k(z)A_0(z)P_k^{-1}(z) + P_k'(z)P_k^{-1}(z)$. From this we obtain that $A_k(z) = \widehat{E}_k/(z-a_k)+$"analytic term at $z = a_k$" with

$$\widehat{E}_k \quad = \quad \begin{pmatrix} -r_{1k}(a_k) & -r_{2k}(a_k) & \ldots & -r_{nk}(a_k) \\ 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix} + \mathrm{diag}\,(n-1, \ldots, 1, 0). \tag{5.6}$$

A straightforward calculation shows that the characteristic equation $\det(\widehat{E}_k - \rho I) = 0$ for the matrix $\widehat{E}_k$ coincides with the indical equation (2.15) for the local exponents of the scalar equation with respect to the singularity a_k. Hence the eigenvalues of $\widehat{E}_k$ are just $\varepsilon_k^{(1)}, \ldots, \varepsilon_k^{(n)}$.

For $k = m$ we make a modification. We define $Y_m(\tilde{z}) = (z-a_m)^{\varkappa_n} P_m(z)W(\tilde{z})$, which is a solution of $Y_m'(\tilde{z}) = A_m(z)Y_m(\tilde{z})$. Again $A_m(z) = \widehat{E}_m/(z-a_m)+$"analytic term at $z = a_m$", but now $\widehat{E}_m$ contains an additional term $\varkappa_n I$. Consequently, $\det(\widehat{E}_m - \varkappa_n I - \rho I) = 0$ coincides with the indical equation for the singularity a_m. Because the assumption on the local exponents is also different, we obtain nevertheless that the eigenvalues of $\widehat{E}_m$ are $\varepsilon_m^{(1)}, \ldots, \varepsilon_m^{(n)}$.

In fact, the converse is also true. The scalar differential equation is Fuchsian at a_k if and only if the system $Y_k'(\tilde{z}) = A_k(z)Y_k(\tilde{z})$ obtained in the above way is Fuchsian at a_k.

Next we are going to analyze the singularity at infinity. We make a change of variables $z \mapsto \xi = (z-a_k)^{-1}$ and introduce $\hat{y}_1(\xi), \ldots, \hat{y}_n(\xi)$ by $\hat{y}_j((z-a_m)^{-1}) = y_j(z)$ for $j = 1, \ldots, n$. These new functions are solutions of a modified scalar linear differential equation. The point $\xi = 0$ corresponds to the point $z = \infty$. By assumption they are not singular points. Hence

$$\widehat{W}(\xi) \;=\; \begin{pmatrix} \hat{y}_1^{(n-1)}(\xi) & \hat{y}_2^{(n-1)}(\xi) & \dots & \hat{y}_n^{(n-1)}(\xi) \\ \vdots & \vdots & & \vdots \\ \hat{y}_1'(\xi) & \hat{y}_2'(\xi) & \dots & \hat{y}_n'(\xi) \\ \hat{y}_1(\xi) & \hat{y}_2(\xi) & \dots & \hat{y}_n(\xi) \end{pmatrix}, \qquad \xi \in D_0, \tag{5.7}$$

is analytic in a neighborhood of $\xi = 0$ and $\det \widehat{W}(0) \neq 0$. We remark that the identity

$$W(z) \;=\; P_m^{-1}(z) S P_m^{-1}(z) \widehat{W}((z-a_m)^{-1}), \qquad z \in D_\infty, \tag{5.8}$$

holds, which can be verified by a direct calculation. Here S is a certain invertible upper triangular $n \times n$ matrix. This matrix does not depend on any parameters and can (in principle) be evaluated explicitly.

Now we are prepared to define the system which will satisfy the conditions of (i). Let

$$U(z) \;=\; (z-a_m)^{\varkappa_n} P_1(z) \cdots P_{m-1}(z) S^{-1} P_m(z), \tag{5.9}$$

and define $Y(\tilde{z}) = U(z) W(\tilde{z})$. Because the function $U(z)$ is analytic and invertible on $S \setminus \{\infty\}$, it follows from the afore-mentioned properties of the system $W'(\tilde{z}) = A_0(z) W(\tilde{z})$ that the new system $Y'(\tilde{z}) = A(z) Y(\tilde{z})$ has singularities only at $a_1, \dots, a_m$ (and possibly an apparent singularity at infinity) and has the same monodromy as the scalar differential equation.

Next we analyze the local behavior of the new system at infinity. Combining (5.8) and (5.9), it follows that

$$\begin{aligned} Y(z) \;&=\; (z-a_m)^{\varkappa_n} P_1(z) \cdots P_{m-1}(z) P_m^{-1}(z) \widehat{W}((z-a_m)^{-1}) \\ &=\; \operatorname{diag}\left(z^{\varkappa_n + (m-2)(n-1)}, z^{\varkappa_n + (m-2)(n-2)}, \dots, z^{\varkappa_n + m - 2}, z^{\varkappa_n}\right) Z_\infty(z) \end{aligned} \tag{5.10}$$

for $z \in D_\infty$, where Z_∞ is analytic and invertible on D_∞. Hence the behavior at infinity is as required for systems of standard form with indices $\varkappa_1, \dots, \varkappa_n$. (Recall that $\varkappa_k - \varkappa_{k+1} = m-2$ is assumed.)

In order to analyze the behavior at the singularity a_k, we make the connection with the systems $Y_k'(\tilde{z}) = A_k(z) Y_k(\tilde{z})$. First write $U(z) = (z-a_m)^{\varkappa_n} \widehat{U}(z) P_1(z) \cdots P_m(z)$, where $\widehat{U}(z) = P_1(z) \cdots P_{m-1}(z) S^{-1} P_{m-1}^{-1}(z) \cdots P_1^{-1}(z)$ can be shown to be analytic and invertible on all of $\mathbb{C}$. In fact, one uses that S is an upper triangular matrix and that $P_1(z) \cdot P_m(z)$ is a diagonal matrix function with particular entries. Now, recalling how $Y_k(\tilde{z})$ and $Y(\tilde{z})$ were obtained from $W(\tilde{z})$, it follows that $Y(\tilde{z}) = U_k(z) Y_k(\tilde{z})$ where $U_k(z)$ is a certain matrix function analytic and invertible on D_{a_k}. Hence the local properties of $Y(\tilde{z})$ and $Y_k(\tilde{z})$ are essentially the same. In particular, $A(z)$ has a simple pole at $z = a_k$ as so has $A_k(z)$, and the residue of $A(z)$ at $z = a_k$ is similar to $\widehat{E}_k$.

Using the characterizations given in Proposition 4.2 and Theorem 4.3, it follows that $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is a system of standard form with "local" data $[\widehat{E}_1, \ldots, \widehat{E}_m]$ and the properties mentioned above. In particular, we obtain that $A(z)$ is of the form (5.2).

The assertion that $\alpha_1 \cdots \alpha_{n-1} \neq 0$ is a consequence of the special structure of $A_0(z)$ given in (5.4) and the fact that $U(z)$ is an upper triangular matrix function. Observe that

$$A(z) \quad = \quad U(z)A_0(z)U^{-1}(z) + U'(z)U^{-1}(z). \tag{5.11}$$

Finally, we need to show that the "local" data is given also by $[E_1, \ldots, E_m]$, i.e., $E_k \sim \widehat{E}_k$. In fact, Proposition 5.3 implies that $\min(M_k) = \min(\widehat{E}_k) = n$. Because by assumption $M_k \sim \exp(-2\pi i E_k)$, we obtain $\min(E_k) = n$. Again by assumption, the eigenvalues of E_k are $\varepsilon_k^{(1)}, \ldots, \varepsilon_k^{(n)}$, hence they are the same as those of $\widehat{E}_k$. Now we can conclude $E_k \sim \widehat{E}_k$.

(i)$\Rightarrow$(ii): The essence of the above construction was the passage from the matrix function $A_0(z)$ to the matrix function $A(z)$ by means of the transformation (5.11) with $U(z)$. The fact that the entries below the first row of $A_0(z)$ are of a special form is reason for the connection of $A_0(z)$ to the scalar equation.

Now consider the singularities $a_1, \ldots, a_m$ and the indices $\varkappa_1, \ldots, \varkappa_n$ as fixed and the functions $q_1(z), \ldots, q_n(z)$ as arbitrary. It follows from the above argumentation that the matrices $A(z)$ obtained by (5.11) are necessarily of the form (5.2). Moreover, a straightforward computation (using the special structure of $U(z)$ and $A_0(z)$) shows that the entries below the first row of $A(z)$ are also of a special form, i.e., they only depend on $a_1, \ldots, a_m$ and $\varkappa_n$ but not on $q_1(z), \ldots, q_n(z)$. Moreover, one obtains that $\alpha_1 \cdots \alpha_{n-1} \neq 0$. In this sense, these entries of $A(z)$ have to be considered as "fixed". In principle, they can be computed.

We want to make the converse transformation with $U^{-1}(z)$, i.e., to pass from $A(z)$ to $A_0(z)$ by means of

$$A_0(z) \quad = \quad U^{-1}(z)A(z)U(z) - U^{-1}(z)U'(z). \tag{5.12}$$

Obviously, the matrix $A_0(z)$ will be only of the desired form (5.4) if the entries below the first row of $A(z)$ are chosen "properly". The crucial point is that if these entries of $A(z)$ are given indeed in the appropriate way, then the matrix $A_0(z)$ will be of the form (5.4) with certain $q_1(z), \ldots, q_n(z)$. This can be shown again by a straightforward computation.

Hence we cannot start with just any matrix $A(z)$ of the form (5.2). Fortunately, the following result is true. Any system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ with $A(z)$ given by (5.2) and $\alpha_1 \cdots \alpha_{n-1} \neq 0$ is equivalent to some system with another matrix $\widehat{A}(z)$ which is of the same form and whose entries below the first row are prescribed arbitrarily.

In order to prove this statement we need only recall Theorem 3.5. We have to prove that there exists a matrix $V(z)$ of the form (3.3) and the properties stated there such that a transformation with $V(z)$ takes the matrix $A(z)$ to the matrix $\widehat{A}(z)$. Note that the entries of $V(z)$ are polynomials of a certain degree. We can split this transformation into several steps. First, let $V(z)$ be a diagonal matrix with suitable diagonal entries such that $A(z)$ is taken into a matrix with entries $\alpha_1, \dots, \alpha_{n-1}$. At the k-step, $k = 1, \dots, n$, we then choose the matrix $V(z)$ to have ones on the main diagonal, suitable polynomials on the k-th diagonal above the main diagonal and zero entries everywhere else. In fact, a straightforward computation (analyzing $A \mapsto VAV^{-1} + V'V^{-1}$) shows that these polynomials can be chosen in such a way that the entries of the $(k-1)$-th diagonal above the main diagonal of $A(z)$ (except the entry in the first row) can be modified in any desired way. At the k-th step, all entries of $A(z)$ below this diagonal remain unaltered. Hence after the n-th step we arrive at a matrix $\widehat{A}(z)$ of the same form and with desired entries below the first row.

After this, we may assume that the matrix $A(z)$ of the system described in (i) is of such a form that the transformation (5.12) gives a matrix $A_0(z)$ which is of the form (5.4) with certain $q_1(z), \dots, q_n(z)$. Hence this new system gives raise to an n-th order linear differential equation. We have to show that this equation has the properties stated in (ii).

In fact, the argumentation given above can be reversed. Using the properties of $U(z)$ (and employing the systems $Y_k'(\tilde{z}) = A_k(z)Y_k(\tilde{z})$ at an intermediate step) it follows that $q_1(z), \dots, q_n(z)$ are analytic on $S \setminus \{\infty\}$ and that $q_j(z)$ has at most a j-th order pole at $z = a_k$. Hence the scalar equation is a Fuchsian equation and it follows also from above that the local exponents are equal to the eigenvalues of the residue of $A(z)$, which is a matrix similar to E_k. Obviously, the monodromy is the same as for the given system.

The assertion that the scalar equation has no singularity at infinity is only slightly more difficult. We know that the solution $Y(\tilde{z})$ of the system can be written as (5.10). Introducing a modified scalar equation by change of variables as above, we see that (5.8) holds with $W(\tilde{z})$ being the solution for the system with $A_0(z)$, or, equivalently, the matrix (5.13) corresponding to the solution of the scalar equation. Combining this with (5.9) and the fact that $Y(\tilde{z}) = U(z)W(\tilde{z})$, it is possible to conclude that $\widehat{W}(\xi)$ is analytic and invertible near $\xi = 0$. Hence $\xi = 0$ is no singular point for the modified scalar equation, and neither is $z = \infty$ a singular point for the desired scalar equation. $\square$

We remark in this connection that the identity

$$\operatorname{trace} E_1 + \ldots + \operatorname{trace} E_m \;=\; \varkappa_1 + \ldots + \varkappa_n \;=\; \frac{n(n-1)}{2}(m-2) + n\varkappa_n, \quad (5.13)$$

which follows from (4.2) and $\varkappa_k - \varkappa_{k+1} = m - 2$, corresponds to

$$\sum_{j=1}^{n} (\varepsilon_1^{(j)} + \varepsilon_2^{(j)} + \ldots + \varepsilon_m^{(j)} - \varkappa_n) \;=\; \frac{n(n-1)}{2}(m-2), \quad (5.14)$$

which is Fuchs' relation (2.16) for the above linear differential equation.

The previous theorem provides us with an implicit description of a class of data which has indices satisfying $\varkappa_k - \varkappa_{k+1} = m-2$. (It does not describe all such data because of the assumption $\alpha_1 \cdots \alpha_{n-1} \neq 0$.) Namely, we may choose indices satisfying $\varkappa_k - \varkappa_{k+1} = m-2$ and prescribe the eigenvalues of $E_1, \ldots, E_m$ under the restrictions (5.13) or (5.14). Then the corresponding linear differential equations, which contain $(m-2)n(n+1)/2-(m-1)n+1$ free parameters, give rise to a class of monodromy $[M_1, \ldots, M_m]$. Although in general there is no explicit description for this monodromy, one knows that this data has the above indices.

The monodromy of linear Fuchsian differential equations is generically irreducible. In fact, if the numbers $\varepsilon_k^{(j)}$ are chosen such that

$$\sum_{k=1}^{m}\sum_{r=1}^{R}\varepsilon_k^{(j_{r,k})} \;\notin\; \mathbb{Z} \tag{5.15}$$

for all possibilities of $1 \leq R < n$ and $1 \leq j_{1,k} < j_{2,k} < \ldots < j_{R,k} \leq n$, then the monodromy of the corresponding differential equation is always irreducible. Hence the condition (5.2) cannot be improved in general. We note, however, that the (non-generic) case of reducible monodromy is also covered by the previous theorem.

The analogue of the condition $\min(M_k) = n$ stated in Proposition 5.3 for scalar linear differential equations is well known. If the difference of any two local exponents of a Fuchsian singularity is not a nonzero integer, then the corresponding monodromy matrix M_k satisfies $\min(M_k) = n$.

6 Some results for the 2×2 matrix case

In this section we consider the 2×2 matrix case with m singularities. For general m we obtain some information about the indices corresponding to given data. For $m = 3$, this information provides us with a complete explicit answer. For $m = 4$, the answer can be given either explicitly or in terms of the monodromy of second order linear Fuchsian differential equations with four singularities.

In order to formulate our results we have to make some preparations related to the "reducibility type" of the data. Let $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ be admissible data of 2×2 matrices. Then the following (exclusive) cases can occur:

(A) $[M_1, \ldots, M_m]$ is irreducible;

(B) $[M_1, \ldots, M_m]$ possesses exactly one non-trivial invariant subspace;

(C) $[M_1, \ldots, M_m]$ possesses at least two non-trivial invariant subspaces.

In all these cases we define

$$\varkappa \;=\; \operatorname{trace} E_1 + \ldots + \operatorname{trace} E_m. \tag{6.1}$$

In case (B), the m-tuple $[M_1, \ldots, M_m]$ is equivalent to

$$\left[\begin{pmatrix} \mu_1^{(1)} & \alpha_1 \\ 0 & \mu_1^{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} \mu_m^{(1)} & \alpha_m \\ 0 & \mu_m^{(2)} \end{pmatrix} \right], \tag{6.2}$$

where $\mu_1^{(1)} \cdots \mu_m^{(1)} = \mu_1^{(2)} \cdots \mu_m^{(2)} = 1$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ satisfy the linear equation

$$\alpha_1 \mu_2^{(2)} \cdots \mu_m^{(2)} \; + \; \mu_1^{(1)} \alpha_2 \mu_3^{(2)} \cdots \mu_m^{(2)} \; + \; \ldots \; + \; \mu_1^{(1)} \cdots \mu_{m-1}^{(1)} \alpha_m \; = \; 0. \tag{6.3}$$

In the representation (6.2), the values of $\mu_k^{(j)}$ are uniquely determined. In fact, $\mu_k^{(1)}$ is the eigenvalue of M_k corresponding to the non-trivial invariant subspace and $\mu_k^{(2)}$ is the other eigenvalue. Let $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$ be the eigenvalues of E_k numbered in such a way that $\mu_k^{(j)} = \exp(-2\pi i \varepsilon_k^{(j)})$. Because of the non-resonance of E_k, their values are again uniquely determined, and so are the following integers,

$$n_1 \; = \; \varepsilon_1^{(1)} + \ldots + \varepsilon_m^{(1)}, \qquad n_2 \; = \; \varepsilon_1^{(2)} + \ldots + \varepsilon_m^{(2)}. \tag{6.4}$$

In case (C), the m-tuple $[M_1, \ldots, M_m]$ is diagonalizable, i.e., it is equivalent to (6.2) with $\alpha_1 = \ldots = \alpha_m = 0$. Here, in contrast, the values of $\mu_k^{(1)}$ and $\mu_k^{(2)}$ can be interchanged. Consequently, the integers n_1 and n_2 are only defined up to change of order. In the description of the main results later on, we will therefore assume without loss of generality that $n_1 \geq n_2$.

The following proposition provides some information about the equivalence classes of m-tuples of the form (6.2)

Proposition 6.1 *Let $[M_1, \ldots, M_m]$ and $[\widetilde{M}_1, \ldots, \widetilde{M}_m]$ be two m-tuples of the form (6.2). Assume that both m-tuples have the same values $\mu_1^{(1)}, \ldots, \mu_m^{(1)}$ and $\mu_1^{(2)}, \ldots, \mu_m^{(2)}$, but possibly different values $\alpha_1, \ldots, \alpha_m$ and $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_m$, respectively. Then these two m-tuples are equivalent if and only if there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $\rho \in \mathbb{C}$ such that*

$$\alpha_k \; = \; \lambda \widetilde{\alpha}_k + \rho(\mu_k^{(1)} - \mu_k^{(2)}) \tag{6.5}$$

for each $k = 1, \ldots, m$.

Proof. The m-tuples $[M_1, \ldots, M_m]$ and $[\widetilde{M}_1, \ldots, \widetilde{M}_m]$ are equivalent if and only if there exists a matrix $C \in G\mathbb{C}^{2\times 2}$ such that $\widetilde{M}_k = C M_k C^{-1}$. If all matrices $M_1, \ldots, M_m$ are scalar matrices, then the assertion is trivial. Otherwise, if at least one M_k is not scalar, one rewrites $\widetilde{M}_k C = C M_k$, and now a straightforward computation shows that C is an upper triangular matrix. Using this, the relation $\widetilde{M}_k = C M_k C^{-1}$ is easily seen to be equivalent to (6.5). □

As a conclusion, an m-tuple of the form (6.2) is diagonalizable if and only if the vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multiple of the vector $(\mu_1^{(1)} - \mu_1^{(2)}, \ldots, \mu_m^{(1)} - \mu_m^{(2)})$.

All m-tuples of the form (6.2) are parameterized by the vector α. Because of the linear condition (6.3), all "admissible" vectors α are taken from an $(m-1)$-dimensional subspace of $\mathbb{C}^m$. The relation (6.5) establishes an equivalence relation between such vectors. The statement of this proposition is that there is a one-to-one correspondence between equivalence classes of such vectors and equivalence classes of such m-tuples. Elaborating on (6.5), it is easy to see that these equivalence classes can be identified with $\mathbb{P}^{m-2}\cup\{0\}$ in case $\mu_k^{(1)}=\mu_k^{(2)}$ for all $k=1,\dots,m$ and with $\mathbb{P}^{m-3}\cup\{0\}$ in case $\mu_k^{(1)}\neq\mu_k^{(2)}$ for some $k=1,\dots,m$. Here $\mathbb{P}^n$ stands for the n-dimensional complex projective space, and $\{0\}$ symbolizes the single equivalence class containing the vector $\alpha=0$. We remark that the singleton $\{0\}$ corresponds exactly to the class of matrices $[M_1,\dots,M_m]$ which are diagonalizable.

In the following theorem, we consider all systems of standard form $Y'(\tilde{z})=A(z)Y(\tilde{z})$ with indices $\varkappa_1,\varkappa_2\in\mathbb{Z}$, $\varkappa_1\geq\varkappa_2$, for which $A(z)$ can be written as

$$A(z) \;=\; \frac{1}{p(z)}\left(\begin{array}{cc} \varkappa_1 z^{m-1}+p_1(z) & p_{12}(z) \\ 0 & \varkappa_2 z^{m-1}+p_2(z)\end{array}\right), \tag{6.6}$$

where $p(z)=(z-a_1)\cdots(z-a_m)$, $p_1,p_2\in\mathcal{P}_{m-2}$, $p_{12}\in\mathcal{P}_{m-2+\varkappa_1-\varkappa_2}$. We will completely characterize the data which is associated to such systems.

Theorem 6.2 *Let $\varkappa_1,\varkappa_2\in\mathbb{Z}$, $\varkappa_1\geq\varkappa_2$, and let $[M_1,\dots,M_m]$ and $[E_1,\dots,E_m]$ be admissible data of 2×2 matrices. Then the following two statements are equivalent:*

(i) $[M_1,\dots,M_m]$ and $[E_1,\dots,E_m]$ is the data associated to some system of the form (6.6).

(ii) $[M_1,\dots,M_m]$ is equivalent to some m-tuple of the form (6.2) with $n_1=\varkappa_1$, $n_2=\varkappa_2$, where n_1 and n_2 are defined by (6.4).

Proof. (i)$\Rightarrow$(ii): Because $A(z)$ is of triangular form we obtain from Lemma 5.1 that there exists a solution $Y(\tilde{z})$ which is also of triangular form. So let us write $Y'(\tilde{z})=A(z)Y(\tilde{z})$ as

$$\left(\begin{array}{cc} y_1'(\tilde{z}) & y_{12}'(\tilde{z}) \\ 0 & y_2'(\tilde{z})\end{array}\right) \;=\; \left(\begin{array}{cc} a_1(z) & a_{12}(z) \\ 0 & a_2(z)\end{array}\right)\left(\begin{array}{cc} y_1(\tilde{z}) & y_{12}(\tilde{z}) \\ 0 & y_2(\tilde{z})\end{array}\right). \tag{6.7}$$

For $j=1,2$, we have

$$a_j(z) \;=\; \frac{\varkappa_j z^{m-1}+p_j(z)}{p(z)} \;=\; \sum_{k=1}^{m}\frac{\varepsilon_k^{(j)}}{z-a_k}, \tag{6.8}$$

where the last equality defines the numbers $\varepsilon_1^{(j)},\dots,\varepsilon_m^{(j)}$. They satisfy $\varkappa_j=\varepsilon_1^{(j)}+\ldots+\varepsilon_m^{(j)}$. Because E_k is similar to the residue of $A(z)$ at $z=a_k$, the numbers $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$ are the eigenvalues of E_k. Once we have shown that $\mu_k^{(j)}=\exp(-2\pi i\varepsilon_k^{(j)})$, it follows that $\varkappa_1=n_1$ and $\varkappa_2=n_2$.

In fact, from (6.8) we obtain that the solutions of $y_j'(\tilde z) = a_j(z) y_j(\tilde z)$ are given by $y_j(\tilde z) = \prod_{k=1}^m (\tilde z - a_k)^{\varepsilon_k^{(j)}}$ up to a redundant constant. Regardless of the precise expression for $y_{12}(\tilde z)$, it follows that the monodromy representation for $Y(\tilde z)$ is given by (6.2) with $\mu_k^{(j)} = \exp(-2\pi i \varepsilon_k^{(j)})$ and with certain α_k. Hence $[M_1, \ldots, M_m]$ is equivalent to (6.2) and the $\mu_k^{(j)}$ and $\varepsilon_k^{(j)}$ are indeed properly ordered.

(ii)$\Rightarrow$(i): We consider all m-tuples of the form (6.2) with fixed $\mu_k^{(j)}$, and we fix also the numbers $\varepsilon_k^{(j)}$ (i.e., the eigenvalues of E_k). Observing that $\varkappa_1 = n_1$ and $\varkappa_2 = n_2$, we introduce the functions $a_1(z)$ and $a_2(z)$ by (6.8).

In what follows we consider all systems of the form (6.7) with those $a_1(z)$ and $a_2(z)$ and with $a_{12}(z) = p_{12}(z)/p(z)$ where $p_{12}(z)$ runs through all polynomials in $\mathcal{P}_{m-2+\varkappa_1-\varkappa_2}$. These systems are indeed of standard form because $\varkappa_1 \geq \varkappa_2$ and because $E_1, \ldots, E_m$ are assumed to be non-resonant. From the first part of this proof we already know that the monodromy representation of the corresponding solutions are of the form (6.2) with certain vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$. It remains to show that if p_{12} ranges through all of $\mathcal{P}_{m-2+\varkappa_1-\varkappa_2}$, then the corresponding vectors α take values in all equivalence classes (defined by the equivalence relation (6.5)).

We first examine the question when two systems with $A(z)$ and $\widetilde{A}(z)$ given by

$$A(z) \;=\; \begin{pmatrix} a_1(z) & p_{12}(z)/p(z) \\ 0 & a_2(z) \end{pmatrix}, \qquad \widetilde{A}(z) \;=\; \begin{pmatrix} a_1(z) & \tilde p_{12}(z)/p(z) \\ 0 & a_2(z) \end{pmatrix}$$

with $p_{12}, \tilde p_{12} \in \mathcal{P}_{m-2+\varkappa_1-\varkappa_2}$ are equivalent. By definition this is the case if and only if there exists a V such that $\widetilde{A} = VAV^{-1} + V'V^{-1}$, where V is of the form

$$V(z) \;=\; \begin{pmatrix} \lambda_1 & v(z) \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ and $v \in \mathcal{P}_{\varkappa_1-\varkappa_2}$ if $\varkappa_1 > \varkappa_2$, and $V(z) = V \in G\mathbb{C}^{2\times 2}$ if $\varkappa_1 = \varkappa_2$. If $\varkappa_1 = \varkappa_2$, then $\widetilde{A} V = VA$ implies that V has to be an upper triangular matrix (except for the case where $A(z)$ is given with $a_1(z) = a_2(z)$ and $p_{12}(z) = 0$, which can easily be dealt with separately). Using this information about $V(z)$, it follows that the above systems are equivalent if and only if $\widetilde{p}_{12} = \lambda p_{12} + v(a_2 - a_1)p + v'p$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $v \in \mathcal{P}_{\varkappa_1-\varkappa_2}$.

This, in turn, can be rephrased by saying that $\widetilde{p}_{12} - \lambda p_{12} \in \operatorname{im} \Xi$ for some $\lambda \in \mathbb{C} \setminus \{0\}$, where $\operatorname{im}\Xi$ is the image of the linear mapping $\Xi : \mathcal{P}_{\varkappa_1-\varkappa_2} \to \mathcal{P}_{m-2+\varkappa_1-\varkappa_2}$, $v \mapsto v(a_2 - a_1)p + v'p$. We conclude that $\dim \operatorname{im} \Xi \leq \varkappa_1 - \varkappa_2 + 1$. If $a_1 = a_2$ (hence $\varkappa_1 = \varkappa_2$), then Ξ is even the zero mapping. Now decompose $\mathcal{P}_{m-2+\varkappa_1-\varkappa_2} = \operatorname{im}\Xi \oplus X$ as a direct sum, and remark that $\dim X \geq m-2$ and even $\dim X \geq m-1$ if $a_1 = a_2$. Finally, we arrive at the following statement: if $p_{12}, \widetilde{p}_{12} \in X$, then the above systems are equivalent if and only if $\widetilde{p}_{12} = \lambda p_{12}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Now choose any basis $p_{12}^{(1)}, \ldots, p_{12}^{(d)}$ in X where $d = \dim X$. Assume that a solution of the system (6.7) with $a_{12} = p_{12}^{(r)}/p$ is given by some $y_{12} = y_{12}^{(r)}$, $r = 1, \ldots, d$. The corresponding monodromy representation is supposed to be given by (6.2) with some vector $\alpha = \alpha^{(r)} \in \mathbb{C}^m$. Now consider an arbitrary polynomial $p_{12} \in X$ and write $p_{12} = \sum \gamma_r p_{12}^{(r)}$. A straightforward computation shows that a solution of (6.7) with $a_{12} = p_{12}/p$ is given with $y_{12} = \sum \gamma_r y_{12}^{(r)}$ (notice that there are certainly further solutions). Moreover, the monodromy representation of this solution is given by (6.2) with $\alpha = \sum \gamma_r \alpha^{(r)}$. In this way, we have defined a linear mapping $\Lambda : X \to \mathbb{C}^m, p_{12} \mapsto \alpha$ from X into the set of all admissible vectors α, which characterize the reducible monodromy.

Now we invoke the fact that there is a one-to-one correspondence between equivalence classes of systems of standard form and equivalence classes of given data (see Corollary 3.6). This fact specialized to the mapping Λ means the following: for $p_{12}, \widetilde{p}_{12} \in X$, the vector $\Lambda(p_{12})$ is equivalent to the vector $\Lambda(\widetilde{p}_{12})$ in the sense of (6.5) if and only if $\widetilde{p}_{12} = \lambda p_{12}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. This allows us easily to conclude that the kernel of Λ is trivial. Hence, in case $a_1 = a_2$, we have $\dim \operatorname{im} \Lambda = d \geq m-1$, and we are done because the set of all admissible vectors α is an $(m-1)$-dimensional subspace of $\mathbb{C}^m$. On the other hand, $a_1 \neq a_2$ means that $\varepsilon_k^{(1)} \neq \varepsilon_k^{(2)}$ for some $k = 1, \ldots, m$. Hence (by the non-resonance assumption) the vector $\mu = (\mu_1^{(1)} - \mu_1^{(2)}, \ldots, \mu_m^{(1)} - \mu_m^{(2)})$, occurring in (6.5), is not the zero vector. Because $\dim \operatorname{im} \Lambda = d \geq m-2$, the assertion is proved if we have shown that $\mu \notin \operatorname{im} \Lambda$. Indeed, if $\Lambda(p_{12}) = \mu$, then the equivalence of μ to the zero vector implies that $p_{12} = 0$. Hence $\mu = 0$, which is a contradiction. □

The reader should realize the meaning of the condition $n_1 = \varkappa_1$ and $n_2 = \varkappa_2$ in the previous theorem. Because of the necessary assumption $\varkappa_1 \geq \varkappa_2$, not all reducible data is covered by (ii) of the theorem, but only those for which $n_1 \geq n_2$. However, the case of diagonalizable data is completely covered.

The next corollary resumes this, and gives in addition some (but in general not a complete) information about the indices in the remaining cases.

Corollary 6.3 *Let $[M_1, \ldots, M_m]$ and $[E_1, \ldots, E_m]$ be admissible data of 2×2 matrices.*

(i) If $[M_1, \ldots, M_m]$ is diagonalizable, or if $[M_1, \ldots, M_m]$ possesses exactly one non-trivial invariant subspace and $n_1 \geq n_2$ holds, then the indices are $\varkappa_1 = n_1$ and $\varkappa_2 = n_2$.

(ii) If $[M_1, \ldots, M_m]$ is irreducible, or if $[M_1, \ldots, M_m]$ possesses exactly one non-trivial invariant subspace and $n_1 < n_2$ holds, then the indices fulfill the conditions:

$$\varkappa_1 - \varkappa_2 \;\leq\; m-2 \quad \textit{and} \quad \varkappa_1 + \varkappa_2 \;=\; \varkappa \tag{6.9}$$

Proof. Part (i) is a conclusion of the implication (ii)⇒(i) of Theorem 6.2. The second condition of part (ii) is just relation (4.2).

Now assume that $\varkappa_1 - \varkappa_2 > m-2$. Then Theorem 4.3 implies that the matrix $A(z)$ of the corresponding system is the form (6.6). Using the implication (i)⇒(ii) of Theorem 6.2, it follows that $[M_1, \ldots, M_m]$ is of the form (6.2) with $n_1 \geq n_2$, contradicting the assumption. $\square$

Hence, for data satisfying the assumption in (i), we have a complete knowledge of the indices. Note that they do not depend on the location of the singularities.

For data satisfying the assumption in (ii), we will give precise information for the cases $m = 3$ and $m = 4$. In case $m = 3$, we need only invoke (6.9).

Corollary 6.4 *Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 2×2 matrices. Assume that $[M_1, M_2, M_3]$ is irreducible, or that $[M_1, M_2, M_3]$ possesses exactly one non-trivial invariant subspace and $n_1 < n_2$ holds.*

(a) If $\varkappa$ is odd, then $\varkappa_1 = (\varkappa+1)/2$ and $\varkappa_2 = (\varkappa-1)/2$.

(b) If $\varkappa$ is even, then $\varkappa_1 = \varkappa_2 = \varkappa/2$.

The results for $m = 3$ presented here are in accordance with the result obtained in [18] for the factorization problem. Actually, they are a generalization of the former result because the matrices E_1, E_2, E_3 can be chosen less restrictive. (We require only non-resonance instead that the real parts of the eigenvalues are contained in an interval of length one.)

Corollary 6.5 *Let $[M_1, \ldots, M_4]$ and $[E_1, \ldots, E_4]$ be admissible data of 2×2 matrices, and let $a_1, \ldots, a_4 \in \mathbb{C}$ be distinct points. Assume that $[M_1, \ldots, M_4]$ is irreducible, or that $[M_1, \ldots, M_4]$ possesses exactly one non-trivial invariant subspace and $n_1 < n_2$ holds. Let $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}$ be the eigenvalues of E_k, $k = 1, \ldots, 4$.*

(a) If $\varkappa$ is odd, then $\varkappa_1 = (\varkappa+1)/2$ and $\varkappa_2 = (\varkappa-1)/2$.

(b) If $\varkappa$ is even, then either $\varkappa_1 = \varkappa_2 = \varkappa/2$, or $\varkappa_1 = \varkappa/2+1$ and $\varkappa_2 = \varkappa/2-1$.

(b) For $\varkappa$ even, the indices are $\varkappa_1 = \varkappa/2+1$ and $\varkappa_2 = \varkappa/2-1$ if and only if $[M_1, \ldots, M_4]$ is the monodromy of some second order linear differential equation with Fuchsian singularities $a_1, \ldots, a_4$ and local exponents given by $\{\varepsilon_k^{(1)}, \varepsilon_k^{(2)}\}$ for $z = a_k$, $k = 1, 2, 3$, and $\{\varepsilon_4^{(1)} + 1 - \varkappa/2, \varepsilon_4^{(2)} + 1 - \varkappa/2\}$ for $z = a_4$.*

Proof. Again, assertions (a) and (b) are an immediate consequence of (6.9). In order to prove (b*) we argue as follows. The indices for the given data are $\varkappa_1 = \varkappa/2+1$ and $\varkappa_2 = \varkappa/2-1$ if and only if this data is associated to some system of standard form for which $A(z)$ can be written as

$$A(z) = \frac{1}{p(z)} \begin{pmatrix} (\varkappa/2+1)z^3 + p_1(z) & p_{12}(z) \\ \alpha & (\varkappa/2-1)z^3 + p_2(z) \end{pmatrix}$$

with $p(z) = (z-a_1)\cdots(z-a_4)$, $\alpha \in \mathbb{C}$, $p_1, p_2 \in \mathcal{P}_2$ and $p_{12} \in \mathcal{P}_4$ (see Theorem 4.3). If $\alpha = 0$, then we can apply Theorem 6.2 and conclude that the monodromy is reducible and $n_1 = \varkappa/2 + 1 > \varkappa/2 - 1 = n_2$, which contradicts our assumption on the data. Hence $\alpha \neq 0$. Now we can use Theorem 5.4, which shows the desired equivalence. $\square$

For the case where assumption (b) of the previous corollary applies, we are led to the monodromy problem for second order Fuchsian differential equations with four singular points. Note that the monodromy depends on the location of the singularities. Hence so will the corresponding indices. Unfortunately, no explicit answer is known to this monodromy problem. Only one particular case can be answered immediately by means of the remark made in the last paragraph of Section 5. Namely, if one of the matrices $M_1, \dots, M_4$ is a scalar matrix (i.e., a multiple of I), then the indices are necessarily $\varkappa_1 = \varkappa_2 = \varkappa/2$.

7 The 3×3 matrix case with 3 singularities

In this section we consider the 3×3 matrix case with 3 singularities. In principle, the results are similar to the 2×2 matrix case with 4 singularities, although it requires some more effort to describe them. In some cases of admissible data, the indices can be determined explicitly, in other cases they depend on the description of the monodromy of third order Fuchsian linear differential equations with three singularities.

7.1 Classification of reducibility type

We start again with some preparations related to the "reducibility type" of the data. Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. Then the following (mutually exclusive) cases can occur:

(A) $[M_1, M_2, M_3]$ is irreducible;

(B-1) $[M_1, M_2, M_3]$ possesses exactly one non-trivial invariant subspace V_1 such that $\dim V_1 = 1$;

(B-2) $[M_1, M_2, M_3]$ possesses exactly one non-trivial invariant subspace V_2 such that $\dim V_2 = 2$;

(B-3) $[M_1, M_2, M_3]$ possesses exactly two non-trivial invariant subspaces V_1 and V_2 such that $\dim V_1 = 1$, $\dim V_2 = 2$ and $V_1 \cap V_2 = \{0\}$;

(C) $[M_1, M_2, M_3]$ possesses exactly two non-trivial invariant subspaces V_1 and V_2 such that $\dim V_1 = 1$, $\dim V_2 = 2$ and $V_1 \subset V_2$;

(C-1) $[M_1, M_2, M_3]$ possesses (at least) two one-dimensional invariant subspaces $V_1^{(a)}$ and $V_1^{(b)}$ and exactly one two-dimensional invariant subspace $V_2 = V_1^{(a)} \oplus V_1^{(b)}$;

(C-2) $[M_1, M_2, M_3]$ possesses (at least) two two-dimensional invariant subspaces $V_2^{(a)}$ and $V_2^{(b)}$ and exactly one one-dimensional invariant subspace $V_1 = V_2^{(a)} \cap V_2^{(b)}$;

(C-3) $[M_1, M_2, M_3]$ possesses (at least) two one-dimensional invariant subspaces and two two-dimensional invariant subspaces, where all one-dimensional invariant subspaces are contained in a two-dimensional subspace and the intersection of all two-dimensional invariant subspaces is a one-dimensional subspace;

(D) $[M_1, M_2, M_3]$ possesses (at least) three one-dimensional invariant subspaces $V_1^{(a)}$, $V_1^{(b)}$ and $V_1^{(c)}$ such that $V_1^{(a)} \oplus V_1^{(b)} \oplus V_1^{(c)} = \mathbb{C}^3$.

The readers may convince themselves that this classification is complete.

In all these cases we define

$$\varkappa \;=\; \operatorname{trace} E_1 + \operatorname{trace} E_2 + \operatorname{trace} E_3. \tag{7.1}$$

In case (B-1), the triple $[M_1, M_2, M_3]$ is equivalent to

$$\left[\begin{pmatrix} \mu_1 & \alpha_1 \\ 0 & M_1^{\#} \end{pmatrix}, \begin{pmatrix} \mu_2 & \alpha_2 \\ 0 & M_2^{\#} \end{pmatrix}, \begin{pmatrix} \mu_3 & \alpha_3 \\ 0 & M_3^{\#} \end{pmatrix}\right] \tag{7.2}$$

with numbers μ_k, 1×2 vectors α_k and 2×2 matrices $M_k^{\#}$, which satisfy $\mu_1\mu_2\mu_3 = 1$, $M_1^{\#}M_2^{\#}M_3^{\#} = I$ and

$$\alpha_1 M_2^{\#} M_3^{\#} + \mu_1 \alpha_2 M_3^{\#} + \mu_1\mu_2\alpha_3 \;=\; 0. \tag{7.3}$$

The triple $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ is irreducible. For each $k = 1, 2, 3$, the value μ_k is uniquely determined and so is the eigenvalue ε_k of E_k that satisfies $\mu_k = \exp(-2\pi i \varepsilon_k)$. Hence the following integers are uniquely defined,

$$\nu \;=\; \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \qquad N \;=\; \varkappa - \nu. \tag{7.4}$$

In case (B-2), the triple $[M_1, M_2, M_3]$ is equivalent to

$$\left[\begin{pmatrix} M_1^{\#} & \alpha_1 \\ 0 & \mu_1 \end{pmatrix}, \begin{pmatrix} M_2^{\#} & \alpha_2 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} M_3^{\#} & \alpha_3 \\ 0 & \mu_3 \end{pmatrix}\right] \tag{7.5}$$

with numbers μ_k, 2×1 vectors α_k and 2×2 matrices $M_k^{\#}$ having the same properties as above, but with (7.3) replaced by

$$\alpha_1 \mu_2\mu_3 + M_1^{\#}\alpha_2\mu_3 + M_1^{\#}M_2^{\#}\alpha_3 \;=\; 0. \tag{7.6}$$

We also define the integers ν and N by (7.4).

In case (B-3), the triple $[M_1, M_2, M_3]$ is equivalent both to (7.2) and (7.5) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Again, the integers ν and N are uniquely defined by (7.4).

In case (C), the triple $[M_1, M_2, M_3]$ is equivalent to

$$\left[\begin{pmatrix} \mu_1^{(1)} & \alpha_1 & \gamma_1 \\ 0 & \mu_1^{(2)} & \beta_1 \\ 0 & 0 & \mu_1^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_2^{(1)} & \alpha_2 & \gamma_2 \\ 0 & \mu_2^{(2)} & \beta_2 \\ 0 & 0 & \mu_2^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_3^{(1)} & \alpha_3 & \gamma_3 \\ 0 & \mu_3^{(2)} & \beta_3 \\ 0 & 0 & \mu_3^{(3)} \end{pmatrix}, \right] \tag{7.7}$$

where the numbers $\mu_k^{(j)}$ are uniquely determined and satisfy $\mu_1^{(j)}\mu_2^{(j)}\mu_3^{(j)} = 1$ for each $j = 1, 2, 3$ and the numbers $\alpha_k, \beta_k, \gamma_k$ satisfy the following conditions:

$$\alpha_1\mu_2^{(2)}\mu_3^{(2)} + \mu_1^{(1)}\alpha_2\mu_3^{(2)} + \mu_1^{(1)}\mu_2^{(1)}\alpha_3 = 0, \tag{7.8}$$

$$\beta_1\mu_2^{(3)}\mu_3^{(3)} + \mu_1^{(2)}\beta_2\mu_3^{(3)} + \mu_1^{(2)}\mu_2^{(2)}\beta_3 = 0, \tag{7.9}$$

$$\mu_1^{(1)}\mu_2^{(1)}\gamma_3 + \mu_1^{(1)}\gamma_2\mu_3^{(3)} + \gamma_1\mu_2^{(3)}\mu_3^{(3)} = -\mu_1^{(1)}\alpha_2\beta_3 - \alpha_1\mu_2^{(2)}\beta_3 - \alpha_1\beta_2\mu_3^{(3)}. \tag{7.10}$$

Note that $[M_1, M_2, M_3]$ is not equivalent to a triple of the form (7.7) where all α_k are zero or all β_k are zero. We introduce the integers n_1, n_2, n_3 by

$$n_j = \varepsilon_1^{(j)} + \varepsilon_2^{(j)} + \varepsilon_3^{(j)}, \qquad j = 1, 2, 3, \tag{7.11}$$

where $\varepsilon_k^{(j)}$ are the eigenvalues of E_k numbered in such a way that $\mu_k^{(j)} = \exp(-2\pi i\varepsilon_k^{(j)})$. The integers n_1, n_2, n_3 are uniquely defined in this case.

In case (C-1), the triple $[M_1, M_2, M_3]$ is equivalent to (7.7) with $\alpha_k = 0$ (but not all β_k or all γ_k can be made zero). The values $\mu_3^{(j)}$ are unique, but the values $\mu_1^{(j)}$ and $\mu_2^{(j)}$ can be interchanged. Hence the integer n_3 is uniquely determined, and we will assume without loss of generality that $n_1 \geq n_2$.

In case (C-2), the triple $[M_1, M_2, M_3]$ is equivalent to (7.7) with $\beta_k = 0$ (but not all α_k or all γ_k can be made zero). The values $\mu_1^{(j)}$ are unique, but the values $\mu_2^{(j)}$ and $\mu_3^{(j)}$ can be interchanged. Hence the integer n_1 is uniquely determined, and we will assume without loss of generality that $n_2 \geq n_3$.

In case (C-3), the triple $[M_1, M_2, M_3]$ is equivalent to (7.7) with $\beta_k = \gamma_k = 0$ (but not all α_k can be made zero). Here the values of $\mu_k^{(j)}$ are again unique for all $j, k = 1, 2, 3$. We define the integers n_1, n_2, n_3 by (7.11), but to remark the different roles that these numbers play, we use the notation

$$\nu_1 = n_1, \quad \nu_2 = n_2, \quad \nu_\# = n_3. \tag{7.12}$$

Notice that (by permuting rows and columns) $[M_1, M_2, M_3]$ is also equivalent to a triple of the form (7.7) with $\alpha_k = \gamma_k = 0$ or with $\alpha_k = \beta_k = 0$. This observation will play some role later on in treating this case. The numbers $\nu_1, \nu_2, \nu_\#$ would then have to be defined appropriately.

Finally, in case (D), the triple $[M_1, M_2, M_3]$ is diagonalizable, i.e., equivalent to (7.7) with $\alpha_k = \beta_k = \gamma_k = 0$. The values of $\mu_k^{(j)}$ may be permuted and we will therefore assume without loss of generality that the numbers n_1, n_2, n_3 defined as above are such that $n_1 \geq n_2 \geq n_3$.

7.2 Systems of triangular form

We will first consider monodromy data which is triangularizable, i.e., which corresponds to the cases (C)–(D). The generic case is (C), in which the following proposition provides some information.

Proposition 7.1 *Let $[M_1, M_2, M_3]$ be a triple of the form (7.7) with $M_1M_2M_3 = I$, and assume that condition (C) is fulfilled. Let $[\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}]$ be another triple of the same form and with the same entries except that $\gamma_1, \gamma_2, \gamma_3$ are replaced by $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$. Then these triples are equivalent if and only if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that*

$$
\begin{array}{llll}
1.) & \widetilde{\gamma}_k = \gamma_k + \gamma(\mu_k^{(1)} - \mu_k^{(3)}) & \textit{in case} & (\mu_k^{(1)})_{k=1}^3 \neq (\mu_k^{(2)})_{k=1}^3 \neq (\mu_k^{(3)})_{k=1}^3, \\
2.) & \widetilde{\gamma}_k = \gamma_k + \gamma(\mu_k^{(2)} - \mu_k^{(3)}) + \alpha\beta_k & \textit{in case} & (\mu_k^{(1)})_{k=1}^3 = (\mu_k^{(2)})_{k=1}^3 \neq (\mu_k^{(3)})_{k=1}^3, \\
3.) & \widetilde{\gamma}_k = \gamma_k + \gamma(\mu_k^{(1)} - \mu_k^{(2)}) + \beta\alpha_k & \textit{in case} & (\mu_k^{(1)})_{k=1}^3 \neq (\mu_k^{(2)})_{k=1}^3 = (\mu_k^{(3)})_{k=1}^3, \\
4.) & \widetilde{\gamma}_k = \gamma_k + \alpha\beta_k + \beta\alpha_k & \textit{in case} & (\mu_k^{(1)})_{k=1}^3 = (\mu_k^{(2)})_{k=1}^3 = (\mu_k^{(3)})_{k=1}^3.
\end{array}
$$

Proof. These triples are equivalent if and only if there exists an invertible matrix S such that $\widetilde{M_k} = S^{-1}M_kS$. It follows that the matrix S maps the invariant subspaces of $[\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}]$ into the invariant subspaces of $[M_1, M_2, M_3]$. Because of the condition (C), $V_1 = \mathbb{C} \oplus \{0\} \oplus \{0\}$ and $V_2 = \mathbb{C} \oplus \mathbb{C} \oplus \{0\}$ are the only non-trivial invariant subspaces of both triples. Hence V_1 and V_2 are also invariant subspaces of S, and this implies that S is of triangular form,

$$S = \begin{pmatrix} s_1 & \alpha & \gamma \\ 0 & s_2 & \beta \\ 0 & 0 & s_3 \end{pmatrix}. \tag{7.13}$$

Now the equality $\widetilde{M_k} = S^{-1}M_kS$ gives the following two conditions,

$$\alpha_k = \frac{s_2}{s_1}\alpha_k + \frac{\alpha}{s_1}(\mu_k^{(1)} - \mu_k^{(2)}), \tag{7.14}$$

$$\beta_k = \frac{s_3}{s_2}\beta_k + \frac{\beta}{s_2}(\mu_k^{(2)} - \mu_k^{(3)}), \tag{7.15}$$

and a third condition, which will be stated later. We first claim that $s_1 = s_2$. Indeed, (7.14) can be rewritten as $0 = (s_2 - s_1)\alpha_k + \alpha(\mu_k^{(1)} - \mu_k^{(2)})$. If $s_1 \neq s_2$, then Proposition 6.1 applied to the triple of 2×2 matrices obtained from $[M_1, M_2, M_3]$ by removing the third rows and columns implies that $[M_1, M_2, M_3]$ is equivalent to a triple of the form (7.7) with $\alpha_k = 0$. This contradicts condition (C). Similarly, we obtain $s_2 = s_3$. A simple thought shows that we may assume $s_1 = s_2 = s_3 = 1$. Hence (7.14), (7.15) simplify to

$$\alpha(\mu_k^{(1)} - \mu_k^{(2)}) = \beta(\mu_k^{(2)} - \mu_k^{(3)}) = 0, \tag{7.16}$$

and the third condition can be expressed as

$$\widetilde{\gamma}_k = \gamma_k + \gamma(\mu_k^{(1)} - \mu_k^{(3)}) - \alpha\beta_k + \beta\alpha_k. \tag{7.17}$$

The conditions (7.16) lead to four distinct cases. If $\mu_k^{(1)} \neq \mu_k^{(2)}$ for some k, then $\alpha = 0$, and if $\mu_k^{(2)} \neq \mu_k^{(3)}$ for some k, then $\beta = 0$. Otherwise, α and β, resp., can be arbitrary. Combining these statements with formula (7.17) proves the assertion. $\square$

We are now able to describe the sets of equivalence classes of triples $[M_1, M_2, M_3]$, which satisfy condition (C) and for which all values appearing in (7.7) are kept fixed except $\gamma_1, \gamma_2, \gamma_3$. We first note that $\gamma_1, \gamma_2, \gamma_3$ have to satisfy the linear condition (7.10).

In case 1, the set of equivalence classes can be identified with $\mathbb{C}$ if $\mu_k^{(1)} \neq \mu_k^{(3)}$ for some k and with $\mathbb{C}^2$ if $\mu_k^{(1)} = \mu_k^{(3)}$ for all k, respectively.

In cases 2 and 3, the sets of equivalence classes are singletons. In fact, as to case 2, the vectors (β_k) and $(\mu_k^{(2)} - \mu_k^{(3)})$ are linearly independent. If they were not, i.e., if (β_k) is a multiple of $(\mu_k^{(2)} - \mu_k^{(3)})$, then Proposition 6.1 applied to the triple of 2×2 matrices obtained from $[M_1, M_2, M_3]$ by removing the first rows and columns implies that $[M_1, M_2, M_3]$ is equivalent to a triple (7.7) with $\beta_k = 0$. This conflicts with condition (C).

In case 4, the set of equivalence classes is a singleton if the vectors (α_k) and (β_k) are linearly independent. Otherwise, this set can be identified with $\mathbb{C}$.

In the next theorem, we consider all system of standard form $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ with indices $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$, for which $A(z)$ can be written as

$$A(z) \;=\; \frac{1}{p(z)} \begin{pmatrix} \varkappa_1 z^2 + p_1(z) & p_{12}(z) & p_{13}(z) \\ 0 & \varkappa_2 z^2 + p_2(z) & p_{23}(z) \\ 0 & 0 & \varkappa_3 z^2 + p_3(z) \end{pmatrix}, \tag{7.18}$$

where $p(z) = (z - a_1)(z - a_2)(z - a_3)$, $p_j \in \mathcal{P}_1$, $p_{jk} \in \mathcal{P}_{1+\varkappa_j - \varkappa_k}$. We will completely characterize the data associated to such systems.

Theorem 7.2 *Let $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{Z}$, $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$, and $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. Then the following two statements are equivalent:*

- *(i) $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is the data associated to some system of the form (7.18).*
- *(ii) $[M_1, M_2, M_3]$ is equivalent to some triple (7.7) with $n_j = \varkappa_j$ for $j = 1, 2, 3$, where the integers n_j are defined by (7.11).*

Proof. (i)$\Rightarrow$(ii): Because $A(z)$ is of triangular form, there exists a solution which is also of triangular form (see Lemma 5.1). Hence we can write $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ as

$$\begin{pmatrix} y_1' & y_{12}' & y_{13}' \\ 0 & y_2' & y_{23}' \\ 0 & 0 & Y_3' \end{pmatrix} \;=\; \begin{pmatrix} a_1 & a_{12} & a_{13} \\ 0 & a_2 & a_{23} \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} y_1 & y_{12} & y_{13} \\ 0 & y_2 & y_{23} \\ 0 & 0 & y_3 \end{pmatrix}. \tag{7.19}$$

For $j = 1, 2, 3$, we have

$$a_j(z) \quad = \quad \frac{\varkappa_j z^2 + p_j(z)}{p(z)} \quad = \quad \sum_{k=1}^{3} \frac{\varepsilon_k^{(j)}}{z - a_k} \tag{7.20}$$

where the numbers $\varepsilon_1^{(j)}, \varepsilon_2^{(j)}, \varepsilon_3^{(j)}$ are defined by the last equality and satisfy $\varkappa_j = \varepsilon_1^{(j)} + \varepsilon_2^{(j)} + \varepsilon_3^{(j)}$. Because E_k is similar to the residue of $A(z)$ at $z = a_k$, the numbers $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}, \varepsilon_k^{(3)}$ are the eigenvalues of E_k. Once it is shown that they are properly ordered, it follows that $\varkappa_j = n_j$ for $j = 1, 2, 3$.

The diagonal entries of the solution $Y(\tilde{z})$ satisfy $y_j'(\tilde{z}) = a_j(z) y_j(\tilde{z})$. Therefore they are given by $y_j(\tilde{z}) = \prod_{k=1}^{3} (\tilde{z} - a_k)^{\varepsilon_k^{(j)}}$ up to a redundant constant. Regardless of the precise expressions for the off-diagonal entries of $Y(\tilde{z})$ it follows that the corresponding monodromy representation is given by (7.7) with $\mu_k^{(j)} = \exp(-2\pi i \varepsilon_k^{(j)})$ and with certain $\alpha_k, \beta_k, \gamma_k$. Hence $[M_1, M_2, M_3]$ is equivalent to (7.7) and the above relation between $\mu_k^{(j)}$ and $\varepsilon_k^{(j)}$ implies that the eigenvalues of E_k are properly numbered.

(ii)$\Rightarrow$(i): Suppose we are given $[M_1, M_2, M_3]$, which is equivalent to a triple of the form (7.7), and assume that the numbers $\varepsilon_k^{(j)}$ and $n_j = \varkappa_j$ are defined appropriately. We introduce $a_j(z)$ by (7.20) and consider systems (7.19) with those a_j and yet unspecified a_{12}, a_{13}, a_{23}. We have to show that there exist a_{12}, a_{13}, a_{23} (being the ratio p_{jk}/p of polynomials with properties as described above) such that the monodromy of this system is given by $[M_1, M_2, M_3]$.

We first consider some trivial cases. Assume that $[M_1, M_2, M_3]$ is equivalent to (7.7) with $\alpha_k = 0$. The proof of Theorem 6.2 tells us that there exists an a_{13} such that the system

$$\begin{pmatrix} y_1' & y_{13}' \\ 0 & y_3' \end{pmatrix} \quad = \quad \begin{pmatrix} a_1 & a_{13} \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} y_1 & y_{13} \\ 0 & y_3 \end{pmatrix} \tag{7.21}$$

has monodromy given by

$$\left[\begin{pmatrix} \mu_1^{(1)} & \gamma_1 \\ 0 & \mu_1^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_2^{(1)} & \gamma_2 \\ 0 & \mu_2^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_3^{(1)} & \gamma_3 \\ 0 & \mu_3^{(3)} \end{pmatrix} \right]. \tag{7.22}$$

Moreover, there exists an a_{23} such that the system

$$\begin{pmatrix} y_2' & y_{23}' \\ 0 & y_3' \end{pmatrix} \quad = \quad \begin{pmatrix} a_2 & a_{23} \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} y_2 & y_{23} \\ 0 & y_3 \end{pmatrix} \tag{7.23}$$

has monodromy given by

$$\left[\begin{pmatrix} \mu_1^{(2)} & \beta_1 \\ 0 & \mu_1^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_2^{(2)} & \beta_2 \\ 0 & \mu_2^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_3^{(2)} & \beta_3 \\ 0 & \mu_3^{(3)} \end{pmatrix} \right]. \tag{7.24}$$

In this connection we remark that (7.22) and (7.24) are indeed "admissible" triples in the sense that the product of the occurring matrices is equal to I. Moreover, $a_{13} = p_{13}/p$ and $a_{23} = p_{23}/p$ where the polynomials p_{13} and p_{23} have the proper degree. Now we consider the system (7.19) with those a_{13} and a_{23} and with $a_{12} = 0$. This system has a solution of triangular form as given in (7.19) with $y_{12} = 0$. The monodromy representation with respect to this solution is of the form (7.7) with desired values $\mu_k^{(j)}$ and $\alpha_k = 0$. However, we may have different values $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$ (instead of the desired values β_k and γ_k). Analyzing the relation between the monodromy of (7.19) and the monodromy of the "subsystems" (7.21) and (7.23), it follows that (7.22) is equivalent to (7.22) with γ_k replaced by $\widetilde{\gamma}_k$ and that (7.24) is equivalent to (7.24) with β_k replaced by $\widetilde{\beta}_k$. We apply Proposition 6.1 and obtain that

$$\widetilde{\gamma}_k = \lambda^{(1)}\gamma_k + \rho^{(1)}(\mu_k^{(1)} - \mu_k^{(3)}), \qquad \widetilde{\beta}_k = \lambda^{(2)}\beta_k + \rho^{(2)}(\mu_k^{(2)} - \mu_k^{(3)}) \tag{7.25}$$

with certain $\lambda^{(1)}, \lambda^{(2)} \in \mathbb{C} \setminus \{0\}$ and $\rho^{(1)}, \rho^{(2)} \in \mathbb{C}$. A simple thought shows that a representation (7.7) with $\alpha_k = 0$ and having above β_k and γ_k as entries is equivalent to the same representation but with above $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$. In fact, the equivalence is established by a certain upper triangular matrix having $(1,2)$-entry equal to zero. Hence, the monodromy of the constructed system (7.19) is given by $[M_1, M_2, M_3]$, and this concludes this case.

The other trivial case, where $[M_1, M_2, M_3]$ is equivalent to a triple of the form (7.7) with $\beta_k = 0$ can be treated in the same way. Here we consider the system (7.19) with $a_{23} = 0$ and $Y_{23} = 0$ and the appropriate "subsystems".

Now assume that $[M_1, M_2, M_3]$ does not fall into these trivial classes. In other words, $[M_1, M_2, M_3]$ satisfies the condition (C) concerning the reducibility type. Arguing similar as above, we find an appropriate function a_{12} such that the system

$$\begin{pmatrix} y_1' & y_{12}' \\ 0 & y_2' \end{pmatrix} = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} y_1 & y_{12} \\ 0 & y_2 \end{pmatrix} \tag{7.26}$$

has monodromy given by

$$\left[\begin{pmatrix} \mu_1^{(1)} & \alpha_1 \\ 0 & \mu_1^{(2)} \end{pmatrix}, \begin{pmatrix} \mu_2^{(1)} & \alpha_2 \\ 0 & \mu_2^{(2)} \end{pmatrix}, \begin{pmatrix} \mu_3^{(1)} & \alpha_3 \\ 0 & \mu_3^{(2)} \end{pmatrix} \right]. \tag{7.27}$$

and we also find a function a_{23} such that the system (7.23) has the monodromy (7.24). With those functions a_{12} and a_{23} and an arbitrary function $a_{13}^{\#}$ we consider the system (7.19). This system has a solution with certain functions y_1, y_2, y_3 and $y_{12}, y_{23}, y_{13}^{\#}$. The monodromy representation of this solution is given by (7.7) with the desired values $\mu_k^{(j)}$, but possibly different values $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k^{\#}$. Analyzing the relation to the "subsystems", for which we know the monodromy, it follows again from Proposition 6.1 that

$$\widetilde{\alpha}_k = \lambda^{(1)}\alpha_k + \rho^{(1)}(\mu_k^{(1)} - \mu_k^{(2)}), \qquad \widetilde{\beta}_k = \lambda^{(2)}\beta_k + \rho^{(2)}(\mu_k^{(2)} - \mu_k^{(3)}) \tag{7.28}$$

with certain $\lambda^{(1)}, \lambda^{(2)} \in \mathbb{C}\setminus\{0\}$ and $\rho^{(1)}, \rho^{(2)} \in \mathbb{C}$. By means of the construction of a suitable triangular matrix (as above), we obtain that $[M_1, M_2, M_3]$ is equivalent to a triple (7.7) having off-diagonal entries $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and certain $\widetilde{\gamma}_k$.

In other word, the monodromy of the system we have constructed so far is "almost" as desired, only the values $\widetilde{\gamma}_k^{\#}$ might not yet be the desired ones $\widetilde{\gamma}_k$. In fact, in cases 2 and 3 and also in the "generic" subcase of case 4, which has been singled out in Proposition 7.1, we are already done because (as has been remarked in the paragraphs following this proposition) there exists only a single equivalence class of such triples. Hence the monodromy of the constructed system is equivalent to $[M_1, M_2, M_3]$.

In the remaining cases we must still modify the system, and we proceed as follows. Again the proof of Theorem 6.2 tells us that there exists a suitable function $a_{13}^{\$}$ such that the system

$$\begin{pmatrix} y_1' & {y_{13}^{\$}}' \\ 0 & y_3' \end{pmatrix} = \begin{pmatrix} a_1 & a_{13}^{\$} \\ 0 & a_3 \end{pmatrix}\begin{pmatrix} y_1 & y_{13}^{\$} \\ 0 & y_3 \end{pmatrix} \tag{7.29}$$

has monodromy given by

$$\left[\begin{pmatrix} \mu_1^{(1)} & \widetilde{\gamma}_1 - \widetilde{\gamma}_1^{\#} \\ 0 & \mu_1^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_2^{(1)} & \widetilde{\gamma}_2 - \widetilde{\gamma}_2^{\#} \\ 0 & \mu_2^{(3)} \end{pmatrix}, \begin{pmatrix} \mu_3^{(1)} & \widetilde{\gamma}_3 - \widetilde{\gamma}_3^{\#} \\ 0 & \mu_3^{(3)} \end{pmatrix}\right]. \tag{7.30}$$

In this connection we remark that both $\widetilde{\gamma}_k$ and $\widetilde{\gamma}_k^{\#}$ satisfy a linear relation (7.10) with the same right hand side. The differences satisfy the homogeneous linear relation, which corresponds to (6.3). Hence the (7.30) is an admissible triple, i.e., the product of the matrices is equal to one.

The monodromy representation of the solution in (7.29) is also given by (7.30) except that the upper right entries are replaced by certain values $\widetilde{\gamma}_k^{\$}$. The relation between these and the former values is (see Proposition 6.1)

$$\widetilde{\gamma}_k^{\$} = \lambda(\widetilde{\gamma}_k - \widetilde{\gamma}_k^{\#}) + \rho(\mu_k^{(1)} - \mu_k^{(3)}) \tag{7.31}$$

with certain $\lambda \in \mathbb{C}\setminus\{0\}$ and $\rho \in \mathbb{C}$.

Now we consider a new system (7.19) where $A(z)$ has the same entries as the old one except that $a_{13}^{\#}$ is replaced by a_{13} where we define $a_{13} := a_{13}^{\#} + \lambda^{-1}a_{13}^{\$}$. It is easy to see that the solution of this new system has the same entries as the old solution except that $y_{13}^{\#}$ is replaced by a certain y_{13}. Moreover, the monodromy representation with respect to this solution is of the form (7.7) with off-diagonal entries $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ (as in the old one) and certain $\widetilde{\gamma}_k^{\&}$.

What remains to show is that this monodromy representation is equivalent to $[M_1, M_2, M_3]$. Because we already pointed out that $[M_1, M_2, M_3]$ is equivalent to (7.7) with off-diagonal entries $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$, we are done as soon as we have shown that $\widetilde{\gamma}_k$ and $\widetilde{\gamma}_k^{\&}$ are related in the way described in Proposition 7.1.

In order to show this, we have to consider the defining relations for the $(1,3)$-entries of the solutions and of the monodromy representations explicitly. As to the old system, we have

$$y_{13}^{\#\prime} = a_1 y_{13}^{\#} + a_{12} y_{23} + a_{13}^{\#} y_3, \tag{7.32}$$

$$y_{13}^{\#}(\sigma_k^{-1}(\tilde{z})) = y_1(\tilde{z})\widetilde{\gamma}_k^{\#} + y_{12}(\tilde{z})\widetilde{\beta}_k + y_{13}^{\#}(\tilde{z})\mu_k^{(3)}, \tag{7.33}$$

where σ_k, $k = 1,2,3$, are the deck transformations. The corresponding relations for the new system are

$$y_{13}' = a_1 y_{13} + a_{12} y_{23} + a_{13} y_3, \tag{7.34}$$

$$y_{13}(\sigma_k^{-1}(\tilde{z})) = y_1(\tilde{z})\widetilde{\gamma}_k^{\&} + y_{12}(\tilde{z})\widetilde{\beta}_k + y_{13}(\tilde{z})\mu_k^{(3)}. \tag{7.35}$$

We also have to state the conditions on the $(1,2)$-entries in (7.29) and (7.30),

$$y_{13}^{\$\,\prime} = a_1 y_{13}^{\$} + a_{13}^{\$} y_3, \tag{7.36}$$

$$y_{13}^{\$}(\sigma_k^{-1}(\tilde{z})) = y_1(\tilde{z})\widetilde{\gamma}_k^{\$} + y_{13}^{\$}(\tilde{z})\mu_k^{(3)}. \tag{7.37}$$

Combining (7.32) and (7.36) and using $a_{13} = a_{13}^{\#} + \lambda^{-1} a_{13}^{\$}$, we obtain that

$$(y_{13}^{\#} + \lambda^{-1} y_{13}^{\$})' = a_1(y_{13}^{\#} + \lambda^{-1} y_{13}^{\$}) + a_{12} y_{23} + a_{13} y_3. \tag{7.38}$$

Comparing this with (7.34) we may conclude without loss of generality that $y_{13} = y_{13}^{\#} + \lambda^{-1} y_{13}^{\$}$. (The solution of the differential equation is not unique but we may take any of them.) Using this identity and combining (7.33) with (7.37), we arrive at

$$y_{13}(\sigma_k^{-1}(\tilde{z})) = y_1(\tilde{z})(\widetilde{\gamma}_k^{\#} + \lambda^{-1}\widetilde{\gamma}_k^{\$}) + y_{12}(\tilde{z})\widetilde{\beta}_k + y_{13}(\tilde{z})\mu_k^{(3)}. \tag{7.39}$$

Comparing this with (7.35) we conclude that

$$\begin{aligned} \widetilde{\gamma}_k^{\&} &= \widetilde{\gamma}_k^{\#} + \lambda^{-1}\widetilde{\gamma}_k^{\$} \\ &= \widetilde{\gamma}_k + \rho\lambda^{-1}(\mu_k^{(1)} - \mu_k^{(3)}), \end{aligned} \tag{7.40}$$

where the last equality follows from (7.31). Now we conclude from Proposition 7.1 that the triple (7.7) with $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k^{\&}$, which is the monodromy representation for the new system, is equivalent to the triple (7.7) with $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$, which in turn is equivalent to $[M_1, M_2, M_3]$. Hence the monodromy of the new system is given by $[M_1, M_2, M_3]$. □

7.3 Systems of block-triangular form

In what follows, we are going to consider 3×3 systems of certain block-triangular forms, which are not of triangular form. There exists two types of them relating to two kinds of block-triangular partitions of the 3×3 matrix function $A(z)$.

In order to describe these systems and the data associated to them, we have first to introduce some definitions and to state some results related to 2×2 systems and the corresponding data.

Let $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ be admissible data of 2×2 matrices. We call this data *quasi-block data* if either

(I) $[M_1^\#, M_2^\#, M_3^\#]$ is irreducible, or,

(II) $[M_1^\#, M_2^\#, M_3^\#]$ possesses exactly one non-trivial invariant subspace and we have $n_1^\# < n_2^\#$, where $n_1^\#$ and $n_2^\#$ are defined by (6.4).

The next result is an immediate consequence of Theorem 6.2. It describes the data of 2×2 systems with three singularities which are not equivalent to a system of triangular form. We remark that such systems have necessarily indices $\varkappa_1^\#, \varkappa_2^\# \in \mathbb{Z}$ for which $\varkappa_1^\# - \varkappa_2^\# \in \{0, 1\}$.

Lemma 7.3 *Let $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ be admissible data of 2×2 matrices. Then the following two statements are equivalent:*

(i) $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ is the data of a 2×2 system of standard form with three singularities which is not equivalent to a system of the form (6.7).

(ii) $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ is quasi-block data.

Proof. Quasi-block data is exactly the data which is not covered by (ii) of Theorem 6.2. As was already remarked in the paragraph after Theorem 6.2, the data which is covered by this theorem is exactly the data for which $[M_1^\#, M_2^\#, M_3^\#]$ is reducible with the condition $n_1^\# \geq n_2^\#$ or is diagonalizable without another condition. $\square$

Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. We call this data $(1,2)$-*quasi-block data* if $[M_1, M_2, M_3]$ is equivalent to a triple

$$\left[\begin{pmatrix} \mu_1 & \alpha_1 \\ 0 & M_1^\# \end{pmatrix}, \begin{pmatrix} \mu_2 & \alpha_2 \\ 0 & M_2^\# \end{pmatrix}, \begin{pmatrix} \mu_3 & \alpha_3 \\ 0 & M_3^\# \end{pmatrix} \right] \tag{7.41}$$

and $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ is quasi-block data. Here μ_k are numbers, α_k are 1×2 vectors, and $M_k^\#$ are 2×2 matrices. The matrices $E_k^\#$ and numbers ε_k are supposed to satisfy $M_k^\# \sim \exp(-2\pi i E_k^\#)$ and $\mu_k = \exp(-2\pi i \varepsilon_k)$ and are chosen in such a way that the eigenvalues of $E_k^\#$ and the number ε_k are exactly the eigenvalues of E_k. We note that for given $M_k^\#$ and μ_k, the number ε_k is uniquely determined and so is the matrix $E_k^\#$ up to similarity (because of the non-resonance of E_k). We also define the integers

$$\nu \;=\; \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \qquad N \;=\; \operatorname{trace}(E_1^\# + E_2^\# + E_3^\#). \tag{7.42}$$

We remark that if $[M_1, M_2, M_3]$ possesses two different one-dimensional invariant subspaces, then $[M_1, M_2, M_3]$ is equivalent to different triples of the form (7.41) (with different μ_k and $M_k^\#$ in particular). In this case we obtain in general also different values for $\varepsilon_k, E_k^\#, \nu$ and N.

Proposition 7.4 *Let $[M_1, M_2, M_3]$ be a triple of the form (7.41). Assume that*

(a) $[M_1^\#, M_2^\#, M_3^\#]$ is not diagonalizable;

(b) If $[M_1^\#, M_2^\#, M_3^\#]$ possess exactly one non-trivial invariant subspace, then

(b1) for at least one k, $M_k^\#$ possesses two different eigenvalues;

(b2) for at least one k, μ_k is not equal to the eigenvalue of $M_k^\#$ which corresponds to this invariant subspace.

Let $[\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3]$ be another triple of the same form (7.41) but with $\alpha_1, \alpha_2, \alpha_3$ replaced by $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3$. Then these triples are equivalent if and only if there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}^{1\times 2}$ such that

$$\alpha_k \quad = \quad \lambda\widetilde{\alpha}_k + cM_k^\# - \mu_k c \qquad\qquad \textit{for each } k = 1, 2, 3. \tag{7.43}$$

Proof. The triples $[M_1, M_2, M_3]$ and $[\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3]$ are equivalent if and only if there exists a matrix $C \in G\mathbb{C}^{3\times 3}$ such that $\widetilde{M}_k = CM_kC^{-1}$. We write C in the same block form as the matrices in the triple (7.41),

$$C \quad = \quad \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \tag{7.44}$$

and inspect the $(2, 1)$-block entry in the resulting equality $\widetilde{M}_kC = CM_k$. We obtain $M_k^\# c_{21} = c_{21}\mu_k$. Because of (a) and (b2), it follows that $c_{21} = 0$. Otherwise, Span $\{c_{21}\}$ is an invariant subspace for all $M_k^\#$ with the eigenvalue μ_k. Hence C is also of block triangular form.

Now we look at the $(2, 2)$-block entry of $\widetilde{M}_kC = CM_k$, and it follows that $M_k^\# c_{22} = c_{22}M_k^\#$. We claim that c_{22} is a scalar matrix. Assume the contrary. Then c_{22} is either similar to a diagonal matrix with two different diagonal entries or it is similar to a Jordan block. We may assume without loss of generality that c_{22} is actually equal to such a diagonal matrix or to a Jordan block. (Otherwise one applies an appropriate similarity transformation simultaneously to c_{22} and $M_1^\#, M_2^\#, M_3^\#$, which will equally result in a contradiction.) By simple computations it follows in the first case that all $M_1^\#, M_2^\#, M_3^\#$ are of diagonal form. In the second case, it follows that all $M_1^\#, M_2^\#, M_3^\#$ are either Jordan blocks or scalar matrices. Both contradict the assumption on the triple $[M_1^\#, M_2^\#, M_3^\#]$.

With this information about the entries of C, it is now easy to conclude that $\widetilde{M}_k = CM_kC^{-1}$ is equivalent to (7.43). □

Now we are able to identify the set of equivalence classes for triples of the form (7.41) with μ_k and $M_k^\#$ being fixed and satisfying the assumptions stated in the previous proposition. All such triples are parameterized by 1×2 vectors $\alpha_1, \alpha_2, \alpha_3$.

It is convenient to introduce the 1×6 vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. First we have to observe the linear condition (7.3) which can be restated as

$$\alpha \begin{pmatrix} M_2^{\#} M_3^{\#} \\ \mu_1 M_3^{\#} \\ \mu_1 \mu_2 I_2 \end{pmatrix} = 0, \tag{7.45}$$

where I_2 is the 2×2 identity matrix. Because the 6×2 matrix appearing in this linear condition has rank 2, the vector α is actually subject to two linear conditions. Hence all "admissible" vectors α are taken from a 4-dimensional linear subspace. The equivalence relation (7.43) can shortly be written as

$$\alpha \;=\; \lambda \widetilde{\alpha} \;+\; c \Big(\; M_1^{\#} - \mu_1 I_2, \quad M_2^{\#} - \mu_2 I_2, \quad M_3^{\#} - \mu_3 I_2 \; \Big) \tag{7.46}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}^{1 \times 2}$. Because of the assumptions in Proposition 7.4, the 2×6 matrix appearing there has either rank 2 or rank 1. It has rank 1 if and only if $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ is equivalent to a triple

$$\left[\begin{pmatrix} \hat{\mu}_1 & \hat{\alpha}_1 \\ 0 & \mu_1 \end{pmatrix}, \begin{pmatrix} \hat{\mu}_2 & \hat{\alpha}_1 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} \hat{\mu}_3 & \hat{\alpha}_1 \\ 0 & \mu_3 \end{pmatrix} \right] \tag{7.47}$$

with certain $\hat{\mu}_k$ and $\hat{\alpha}_k$. In other words, the above matrix has rank 1 if and only if $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ possesses exactly one non-trivial invariant subspace and, for each $k = 1, 2, 3$, the number μ_k is equal to the eigenvalue of $M_k^{\#}$ which does not correspond to this invariant subspace. The second part in (7.46) gives a "linear" factorization of vector spaces and thus yields either a two-dimensional or a three-dimensional quotient space. Finally, there is another "multiplicative" equivalence. So we arrive at the statement that the set of equivalence classes of such triples $[M_1, M_2, M_3]$ can be identified either with $\mathbb{P}^1 \cup \{0\}$ or with $\mathbb{P}^2 \cup \{0\}$ (corresponding to the above distinction), where $\mathbb{P}^n$ is the n-dimensional complex projective space and $\{0\}$ denotes the single equivalence class containing the vector $\alpha = 0$. This single equivalence class consists precisely of the triples which are "block-diagonalizable".

Next we consider systems $Y'(\tilde{z}) = A(z) Y(\tilde{z})$ of standard form with indices $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$ for which $A(z)$ can be written as

$$A(z) \;=\; \frac{1}{p(z)} \begin{pmatrix} \varkappa_1 z^2 + p_1(z) & p_{12}(z) & p_{13}(z) \\ 0 & \varkappa_2 z^2 + p_2(z) & p_{23}(z) \\ 0 & p_{32}(z) & \varkappa_3 z^2 + p_3(z) \end{pmatrix}, \tag{7.48}$$

where $p(z) = (z - a_1)(z - a_2)(z - a_3)$, $p_k \in \mathcal{P}_1$, $p_{jk} \in \mathcal{P}_{1 + \varkappa_j - \varkappa_k}$. If such a system is not equivalent to a system of the form (7.18), then it will be called a *system of* $(1,2)$*-block form.* We remark that necessarily $\varkappa_2 - \varkappa_3 \leq 1$ because otherwise $p_{32}(z) = 0$ and the system is of the form (7.18).

The next theorem describes the data which is associated to such systems.

Theorem 7.5 *Let $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{Z}$, $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$, and assume $\varkappa_2 - \varkappa_3 \leq 1$. Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. Then the following two statements are equivalent:*

(i) $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is the data associated to a system of $(1,2)$-block form with indices $\varkappa_1, \varkappa_2, \varkappa_3$.

(ii) $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is $(1,2)$-quasi-block data, and $\nu = \varkappa_1$, $N = \varkappa_2 + \varkappa_3$, where ν and N are defined by (7.42).

Proof. (i)$\Rightarrow$(ii): Because $A(z)$ is of block-triangular form, there exists a solution which is also of block triangular form. Hence let us write $Y'(\tilde{z}) = A(z) = Y(\tilde{z})$ as

$$\begin{pmatrix} y_1'(\tilde{z}) & y_{12}'(\tilde{z}) \\ 0 & Y_2'(\tilde{z}) \end{pmatrix} = \begin{pmatrix} a_1(z) & a_{12}(z) \\ 0 & A_2(z) \end{pmatrix} \begin{pmatrix} y_1(\tilde{z}) & y_{12}(\tilde{z}) \\ 0 & Y_2(\tilde{z}) \end{pmatrix}. \tag{7.49}$$

We first consider the scalar subsystem $y_1'(\tilde{z}) = a_1(z)y_1(\tilde{z})$. We know that $a_1(z) = (\varkappa_1 z^2 + p_1(z))/p(z)$. If $\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote the residues of $a_1(z)$ at $z = a_1, a_2, a_3$, then the monodromy is given by $[\mu_1, \mu_2, \mu_3]$, where $\mu_k = \exp(-2\pi i \varepsilon_k)$. Moreover, it follows that $\varkappa_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

Now we consider the 2×2 subsystem $Y_2'(\tilde{z}) = A_2(z)Y_2(\tilde{z})$, where $A_2(z)$ is given by the lower right block in (7.48). We claim that this system is not equivalent to a system of the form (6.7). Indeed, if it were equivalent, then a simple thought shows that the system $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ is equivalent to a system of the form (7.18), which contradicts the assumption that it is a system of $(1,2)$-block form. Hence we are in a position to apply Lemma 7.3. We obtain that the data of the system $Y_2'(\tilde{z}) = A_2(z)Y_2(\tilde{z})$, $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$, is quasi-block data. Note that $M_k^\# \sim \exp(-2\pi i E_k^\#)$ and that $E_k^\#$ is similar to the residue of $A_2(z)$ at $z = a_k$. Hence, in particular, $\varkappa_2 + \varkappa_3 = \operatorname{trace}(E_1^\# + E_2^\# + E_3^\#)$.

Combining the information about the two subsystems, we obtain immediately that the monodromy data $[M_1, M_2, M_3]$ of the original system is equivalent to a triple (7.41) with certain $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}^{1\times 2}$. The eigenvalues of the matrix E_k, which is similar to the residue of $A(z)$ at $z = a_k$, are given by the number ε_k and the eigenvalues of $E_k^\#$ because they are the residues of $a_1(z)$ and $A_2(z)$, respectively. Hence, since the data of the 2×2 subsystem is quasi-block data, the data of the original system is $(1,2)$-quasi-block data. Moreover, from the values for $\varkappa_1$ and $\varkappa_2 + \varkappa_3$ given above, it follows that $\varkappa_1 = \nu$ and $\varkappa_2 + \varkappa_3 = N$.

(ii)$\Rightarrow$(i): Assume that $[M_1, M_2, M_3]$ is a triple of the form (7.41) such that (on defining ε_k and $E_k^\#$ as well as ν and N by (7.42)) the appropriate conditions for $(1,2)$-quasi-block data are satisfied. In particular, $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ is quasi-block data, and $\nu = \varkappa_1$, $N = \varkappa_2 + \varkappa_3$. We have to show that there exists a system of $(1,2)$-block form which has monodromy $[M_1, M_2, M_3]$, the indices $\varkappa_1, \varkappa_2, \varkappa_3$, and the residues of $A(z)$ at $z = a_k$ are similar to E_k.

We first define

$$a_1(z) \;=\; \sum_{k=1}^{3} \frac{\varepsilon_k}{z-a_k} \;=\; \frac{\varkappa_1 z^2 + p_1(z)}{p(z)}, \tag{7.50}$$

noting that $\varkappa_1 = \nu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Furthermore, because $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ and $[E_1^{\#}, E_2^{\#}, E_3^{\#}]$ is quasi-block data, Lemma 7.3 shows that there exists a 2×2 system having this data with singularities a_1, a_2, a_3 such that this system is not equivalent to a system of the form (6.7). Denote this system by $Y_2'(\tilde{z}) = A_2(z)Y_2(\tilde{z})$, and thus define the matrix function $A_2(z)$. Let $\varkappa_2^{\#}$ and $\varkappa_3^{\#}$ be the indices of this 2×2 system. We claim that $\varkappa_2 = \varkappa_2^{\#}$ and $\varkappa_3 = \varkappa_3^{\#}$. Indeed, because the 2×2 system is not of triangular form, we have $0 \le \varkappa_2^{\#} - \varkappa_3^{\#} \le 1$, and by assumption we also have $0 \le \varkappa_2 - \varkappa_3 \le 1$. Moreover, $\varkappa_2 + \varkappa_3 = N = \operatorname{trace}(E_1^{\#} + E_2^{\#} + E_3^{\#}) = \varkappa_2^{\#} + \varkappa_3^{\#}$, where the last relation follows from formula (4.2) applied in the case of the 2×2 system. These relations for the integers prove the stated equality.

Now we consider systems $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ given by (7.49) with those $a_1(z)$ and $A_2(z)$ and yet unspecified $a_{12}(z)$, the entries of which have, of course, to be the ratio of appropriate polynomials. We note that these systems are of the form (7.48). They are also systems of standard form with indices $\varkappa_1, \varkappa_2, \varkappa_3$, because, in particular, $\varkappa_1 \ge \varkappa_2 \ge \varkappa_3$ is assumed. The monodromy representation of a suitable solution of these systems is given by (7.41) with certain $\alpha_1, \alpha_2, \alpha_3$. The residue $\widehat{E}_k$ of $A(z)$ at $z = a_k$ has the eigenvalues given by ε_k and the eigenvalues of $E_k^{\#}$. Hence, as the eigenvalues of E_k are also given in this way, it follows that E_k is similar to $\widehat{E}_k$. (In case of coinciding eigenvalues, one has also to employ that $M_k \sim \exp(-2\pi i E_k)$ and $M_k \sim \exp(-2\pi i \widehat{E}_k)$.)

What remains to show is, firstly, that the systems constructed in this way are indeed of $(1,2)$-block form, i.e., that they are not equivalent to systems (7.18). Secondly, one has to show that if $a_{12}(z)$ runs through all admissible functions, then corresponding vectors $\alpha_1, \alpha_2, \alpha_3$ appearing in the monodromy representation (7.41) run through all possibilities modulo the equivalence described in Proposition 7.4.

Let us turn to the first problem. Assume the contrary, namely that our system with $A(z)$ is equivalent to a system with $\widetilde{A}(z)$ being of the form (7.18). Theorem 3.5 says that both systems have the same indices and that the equivalence is established by a matrix function $V(z)$, which has a particular structure. We also know that the 2×2 subsystem $A_2(z)$ of $A(z)$ is not equivalent to a triangular system, and we are going to show a contradiction to this assertion. If $\varkappa_2 > \varkappa_3$, then the $(3,1)$- and $(3,2)$-entries of $V(z)$ vanish. Using $A(z) = V^{-1}(z)\widetilde{A}(z)V(z) - V^{-1}(z)V'(z)$, it follows that $A(z)$ has $(3,2)$-entry equal to zero, which is a contradiction. If $\varkappa_1 > \varkappa_2$, then the $V(z)$ is of the same block structure as $A(z)$. Now using $\widetilde{A}(z) = V(z)A(z)V^{-1}(z) + V'(z)V^{-1}(z)$, it is easy to see that the lower right 2×2 submatrix in $V(z)$ establishes an equivalence between the subsystem $A_2(z)$ and a triangular 2×2 system, namely the one which is given by the lower right 2×2 matrix in $\widetilde{A}(z)$. In the remaining case, $\varkappa_1 = \varkappa_2 = \varkappa_3$, the function $V(z)$ is a constant matrix V. As before, let $\widehat{E}_k$

be the residues of $A(z)$ at $z = a_k$, and let $\widetilde{E}_k$ be those of $\widetilde{A}(z)$. The equivalence between $A(z)$ and $\widetilde{A}(z)$ can now be restated as $\widetilde{E}_k = V\widehat{E}_kV^{-1}$. We claim that each one-dimensional invariant subspace of $[\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3]$ is contained in some two-dimensional invariant subspace of this triple. Indeed, this is obvious if this triple has at least two one-dimensional invariant subspaces. If it has exactly one, then it is necessarily the space $\mathbb{C} \oplus \{0\} \oplus \{0\}$, which is contained in $\mathbb{C} \oplus \mathbb{C} \oplus \{0\}$ (because of the triangular form of $\widetilde{E}_k$). The above relation between $\widetilde{E}_k$ and $\widehat{E}_k$ establishes a one-to-one correspondence between invariant subspaces of the previous triple and $[\widehat{E}_1, \widehat{E}_2, \widehat{E}_3]$. Hence each one-dimensional invariant subspace of the latter triple is contained in some two-dimensional invariant subspace. A simple thought shows that the 2×2 matrices obtained from $[\widehat{E}_1, \widehat{E}_2, \widehat{E}_3]$ by removing the first rows and columns are reducible. But those matrices are the residues of the system $A_2(z)$, and thus we obtain a contradiction.

Finally, we turn to the problem of showing that if a_{12} runs through all admissible vector functions, then the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^{1\times 6}$ runs through all possibilities modulo the equivalence stated in Proposition 7.4 (see also (7.45) and (7.46)). Before we have to show that the assumptions of Proposition 7.4 are satisfied.

We know that $[M_1^\#, M_2^\#, M_3^\#]$ and $[E_1^\#, E_2^\#, E_3^\#]$ is quasi-block data. Hence $[M_1^\#, M_2^\#, M_3^\#]$ is not diagonalizable. If this triple possesses one non-trivial invariant subspace and all $M_k^\#$ have only a single eigenvalue, then so have $E_k^\#$. It follows that $n_1^\# = n_2^\#$, which is a contradiction (see (II) in the definition of quasi-block data). As to condition (b2), assume that (b2) is not satisfied. Then it is easy to see (using the nonresonance of E_k) that $\nu = n_1^\#$. Because $N = \operatorname{trace}(E_1^\# + E_2^\# + E_3^\#) = n_1^\# + n_2^\#$ and $n_1^\# < n_2^\#$ by (II) of the definition of quasi-block data, it follows that $N > 2n_1^\#$. On the other hand, $\varkappa_1 = \nu$ and $\varkappa_2 + \varkappa_3 = N$. We obtain that $\varkappa_1 = n_1^\#$ and $\varkappa_2 + \varkappa_3 > 2n_1^\#$. Hence $\varkappa_2 + \varkappa_3 > 2\varkappa_1$, which conflicts with $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$. So we have proved that the assumptions of Proposition 7.4 are satisfied.

Next we examine the question when two system with $A(z)$ and $\widetilde{A}(z)$ given by (7.49) with $a_1(z)$ and $A_2(z)$ being introduced above, but with possibly different functions $a_{12}(z)$ and $\tilde{a}_{12}(z)$, respectively, are equivalent. Assume that

$$a_{12}(z) = \left(\frac{p_{12}(z)}{p(z)}, \frac{p_{13}(z)}{p(z)}\right), \qquad \tilde{a}_{12}(z) = \left(\frac{\tilde{p}_{12}(z)}{p(z)}, \frac{\tilde{p}_{13}(z)}{p(z)}\right), \tag{7.51}$$

where $p_{12}, \tilde{p}_{12} \in \mathcal{P}_{1+\varkappa_1-\varkappa_2}$ and $p_{13}, \tilde{p}_{13} \in \mathcal{P}_{1+\varkappa_1-\varkappa_3}$. This is the case (see Theorem 3.5) if there exists a function $V(z)$ of a particular form such that $\widetilde{A} = VAV^{-1} + V'V^{-1}$.

We first claim that the $(2,1)$- and $(3,1)$-entries of $V(z)$, V_{21} and V_{31}, are equal to zero. This is obvious in the case $\varkappa_1 > \varkappa_2$. In case $\varkappa_1 = \varkappa_2 > \varkappa_3$, we have $V_{31} = V_{32} = 0$ and the upper left 2×2 matrix of $V(z)$ is constant. In order to show that $V_{21} = 0$, we write $\widetilde{A}V = VA + V'$, and inspect the $(3,1)$-entries. It is equal to zero for V' and also for VA. The $(3,1)$-entry of $\widetilde{A}V$ equals V_{21} times the $(3,2)$-entry of $\widetilde{A}(z)$. We conclude that $V_{21} = 0$ because otherwise the $(3,2)$-entry of $\widetilde{A}(z)$ is zero, which contradicts the assumption that $\widetilde{A}(z)$ is not of triangular form. In the case, $\varkappa_1 = \varkappa_2 = \varkappa_3$, we have $\widetilde{A}(z) = VA(z)V^{-1}$ with a constant matrix V. This can be rephrased in terms of the residues $\widehat{E}_k$ and $\widetilde{E}_k$ of $A(z)$ and $\widetilde{A}(z)$, respectively, namely $\widetilde{E}_k = V\widehat{E}_k V^{-1}$. The triples $[\widehat{E}_1, \widehat{E}_2, \widehat{E}_3]$ and $[\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3]$ possess exactly one one-dimensional invariant subspace $\mathbb{C} \oplus \{0\} \oplus \{0\}$. (If there were another, then we could argue similar to above and conclude that the 2×2 matrices obtained from these triples by removing the first rows and columns are reducible.) The matrix V must map this invariant subspace into itself. Hence it is also an invariant subspace of V, which implies that $V_{21} = V_{31} = 0$.

So we have shown that

$$V(z) \;=\; \begin{pmatrix} v_1 & v(z) \\ 0 & V_2(z) \end{pmatrix}, \tag{7.52}$$

with the same block structure as $A(z)$ and $\widetilde{A}(z)$ and with entries having certain properties. The next claim is that $V_2(z)$ is a constant scalar matrix. From $\widetilde{A} = VAV^{-1} + V'V^{-1}$, it follows that $A_2 = V_2A_2V_2^{-1} + V_2'V_2^{-1}$, and we know that the 2×2 system with $A_2(z)$ is not equivalent to a triangular system. If the indices $\varkappa_2, \varkappa_3$ coincide, then $V_2(z)$ is a constant matrix, and the above can be rephrased in terms of the residues $\widehat{E}_k^{\#}$ of $A_2(z)$. Namely, $\widehat{E}_k^{\#} = V_2\widehat{E}_k^{\#}V_2^{-1}$. If V_2 is not a scalar matrix, then we can argue similar as in the second paragraph of the proof of Proposition 7.4 and conclude that the triple $[\widehat{E}_1^{\#}, \widehat{E}_2^{\#}, \widehat{E}_3^{\#}]$ is reducible, which contradicts the assumption on $A_2(z)$. In case $\varkappa_2 - \varkappa_3 = 1$, $V_2(z)$ is of triangular form with constant values on the diagonal. Now a straightforward analysis of $A_2 = V_2A_2V_2^{-1} + V_2'V_2^{-1}$ (using that $A_2(z)$ is not triangular) implies that V_2 is a scalar constant matrix.

Having this information about $V(z)$, we can now reformulate the condition $\widetilde{A} = VAV^{-1} + V'V^{-1}$ as follows. The systems $A(z)$ and $\widetilde{A}(z)$ are equivalent if and only if

$$\tilde{a}_{12}(z) \;=\; \lambda a_{12}(z) + v(z)A_2(z) - a_1(z)v(z) + v'(z), \tag{7.53}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and some $v(z) = (v_{12}(z), v_{13}(z))$ where $v_{12} \in \mathcal{P}_{\varkappa_1-\varkappa_2}$ and $v_{13} \in \mathcal{P}_{\varkappa_1-\varkappa_3}$. Referring to (7.51) this can be expressed as

$$(\tilde{p}_{12}, \tilde{p}_{13}) \;=\; \lambda(p_{12}, p_{13}) + p(v_{12}, v_{13})A_2 - a_1 p(v_{12}, v_{13}) + p(v_{12}', v_{13}'). \tag{7.54}$$

Hence the two systems with (p_{12}, p_{13}) and $(\tilde{p}_{12}, \tilde{p}_{13})$ are equivalent if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$(\tilde{p}_{12}, \tilde{p}_{13}) - \lambda(p_{12}, p_{13}) \;\in\; \operatorname{im}\Xi, \tag{7.55}$$

where $\operatorname{im}\Xi$ is the image of the linear mapping defined by

$$\begin{aligned} \Xi \;:\; & \mathcal{P}_{\varkappa_1-\varkappa_2} \oplus \mathcal{P}_{\varkappa_1-\varkappa_3} \to \mathcal{P}_{1+\varkappa_1-\varkappa_2} \oplus \mathcal{P}_{1+\varkappa_1-\varkappa_3} \\ & (v_{12}, v_{13}) \mapsto p(v_{12}, v_{13})A_2 - a_1 p(v_{12}, v_{13}) + p(v_{12}', v_{13}'). \end{aligned} \tag{7.56}$$

Now we decompose $\mathcal{P}_{1+\varkappa_1-\varkappa_2} \oplus \mathcal{P}_{1+\varkappa_1-\varkappa_3} = \operatorname{im}\Xi \oplus X$ as a direct sum. We arrive at the following statement: If $(p_{12}, p_{13}) \in X$ and $(\tilde{p}_{12}, \tilde{p}_{13}) \in X$, then the corresponding systems are equivalent if and only if $(\tilde{p}_{12}, \tilde{p}_{13}) = \lambda(p_{12}, p_{13})$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Obviously, we have $\dim X \geq 2$. However, we claim that $\dim X \geq 3$ in the case where the 2×6 matrix appearing in (7.46) has rank 1, i.e., if $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ is equivalent to a triple (7.47). Indeed, this is shown as soon as one has shown that the kernel of Ξ is non-trivial. For this we consider the 2×2 system $\widehat{Y}_2'(\tilde{z}) = \widehat{A}_2(z)\widehat{Y}_2(\tilde{z})$ with $\widehat{A}_2(z) = A_2(z) - a_1(z)I_2$. It is easy to see that this system is also of standard form with indices $\varkappa_2 - \varkappa_1$ and $\varkappa_3 - \varkappa_1$, and with data $[M_1^{\#}\mu_1^{-1}, M_2^{\#}\mu_2^{-1}, M_3^{\#}\mu_3^{-1}]$ and $[E_1^{\#} - \varepsilon_1 I_2, E_2^{\#} - \varepsilon_2 I_2, E_3^{\#} - \varepsilon_3 I_2]$. The solution of this system is given by $\widehat{Y}_2(\tilde{z}) = Y_2(\tilde{z})y_1^{-1}(\tilde{z})$. From $\widehat{Y}_2' = \widehat{A}_2\widehat{Y}_2$ it follows immediately that $\widehat{Y}_2^{-1}\widehat{A}_2 + (\widehat{Y}_2^{-1})' = 0$ by multiplying on both sides with $\widehat{Y}_2^{-1}$. We may assume that the monodromy representation of $\widehat{Y}_2(\tilde{z})$ is given by

$$\left[\begin{pmatrix} \hat{\mu}_1\mu_1^{-1} & \hat{\alpha}_1\mu_1^{-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \hat{\mu}_2\mu_2^{-1} & \hat{\alpha}_1\mu_2^{-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \hat{\mu}_3\mu_3^{-1} & \hat{\alpha}_1\mu_3^{-1} \\ 0 & 1 \end{pmatrix}\right]. \tag{7.57}$$

Now we define (v_{12}, v_{13}) to be the last row in $\widehat{Y}_2^{-1}$. Obviously, this vector function satisfies $(v_{12}, v_{13})(A_2(z) - a_1(z)I_2) + (v_{12}', v_{13}') = 0$. What remains to show is that v_{12} and v_{13} are polynomials of appropriate degree. From the above monodromy representation for $\widehat{Y}_2$, it follows that (v_{12}, v_{13}) is a single valued function. This function is analytic on $\mathbb{C}\setminus\{a_1, a_2, a_3\}$. Because of the non-resonance of $[E_1, E_2, E_3]$ and because μ_k is an eigenvalue of $M_k^{\#}$, it follows that ε_k is an eigenvalue of $E_k^{\#}$. Hence the matrices $E_k^{\#} - \varepsilon_k I_2$ have an eigenvalue equal to zero and are also non-resonant. But these matrices describe the local behavior of $\widehat{Y}_2$ (and of its inverse) near the point a_k. A straightforward computation (using the fact that (v_{12}, v_{13}) is single valued) shows that (v_{12}, v_{13}) is actually analytic at $z = a_k$. Hence we have shown that (v_{12}, v_{13}) is entire analytic. That v_{12} and v_{13} are polynomials of degree not greater than $\varkappa_1 - \varkappa_2$ and $\varkappa_1 - \varkappa_3$, respectively, can be obtained by employing the behavior of $\widetilde{Y}_2$ at infinity, which is described by the integers $\varkappa_2 - \varkappa_1$ and $\varkappa_3 - \varkappa_1$. In summary, we can conclude that $(v_{12}, v_{13}) \neq 0$ lies in the kernel of Ξ.

Similar as in the proof of Theorem 6.2, we can define a linear mapping $\Lambda : X \to \mathbb{C}^{1\times 6}$ from the space X into the set of all "admissible" vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ (see also (7.45)). This mapping is characterized by the following property. For a system of the form (7.49) with above defined $a_1(z)$ and $A_2(z)$ and with a_{12} given by (7.51), where $(p_{12}, p_{13}) \in X$, the corresponding monodromy is given by (7.41), where $\alpha_1, \alpha_2, \alpha_3$ are given by $\alpha = (\alpha_1, \alpha_2, \alpha_3) = \Lambda((p_{12}, p_{13}))$.

We have a one-to-one correspondence between equivalence classes of systems of standard form and equivalence classes of data (see Corollary 6.3). In our situation we can apply Proposition 7.4 (see also (7.46)) and what has been stated above about the equivalence of the systems under consideration. We obtain the following statement. For $(p_{12}, p_{13}), (\tilde{p}_{12}, \tilde{p}_{13}) \in X$, the vectors $\Lambda((p_{12}, p_{13}))$ and $\Lambda((\tilde{p}_{12}, \tilde{p}_{13}))$ are equivalent in the sense of (7.46) (or, (7.43)) if and only if there exists a $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$(\tilde{p}_{12}, \tilde{p}_{13}) = \lambda(p_{12}, p_{13})$.

The kernel of Λ is trivial. Indeed, $\Lambda((p_{12}, p_{13})) = 0 = \Lambda(0)$ implies immediately $(p_{12}, p_{13}) = 0$. Hence $\dim \operatorname{im} \Lambda \geq 2$ or even $\dim \operatorname{im} \Lambda \geq 3$ depending on whether the 2×6 matrix appearing in (7.46) has rank 2 or 1. So we are done as soon as we have shown that the sum of $\operatorname{im} \Lambda$ and the following linear subspace of $\mathbb{C}^{1\times 6}$,

$$\left\{ c \left(\begin{array}{ccc} M_1^{\#} - \mu_1 I_2, & M_2^{\#} - \mu_2 I_2, & M_3^{\#} - \mu_3 I_2 \end{array} \right) \;:\; c \in \mathbb{C}^{1\times 2} \right\}, \tag{7.58}$$

exhausts the whole four-dimensional subspace of $\mathbb{C}^{1\times 6}$ consisting of all admissible vectors α (see again (7.46)). As we have estimates for the dimensions, this follows from the statement that the intersection of (7.58) with $\operatorname{im} \Lambda$ contains only the zero vector. Indeed, if $\alpha = \Lambda((p_{12}, p_{13}))$ is contained in the above set, then α is equivalent to $0 = \Lambda(0)$, whence again follows that $(p_{12}, p_{13}) = 0$. Hence $\alpha = 0$. $\Box$

Finally, we are going to consider data and systems which are also of block form, but where the block partition of the matrices are $(2, 1)$ rather than $(1, 2)$. The proofs of the corresponding statements are analogous, and will therefore be omitted. However, we will state definitions and the results completely.

Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. We call this data $(2, 1)$-*quasi-block data* if $[M_1, M_2, M_3]$ is equivalent to a triple

$$\left[\begin{pmatrix} M_1^{\#} & \alpha_1 \\ 0 & \mu_1 \end{pmatrix}, \begin{pmatrix} M_2^{\#} & \alpha_2 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} M_3^{\#} & \alpha_3 \\ 0 & \mu_3 \end{pmatrix} \right] \tag{7.59}$$

and $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ and $[E_1^{\#}, E_2^{\#}, E_3^{\#}]$ is quasi-block data. Here μ_k are numbers, α_k are 2×1 vectors, and $M_k^{\#}$ are 2×2 matrices. The matrices $E_k^{\#}$ and numbers ε_k are supposed to satisfy $M_k^{\#} \sim \exp(-2\pi i E_k^{\#})$ and $\mu_k = \exp(-2\pi i \varepsilon_k)$ and are chosen in such a way that the eigenvalues of $E_k^{\#}$ and the number ε_k are the exactly eigenvalues of E_k. For given $M_k^{\#}$ and μ_k, the number ε_k is uniquely determined and so is the matrix $E_k^{\#}$ up to similarity (because of the non-resonance of E_k). Again we define the integers

$$\nu \;=\; \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \qquad N \;=\; \operatorname{trace}(E_1^{\#} + E_2^{\#} + E_3^{\#}). \tag{7.60}$$

As above, we remark that if $[M_1, M_2, M_3]$ possesses two different two-dimensional invariant subspaces, then $[M_1, M_2, M_3]$ is equivalent to different triples of the form (7.59) (with different μ_k and $M_k^{\#}$ in particular). In general, we obtain also different values for $\varepsilon_k, E_k^{\#}, \nu$ and N.

The counterpart to Proposition 7.4 is the following result.

Proposition 7.6 *Let $[M_1, M_2, M_3]$ be a triple of the form (7.59). Assume that*

(a) $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ is not diagonalizable;

(b) If $[M_1^{\#}, M_2^{\#}, M_3^{\#}]$ possess exactly one non-trivial invariant subspace, then

(b1) for at least one k, $M_k^\#$ possesses two different eigenvalues;

(b2) for at least one k, μ_k is not equal to the eigenvalue of $M_k^\#$ which does not correspond to this invariant subspace.

Let $[\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3]$ be another triple of the same form (7.59) but with $\alpha_1, \alpha_2, \alpha_3$ replaced by $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3$. Then these triples are equivalent if and only if there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}^{2\times 1}$ such that

$$\alpha_k \;=\; \lambda \widetilde{\alpha}_k + M_k^\# c - c\mu_k \qquad \textit{for each } k = 1, 2, 3. \tag{7.61}$$

The set of equivalence classes for triples of the form (7.59) with μ_k and $M_k^\#$ being fixed can be identified similar as before. All such triples are parameterized by 2×1 vectors $\alpha_1, \alpha_2, \alpha_3$, and we introduce the 6×1 vector $\alpha = (\alpha_1^T, \alpha_2^T, \alpha_3^T)^T$. We have to take the linear condition (7.6) into account. It can be restated as

$$\begin{pmatrix} \mu_2\mu_3 I_2, & M_1^\# \mu_3, & M_1^\# M_2^\# \end{pmatrix} \alpha = 0. \tag{7.62}$$

The 2×6 matrix appearing here has rank 2. Hence the vector α is again subject to two linear conditions, and all "admissible" vectors α are taken from a 4-dimensional linear subspace of $\mathbb{C}^{6\times 1}$. The equivalence relation (7.61) can be rewritten as

$$\alpha \;=\; \lambda \widetilde{\alpha} \;+\; \begin{pmatrix} M_1^\# - \mu_1 I_2 \\ M_2^\# - \mu_2 I_2 \\ M_3^\# - \mu_3 I_2 \end{pmatrix} c \tag{7.63}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}^{2\times 1}$. The 6×2 matrix appearing here has again either rank 2 or rank 1. It has rank one if and only if $[M_1^\#, M_2^\#, M_3^\#]$ possesses exactly one non-trivial invariant subspace and, for each $k = 1, 2, 3$, the number μ_k is equal to the eigenvalue of $M_k^\#$ which corresponds to this invariant subspace. In conclusion, we obtain that the set of equivalence classes of the above triples can be identified with either $\mathbb{P}^1 \cup \{0\}$ or with $\mathbb{P}^2 \cup \{0\}$. The single equivalence class containing the vector $\alpha = 0$ consists of the triples which are "block-diagonalizable".

Next we consider systems $Y'(\tilde{z}) = A(z)Y(\tilde{z})$ of standard form with indices $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$ for which $A(z)$ can be written as

$$A(z) \;=\; \frac{1}{p(z)} \begin{pmatrix} \varkappa_1 z^2 + p_1(z) & p_{12}(z) & p_{13}(z) \\ p_{21}(z) & \varkappa_2 z^2 + p_2(z) & p_{23}(z) \\ 0 & 0 & \varkappa_3 z^2 + p_3(z) \end{pmatrix}, \tag{7.64}$$

where $p(z) = (z - a_1)(z - a_2)(z - a_3)$, $p_k \in \mathcal{P}_1$, $p_{jk} \in \mathcal{P}_{1+\varkappa_j - \varkappa_k}$. If such a system is not equivalent to a system of the form (7.18), then it will be called a *system of* $(2,1)$-*block form.* Note that necessarily $\varkappa_2 - \varkappa_3 \leq 1$ because otherwise $p_{32}(z) = 0$ and the system is of the form (7.18).

The data which is associated to such systems is described in the following theorem.

Theorem 7.7 *Let $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{Z}$, $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$, and assume $\varkappa_1 - \varkappa_2 \leq 1$. Let $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ be admissible data of 3×3 matrices. Then the following two statements are equivalent:*

(i) $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is the data associated to a system of $(2,1)$-block form with indices $\varkappa_1, \varkappa_2, \varkappa_3$.

(ii) $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is $(2,1)$-quasi-block data, and $N = \varkappa_1 + \varkappa_2$, $\nu = \varkappa_3$, where ν and N are defined by (7.60).

We remark that the cases of $(1,2)$-block form systems and of $(2,1)$-block form systems are not exclusive. They overlap exactly if $\varkappa_1 = \varkappa_2 = \varkappa_3$ and if the system is equivalent to a system of a "block-diagonal form". This can be deduced from the argumentation in the proof of the main results given below.

7.4 The main results in the case $n = m = 3$

The main results for the case $m = n = 3$, i.e., the identification of the indices corresponding to given data is given in the following theorem. We have to resort to the classification of the reducibility type of $[M_1, M_2, M_3]$ and the definition of the various integers $\varkappa$, ν, N, n_1, n_2, n_3, ν_1, ν_2, $\nu_\#$ stated at the beginning of this section. Recall that we have assumed $n_1 \geq n_2$ in case (C-1), $n_2 \geq n_3$ in case (C-2), and $n_1 \geq n_2 \geq n_3$ in case (D) without lost of generality. In the other cases, the integers of interest were defined uniquely.

Theorem 7.8 *Let $[M_1, M_2, M_3]$ be admissible data of 3×3 matrices. Then in the following cases the indices $\varkappa_1, \varkappa_2, \varkappa_3 \in \mathbb{Z}$, $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$, are as follows.*

(1) In cases (B-1) or (B-3) with $2\nu \geq N$, the indices are

$$\varkappa_1 = \nu,\ \varkappa_2 = N/2,\ \varkappa_3 = N/2 \text{ if } N \text{ is even, or,}$$
$$\varkappa_1 = \nu,\ \varkappa_2 = (N+1)/2,\ \varkappa_3 = (N-1)/2 \text{ if } N \text{ is odd.}$$

(2) In cases (B-2) or (B-3) with $2\nu \leq N$, the indices are

$$\varkappa_1 = N/2,\ \varkappa_2 = N/2,\ \varkappa_3 = \nu \text{ if } N \text{ is even, or,}$$
$$\varkappa_1 = (N+1)/2,\ \varkappa_2 = (N-1)/2,\ \varkappa_3 = \nu \text{ if } N \text{ is odd.}$$

(3) In case (C) with $n_1 \geq n_2 \geq n_3$ or in case (C-1) with $n_2 \geq n_3$ or in case (C-2) with $n_1 \geq n_2$ or in case (D), the indices are

$$\varkappa_1 = n_1,\ \varkappa_2 = n_2,\ \varkappa_3 = n_3.$$

(4) In case (C-3) with $\nu_1 \geq \nu_2$, the indices are

$$\varkappa_1 = \nu_1,\ \varkappa_2 = \nu_2,\ \varkappa_3 = \nu_\# \text{ if } \nu_2 \geq \nu_\#, \text{ or,}$$
$$\varkappa_1 = \nu_1,\ \varkappa_2 = \nu_\#,\ \varkappa_3 = \nu_2 \text{ if } \nu_1 \geq \nu_\# \geq \nu_2, \text{ or,}$$
$$\varkappa_1 = \nu_\#,\ \varkappa_2 = \nu_1,\ \varkappa_3 = \nu_2 \text{ if } \nu_\# \geq \nu_1.$$

(5) *In cases (C) or (C-1) with* $n_2 < n_3$ *and* $2n_1 \geq n_2 + n_3$, *the indices are*

$$\varkappa_1 = n_1,\ \varkappa_2 = (n_2 + n_3)/2,\ \varkappa_3 = (n_2 + n_3)/2 \text{ if } n_2 + n_3 \text{ is even, or,}$$
$$\varkappa_1 = n_1,\ \varkappa_2 = (n_2 + n_3 + 1)/2,\ \varkappa_3 = (n_2 + n_3 - 1)/2 \text{ if } n_2 + n_3 \text{ is odd.}$$

(6) *In cases (C) or (C-2) with* $n_1 < n_2$ *and* $n_1 + n_2 \geq 2n_3$, *the indices are*

$$\varkappa_1 = (n_1 + n_2)/2,\ \varkappa_2 = (n_1 + n_2)/2,\ \varkappa_3 = n_3 \text{ if } n_1 + n_2 \text{ is even, or,}$$
$$\varkappa_1 = (n_1 + n_2 + 1)/2,\ \varkappa_2 = (n_1 + n_2 - 1)/2,\ \varkappa_3 = n_3 \text{ if } n_1 + n_2 \text{ is odd.}$$

(7) *In case (C-3) with* $\nu_1 < \nu_2$ *and* $2\nu_\# \geq \nu_1 + \nu_2$, *the indices are*

$$\varkappa_1 = \nu_\#,\ \varkappa_2 = (\nu_1 + \nu_2)/2,\ \varkappa_3 = (\nu_1 + \nu_2)/2, \text{ if } \nu_1 + \nu_2 \text{ is even, or,}$$
$$\varkappa_1 = \nu_\#,\ \varkappa_2 = (\nu_1 + \nu_2 + 1)/2,\ \varkappa_3 = (\nu_1 + \nu_2 - 1)/2, \text{ if } \nu_1 + \nu_2 \text{ is odd.}$$

(8) *In case (C-3) with* $\nu_1 < \nu_2$ *and* $\nu_1 + \nu_2 \geq 2\nu_\#$, *the indices are*

$$\varkappa_1 = (\nu_1 + \nu_2)/2,\ \varkappa_2 = (\nu_1 + \nu_2)/2,\ \varkappa_3 = \nu_\# \text{ if } \nu_1 + \nu_2 \text{ is even, or,}$$
$$\varkappa_1 = (\nu_1 + \nu_2 + 1)/2,\ \varkappa_2 = (\nu_1 + \nu_2 - 1)/2,\ \varkappa_3 = \nu_\# \text{ if } \nu_1 + \nu_2 \text{ is odd.}$$

In the remaining cases, i.e.,

(9) *in case (A),*

(10) *in case (B-1) with* $2\nu < N$,

(11) *in case (B-2) with* $2\nu > N$,

(12) *in case (C) with* $n_1 + n_2 < 2n_3$ *and* $2n_1 < n_2 + n_3$,

(13) *in case (C-1) with* $2n_1 < n_2 + n_3$,

(14) *in case (C-2) with* $n_1 + n_2 < 2n_3$,

the indices are determined by the following statements:

(a) *If* $\varkappa \equiv 1 \bmod 3$, *then* $\varkappa_1 = (\varkappa + 2)/3$, $\varkappa_2 = \varkappa_3 = (\varkappa - 1)/3$.

(b) *If* $\varkappa \equiv -1 \bmod 3$, *then* $\varkappa_1 = \varkappa_2 = (\varkappa + 1)/3$, $\varkappa_3 = (\varkappa - 2)/3$.

(c) *If* $\varkappa \equiv 0 \bmod 3$, *then either* $\varkappa_1 = \varkappa_2 = \varkappa_3 = \varkappa/3$ *or* $\varkappa_1 = \varkappa/3 + 1$, $\varkappa_2 = \varkappa/3$, $\varkappa_3 = \varkappa/3 - 1$.

(c*) *If* $\varkappa \equiv 0 \bmod 3$, *then the indices are* $\varkappa_1 = \varkappa/3 + 1$, $\varkappa_2 = \varkappa/3$, $\varkappa_3 = \varkappa/3 - 1$ *if and only if* $[M_1, M_2, M_3]$ *is the monodromy of some third order linear differential equation with Fuchsian singularities* a_1, a_2, a_3 *and local exponents given by* $\{\varepsilon_k^{(1)}, \varepsilon_k^{(2)}, \varepsilon_k^{(3)}\}$ *for* $z = a_k$, $k = 1, 2$, *and* $\{\varepsilon_3^{(1)} + 1 - \varkappa/3, \varepsilon_k^{(2)} + 1 - \varkappa/3, \varepsilon_k^{(3)} + 1 - \varkappa/3\}$ *for* $z = a_3$, *where* $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}, \varepsilon_k^{(3)}$ *are the eigenvalues of* E_k, $k = 1, 2, 3$.

Proof. First of all, the reader can easily verify that the classification into the cases (1)–(14) is complete. Some of these cases do overlap (namely, case (B-3) with $2\nu = N$ is stated in (1) and (2), case (C-3) with $\nu_1 \geq \nu_2$ and $\nu_1 = \nu_\#$ or $\nu_2 = \nu_\#$ is stated multiply in (4), and case (C-3) with $\nu_1 < \nu_2$ and $\nu_1 + \nu_2 = 2\nu_\#$ is given in both (7) and (8)), but in these cases, the description of the partial indices is the same. We remark in particular that the cases given in (9)–(14) are exactly the cases which are not contained in (1)– (8).

In Theorem 7.2 we have completely characterized the data which is associated to systems of the form (7.18). We obtain that given data is associated to such a system if and only if $[M_1, M_2, M_3]$ is equivalent to a certain triple (7.7) such that for the corresponding integers n_1, n_2, n_3 (which are defined by resorting to the properly numbered eigenvalues of E_1, E_2, E_3) the inequality $n_1 \geq n_2 \geq n_3$ holds. This inequality follows from the statement that $\varkappa_k = n_k$ for $k = 1, 2, 3$, and from the ordering $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$. We note that in some cases, there exist essentially different ways to establish an equivalence between $[M_1, M_2, M_3]$ and a triangular triple (7.7), which in turn results into different definitions of n_1, n_2, n_3. The statement (i) of Theorem 7.2 is true if and only if such a relationship exists.

Examining the different reducibility types for a triple (7.7) under the condition $n_1 \geq n_2 \geq n_3$ we arrive at the following cases. First of all, only the reducibility types (C)–(D) are covered. Case (C) is covered exactly for $n_1 \geq n_2 \geq n_3$ as the enumeration of these integers is unique. Case (C-1) is covered exactly for $n_2 \geq n_3$ as n_3 is defined uniquely and $n_1 \geq n_2$ without loss of generality. Similarly, case (C-2) is covered exactly for $n_1 \geq n_2$ as n_1 is unique and $n_2 \geq n_3$ without loss of generality. The case (D) is obviously covered completely. So we arrive at the statement (3).

However, there are some other possibilities leading to case (C-3) and statement (4). In case (C-3), the integers ν_1, ν_2, ν_3 are uniquely defined, but an equivalence with a triple (7.7) can be established in three different ways. The first way is if the triple has entries $\beta_k = \gamma_k = 0$. This gives $n_1 = \nu_1$, $n_2 = \nu_2$, $n_3 = \nu_\#$, and hence the first subcase in (4). The second possibility is if the triple has entries $\alpha_k = \beta_k = 0$. Here we obtain $n_1 = \nu_1$, $n_2 = \nu_\#$, $n_3 = \nu_2$, which is the second subcase in (4). Finally, we may have $\alpha_k = \gamma_k = 0$, which leads to $n_1 = \nu_\#$, $n_2 = \nu_1$, $n_3 = \nu_2$, and the third subcase in (4).

We have now identified all the data associated to system (7.18) in terms of the classification established at the beginning of this section.

Next we do the same for the systems of $(1, 2)$-block form (see (7.48)). The data of such systems has been described in Theorem 7.5. With the numbers ν and N defined by (7.42) we obtain that the indices are $\varkappa_1 = \nu$, $\varkappa_2 = [(N + 1)/2]$, and $\varkappa_3 = [N/2]$. The latter follows from the fact that $\varkappa_2 + \varkappa_3 = N$ and $0 \leq \varkappa_2 - \varkappa_3 \leq 1$. The ordering $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$ can be rephrased by $\nu \geq [(N + 1)/2]$, which is equivalent to $2\nu \geq N$. We arrive at conclusion that the data of such systems is precisely the data which is of $(1, 2)$-quasi-block form and satisfies $2\nu \geq N$.

Now we have to examine what does this mean in terms of the classification given at the beginning of this section. In the triple (7.41), there appears a triple

$[M_1^\#, M_2^\#, M_3^\#]$ which along with $[E_1^\#, E_2^\#, E_3^\#]$ has to be quasi-block data. We have to distinguish the cases (I) and (II) stated in the definition of quasi-block data.

In case (I), the triple (7.41) possesses exactly one one-dimensional invariant subspace $\mathbb{C} \oplus \{0\} \oplus \{0\}$ and no two-dimensional invariant subspace containing it. Hence the data $[M_1, M_2, M_3]$ and $[E_1, E_2, E_3]$ is precisely the data of reducibility type (B-1) or (B-3) with the condition $2\nu \geq N$, where ν and N are now defined by (7.4). In fact, these numbers are the same as the former ones defined in (7.48). So we arrive at the statement (1).

In case (II), we may assume $[M_1^\#, M_2^\#, M_3^\#]$ to be of triangular form. We have $N = n_1^\# + n_2^\#$ and $n_1^\# < n_2^\#$ with the integers $n_1^\#$ and $n_2^\#$ defined in terms of the eigenvalues of $[E_1^\#, E_2^\#, E_3^\#]$. Hence the data is given by triples (7.48) with $2\nu \geq n_1^\# + n_2^\#$ and $n_1^\# < n_2^\#$. These triples are of reducibility type (C), (C-1) or (C-3). We do not obtain (C-2) or (D) because of the non-diagonalizability of $[M_1^\#, M_2^\#, M_3^\#]$. In case of reducibility type (C), we have $n_1 = \nu$, $n_2 = n_1^\#$ and $n_3 = n_2^\#$, which gives the condition $2n_1 \geq n_2 + n_3$ and $n_2 < n_3$. In case (C-1) we have the same identification, but in addition we may also have (interchanging n_1 and n_2) $n_2 = \nu$, $n_1 = n_1^\#$ and $n_3 = n_2^\#$. This would give the condition $2n_2 \geq n_1 + n_3$ and $n_1 < n_3$, which, however, conflicts with the assumption $n_1 \geq n_2$. So we arrive at the statement (5).

In case (C-3), the only possibility not contradicting the non-diagonalizability of $[M_1^\#, M_2^\#, M_3^\#]$ is if the triple $[M_1, M_2, M_3]$ is equivalent to a triple (7.7) with $\alpha_k = \gamma_k = 0$. This results in the identification $\nu_\# = \nu$, $\nu_1 = n_1^\#$ and $\nu_2 = n_2^\#$. Hence we obtain the conditions $2\nu_\# \geq \nu_1 + \nu_2$ and $\nu_1 < \nu_2$, and this gives the statement (7).

So we have also settled the cases of data corresponding to systems of $(1, 2)$-block form.

The analysis of the data corresponding to systems of $(2, 1)$-block form (7.64) is completely analogous and will therefore be omitted. Here we arrive at the statements (2), (6) and (8) instead of (1), (5) and (7).

Finally, we have to examine the indices for the data in the remaining cases. We know that this data is associated to certain systems which are neither of $(1, 2)$-block form, $(2, 1)$-block form nor of triangular form.

It follows that $\varkappa_1 - \varkappa_2 \leq 1$ because otherwise (by Theorem 4.3) the $(2, 1)$- and $(3, 1)$-entries of the matrix $A(z)$ of the system vanish. Similarly, it follows that $\varkappa_2 - \varkappa_3 \leq 1$. From the fact that $\varkappa_1 \geq \varkappa_2 \geq \varkappa_3$ and $\varkappa_1 + \varkappa_2 + \varkappa_3 = \varkappa$, we immediately obtain the assertions (a), (b) and (c).

In order to prove (c*), we have to examine when the indices are such that $\varkappa_1 - \varkappa_2 = \varkappa_2 - \varkappa_3 = 1$. Again by Theorem 4.3, this is exactly the case if the system is of the form

$$A(z) = \frac{1}{p(z)} \begin{pmatrix} (\varkappa/3 + 1)z^2 + p_1(z) & p_{12}(z) & p_{13}(z) \\ \alpha & \varkappa/3 z^2 + p_2(z) & p_{23}(z) \\ 0 & \beta & (\varkappa/3 - 1)z^2 + p_3(z) \end{pmatrix},$$

where $p(z) = (z - a_1)(z - a_2)(z - a_3)$, $p_k \in \mathcal{P}_1$, $p_{jk} \in \mathcal{P}_{1+\varkappa_j - \varkappa_k}$ and $\alpha, \beta \in \mathbb{C}$. If $\alpha = 0$ or $\beta = 0$, then $A(z)$ is of block-triangular or even triangular form, which has been excluded. Hence $\alpha\beta \neq 0$. (In fact, if $\alpha\beta \neq 0$, then $A(z)$ is not equivalent to a system of triangular or block-triangular form.) Applying Theorem 5.4, we obtain the desired assertion (c*). $\square$

In the cases where assumption (c) of the previous theorem applies, we are led to the monodromy problem for third order Fuchsian differential equations with three singular points. Here the monodromy does not depend on the location of the singularities, and neither do the corresponding indices. Again, as far as the authors know, no explicit answer exists to this monodromy problem. However, some particular cases can be tackled directly by help of the remark made in the last paragraph of Section 5. Namely, if one of the matrices M_1, M_2 or M_3 has the property that $\min(M_k) \leq 2$, then the indices are necessarily $\varkappa_1 = \varkappa_2 = \varkappa_3 = \varkappa/3$. We note that in the cases (9)–(14), matrices $[M_1, M_2, M_3]$ with this property really exist.

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References

[1] D.V. Anosov, A.A. Bolibruch, *The Riemann-Hilbert problem*, Aspects of Mathematics **E 22**, Vieweg, Braunschweig, 1994.

[2] G.D. Birkhoff, *Collected mathematical papers*, Vol. I, Dover Publications, New York, 1968.

[3] A.A Bolibruch, The Riemann-Hilbert problem, *Russian Math. Surveys* **45:2** (1990), 1–58.

[4] A.A Bolibruch, Vector bundles associated with monodromies and asymptotics of Fuchsian systems, *J. Dynam. Control Systems* **1** (1995), no. 2, 229–252.

[5] A.A Bolibruch, Hilbert's twenty-first problem for linear Fuchsian systems, *Trudy Mat. Inst. Steklov.* **206** (1994), pp. 160.

[6] A.A Bolibruch: private communication (paper to appear).

[7] K.F. Clancey, Gohberg, I., *Factorization of Matrix Functions and Singular Integral Operators*, Birkhäuser, Basel and Boston, 1981.

[8] E.A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, New York, 1955.

[9] O. Förster, *Riemannsche Flächen*, Springer, Berlin, 1977.

[10] I. Gohberg, N. Krupnik, *One-Dimensional Linear Singular Integral Equations. Introduction*, Birkhäuser, Basel-Boston, 1992.

[11] G.M. Goluzin, *Geometrical Function Theory of Complex Variables*, Nauka, Moscow, 1966.

[12] J. Leiterer, Banach coherence and Oka's principle, *Contributions to the School on Global Analysis (Ludwigsfelde, 1976), Part 2*, Akad. Wiss. DDR, Berlin, 1977, 63–129.

[13] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, New York, 1964.

[14] E.L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1944.

[15] G.S. Litvinchuk, I.M. Spitkovsky, Factorization of measurable matrix functions, *Operator Theory: Advances and Applications* **25**, Birkhäuser, Basel, 1987.

[16] J. Plemelj, *Problems in the sense of Riemann and Klein*, John Wiley & Sons, New York, 1964.

[17] H. Röhrl, Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen, *Math. Ann.* **133** (1957), 1–25.

[18] I.M. Spitkovsky, A.M. Tashbaev, Factorization of certain piecewise constant matrix functions and its applications, *Math. Nachr.* **151** (1991), 241–261.

[19] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, N.J., 1951.

[20] E.I. Zverovich, L.A. Khvoschinskaya, Solution of a generalized Riemann problem, *Proceedings of a commemorative seminar on boundary value problems*, Minsk, Belorussian University (1985), 153–158.