PARUSIŃSKI'S "KEY LEMMA" VIA ALGEBRAIC GEOMETRY

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Abstract. We give a new algebro-geometric proof of the "Key Lemma" of Parusiński.

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1. Introduction

The purpose of this paper is to give a new proof of the "Key Lemma" of Parusiński; see [Pa, pp. 202–203] and Theorem 1.1 below.

In order to place this result in its proper context, we begin with a brief introduction to the subject of Lipschitz stratifications. Two germs of varieties X, X' at a point O in $\mathbb{R}^n$ are called *Lipschitz equivalent* if there is a germ of a Lipschitz homeomorphism between two neighborhoods of O in $\mathbb{R}^n$ which transforms X into X'. Siebenmann and Sullivan [SS, p. 511] asked whether or not there are only countably many equivalence classes of singular points of analytic varieties up to Lipschitz equivalence. Intuitively, the question is, whether the classification of analytic singularities up to Lipschitz equivalence has continuous moduli.

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Lipschitz equivalence appeared for the following reasons. On the one hand, it is well-known that the analytic equivalence has continuous moduli: for example, the union of four curves in the plane intersecting at a given point x, has a continuous invariant, the double ratio of the tangent lines at x. For this reason, analytic equivalence is too delicate for classification purposes. On the other hand, there are no moduli of singularities up to homeomorphism: any family of singularities can be Whitney stratified [Wh], and Thom's Isotopy Lemma [Ma] shows that the singularities at any two points of the same Whitney stratum are homeomorphic.

However, the homeomorphism classification is very coarse, as it captures only the topological properties of the singular set. It was hoped that a Lipschitz classification, while capturing some of the analytic properties, would have no continuous moduli. The analog of a Whitney stratification is a Lipschitz stratification, i.e., a stratification with the property that for any point of a stratum S there is a neighborhood which has a Lipschitz product structure along S.

It turned out that Lipschitz stratifications are also important in the study of L^2 and L^p cohomology: if a stratification is Lipschitz then L^2 and L^p cohomology are locally constant along each stratum [C] (see also [Y]). T. Ohsawa told the second author in November 1993 that he could fix all the difficulties in his unsuccessful attempt [Oh] at the Cheeger—Goresky—MacPherson conjecture, with the exception of the existence of stratifications satisfying certain even stronger Lipschitz properties, which are not yet known to exist.

The existence of a complex-analytic Lipschitz stratification for a germ of a complex-analytic space was proved by Mostowski [Most]; however, this trailblazing paper is so difficult, that many experts do not understand all of its details. Parusiński [Pa] proved the existence of Lipschitz stratifications in the class of semianalytic sets; his arguments are simpler than those of [Most]. Parusiński's approach relies on the following result, which he calls the "Key Lemma".

Theorem 1.1. *(Parusiński* [Pa, pp. 202–203]*) For any positive integer n, there is a finite set of homogeneous symmetric polynomials $W_1, \dots, W_N \in \mathbb{Z}[x_1, \dots, x_n]$ and a constant $M > 0$ such that*

$$\Big|\frac{dx_i}{x_i}(p, v)\Big| \leq M \max_{j=1,\dots,N} \Big|\frac{dW_j}{W_j}(p, v)\Big| \tag{1.1}$$

for all $p \in \mathbb{C}^n$ and $v \in T_p\mathbb{C}^n$ for which both sides are defined. Here, for any $P \in \mathbb{C}[x_1, \dots, x_n]$ we view the meromorphic differential form $\frac{dP}{P}$ on $\mathbb{C}^n$ as a densely defined function on the total space of the tangent bundle $T\mathbb{C}^n$.

The proof of the "Key Lemma" given in [Pa, Section 6] is quite difficult, being apparently the hardest part of [Pa]. The purpose of this paper is to show that Theorem 1.1, in spite of its analytic appearance, has a natural proof in the framework of algebraic geometry. Our argument is an application of the results of [RY] about resolving group actions on algebraic

varieties; these results, in turn, rely on canonical resolution of singularities. The necessary background material from [RY] is summarized at the beginning of Section 4.

We remark that Parusiński proves the inequality (1.1) under the the additional assumption that $dV(p,v) = 0$ if $V(p) = 0$ for every V belonging to finite set $\mathcal{V}$ of polynomials. Since this additional requirement does not affect a dense Zariski open subset of $T\mathbb{C}^m$ (given by $V(p) \neq 0$ for every $V \in \mathcal{V}$), it can be dropped. We also note that the statement of the Key Lemma in [Pa] only asserts the existence of polynomials $W_1, \ldots, W_N$ with real coefficients; however, the construction given in the proof there produces polynomials over $\mathbb{Z}$. Thus, while Theorem 1.1 appears to be stronger than the "Key Lemma" in [Pa], the two are, in fact, equivalent.

2. Preliminaries

Notational conventions. All algebraic varieties considered in this paper, are assumed to be irreducible and definied over a field k of characteristic 0. The base field k is not assumed to be algebraically closed; the two cases of interest to us are $k = \mathbb{Q}$ and $k = \mathbb{C}$. By a point of a variety $x \in X$, we shall always understand a closed point; the local ring of x will be denoted by $\mathcal{O}_{x,X}$, its maximal ideal by $\mathfrak{m}_x$, its residue field by $k(x) = \mathcal{O}_{x,X}/\mathfrak{m}_x$, and its tangent space by $T_x(X) = (\mathfrak{m}_x/(\mathfrak{m}_x)^2)^*$. The ring of regular functions on a Zariski open subset $U \subset X$ will be denoted by $\mathcal{O}_X(U)$. The sheaf of regular functions on X will be denoted by $\mathcal{O}_X$. Given an embedding $k \subset \mathbb{C}$, we define $X_{\mathbb{C}} = X \otimes_k \mathbb{C}$; for any rational function f on X, we shall denote the corresponding rational function on $X_{\mathbb{C}}$ by f as well.

Throughout this paper G will be a finite group. A G-variety X is a variety with a regular action of G, $G \times X \longrightarrow X$, where $G \times X$ is understood as the disjoint union of $|G|$ copies of X. We will always assume that the G-action is faithful, i.e., every nonidentity element of G acts nontrivially. By a morphism (respectively, rational map, birational isomorphism) of G-varieties we shall mean a G-equivariant morphism (respectively, rational map, birational isomorphism).

"Naive" vs. "honest" stabilizers. For a point x in a G-variety X, we define its "naive" stabilizer $\mathrm{NStab}(x)$ as the set of all $g \in G$ which preserve x. If k is not algebraically closed, the residue field $k(x)$ may be a nontrivial finite extension of k, and G may act on it nontrivially; we define the "honest" stabilizer $\mathrm{Stab}(x)$ as the set of all $g \in \mathrm{NStab}(x)$ that act on $k(x)$ trivially. In other words, $\mathrm{Stab}(x)$ is the stabilizer of x in the scheme-theoretic sense; cf. [MFK, Definition 0.4]. Both $\mathrm{NStab}(x)$ and $\mathrm{Stab}(x)$ are subgroups of G; they are sometimes called the *decomposition group* and the *inertia group* respectively.

If $\overline{k}$ is the algebraic closure of k, then x is represented by a set of "conjugate" points of the variety $X_{\overline{k}} = X \otimes_k \overline{k}$ ("geometric points" of X), one for each embedding $k(x) \hookrightarrow \overline{k}$. The "honest" stabilizer $\mathrm{Stab}(x)$ fixes each

of these points while $\mathrm{NStab}(x)$ permutes them. The case where $\mathrm{Stab}(x)$ is strictly smaller than $\mathrm{NStab}(x)$, is quite common. For example, consider the action of $G = \mathbb{Z}/2\mathbb{Z}$ on the affine line $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[t]$: the nontrivial element of this group acts by $t \mapsto -t$. For any $x \in \mathbb{A}^1_{\mathbb{Q}} - \{0\}$, $\mathrm{Stab}(x) = \{1\}$. On the other hand, $\mathrm{NStab}(x) = G$ iff x corresponds to the ideal in $\mathbb{Q}[t]$ generated by an irreducible polynomial of the form $q(t^2)$ (e.g., $t^2 + 1$).

This phenomenon is entirely arithmetic; we are concerned with it here because the symmetric polynomials $W_1, \dots, W_N$ in Theorem 1.1 are asserted to have integer coefficients. A reader who is only interested in the existence of such polynomials in $\mathbb{C}[x_1, \dots, x_n]$ may skip the rest of this section and assume that "naive" stabilizers always coincide with "honest" ones in the sequel.

Semi-linear representations and skew group rings. Let X be a G-variety and let $x \in X$. The "naive stabilizer" $\mathrm{NStab}(x)$ acts upon $T_x(X)^* = \mathfrak{m}_x/\mathfrak{m}_x^2$. However, if $\mathrm{NStab}(x)$ is strictly larger than $\mathrm{Stab}(x)$ then this action is not linear over $k(x)$ but rather "semi-linear" in the following sense.

Definition 2.1. Suppose a finite group H acts by automorphisms on a field K. A *semi-linear representation* of H over K is a K-vector space V with a K^H-linear action of H on V having the property $g(\lambda v) = g(\lambda)g(v)$ for any $g \in H$, $\lambda \in K$ and $v \in V$.

For the rest of this section we shall assume that K is a field, K^* is the multiplicative group of K, H is a finite group acting on K by automorphisms, and H' is the kernel of this action. In the subsequent applications we will take $K = k(x)$, $H = \mathrm{NStab}(x)$ and $H' = \mathrm{Stab}(x)$.

Recall that skew group algebra $K * H$ is generated by K and H with the relations $ha = h(a)h$ for any $h \in H$ and $a \in K$; alternatively, it is the space of all formal sums $\sum_{h \in H} a_h h$ where the coefficients a_h are elements of K and multiplication is defined by the formula $(\sum_{h \in H} a_h h)(\sum_{h \in H} b_h h) = \sum_{h \in H} c_h h$ where $c_h = \sum_{h_1 h_2 = h} a_{h_1} \cdot h_1(b_{h_2})$.

It is immediate from Definition 2.1 that a semi-linear representation of H is the same thing as a $(K * H)$-module. (All modules in this paper are understood to be left modules.)

Remark 2.2. Note that $V = K$ has a natural structure of a $(K * H)$-module. This module contains a vector $1 \in K$ which is fixed by H.

Recall that by the Wedderburn Structure Theorem every semisimple ring R is a direct product of simple rings, called *the simple components* of R; see, e.g., [Bo, Theorem VIII.5.1].

Lemma 2.3. *$K * H$ is a semisimple K^H-algebra with at most $|H'|$ simple components. (Here H' is the kernel of the H-action on K, as above.)*

Proof. Semisimplicity of $K * H$ is proved by the same averaging argument as the usual Maschke's theorem; for details see, e.g., [Mont, Theorem 0.1 and Corollary 0.2].

Denote the simple components of $K*H$ by $S_1, \ldots, S_m$. Then $Z(K*H) = Z(S_1) \times \cdots \times Z(S_m)$, where $Z(A)$ denotes the center of A. Here m is the number of simple components of $Z(K*H)$; we will prove that $m \leq |H'|$ by showing that $Z(K*H)$ is a direct product of $\leq |H'|$ fields. It is easy to see that

$$Z(K*H) \subset \Big\{ \sum_{h' \in H'} a_{h'} h' \;\Big|\; a_{h'} \in K \Big\} = K[H'] \,,$$

where $Z(A)$ denotes the center of the ring A, and the (usual) group algebra $K[H']$ is naturally embedded in $K*H$. (Indeed, for every $s \in K*H - K[H']$ we can choose $a \in K$ so that $sa \neq as$.) Consequently,

$$Z(K*H) \subset Z(K[H']) \,. \tag{2.1}$$

Recall that the group ring $K[H']$ is a semisimple K-algebra of dimension $|H'|$. Thus $K[H']$ has $\leq |H'|$ simple components and, consequently, $Z(K[H'])$ is a direct product of $\leq |H'|$ fields. Since $Z(K*H)$ is a direct product of m fields, it has m mutually orthogonal idempotents and, in view of (2.1), so does $Z(K[H'])$. This is only possible if $m \leq |H'|$, as claimed. □

We are now ready to describe the particular type of skew group rings we will encounter in the sequel.

Proposition 2.4. *Suppose that H' is an abelian group of exponent e, K contains a primitive eth root of unity and $\chi_1, ..., \chi_m$ is a set of generators for the dual group $(H')^* = \mathrm{Hom}(H', K^*)$. Assume further that for each i there is a one-dimensinal semi-linear representation V_i of H over K such that $h'(v) = \chi_i(h')v$ for every $v \in V_i$ and every $h' \in H'$. Then:*

(a) *For every $\chi \in (H')^*$, there exists a unique semi-linear representation V_χ of H such that $\dim_K(V_\chi) = 1$ and $h'(v) = \chi(h')v$ for every $v \in V_\chi$ and every $h' \in H'$.*
(b) *Every simple $(K*H)$-module is isomorphic to V_χ for some $\chi \in (H')^*$.*

Proof. Note that if V_1 and V_2 are semi-linear representations of H over K then so is $V_1 \otimes_K V_2$. Indeed,

$$h(\lambda v_1) \otimes h(v_2) = h(\lambda) \cdot (h(v_1) \otimes h(v_2)) = h(v_1) \otimes h(\lambda v_2) \,.$$

We can thus construct V_χ as follows. Write $\chi \in (H')^*$ as $\chi = \chi_1^{l_1} \ldots \chi_m^{l_m}$ for some nonnegative integers $l_1, \ldots, l_m$. Now set

$$V_\chi = V_1^{\otimes l_1} \otimes_K \cdots \otimes_K V_m^{\otimes l_m} \,.$$

The subgroup H' acts on V_χ by the character χ, as desired. By our construction $\dim_K(V_\chi) = 1$; consequently, V_χ is an simple $(K*H)$-module. Note that the $(K*H)$-modules V_χ are pairwise nonisomorphic because H' acts on them by different characters.

The isomorphism classes of simple $(K*H)$-modules are in 1—1 correspondence with the simple components of $K*H$; see [Bo, Proposition VIII.5.11].

Thus Lemma 2.3 implies that $K * H$ has $\leq |H'|$ nonisomorphic simple modules. On the other hand, we have constructed $|H'|$ nonisomorphic simple modules V_χ. This proves (b) and the uniqueness of V_χ in (a). □

Remark 2.5. One can show that, under the assumptions of Proposition 2.4, H is a semidirect product of H' and H/H', where the the action of H/H' on H' is given by embedding H' into $(K^*)^m$ via $h' \longrightarrow (\chi_1(h'), \dots, \chi_m(h'))$.

3. Reduction to an algebro-geometric problem

Reduction 3.1. *In the setting of Theorem 1.1 it is enough to construct symmetric polynomials $W_1, \dots, W_N$ with rational coefficients so that* (1.1) *is satisfied.*

Indeed, after multiplying $W_1, \dots, W_N \in \mathbb{Q}[x_1, \dots, x_n]$ by a suitable integer constant, we will obtain a desired collection of symmetric polynomials with integer coefficients. □

Reduction 3.2. *In the setting of Theorem 1.1 it is enough to choose $W_1, \dots, W_N$ so that inequality* (1.1) *holds for $i = 1$ only.*

To see this, note that the natural S_n-action transitively permutes the differentials dx_i/x_i and leaves the differentials dW_j/W_j invariant. Thus if inequality (1.1) holds for $i = 1$ then it holds for every i (with the same M and $W_1, \dots, W_N$). □

Next we would like to restate (1.1) as an inequality involving densely defined functions on the tangent bundle of $\mathbb{P}^{n-1}_{\mathbb{C}}$ rather than the tangent bundle of $\mathbb{C}^n$. The reason this will be helpful in the sequel is that $\mathbb{P}^{n-1}_{\mathbb{C}}$ is compact in the metric topology; this will allow us to pass from local to global estimates. Note that inequality (1.1) cannot be "projectivised" directly because dP/P is not a pullback of any differential on $\mathbb{P}^{n-1}_{\mathbb{C}}$, even if P is a homogeneous polynomial in $x_1, \dots, x_n$. On the other hand, if P and Q are homogeneous polynomials of the same degree then the difference $dP/P - dQ/Q$ equals dr/r, where $r = P/Q$ is a well-defined rational function on $\mathbb{P}^{n-1}_{\mathbb{C}}$. If we choose Q to be symmetric, then for the purposes of proving Theorem 1.1, replacing dP/P by dr/r will not cause any harm. The argument of Reduction 3.4 below makes this observation precise.

Proposition 3.3. *Let X be a projective G-variety over a field $k \subset \mathbb{C}$, and let f be a rational function on X. Then there exist G-invariant rational functions $\beta_1, \dots, \beta_m$ on X and a constant $K > 0$ such that*

$$\left|\frac{df}{f}(p, v)\right| \leq K \max_{j=1,\dots,m} \left|\frac{d\beta_j}{\beta_j}(p, v)\right| \tag{3.1}$$

for any $(p, v) \in T(X_{\mathbb{C}})$ such that p is a smooth point of $X_{\mathbb{C}}$ and does not lie on the divisors of $f, \beta_1, \dots, \beta_m$.

Reduction 3.4. *Proposition 3.3 $\Longrightarrow$ Theorem 1.1.*

Indeed, apply Proposition 3.3 to the case where $X = \mathbb{P}^{n-1}_{\mathbb{Q}}$ (so that $k = \mathbb{Q}$), $G = S_n$ acts on $\mathbb{P}^{n-1}_{\mathbb{Q}}$ by permutations of the homogeneous coordinates $x_1, \ldots, x_n$, $f = x_1/s_1$ and $s_1 = x_1 + \cdots + x_n$. Write each β_j as a quotient of two homogeneous polynomials (of the same degree) in $x_1, \ldots, x_n$:

$$\beta_1 = \frac{W_1}{W_2}, \ldots, \beta_m = \frac{W_{2m-1}}{W_{2m}}\,.$$

We claim that the polynomials $W_1, \ldots, W_{2m}, W_{2m+1} \stackrel{\text{def}}{=} s_1$ have the property asserted in Theorem 1.1. Indeed, since $df/f = dx_1/x_1 - ds_1/s_1$ and $d\beta_i/\beta_i = dW_{2i-1}/W_{2i-1} - dW_{2i}/W_{2i}$, inequality (3.1) translates into

$$\begin{aligned}\left|\left(\frac{dx_1}{x_1} - \frac{ds_1}{s_1}\right)(p,v)\right| &\le K \max_{i=1,\ldots,m}\left|\left(\frac{dW_{2i-1}}{W_{2i-1}} - \frac{dW_{2i}}{W_{2i}}\right)(p,v)\right| \\ &\le 2K \max_{j=1,\ldots,2m}\left|\frac{dW_j}{W_j}(p,v)\right|\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\frac{dx_1}{x_1}(p,v)\right| &\le 2K \max_{j=1,\ldots,2m}\left|\frac{dW_j}{W_j}(p,v)\right| + \left|\frac{ds_1}{s_1}(p,v)\right| \\ &\le (2K+1) \max_{j=1,\ldots,2m+1}\left|\frac{dW_j}{W_j}(p,v)\right|\,.\end{aligned}$$

This means that (1.1) holds for $i = 1$, with $M = 2K + 1$ and $N = 2m + 1$. In view of Reduction 3.2 this proves Theorem 1.1. □

Reduction 3.5. Note that the statement of Proposition 3.3 involves only rational functions on X. This suggests that in the course of proving Proposition 3.3 we may replace X by a birationally equivalent G-variety. More precisely,

Suppose X and X' are birationally isomorphic G-varieties over $k \subset \mathbb{C}$. If Proposition 3.3 holds for X and $f \in k(X)$ then it holds for X' and the same $f \in k(X') = k(X)$.

Indeed, X and X' have isomorphic Zariski-open subsets U and U'. After passing to smaller subsets if necessary, we may assume that U and U' are smooth and do not intersect the divisors of $f, \beta_1, \ldots, \beta_n$ on X and X' respectively. Thus if inequality (3.1) holds for every (p, v) such that $p \in U_{\mathbb{C}}$ and $v \in T_p(X_{\mathbb{C}})$ then

$$\left|\frac{df}{f}(p', v')\right| \le K \max_{j=1,\ldots,m}\left|\frac{d\beta_j}{\beta_j}(p', v')\right|$$

for every (p', v') such that $p' \in U'_{\mathbb{C}}$ and $v' \in T_{p'}(X'_{\mathbb{C}})$. The subset $U'_{\mathbb{C}}$ is dense in $X'_{\mathbb{C}}$ with respect to metric topology; hence, by continuity the same inequality (with the same β_j and the same K) holds for every $(p', v') \in T(X'_{\mathbb{C}})$ such that p' is a smooth point of $X'_{\mathbb{C}}$ and does not lie in the union of divisors of $f, \beta_1, \ldots, \beta_m$. This means that Proposition 3.3 holds for X' as claimed. □

Definition 3.6. Let X be a projective G-variety and f be a rational function on X. We shall say that the pair (X, f) has property (*) if there is a Zariski open covering $X = \bigcup_i U_i$ and, for each i, rational functions $\beta_{i1}, \dots, \beta_{i,q_i} \in k(X)^G$ and regular functions $\gamma_{i1}, \dots, \gamma_{i,q_i} \in \mathcal{O}_X(U_i)$ such that

$$\frac{df}{f} = \gamma_{i1}\frac{d\beta_{i1}}{\beta_{i1}} + \dots \gamma_{i,q_i}\frac{d\beta_{i,q_i}}{\beta_{i,q_i}} \,. \tag{3.2}$$

In other words, the pair (X, f) has property (*) if df/f is a global section of the sheaf of differentials on X generated over $\mathcal{O}_X$ by $d\beta/\beta$, as β ranges over some finite subset of $k(X)^G$ (or, equivalently, as β ranges over all of $k(X)^G$).

Reduction 3.7. *Proposition 3.3 holds, assuming the pair (X, f) that appears there, has property (*).*

Indeed, we have an open covering $X_{\mathbb{C}} = \bigcup_i U_{i,\mathbb{C}}$ (with respect to both Zariski and metric topology), and the equality (3.2) holds in $U_{i,\mathbb{C}}$ where γ_{ij} can be viewed as continuous functions on $U_{i,\mathbb{C}}$ with respect to metric topology.

It follows that any point $x \in X$ (whether smooth or not) has a neighborhood U_x (in the metric topology) such that $U_x \subset U_{i_x,\mathbb{C}}$ for some i_x, and

$$\left|\frac{df}{f}(p, v)\right| \le K_x \max_{j=1,\dots,q_{i_x}} \left|\frac{d\beta_{i_x,j}}{\beta_{i_x,j}}(p, v)\right|$$

whenever $v \in T_p(X)$ and p is a smooth point of U_x which does not lie on the divisors of $f, \beta_{i_x,1}, \dots, \beta_{i_x,q_{i_x}}$. The open sets U_x form a cover of X. Since X compact in the metric topology, we can choose a finite subcover $U_{x_1}, \dots, U_{x_r}$. Now if $K > K_{x_1}, \dots, K_{x_r}$ then

$$\left|\frac{df}{f}(p, v)\right| \le K \max_{i,j} \left|\frac{d\beta_{ij}}{\beta_{ij}}(p, v)\right| \,.$$

This shows that Proposition 3.3 holds. □

We have thus shown that Theorem 1.1 is a consequence of the following, purely algebraic statement (see Reductions 3.4, 3.5 and 3.7).

Proposition 3.8. *Let G be a finite group, X a projective G-variety, and $f \in k(X)$. Then there exists a (regular) birational morphism $\pi\colon X' \longrightarrow X$ of G-varieties such that the pair $(X', \pi^*(f))$ has property (*) (see Definition 3.6).*

A proof of Proposition 3.8 (and thus of Theorem 1.1) will be given in the next section. The idea of the proof can be seen from the simplest example where $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$, k contains the mth roots of unity, $G = \mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{A}^1$ linearly by a faithful character, and $f = t$. In this case we can take $\beta = t^m \in k(X)^G$, and the equality $df/f = \frac{1}{m}d\beta/\beta$ shows that (X, f) has property (*).

4. G-VARIETIES IN STANDARD FORM

The results of [RY].

Definition 4.1. ([RY, Definition 3.1]) We say that a generically free G-variety X is *in standard form with respect to a divisor* Y if

(a) X is smooth and Y is a normal crossing divisor on X,
(b) the G-action on $X - Y$ is free, and
(c) for every $g \in G$ and for every irreducible component Y_0 of Y either $g(Y_0) = Y_0$ or $g(Y_0) \cap Y_0 = \emptyset$.

Theorem 4.2. *(*[RY, Corollary 3.6]*) Let X be a G-variety and $Y \subset X$ be a Zariski closed G-invariant subvariety such that the action of G on $X - Y$ is free. Then there is a sequence of blowups*

$$\pi\colon X_n \xrightarrow{\pi_n} X_{n-1} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X \tag{4.1}$$

with smooth G-invariant centers $C_i \subset X_i$ such that X_n is in standard form with respect to a divisor $\tilde{Y}$ containing $\pi^{-1}(Y)$.

Theorem 4.3. *Let X be a G-variety in standard form with respect to a divisor Y, let x be a point of X, let $Y_1, \dots, Y_m$ be the irreducible components of Y passing though x, and let $W = Y_1 \cap \cdots \cap Y_m$. Then*

(a) *(*[RY, Theorem 4.1]*)* $\mathrm{Stab}(x)$ *is commutative.*
(b) *(*[RY, Remark 4.4]*) The action of* $\mathrm{Stab}(x)$ *in the normal space to W at x is faithful and decomposes into the sum of one-dimensional representations as follows:*

$$T_x(X)/T_x(W) = \bigoplus_{i=1}^{m} \frac{T_x(Y_1) \cap \cdots \cap \widehat{T_x(Y_i)} \cap \cdots \cap T_x(Y_m)}{T_x(W)} \,. \tag{4.2}$$

(c) *(*[RY, Remark 4.5]*) Let e be the exponent of* $\mathrm{Stab}(x)$*. Then the residue field $k(x)$ of x contains the eth roots of unity.*

Remark 4.4. Under the assumptions of Theorem 4.3, set $H = \mathrm{NStab}(x)$ and $H' = \mathrm{Stab}(x)$. Recall that H acts on $T_x(X)$ semi-linearly; see Definition 2.1. Property (c) of Definition 4.1 implies that Y_i is preserved by the action of H, and hence, the subspace $T_x(Y_i)$ is H-invariant for each i. It follows that all spaces appearing in (4.2), are $(k(x) * H)$-modules.

The conormal space $(T_x(X)/T_x(Y_i))^*$ is dual to the ith summand in (4.2); it is a $(k(x)*H)$-module of dimension 1 over $k(x)$, and H' acts on it by a character. Denote this character by ξ_i. Theorem 4.3(b) implies that the characters $\xi_1, \dots, \xi_m$ generate the dual group $(H')^*$. In view of Theorem 4.3(c), this shows that Proposition 2.4 applies in this setting.

A local coordinate system. Suppose that X is an algebraic variety and x is a point of X. Recall that $u_1, \dots, u_n \in \mathfrak{m}_x$ are said to form a local coordinate system on X at x if their classes modulo $\mathfrak{m}_x^2$ form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $k(x)$-vector space.

Proposition 4.5. *Let X be a quasiprojective G-variety in standard form with respect to a divisor Y. Suppose $Y_1, \dots, Y_m$ are the irreducible components of Y passing though a point x of X. Then there exists a local coordinate system $u_1, \dots, u_m, v_1, \dots, v_l$ at x with the following properties:*

(a) *Let $\overline{u}_i = u_i \bmod \mathfrak{m}_x^2$ and $\overline{v}_j = v_j \bmod \mathfrak{m}_x^2$. Then each of $\overline{u}_1, \dots, \overline{u}_m, \overline{v}_1, \dots, \overline{v}_l$ generates a one-dimensional H-invariant $k(x)$-subspace of $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

(b) *For every $i = 1, \dots, m$ and every $g \in G$, u_i is a local equation of $g(Y_i)$ at gx.*

(c) *For every $j = 1, \dots, l$ there are integers $e_{j1}, \dots, e_{jm} \geq 0$ such that $h_j = u_1^{e_{j1}} \dots u_m^{e_{jm}} v_j$ is a G-invariant rational function on X.*

Proof. We begin by constructing $u_1, \dots, u_m$. Consider the divisor $D_i = \sum g(Y_i)$, where each summand of the form $g(Y_i)$ (for some $g \in G$) appears in this sum exactly once. Since X is quasiprojective, the divisor D_i can be "moved off" any finite set of points (see [Sh, Theorem III.1.1]); in particular it can be "moved off" the orbit Gx. In other words, for every $i = 1, \dots, m$ there is a rational function u_i on X such that the support of the divisor $D_i - (u_i)$ does not intersect Gx. It is now easy to see that $u_1, \dots, u_m$ satisfy (a) and (b).

Next we turn to the construction of $v_1, \dots, v_l$. Each $\overline{u}_i$ generates a one-dimensional $k(x)$-subspace $<\overline{u}_i> = (T_x(X)/T_x(Y_i))^* \subset \mathfrak{m}_x/\mathfrak{m}_x^2$. We have seen in Remark 4.4 that $<\overline{u}_i>$ is H-invariant; H' acts on it by the character ξ_i. In view of Lemma 2.3 and Proposition 2.4, $\mathfrak{m}_x/\mathfrak{m}_x^2$ can be decomposed into a direct sum of $(k(x) * H)$-submodules which are one-dimensional over $k(x)$ and include $<\overline{u}_1>, \dots, <\overline{u}_m>$. Let $\overline{v}_1, \dots, \overline{v}_l \in \mathfrak{m}_x/\mathfrak{m}_x^2$ be generators of the other submodules of this decomposition. Denote the character of H' associated to $\overline{v}_j$ by η_j.

By Remark 4.4, $\xi_1, \dots, \xi_m$ generate the dual group $(H')^*$; consequently, each η_j can be written in the form

$$\eta_j = \xi_1^{-e_{j1}} \dots \xi_m^{-e_{jm}}$$

for some integers $e_{j1}, \dots, e_{jm}$. We may assume without loss of generality that each $e_{ji} \geq 0$. Note that $\overline{h}_j = \overline{u}_1^{e_{j1}} \dots \overline{u}_m^{e_{jm}} \overline{v}_j$ is an H'-invariant element of $\mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$, where $d_j = e_{j1} + \dots + e_{jm} + 1$. Clearly, $\overline{h}_j$ generates an H-invariant one-dimensional $k(x)$-subspace $<\overline{h}_j> \subset \mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$ on which H' acts trivially. By the uniqueness statement in Proposition 2.4(a), $<\overline{h}_j> \cong k(x)$ as $(k(x) * H)$-modules, and by Remark 2.2, after replacing $\overline{v}_j$ by $\lambda \overline{v}_j$ for some $\lambda \in k(x)$, we may assume

$$\overline{h}_j = \overline{u}_1^{e_{j1}} \dots \overline{u}_m^{e_{jm}} \overline{v}_j \text{ is an } H\text{-invariant element of } \mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}. \tag{4.3}$$

We claim that we can choose $v_1, \dots, v_l \in \mathfrak{m}_x$ so that $\overline{v}_j = v_j \bmod \mathfrak{m}_x^2$ and each $h_j = u_1^{e_{j1}} \dots u_m^{e_{jm}} v_j$ is G-invariant. If we can do this, then $u_1, \dots, u_m, v_1, \dots, v_l$ will clearly satisfy the requirements of the proposition.

To prove the claim, let $R = \bigcap_{y\in Gx} \mathcal{O}_{y,X}$ be the ring of rational functions on X that are well-defined on Gx and let $\mathcal{I}_{Gx} = \bigcap_{y\in Gx} \mathfrak{m}_y$ be a G-invariant ideal in R consisting of all elements that vanish on Gx. By our choice of $u_1, \ldots, u_m$,

$$g^*(u_i)/u_i \text{ is defined and invertible at every point of } Gx \tag{4.4}$$

for every $g \in G$, and every $i = 1, \ldots, m$. Consequently, $u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathcal{I}_{Gx}$ is a G-invariant ideal of R.

For each $y \in Gx$, the functions $u_1, \ldots, u_m$ vanish at y, and hence, $u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathcal{I}_{Gx} \subset \mathfrak{m}_y^{d_j}$. Consider the G-equivariant projection map

$$\psi\colon u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathcal{I}_{Gx} \longrightarrow \bigoplus_{y\in Gx} \mathfrak{m}_y^{d_j}/\mathfrak{m}_y^{d_j+1} .$$

If $v \in \mathcal{I}_{Gx}$ then $\psi(u_1^{e_{j1}} \ldots u_m^{e_{jm}} v)$ depends only on the image of v in $\bigoplus_{y\in Gx} \mathfrak{m}_y/\mathfrak{m}_y^2$. Moreover, since X is quasiprojective, the finite set Gx lies in an affine open subset of X and hence, by the Chinese Remainder Theorem, the projection map $\mathcal{I}_{Gx} \longrightarrow \bigoplus_{y\in Gx} \mathfrak{m}_y/\mathfrak{m}_y^2$ is surjective. Thus

$$\mathrm{Im}(\psi) = \bigoplus_{y\in Gx} \left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_y\right) \bmod \mathfrak{m}_y^{d_j+1} \subset \bigoplus_{y\in Gx} \mathfrak{m}_y^{d_j}/\mathfrak{m}_y^{d_j+1} .$$

We shall denote elements of $\mathrm{Im}(\psi)$ by $a = (a_y \mid y \in Gx)$, where $a_y \in \left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_y\right) \bmod \mathfrak{m}_y^{d_j+1}$. Recall that by (4.3), $\overline{h}_j$ is a nonzero H-invariant element of $\left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_x\right) \bmod \mathfrak{m}_x^{d_j+1}$. Let a_j be the element of $\mathrm{Im}(\psi)$ such that $(a_j)_y = (g^{-1})^*(\overline{h}_j)$, where $y = gx$; in view of (4.4), we have here

$$(g^{-1})^*[\left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_x\right) \bmod \mathfrak{m}_x^{d_j+1}] = \left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_y\right) \bmod \mathfrak{m}_y^{d_j+1} ,$$

and hence, $(a_j)_y \in \left(u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathfrak{m}_y\right) \bmod \mathfrak{m}_y^{d_j+1}$. Note that since $\overline{h}_j$ is H-invariant, $(a_j)_y$ is independent of the choice of g. By our construction a_j is G-invariant and $(a_j)_x = \overline{h}_j \in \mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$.

Recall that in characteristic 0, every morphism of (not necessarily finite-dimensional) G-vector spaces over k splits. Thus ψ has a G-equivariant k-linear splitting and consequently, there exists a G-invariant element $h_j = u_1^{e_{j1}} \ldots u_m^{e_{jm}} v_j \in u_1^{e_{j1}} \ldots u_m^{e_{jm}} \mathcal{I}_{Gx}$ such that $\psi(h_j) = a_j$. In particular, $\overline{h}_j = \overline{u}_1^{e_{j1}} \ldots \overline{u}_m^{e_{jm}} \overline{v}_j = h_j \bmod \mathfrak{m}_x^{d_j+1}$ and hence $\overline{v}_j = v_j \bmod \mathfrak{m}_x^2$. This proves the claim and thus shows that $u_1, \ldots, u_m, v_1, \ldots, v_l$ have the required properties. □

Property (*). We are now ready to examine property (*) of Definition 3.6 more closely.

Lemma 4.6. *Suppose X be a quasiprojective G-variety in standard form with respect to a divisor Y, $x \in X$, $u_1, \ldots, u_m$, $v_1, \ldots, v_l$, and $h_1, \ldots, h_l$, are*

as in Proposition 4.5, and $w_i = \prod_{g\in G} g^*(u_i)$. *Let* f *be a rational function on* X *whose divisor is supported on* Y. *Then*

$$\frac{df}{f} \in \mathcal{M} ,$$

where $\mathcal{M}$ *is the* $\mathcal{O}_{x,X}$*-module generated by* dw_i/w_i *and* dh_j/h_j *with* $i = 1,\dots,m$ *and* $j = 1,\dots,l$.

Note that w_i and h_j are G-invariant rational functions on X for every $i = 1,\dots,m$ and $j = 1,\dots,l$.

Proof. Let $(\Omega^1_X)_x$ be the $\mathcal{O}_{x,X}$-module of germs at x of regular differential forms on X. Since X is smooth, $(\Omega^1_X)_x$ is a free $\mathcal{O}_{x,X}$-module generated by $du_1,\dots,du_m, dv_1,\dots,dv_l$. Let $\mathcal{M}'$ be the $\mathcal{O}_{x,X}$-module generated by du_i/u_i, and dv_j/v_j, where $i = 1,\dots,m$ and $j = 1,\dots,l$. Clearly, $(\Omega^1_X)_x \subset \mathfrak{m}_x\mathcal{M}'$.

Since the divisor of f is supported on Y, locally near x it is a union of smooth hypersurfaces of the form $\{u_i = 0\}$. This means that $f = au_1^{e_1}\dots u_m^{e_m}$ for some $a \in \mathcal{O}_{x,X} - \mathfrak{m}_x$ and $e_1,\dots,e_m \geq 0$ and, hence,

$$\frac{df}{f} = \frac{da}{a} + e_1\frac{du_1}{u_1} + \dots + e_m\frac{du_m}{u_m} .$$

Since $da/a \in (\Omega^1_X)_x$, we conclude that df/f is an $\mathcal{O}_{x,X}$-linear combination of du_i/u_i and dv_j; in particular, $df/f \in \mathcal{M}'$.

We claim that $\mathcal{M} = \mathcal{M}'$; this would imply that $df/f \in \mathcal{M}$, as desired. It is clear that $\mathcal{M} \subset \mathcal{M}'$. It remains to prove the opposite inclusion, i.e., it remains to prove that dw_i/w_i and dh_j/h_j generate $\mathcal{M}'$, as i ranges from 1 to m and j ranges from 1 to l.

Note that by our choice of $u_1,\dots,u_m$, we can write $w_i = a_i u_i^{|G|}$ for some $a_i \in \mathcal{O}_{X,x} - \mathfrak{m}_x$; see (4.4). Thus

$$\frac{dw_i}{w_i} = \frac{da_i}{a_i} + |G|\frac{du_i}{u_i} \equiv |G|\,\frac{du_i}{u_i} \pmod{(\Omega^1_X)_x} .$$

In particular, since $(\Omega^1_X)_x \subset \mathfrak{m}_x\mathcal{M}'$, we conclude that

$$\frac{dw_i}{w_i} \equiv |G|\,\frac{du_i}{u_i} \pmod{\mathfrak{m}_x\mathcal{M}'} . \tag{4.5}$$

On the other hand, since $h_j = u_1^{e_{j1}}\dots u_m^{e_{jm}}v_j$, we have

$$\frac{dh_j}{h_j} = \frac{dv_j}{v_j} + \sum_i e_{ji}\frac{du_i}{u_i} . \tag{4.6}$$

Examining (4.5) and (4.6), we see that dw_i/w_i $(i = 1,\dots,m)$, and dh_j/h_j $(j = 1,\dots,l)$ generate $\mathcal{M}'/\mathfrak{m}_x\mathcal{M}'$. Consequently, by Nakayama's lemma these elements generate $\mathcal{M}'$. Thus $\mathcal{M}' = \mathcal{M}$, as claimed. □

Proposition 4.7. *Let* X *be a projective variety in standard form with respect to a divisor* Y *and let* f *be a rational function on* X *whose divisor is supported on* Y. *Then the pair* (X, f) *has property (*); see Definition 3.6.*

Proof. By Zariski compactness, it is enough to show that for any $x \in X$, there exist $\beta_1, \ldots, \beta_q \in k(X)^G$ and $\gamma_1, \ldots, \gamma_q \in \mathcal{O}_{x,X}$ such that

$$\frac{df}{f} = \gamma_1 \frac{d\beta_1}{\beta_1} + \ldots \gamma_q \frac{d\beta_q}{\beta_q} .$$

In other words, df/f lies in the $\mathcal{O}_{x,X}$-module generated by $d\beta/\beta$, as β ranges over some finite subset of $k(X)^G$. The last assertion is immediate from Lemma 4.6. □

Proof of Theorem 1.1. As we showed in Section 3, Theorem 1.1 follows from Proposition 3.8. We thus proceed with the proof of Proposition 3.8,

Let X be a projective G-variety and $f \in k(X)$. Let X_0 be the subvariety of all points in X with nontrivial stabilizers. Let Y be the union of X_0 and (the supports of) the divisors of $g^*(f)$ for every $g \in G$; it is a G-invariant Zariski closed subvariety of X. By Theorem 4.2 there exists a birational morphism $\pi\colon X' \longrightarrow X$ and a divisor $Y' \subset Y$, such that X' is in standard form with respect to Y' and $\pi^{-1}(Y) \subset Y'$. Note that the divisor of $\pi^*(f)$ is contained in $\pi^{-1}(Y)$ and thus in Y'.

Proposition 3.8 (and thus Theorem 1.1) now follows from Proposition 4.7, which asserts that the pair $(X', \pi^*(f))$ has property (*). □

References

[Bo] N. Bourbaki, Algèbre, Hermann, Paris.

[C] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proceedings of Symposia in Pure Mathematics, 36 (1980), 91–146.

[Ma] J. Mather, *Stratifications and mappings*, in "Dynamical systems", Proc. Sympos., Univ. Bahia, Salvador 1971, edited by M. M. Peixoto, Academic Press, 1973, 195–232.

[Mont] S. Montgomery, *Fixed points of finite automorphism groups of associative rings*, Lect. Notes in Math. **818**, Springer-Verlag, 1980.

[Most] T. Mostowski. *Lipschitz equisingularity*, Dissertationes Math. 243 (1985).

[MFK] D. Mumford, J. Fogarty and F. Kirwan. Geometric invariant theory. Third enlarged edition, Springer, 1994.

[Oh] T. Ohsawa, *On the L^2 cohomology of complex spaces*, Math. Z. 209 (1992), 519–530.

[Pa] A. Parusiński, *Lipschitz properties of semianalytic sets*, Ann. Inst. Fourier, Grenoble 38 (1988), 189–213.

[RY] Z. Reichstein, B. Youssin, *Essential dimensions of algebraic groups and a resolution theorem for G-varieties*, with an appendix by J. Kollár and E. Szabó, preprint. Available at http://ucs.orst.edu/~reichstz/pub.html.

[Sh] I. R. Shafarevich, Basic algebraic geometry, Springer-Verlag, Heidelberg, 1974.

[SS] L. Siebenmann and D. Sullivan, *On complexes that are Lipschitz manifolds*, in "Geometric Topology", edited by J. C. Cantrell, p. 503–525, Academic Press, 1979.

[Wh] H. Whitney, *Tangents to an analytic variety*, Annals of Math., 81 (1964), 496–549.

[Y] B. Youssin, *L^p cohomology of cones and horns*, J. Diff. Geom. 39 (1994), 559–603.

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