Rational Hypergeometric Functions

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Abstract

Multivariate hypergeometric functions associated with toric varieties were introduced by Gel'fand, Kapranov and Zelevinsky. Singularities of such functions are discriminants, that is, divisors projectively dual to torus orbit closures. We show that most of these potential denominators never appear in rational hypergeometric functions. We conjecture that the denominator of any rational hypergeometric function is a product of resultants, that is, a product of special discriminants arising from Cayley configurations. This conjecture is proved for toric hypersurfaces and for toric varieties of dimension at most three. Toric residues are applied to show that every toric resultant appears in the denominator of some rational hypergeometric function.

1 Introduction

Which rational functions in n variables are hypergeometric functions? Which denominators appear in such rational hypergeometric functions? Our aim is to answer these questions for the multivariate hypergeometric functions introduced by Gel'fand, Kapranov and Zelevinsky [8, 9, 16]. These functions are defined by a system of linear partial differential equations, associated to any integer $d \times s$-matrix $A = (a_{ij})$ and any complex vector $\beta \in \mathbb{C}^d$:

Definition 1.1. The *A-hypergeometric system of degree $\beta \in \mathbb{C}^d$* is the left ideal $H_A(\beta)$ in the Weyl algebra $\mathbb{C}\langle x_1, \ldots, x_s, \partial_1, \ldots, \partial_s\rangle$ generated by

$$\text{the toric operators} \quad \partial^u - \partial^v \quad \text{for} \quad u, v \in \mathbb{N}^s \quad \text{with} \quad A \cdot u = A \cdot v, \tag{1.1}$$

$$\text{and the Euler operators} \quad \sum_{j=1}^{s} a_{ij} x_j \partial_j - \beta_i \quad \text{for} \quad i = 1, \ldots, d. \tag{1.2}$$

A function $f(x_1,\ldots,x_s)$, holomorphic on an open set $U \subset \mathbb{C}^s$, is said to be A-hypergeometric of degree β if it is annihilated by the left ideal $H_A(\beta)$.

Throughout this paper we use the multi-exponent notation $\partial^u = \prod_{j=1}^{s} \partial_j^{u_j}$. We shall assume that the rank of the matrix A equals d, the column vectors a_j of A are distinct, and the vector $(1,1,\ldots,1)$ lies in the row span of A.

These hypotheses greatly simplify our exposition, but our main results remain valid without them. The last hypothesis means that the toric ideal

$$I_A \;:=\; \langle\, \xi^u - \xi^v \;:\; A\cdot u = A\cdot v \,\rangle \quad\subset\quad \mathbb{C}[\xi_1,\ldots,\xi_s]$$

is homogeneous with respect to total degree and defines a projective toric variety $X_A \subset \mathbb{P}^{s-1}$, and the columns of A represent a configuration $\{a_1,\ldots,a_s\}$ of s distinct points in affine $(d-1)$-space. This condition ensures that the system $H_A(\beta)$ has only regular singularities ([8], [16, Theorem 2.4.11]). A detailed analysis of the non-regular case was carried out by Adolphson [1].

The system $H_A(\beta)$ is always holonomic. Its holonomic rank $r_A(\beta)$ coincides with the dimension of the space of local holomorphic solutions in $\mathbb{C}^s\backslash \mathrm{Sing}(H_A(\beta))$. If I_A is Cohen-Macaulay or β is generic in $\mathbb{C}^d$, then

$$r_A(\beta) \quad=\quad \mathrm{degree}(X_A) \quad=\quad \mathrm{vol}(\mathrm{conv}(A)), \tag{1.3}$$

the normalized volume of the lattice polytope $\mathrm{conv}(A) = \mathrm{conv}\{a_1,\ldots,a_s\}$. The inequality $r_A(\beta) \geq \mathrm{vol}(\mathrm{conv}(A))$ holds without any assumptions on A and β. See [1], [8], [16] for proofs and details. If $d=2$, i.e. when X_A is a curve, then (1.3) holds for all $\beta \in \mathbb{C}^2$ if and only if I_A is Cohen-Macaulay [4].

The irreducible components of $\mathrm{Sing}(H_A(\beta))$ are the hypersurfaces defined by the A'-discriminants $D_{A'}$, where A' runs over *facial subsets* of A, or, equivalently, $X_{A'}$ runs over closures of torus orbits on X_A. The *A-discriminant* D_A is the irreducible polynomial defining the dual variety of the toric variety X_A, with the convention $D_A = 1$ if that dual variety is not a hypersurface; see [1], [8], [10]. Note that for a singleton $A' = \{a_j\}$ we have $D_{A'} = x_j$.

Consider any rational A-hypergeometric function of degree β,

$$f(x_1,\ldots,x_s) \quad=\quad \frac{P(x_1,\ldots,x_s)}{Q(x_1,\ldots,x_s)}, \tag{1.4}$$

where P and Q are relatively prime polynomials. The denominator equals

$$Q(x_1,\ldots,x_s) \qquad=\quad \prod_{A'} D_{A'}(x_1,\ldots,x_s)^{i_{A'}}, \tag{1.5}$$

where A' runs over facial subsets of A and the $i_{A'}$ are non-negative integers.

Our long-term goal is to classify all rational A-hypergeometric functions. For toric curves this was done in [4]: if $d = 2$, every rational A-hypergeometric function is a Laurent polynomial. Here we generalize this result to higher-dimensional toric varieties, by studying rational A-hypergeometric functions which are not Laurent polynomials. We note that, by [16, Lemma 3.4.10], A-hypergeometric polynomials exist for all toric varieties X_A .

We call the matrix A *gkz-rational* if the A-discriminant D_A is not a monomial and appears in the denominator (1.5) of some rational A-hypergeometric function (1.4). The smallest example of a gkz-rational configuration is

$$A \quad = \quad \Delta_1 \times \Delta_1 \quad = \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \tag{1.6}$$

since $1/(x_1x_2 - x_3x_4)$ is A-hypergeometric of degree $\beta = (-1,-1,-1)^T$. Note that (1.6) encodes the *Gauss hypergeometric function* ${}_2F_1$ [16, §1.3].

More generally, the product of simplices $A = \Delta_p \times \Delta_q$ is gkz-rational if and only if $p = q$. The Segre variety $X_A = \mathbb{P}^q \times \mathbb{P}^q$ is projectively dual to the $(q+1) \times (q+1)$-determinant, and the reciprocal of this determinant is a rational A-hypergeometric function. Consider by contrast the configuration $A = 2 \cdot \Delta_q$. The toric variety X_A is the quadratic Veronese embedding of $\mathbb{P}^q$, whose projectively dual hypersurface is the discriminant of a quadratic form,

$$D_A \quad = \quad \det \begin{pmatrix} 2x_{00} & x_{01} & x_{02} & \cdots & x_{0q} \\ x_{01} & 2x_{11} & x_{12} & \cdots & x_{1q} \\ x_{02} & x_{12} & 2x_{22} & \cdots & x_{2q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{0q} & x_{1q} & x_{2q} & \cdots & 2x_{qq} \end{pmatrix}. \tag{1.7}$$

Theorem 1.2 below implies that the classical ("dense") discriminants such as (1.7) do not appear in the denominators of rational hypergeometric functions. In other words, multiples of simplices, $A = r \cdot \Delta_q$, are never gkz-rational.

In Section 2 we resolve the case of *circuits*, that is, matrices A whose kernel is spanned by a single vector $b = b_+ - b_- \in \mathbb{Z}^s$. We call A *balanced* if the positive part b_+ is a coordinate permutation of the negative part b_-, and we show that A is balanced if and only if A is gkz-rational. In Section 3 we study arbitrary configurations A, and we prove the following theorem.

Theorem 1.2. *If the configuration A contains an unbalanced circuit which does not lie in any proper facial subset of A, then A is not gkz-rational.*

This implies that gkz-rational configurations are rare; for instance, they have no interior points. Hence, reflexive polytopes [2] are not gkz-rational, and sufficiently ample embeddings of any toric variety are not gkz-rational.

In order to formulate a conjectural characterization of gkz-rational configurations, we recall the following construction from [10]. Let $A_0, A_1, \ldots, A_r$ be vector configurations in $\mathbb{Z}^r$. Their *Cayley configuration* is defined as

$$A \quad = \quad \{e_0\} \times A_0 \,\cup\, \{e_1\} \times A_1 \,\cup\, \cdots \,\cup\, \{e_r\} \times A_r \,\subset\, \mathbb{Z}^{r+1} \times \mathbb{Z}^r, \qquad (1.8)$$

where $e_0, \ldots, e_r$ is the standard basis of $\mathbb{Z}^{r+1}$.

We call A *essential* if the Minkowski sum $\sum_{i \in I} A_i$ has affine dimension at least $|I|$ for every proper subset I of $\{0, \ldots, r\}$. Cayley configurations are very special. For instance, a configuration A in the plane ($d = 3$) is a Cayley configuration if and only if A lies on two parallel lines; such an A is essential if and only if each line contains at least two points.

Conjecture 1.3. *An arbitrary configuration A is gkz-rational if and only if A is affinely isomorphic to an essential Cayley configuration (1.8).*

This conjecture can be reformulated as follows. The discriminant of an essential Cayley configuration coincides with the sparse resultant $R_{A_0, A_1, \ldots, A_r}$; see [10, §8.1.1]. That resultant characterizes the solvability of a sparse polynomial system $f_0 = f_1 = \cdots = f_r = 0$ with support $(A_0, A_1, \ldots, A_r)$,

$$f_i(t_1, \ldots, t_r) \quad = \quad \sum_{a \in A_i} x_a t^a, \qquad i = 0, 1, \ldots, r.$$

By Corollary 5.2, Conjecture 1.3 is equivalent to the following:

Conjecture 1.4. *A discriminant D_A appears in the denominator of a rational A-hypergeometric function if and only if D_A is a resultant $R_{A_0, A_1, \ldots, A_r}$.*

Being a resultant among discriminants is being a needle in a haystack. None of the univariate or classical discriminants such as (1.7) are resultants. On the other hand, consider two triples of equidistant points on parallel lines,

$$A \quad = \quad \{e_0\} \times A_0 \,\cup\, \{e_1\} \times A_1 \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}. \qquad (1.9)$$

This is the Cayley configuration of $A_0 = A_1 = \{0, 1, 2\}$. The variety X_A is a rational normal scroll in $\mathbb{P}^5$. Its discriminant D_A is the Sylvester resultant

$$R_{A_0,A_1} \;=\; x_1^2x_6^2 - x_1x_2x_6x_5 - 2x_1x_3x_4x_6 + x_1x_3x_5^2 + x_2^2x_4x_6 - x_2x_3x_4x_5 + x_3^2x_4^2$$

of the quadrics $F_0 = x_1u_1^2 + x_2u_1u_2 + x_3u_2^2$ and $F_1 = x_4u_1^2 + x_5u_1u_2 + x_6u_2^2$. The following theorem is the second main result in this paper.

Theorem 1.5. *The if directions of Conjectures 1.3 and 1.4 hold. The only-if directions hold for $d \leq 4$, that is, for toric varieties X_A of dimension ≤ 3.*

The proof of the only-if direction is given in Section 4. It consists of a detailed combinatorial case analysis based on Theorem 1.2. The proof of the if direction, given in Section 5, is based on the notion of *toric residues* introduced by Cox [7], and on our earlier results in [6] about their denominators.

An example of a toric residue is the rational A-hypergeometric function

$$\frac{1}{(2\pi i)^2}\int_\Gamma \frac{u_1\,u_2}{F_0(u_1,u_2)\cdot F_1(u_1,u_2)}\,du_1\wedge du_2 \quad = \quad \frac{x_1x_6 - x_3x_4}{R_{A_0,A_1}}. \qquad (1.10)$$

Here A is the configuration (1.9) and Γ is a suitable 2-cycle in $\mathbb{C}^2$. Such integrals can be evaluated by a single Gröbner basis computation; see [5].

2 Circuits

We fix a configuration A which is a circuit, that is, A is a $d \times (d+1)$-matrix whose integer kernel is spanned by a vector $b = (b_0, b_1, \ldots, b_d)$ all of whose coordinates b_i are non-zero. After relabeling, we may assume

$$b_j > 0 \text{ for } j = 0, \ldots, m-1 \quad \text{ and } \quad b_j < 0 \text{ for } j = m, \ldots, d, \qquad (2.1)$$

so that $b_+ = (b_0, \ldots, b_{m-1}, 0, \ldots, 0)$ and $b_- = (0, \ldots, 0, -b_m, \ldots, -b_d)$. The toric variety X_A is a hypersurface in $\mathbb{P}^d$, defined by the principal ideal

$$I_A \quad = \quad \langle\, \xi^{b_+} - \xi^{b_-} \,\rangle \quad = \quad \langle\, \xi_0^{b_0}\cdots\xi_{m-1}^{b_{m-1}} \;-\; \xi_m^{-b_m}\cdots\xi_d^{-b_d} \,\rangle.$$

In this section we shall prove our main conjecture for the case of circuits.

Theorem 2.1. *Conjectures 1.3 and 1.4 are true for toric hypersurfaces.*

A function $f(x_0, x_1, \ldots, x_d)$ is A-hypergeometric if it is A-homogeneous (satisfies (1.2) for some β) and annihilated by the homogeneous toric operator

$$\partial^{b_+} - \partial^{b_-} \quad = \quad \partial_0^{b_0} \cdots \partial_{m-1}^{b_{m-1}} \; - \; \partial_m^{-b_m} \cdots \partial_d^{-b_d}. \qquad (2.2)$$

The order ρ of this operator equals the holonomic rank of $H_A(\beta)$:

$$\rho := = \; b_0 + \cdots + b_{m-1} \; = \; -b_m - \cdots - b_d \; = \; \mathrm{vol}(\mathrm{conv}(A)) \; = \; r_A(\beta).$$

This holds for all $\beta \in \mathbb{C}^d$ since the principal ideal $I_A = \langle \xi^{b_+} - \xi^{b_-} \rangle$ is Cohen-Macaulay. The toric hypersurface X_A is projectively self-dual. Indeed, by [10, Proposition 9.1.8], the A-discriminant of the circuit A equals

$$D_A \quad = \quad x^{b_-} - \lambda x^{b_+} \qquad \text{where } \lambda = (-1)^\rho b_-^{b_-}/b_+^{b_+}. \qquad (2.3)$$

Recall that the circuit A is *balanced* if $d = 2m - 1$ and, after reordering if necessary, $b_i = -b_{m+i}$ for $i = 0, \ldots, m-1$. Otherwise, we call A *unbalanced*. Note that the configuration (1.6) is a balanced circuit with $b = (1, 1, -1, -1)$.

Lemma 2.2. *Let A be a balanced circuit. Then the rational function $1/D_A$ is A-hypergeometric.*

Proof. Balanced implies $b_-^{b_-} = b_+^{b_+}$ and $\lambda = (-1)^\rho$. Consider the expansion

$$\frac{1}{D_A} \quad = \quad x^{-b_-} \cdot \frac{1}{1 - (-1)^\rho x^b} \quad = \quad \sum_{n=0}^{\infty} (-1)^{\rho n} x^{n b_+ - (n+1) b_-}.$$

For this series to be annihilated by (2.2) it is necessary and sufficient that

$$\prod_{i=0}^{m-1} \prod_{j=1}^{b_i} (n b_i + j) \quad = \quad \prod_{i=m}^{s} \prod_{j=1}^{-b_i} (n(-b_i) + j) \quad \text{for all } n \geq 0.$$

This identity holds if and only if the circuit A is balanced. ∎

This lemma implies that balanced circuits are gkz-rational. The main result in this section is the following converse to this statement. For an arbitrary configuration A, we say that A is *weakly gkz-rational* if there exists a rational A-hypergeometric function which is not a Laurent polynomial.

Theorem 2.3. *Let A be a circuit in $\mathbb{Z}^d$. Then the following are equivalent: (1) A is balanced; (2) A is gkz-rational; (3) A is weakly gkz-rational.*

Proof. The implication from (1) to (2) follows from the previous lemma. The equivalence of (2) and (3) holds because every proper facial subset A' of A is affinely independent. Hence the only non-constant A'-discriminants arising from facial subsets A' arise from vertices $A' = \{a_j\}$, in which case $D_{A'} = x_j$.

It remains to prove the implication from (2) to (1). Suppose that A is gkz-rational. Consider a non-Laurent rational A-hypergeometric function and expand it as a Laurent series with respect to increasing powers of x^b. It follows from the results in [16, §3.4] that this series is the sum of a Laurent polynomial and a *canonical A-hypergeometric series* of the following form:

$$F(x) \quad = \quad x^c \cdot \sum_{n=0}^{\infty} (-1)^{\rho n} \, \frac{\prod_{j \geq m}(-c_j - nb_j - 1)!}{\prod_{j<m}(c_j + nb_j)!} \, x^{nb}. \tag{2.4}$$

Here $c = (c_0, c_1, \ldots, c_d)$ is a suitable integer vector. The series $F(x)$ represents a rational function. We may view the series on the right-hand side of (2.4) as defining a rational function of the single variable $t = x^b = x^{b_+}/x^{b_-}$:

$$\varphi(t) \quad = \quad \sum_{n=0}^{\infty} (-1)^{\rho n} \, \frac{\prod_{j \geq m}(-c_j - nb_j - 1)!}{\prod_{j<m}(c_j + nb_j)!} \, t^n \tag{2.5}$$

The A-discriminant equals $D_A = x^{b_-}(1 - \lambda t)$ where $\lambda = (-1)^\rho \, b_-^{b_-}/b_+^{b_+}$. This implies that the rational function $\varphi(t)$ may be written as a quotient

$$\varphi(t) \quad = \quad \frac{P(t)}{(1 - \lambda t)^{k+1}},$$

where $P(t)$ is a polynomial and $k \in \mathbb{N}$. It follows from [17, Corollary 4.3.1] that the coefficients of the series (2.5) must be of the form λ^n times a polynomial in n. That is, the following expression is a polynomial in n:

$$\gamma(n) \quad := \quad \lambda^{-n} \cdot \frac{\prod_{j \geq m}(-c_j - nb_j - 1)!}{\prod_{j<m}(c_j + nb_j)!}$$

The rational function $\mu(z) := \gamma(z+1)/\gamma(z)$ satisfies the following general identity [15, Lemma 2.1] for any fixed complex number z_0:

$$\sum_{\alpha \in z_0 + \mathbb{Z}} \operatorname{ord}_\alpha(\mu) \quad = \quad 0. \tag{2.6}$$

Our rational function $\mu(z)$ has its poles among the points

$$-(\frac{c_j}{b_j}+\frac{1}{b_j}),\ -(\frac{c_j}{b_j}+\frac{2}{b_j}),\ \ldots,\ -(\frac{c_j}{b_j}+1)\ ;\quad j=0,\ldots,m-1,$$

and its zeroes among

$$-\frac{c_j}{b_j},\ -(\frac{c_j}{b_j}+\frac{1}{b_j}),\ \ldots,\ -(\frac{c_j}{b_j}+\frac{-b_j-1}{b_j})\ ;\quad j=m,\ldots,d.$$

We may assume $b_0=\max\{b_j\ ;j=0,\ldots,m-1\}$ and $-b_m=\max\{-b_j\ ;j=m,\ldots,d\}$. Suppose now that $b_0>-b_m$. Then, $\mu(z)$ has a pole at a point p/b_0 with p and b_0 coprime, but since none of the zeroes may be of this form, this contradicts (2.6). A symmetric argument leads to a contradiction if we assume $b_0<-b_m$. This implies that $b_0=-b_m$ and therefore

$$\gamma(n)\cdot\frac{(c_0+nb_0)!}{(-c_m-nb_m-1)!}$$

is also rational function of n. Consequently, we can iterate our argument to conclude that, after reordering, $b_i=-b_{m+i}$ for all $i=0,\ldots,m-1$. ∎

Remark 2.4. The above results imply that a circuit A is gkz-rational if and only if the specific rational function $1/D_A$ is A-hypergeometric. The same statement is false for non-circuits. For instance, for the gkz-rational configuration in (1.9), the function $1/D_A=1/R_{A_0,A_1}$ is not A-hypergeometric.

Let us now return to the result stated at the beginning of this section.

Proof of Theorem 2.1. Theorem 2.3 and the lemma below imply Conjecture 1.3. The equivalence of Conjectures 1.3 and 1.4 will be shown in Section 5. ∎

Lemma 2.5. *A circuit A is balanced if and only if it is affinely isomorphic to an essential Cayley configuration (1.8).*

Proof. We first prove the if-direction. Let A be an essential Cayley configuration which is a circuit. Each A_i must consist of a pair of vectors in $\mathbb{Z}^r$, so that A becomes an $(2r+1)\times(2r+2)$-matrix. The first $r+1$ rows of A show that the kernel of A is spanned by a vector $b=(b_0,-b_0,b_1,-b_1,\ldots,b_r,-b_r)$.

This means that A is balanced. Conversely, if A is balanced then we can apply left multiplication by an element of $GL(d, \mathbb{Q})$ to get isomorphically

$$A = \begin{pmatrix} I_m & I_m \\ 0 & \tilde{A} \end{pmatrix}$$

where $\tilde{A}$ is an $(m-1) \times m$ integral matrix of rank $m-1$. By permuting columns we see that A is an essential Cayley configuration for $m = r+1$. ∎

3 The General Case

In this section we prove Theorem 1.2. A configuration A is called *non-degenerate* if the A-discriminant D_A is neither equal to 1 nor a variable. Circuits are non-degenerate by (2.3). Recall that D_A is a variable if and only if A is a point. A subconfiguration $B \subseteq A$ is called *spanning* if B is not contained in any proper facial subset of A. If the dimension of B is equal to the dimension of A then B is spanning, but the converse is not true. For instance, the vertex set of an octahedron contains spanning circuits but no full-dimensional circuits.

The condition $D_A = 1$ means that the dual variety to the toric variety X_A is not a hypersurface. No combinatorial characterization of this condition is presently known. A necessary condition is given in the following proposition. That condition is not sufficient: the skew prisms in (4.6) contain no spanning circuit but $D_A \neq 1$. Note that $D_A = 1$ for the regular prism $A = \Delta_1 \times \Delta_2$.

Proposition 3.1. *If A contains a subconfiguration B which is spanning and non-degenerate then A is non-degenerate. In particular, A is non-degenerate if it contains a spanning circuit.*

Proof. Proceeding by induction, it suffices to consider the case when B is obtained from A by removing a single point, say, $B = A \backslash \{a_s\}$. Since B is not contained in any face of A, and B is a facial subset of itself, the following lemma tells us that the B-discriminant D_B divides $D_A|_{x_s=0}$. Since D_B is not a monomial, this implies that D_A is not a monomial. ∎

Lemma 3.2. *Let $a_s \in A$, x_s the corresponding variable, and B' a facial subset of $B = A \backslash \{a_s\}$ which does not lie in any proper facial subset of A. Then the B'-discriminant $D_{B'}$ divides the specialized A-discriminant $D_A|_{x_s=0}$.*

Proof. Let $f = \sum_{j=1}^{s} x_j t^{a_j}$ be a generic polynomial with support A. By [11, Theorem 5.10], the *principal A-determinant* is the specialization

$$E_A \quad = \quad R_A\big(t_1 \frac{\partial f}{\partial t_1}, \ldots, t_d \frac{\partial f}{\partial t_d}\big),$$

where R_A denotes the *A-resultant*; see [10, §8.1]. The irreducible factorization of the principal A-determinant ranges over the facial subsets A' of A,

$$E_A \quad = \quad \prod_{A'} D_{A'}^{m_{A'}}, \tag{3.1}$$

where $m_{A'}$ are certain positive integers [10, Theorem 10.1.2].

Let $w \in \mathbb{Z}^s$ be the weight vector with $w_s = -1$ and $w_j = 0$ for $j \neq s$. The initial form of the principal A-determinant with respect to w can be factored in two different ways:

$$in_w(E_A) \quad = \quad \prod_{A'} (in_w D_{A'})^{m_{A'}} \;=\; \prod_{C \text{ facet of } \Delta_w} (E_C)^{n_C}.$$

Here Δ_w is the coherent polyhedral subdivision of A defined by w and the n_C are certain positive integers. The first formula comes from (3.1) and the second formula comes from [10, Theorem 10.1.12]. Since B' is a facial subset of $A \backslash \{a_s\}$, it is also a cell of the subdivision Δ_w, and hence $D_{B'}$ divides E_C for the facet $C = A \backslash \{a_s\}$ of Δ_w. We conclude that $D_{B'}$ divides $in_w(D_{A'})$ for some facial subset A' of A. If $D_{B'} \neq 1$, this implies that $B' \subseteq A'$ because $D_{B'}$ involves all the variables associated with points in B'. By our hypothesis, the only facial subset of A which contains B' is A itself. Therefore $D_{B'}$ divides $in_w(D_A) \;=\; D_A|_{x_s=0}$. ∎

We also need the following lemma from commutative algebra whose proof was shown to us by Mircea Mustaţă:

Lemma 3.3. *Let $\mathcal{R}$ be a unique factorization domain with field of fractions K, and let $f(t) = \sum_{i=0}^{m} a_i \cdot t^i$ and $g(t) = \sum_{j=0}^{n} b_j \cdot t^j$ be relatively prime elements in the polynomial ring $\mathcal{R}[t]$. Assume that $b_0 \neq 0$ and consider the Taylor series expansion of the ratio f/g:*

$$\frac{f(t)}{g(t)} \;=\; \sum_{\ell=0}^{\infty} c_\ell \cdot t^\ell \quad \text{in } K[[t]].$$

If all the Taylor coefficients c_ℓ lie in $\mathcal{R}$, then b_0 is a unit in $\mathcal{R}$.

Proof. Let p be any prime element in $\mathcal{R}$. We must show that p does not divide b_0. Consider the localization $\mathcal{R}[t]_{\langle p,t\rangle}$ of $\mathcal{R}[t]$ at the prime ideal $\langle p,t\rangle$. The power series ring $\mathcal{R}[[t]]$ is the completion of the local ring $\mathcal{R}[t]_{\langle p,t\rangle}$ with respect to the $\langle t\rangle$-adic topology. By assumption, the polynomial f lies in the principal ideal generated by g in $\mathcal{R}[[t]]$. The basic flatness property of completions, as stated in [13, §8, p.63], implies that f lies in the principal ideal generated by g in $\mathcal{R}[t]_{\langle p,t\rangle}$. Since f and g are relatively prime in $\mathcal{R}[t]$, we conclude that g is a unit in $\mathcal{R}[t]_{\langle p,t\rangle}$ and so b_0 is not divisible by p. ∎

We are now prepared to prove the first theorem stated in the introduction.

Proof of Theorem 1.2. Suppose $A = \{a_1, \ldots, a_s\}$ is a gkz-rational configuration and let $f = P/Q$ be a rational A-hypergeometric function of degree $\beta \in \mathbb{Z}^d$, where $P, Q \in \mathbb{C}[x_1, \ldots, x_s]$ are relatively prime, and the A-discriminant D_A is not a monomial and divides Q. We claim that any spanning circuit Z of A is balanced. We shall prove this by induction on s. If $s = d+1$, then we are done by Theorem 2.3. We may assume that A is not a circuit and therefore Z is a proper subset of A. Suppose $a_s \in A \backslash Z$, and set $t = x_s$, $\tilde{A} = \{a_1, \ldots, a_{s-1}\}$, $\tilde{x} = (x_1, \ldots, x_{s-1})$. We may expand the rational A-hypergeometric function $f(x) = f(\tilde{x}; t)$ as

$$f(\tilde{x}; t) \quad = \quad \sum_{\ell \geq \ell_0} R_\ell(\tilde{x}) \cdot t^\ell \,, \qquad (3.2)$$

where each $R_\ell(\tilde{x})$ is a rational $\tilde{A}$-hypergeometric function of degree $\beta - \ell \cdot a_s$.

Let A' denote the unique smallest facial subset of $\tilde{A} = A \backslash \{a_s\}$ which contains the circuit Z. Then Z is a spanning circuit in A'. Proposition 3.1 implies that its discriminant $D_{A'}$ is not a monomial. Lemma 3.2 implies that $D_{A'}$ divides $in_t(Q)$, the lowest coefficient of Q with respect to t.

We now apply Lemma 3.3 to the domain $\mathcal{R} = \mathbb{C}[\tilde{x}, \tilde{x}^{-1}]_{<D_{A'}>}$, the localization of the Laurent polynomial ring at the principal prime ideal $\langle D_{A'}\rangle$. Since $in_t(Q)$ is not a unit in $\mathcal{R}$, we conclude that some Taylor coefficient $R_\ell(\tilde{x})$ lies in the field of fractions of $\mathcal{R}$ but not in $\mathcal{R}$ itself. This means that $D_{A'}$ divides the denominator of $R_\ell(\tilde{x})$. We have found a rational $\tilde{A}$-hypergeometric function whose denominator contains the non-trivial factor $D_{A'}$. It follows by induction that the spanning circuit Z of A' is balanced. ∎

Recall that a configuration A is called *weakly gkz-rational* if there exists a rational A-hypergeometric function which is not a Laurent polynomial. It is called *gkz-rational* if the A-discriminant D_A is not a monomial and appears in the denominator of a rational A-hypergeometric function.

Proposition 3.4. *A configuration A is weakly gkz-rational if and only if some facial subset A' of A is gkz-rational.*

Proof. If A' is a facial subset of A then every A'-hypergeometric function $f(x)$ is also A-hypergeometric. Indeed, $f(x)$ is obviously A-homogeneous, but it is also annihilated by the toric operators $\partial^u - \partial^v$ because the support of ∂^u lies in $\{\partial_i : a_i \in A'\}$ if and only if the support of ∂^v lies in $\{\partial_i : a_i \in A'\}$. This proves the if-direction. For the only-if direction, suppose that A is weakly gkz-rational and let $f(x) = P(x)/Q(x)$ be a non-Laurent rational hypergeometric function. There exists a facial subset A' of A such that $D_{A'}$ is not a monomial and divides $Q(x)$. Our goal is to show that A' is gkz-rational. We proceed by induction on the cardinality of $A \backslash A'$. There is nothing to show if $A = A'$. Let $a_s \in A \backslash A'$ and form the series expansion as in (3.2). By applying Lemma 3.3 as in the proof of Theorem 1.2, we construct a rational $(A \backslash \{a_s\})$-hypergeometric function whose denominator is a multiple of the A'-discriminant $D_{A'}$. This proves our claim, by induction. ∎

We close this section with two corollaries which demonstrate the scope of Theorem 1.2. They show that gkz-rational configurations A are very special.

Corollary 3.5. *A gkz-rational configuration A has no interior point.*

Proof. Let a_1 be an interior point of A, and let Z' be a minimal size subset of $A \backslash \{a_1\}$ which contains a_1 in its relative interior. Then $Z = Z' \cup \{a_1\}$ is a circuit of A which is spanning and not balanced. ∎

Corollary 3.5 can be rephrased, using Khovanskii's genus formula [12], into the language of algebraic geometry as follows. If a projective toric variety X_A is gkz-rational, then the generic hyperplane section of X_A has arithmetic genus 0. Clearly, this fails if X_A is embedded by a sufficiently ample line bundle, and also in the case of special interest in mirror symmetry (see [2]).

Corollary 3.6. *The configuration A is not gkz-rational if A is the set of lattice points in a reflexive polytope, or A is the set of lattice points in a polytope of the form $s \cdot \mathcal{P}$, where $\mathcal{P}$ is any lattice polytope and $s > \dim(\mathcal{P})$.*

Proof. Reflexive polytopes possess exactly one interior point. If s is bigger than the ambient dimension then s times any lattice polytope contains an interior point. ∎

4 Low dimensions

In this section we present the complete classification of gkz-rational configurations for $d \leq 4$. Note that the $d = 1$ case is trivial since we disallow repeated points. If we did allow them then $A = (\,1\ 1\ 1\ \cdots\ 1\,)$ would be gkz-rational for $s \geq 2$ because the function $1/(x_1 + x_2 + \cdots + x_s)$ is A-hypergeometric.

Toric curves ($d = 2$) are never gkz-rational. This was shown in [4, Theorem 1.10]. We rederive this result as follows. Write the configuration as

$$A \;=\; \begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_s \end{pmatrix} \;; \quad k_1 < k_2 < \cdots < k_s \,.$$

Every circuit $Z \subseteq A$ consists of three collinear points:

$$Z \;=\; \begin{pmatrix} 1 & 1 & 1 \\ k_a & k_b & k_c \end{pmatrix} \;; \quad k_a < k_b < k_c \,.$$

Such a 1-dimensional circuit is never balanced. Theorem 1.2 implies that A is not gkz-rational. In what follows we prove the only-if part of Theorem 1.5.

Theorem 4.1. *Let A be an integer matrix with $d \leq 4$ rows. If A is gkz-rational then A is affinely isomorphic to an essential Cayley configuration.*

Proof. It suffices to prove the following two assertions:

• If A is a configuration on the line ($d = 2$) or in 3-space ($d = 4$) then A is not gkz-rational.

• If A is a configuration in the plane ($d = 3$) then A is gkz-rational if and only if the points of A lie on two parallel lines with each line containing at least two points from A.

The case $d = 2$ was proved above. We first assume $d = 3$. If the points of A lie on two parallel lines then we can write their coordinates as follows:

$$A \quad = \quad \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & k_1 & \cdots & k_m & 0 & \ell_1 & \cdots & \ell_n \end{pmatrix} \tag{4.1}$$

Thus A is the Cayley configuration of two one-dimensional configurations. The construction in the next section shows that A is gkz-rational for $m, n \geq 1$.

Conversely, suppose that A does not lie on two parallel lines. We may further assume that A contains no unbalanced spanning circuit by Theorem 1.2. One example of a configuration satisfying these requirements is

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}. \tag{4.2}$$

The toric variety X_A is the Veronese surface in $\mathbb{P}^5$. Its dual variety is the hypersurface defined by the discriminant of a ternary quadratic form

$$D_A \quad = \quad \det \begin{pmatrix} 2x_1 & x_2 & x_4 \\ x_2 & 2x_3 & x_5 \\ x_4 & x_5 & 2x_6 \end{pmatrix}. \tag{4.3}$$

Suppose there exists a rational A-hypergeometric function $f(x) = P(x)/Q(x)$ with Q a multiple of D_A. Let A' be the configuration obtained from A by removing the fourth and fifth columns. Setting $x_4 = x_5 = 0$ in D_A yields $(4x_1x_3 - x_2^2) \cdot x_6$. We can argue as in the proof of Theorem 1.2 and construct a rational A'-hypergeometric function whose denominator contains the binomial factor. Proposition 3.4 would imply that the configuration consisting of the first three columns of A is gkz-rational, and this is a contradiction to Theorem 2.3. Hence the configuration A in (4.2) is not gkz-rational.

Another configuration to be considered is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & p & q & 0 & 0 \\ 0 & 0 & 0 & p & q \end{pmatrix}, \tag{4.4}$$

where $1 \leq p \leq q$ are relatively prime integers. The only spanning circuit of A is the balanced circuit $\{a_2, a_3, a_4, a_5\}$. Consider the subset $A' = \{a_1, a_2, a_3\}$ which is an unbalanced circuit on the boundary of A. The A-discriminant is an irreducible homogeneous polynomial of degree $q^2 - p^2$ which looks like

$$D_A \quad = \quad x_5^{p(q-p)} \cdot D_{A'}(x_1, x_2, x_3)^{q-p} + \text{terms containing } x_4.$$

Applying the expansion technique with respect to x_4, we get a rational A'-hypergeometric function whose denominator contains $D_{A'}$. This contradicts Theorem 2.3. Hence the configuration A in (4.4) is not gkz-rational.

Our assertion for $d = 3$ now follows from the subsequent lemma of combinatorial geometry. Note that four points in the plane, with no three collinear, lie on two parallel lines if and only if they form a balanced circuit. ∎

Lemma 4.2. *Let B be a planar configuration without interior points such that every four-element subset of B lies on two parallel lines. Then either B lies on two parallel lines, or B is affinely equivalent to (4.2) or (4.4) or to the vertices of a regular pentagon, in which case B has irrational coordinates.*

Proof. We may assume without loss of generality that the origin O lies in B and is a vertex of the convex hull $\mathrm{conv}(B)$. Let a and b be the points of B closest to O along the edges of $\mathrm{conv}(B)$ adjacent to O. Let $c = a + b$. Any other point $x \in B$ must be of the form $r_1 \cdot a$, or $r_2 \cdot b$, or $a + r_3 \cdot b$, or $b + r_4 \cdot a$, where r_1, r_2, r_3, r_4 are positive real numbers and $r_1, r_2 > 1$.

If $c \in B$, then only the first two cases may occur. Indeed, suppose $x = a + r_3 \cdot b \in B$, then either all the points lie on two parallel lines or there exists a point $y = r_1 \cdot a$ or $y = b + r_4 \cdot a$ in B. It is easy to check that in all of these cases, B has an interior point. Suppose then that $x_1 = r_1 \cdot a \in B$ and $x_2 = r_2 \cdot b \in B$. We have $r_1 = r_2$ since the subset $\{a, b, x_1, x_2\}$ lies on two parallel lines. If $r_1 \neq 2$ then the subset $\{O, c, x_1, x_2\}$ contradicts the assumption. Hence, if $c \in B$, either all the points lie on two parallel lines, or $r_1 = r_2 = 2$ which means that B is affinely equivalent to (4.2).

On the other hand, if $c \notin B$ and there exists a point $x_1 = a + r_3 b \in B$, then either all the points lie on two parallel lines or B contains a point of the form $x_2 = r_1 a$ or $x_2 = b + r_4 a$. Since $r_3 \neq 1$, in all of these cases B contains an unbalanced circuit, or $B = \{a, O, b, x_1, x_2\}$ is affinely equivalent to the vertex set of a regular pentagon. The only remaining possibility is that all points of B be multiples of either a or b. But if $x_1 = r_1 a$ and $x_2 = r_2 b$ are in B then $\{a, b, x_1, x_2\}$ is unbalanced unless $r_1 = r_2$. Hence the only possible configuration not containing the point c is affinely equivalent to (4.4) ∎

We note that the argument in the paragraph following (4.3) works also for the discriminant (1.7) of any quadratic form. Hence $A' = 2 \cdot \Delta_q$ is not gkz-rational for any q. Since A' is a spanning subconfiguration of $A = r \cdot \Delta_q$ for all $r \geq 2$, we conclude the following result which was stated in the introduction.

Proposition 4.3. *Multiples of simplices, $A = r \cdot \Delta_q$, are never gkz-rational.*

We now proceed to discuss configurations in affine 3-space ($d = 4$). Let us begin by stating the relevant fact of combinatorial geometry in this case.

Lemma 4.4. *Let B be a 3-dimensional point configuration which is not a pyramid and such that every 2-dimensional circuit is balanced and no 3-dimensional circuit exists. Then either B lies on three parallel lines, or B is affinely equivalent to a subconfiguration of $\{O, P, Q, R, cP, cQ, cR\}$ for some points P, Q, R and some $c \in \mathbb{R}$.*

Proof. Choose five points from our configuration which are not in a plane. They have the form $\{A_1, A_2, B_1, B_2, C\}$ where the lines $\overline{A_1A_2}$ and $\overline{B_1B_2}$ are parallel and C lies outside the plane $\Pi = \overline{A_1A_2B_1B_2}$. Suppose that our configuration is not on three parallel lines. There exists a point $D \notin \Pi$ such that the line $\overline{CD}$ is not parallel to the lines $\overline{A_1A_2}$ and $\overline{B_1B_2}$. If, under this hypothesis, the line $\overline{CD}$ is still parallel to the plane Π, then we have created a 3-dimensional circuit, a contradiction. Therefore the line $\overline{CD}$ meets the plane Π in a point which we call the origin O. The origin O must be equal to either $\overline{A_1B_1} \cap \overline{A_2B_2}$ or $\overline{A_1B_2} \cap \overline{A_2B_1}$; otherwise we would have created a 3-dimensional circuit. From this requirement we conclude that the configuration $\{O, A_1, B_1, C, A_2, B_2, D\}$ is affinely equivalent to $\{O, P, Q, R, cP, cQ, cR\}$.

It remains to be seen that O is the only point that may be added to the configuration $\{P, Q, R, cP, cQ, cR\}$ without creating either a 3-dimensional circuit or an unbalanced 2-dimensional circuit. A point not a multiple of P, Q or R obviously creates a 3-dimensional circuit. A multiple of P, Q or R creates an unbalanced 2-dimensional circuit, unless it is the origin. ■

Proof of Theorem 4.1 (continued). Let A be a configuration in affine 3-space. We shall prove that A is not gkz-rational. In view of Theorem 1.2, we may assume that A contains no unbalanced spanning circuit. This implies that A contains no 3-dimensional circuit, because such a circuit involves five points and, five being an odd number, that circuit would be unbalanced.

Suppose that A contains an unbalanced 2-dimensional circuit Z. Then Z lies in a facet of A. There must be at least two distinct points P and Q of A which do not lie in that facet. Otherwise, A is a pyramid and the A-discriminant is 1. If the line $\overline{PQ}$ is parallel to the plane spanned by Z then, since Z is unbalanced, some triangle in Z has all of its three edges skew to $\overline{PQ}$. This triangle together with P and Q forms a 3-dimensional circuit, a contradiction. Hence the line $\overline{PQ}$ intersects the plane spanned by Z. Some triangle in Z has the property that none of the lines spanned by its edges contains that intersection point. Again, this triangle together with P and Q forms a 3-dimensional circuit.

We conclude that A has no 3-dimensional circuit and every 2-dimensional circuit of A is balanced. Lemma 4.4 tells us what the possibilities are. If A lies on three parallel lines, then $D_A = 1$ and thus A is not gkz-rational. It remains to examine the special configurations $\{O, P, cP, Q, cQ, R, cR\}$. An affine transformation moves the points P, Q and R onto the coordinate axes, so that our configuration has the matrix form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & q & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & -p \end{pmatrix} \tag{4.5}$$

where p and q are relatively prime integers, and $q > 0$. The subconfiguration consisting of the last six columns is spanning. We shall prove that it is non-degenerate and not gkz-rational. Our usual deletion technique then implies that the bigger configuration (4.5) is also not gkz-rational. It therefore suffices to consider the following 4×6-matrix

$$A \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ q & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & q & -p \end{pmatrix}. \tag{4.6}$$

We shall distinguish the two cases $p > 0$ and $p < 0$. If $p > 0$ then A represents an octahedron, and if $p < 0$ then A represents a triangular prism.

We shall present a detailed proof for the octahedron case $p > 0$. The proof technique to be employed was shown to us by Laura Matusevich. We first note that the A-discriminant D_A is a homogeneous irreducible polynomial of degree $(p+q)^2$. The Newton polytope of D_A is a simplex with vertices corresponding to the monomials:

$$(x_1^p x_2^q)^{p+q}, \quad (x_3^p x_4^q)^{p+q}, \quad (x_5^p x_6^q)^{p+q}.$$

Suppose A is gkz-rational. There is a rational function $f(x) = P(x)/Q(x)$, with P, Q relatively prime polynomials, such that the A-discriminant D_A divides Q, and f is A-hypergeometric of some degree β. For $u \in \mathbb{N}^6$, the derivative $\partial^u f$ is A-hypergeometric of degree $\beta - A \cdot u$ and has D_A in its denominator. Replacing f by $\partial^u f$ for suitable u, we may assume that the A-degree of f is of the form $\beta = (K, 0, 0, 0)$ for some negative integer K.

We expand f around the vertex $(x_5^p x_6^q)^{p+q}$ of the Newton polytope of D_A. This results in a convergent Taylor series in the new variables

$$u \;:=\; (-1)^{p+q}\,\frac{x_1^p x_2^q}{x_5^p x_6^q} \quad \text{and} \quad v \;:=\; (-1)^{p+q}\,\frac{x_3^p x_4^q}{x_5^p x_6^q}\,.$$

That hypergeometric series equals, up to a constant,

$$\frac{1}{(x_5)^{pk}(x_6)^{qk}} \sum_{m,n\geq 0} \frac{(p(m+n+k)-1))!(q(m+n+k)-1))!}{(np)!(nq)!(mp)!(mq)!}\, u^m v^n\,,$$

for an appropriate positive integer k. The coefficients of this series can be derived directly from the toric operators (1.1) arising from A. It is one of the canonical series described for general A in [16, §3.4]. The series

$$\Psi(u,v) \quad = \quad \sum_{m,n\geq 0} \frac{(p(m+n+k)-1))!(q(m+n+k)-1))!}{(np)!(nq)!(mp)!(mq)!}\, u^m v^n \tag{4.7}$$

represents a rational function in two variables.

We denote by $F(m,n)$ the coefficient of $u^m v^n$ in the series (4.7). Note that $F(0,0)\neq 0$. Since Ψ is rational, there exist positive integers N, M, and constants $c_{ij}\in\mathbb{C}$, $0\leq i,j\leq N$, such that $c_{00}\neq 0$ and

$$\sum_{i,j=0}^{N} c_{ij} F(m+i,n+j) \quad = \quad 0 \quad \text{holds for all } m,n\geq M.$$

If we divide $F(m+i,n+j)$ by $F(m,n)$ then we get a rational function in m and n. Hence the following is an identity of rational functions in m and n:

$$\sum_{i,j=0}^{N} c_{ij} \frac{F(m+i,n+j)}{F(m,n)} \quad = \quad 0 \tag{4.8}$$

Let $R(m,n)$ and $S(m,n)$ denote the incremental quotients:

$$R(m,n) \;:=\; \frac{F(m+1,n)}{F(m,n)}\;; \quad S(m,n) \;:=\; \frac{F(m,n+1)}{F(m,n)}\,.$$

If $a,b\in\mathbb{N}$ and we set $\mu=m+n$, $c=a+b$, we have

$$R(m+a,n+b) \;=\; \frac{\prod_{j=0}^{p-1}(p(\mu+c+k)+j)\prod_{j=0}^{q-1}(q(\mu+c+k)+j)}{\prod_{j=1}^{p}(p(m+a)+j)\prod_{j=1}^{q}(q(m+a)+j)}\,, \tag{4.9}$$

$$S(m+a,n+b) \;=\; \frac{\prod_{j=0}^{p-1}(p(\mu+c+k)+j)\prod_{j=0}^{q-1}(q(\mu+c+k)+j)}{\prod_{j=1}^{p}(p(n+b)+j)\prod_{j=1}^{q}(q(n+b)+j)}. \quad (4.10)$$

Given now $0 \leq i,j \leq N$ with $i+j>0$ we have

$$\frac{F(m+i,n+j)}{F(m,n)} \;=\; \prod_{a=0}^{i-1} R(m+a,n+j) \cdot \prod_{b=0}^{j-1} S(m,n+b)$$

Note that either $R(m,n)$ or $S(m,n)$ is a factor in the above product. Consider now the point

$$(m_0,n_0) \;:=\; (-\frac{p-1}{p}-k-\alpha,\alpha)\,,$$

where α is an irrational number. We have $p(m_0+n_0+k) = -(p-1)$ and therefore both $R(m,n)$ and $S(m,n)$ vanish at (m_0,n_0). On the other hand, since α is irrational, none of the denominators in (4.9) or (4.10) may vanish at (m_0,n_0). Evaluating the left-hand side of (4.8) at (m_0,n_0) yields $c_{00}=0$ which is impossible.

We have shown that the matrix A in (4.6) is not gkz-rational for $p>0$. The proof of non-rationality in the triangular prism case ($p<0$), provided to us by Laura Matusevich, is analogous and will be omitted here. In summary, we conclude that every 3-dimensional configuration is not gkz-rational. ∎

5 Toric residues

In this section we present an explicit construction of non-Laurent rational hypergeometric functions. This will prove the if-direction of Conjectures 1.3 and 1.4 as promised in Theorem 1.5. At the end of Section 5 we state further open problems concerning residues and rational hypergeometric functions. We begin with the "Cayley trick" for representing resultants as discriminants.

Proposition 5.1. *Let A be a Cayley configuration (1.8). If A is essential then the resultant $R_{A_0,\dots,A_r}$ is non-constant and equals the discriminant D_A.*

Proof. The identity $R_{A_0,\dots,A_r} = D_A$ was proved in [10, Proposition 9.1.7] under the more restrictive hypothesis that the configurations $A_0,\dots,A_r$ are all full-dimensional; see [10, Hypothesis (1) on page 252]. The argument given in that proof shows that $R_{A_0,\dots,A_r} \neq 1$ suffices to imply $R_{A_0,\dots,A_r} = D_A$. On the other hand, the condition of A being essential appears in [14, equation (2.9)], and [14, Corollary 2.4] shows that it is equivalent to $R_{A_0,\dots,A_r} \neq 1$. ∎

Corollary 5.2. *Conjecture 1.3 and Conjecture 1.4 are equivalent.*

Proof. We must show that a non-degenerate configuration B is affinely isomorphic to an essential Cayley configuration if and only if its discriminant D_B equals the mixed resultant $R_{A_0,\ldots,A_r}$ of some tuple of configurations $(A_0,\ldots,A_r)$. The only-if direction is the content of Proposition 5.1. For the converse, suppose $D_B = R_{A_0,\ldots,A_r} \neq 1$. Let A be the Cayley configuration of $A_0,\ldots,A_r$. Then $D_A = D_B$. In other words, the toric varieties X_A and X_B in $\mathbb{P}^{s-1}$ have the same dual variety, namely, the hypersurface defined by $D_A = D_B$. The Biduality Theorem [10, Theorem 1.1.1] shows that $X_A = X_B$, and this implies that A and B are affinely isomorphic. ∎

We next review the construction of the *toric residue* associated with a toric variety X_Δ. This was introduced by Cox [7] and further developed in [3], [5], [6]. Here Δ is the set of all lattice points in a full-dimensional convex polytope in $\mathbb{R}^r$. We consider $r+1$ Laurent polynomials $f_0, f_1, \ldots, f_r$ supported in Δ with generic complex coefficients:

$$f_j(t) \;=\; \sum_{m\in\Delta} x_{jm}\, t^m \;, \qquad j = 0,1,\ldots,r\,. \tag{5.1}$$

Given an interior lattice point $a \in \mathrm{Int}((r+1)\cdot\Delta)$ and an index $i \in \{0,\ldots,r\}$, consider the total sum of Grothendieck point residues:

$$\mathrm{Res}^{\Delta}_{f}(t^a) \;:=\; (-1)^i \sum_{\xi\in V_i} \mathrm{Res}_\xi\left(\frac{t^a/f_i}{f_0\cdots f_{i-1}f_{i+1}\cdots f_r}\,\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_r}{t_r}\right)\,, \tag{5.2}$$

where $V_i \;=\; \{t\in(\mathbb{C}^*)^r \,:\, f_0(t)=\cdots=f_{i-1}(t)=f_{i+1}(t)=\cdots=f_r(t)=0\}$. It is shown in [3, Theorem 0.4] that the expression (5.2) is independent of i and agrees with the residue defined by Cox in [7]. We refer to [7, §6] and [3, §5] for integral representations such as (1.10) of the toric residue $\mathrm{Res}^{\Delta}_{f}(t^a)$.

The toric residue $\mathrm{Res}^{\Delta}_{f}(t^a)$ is a rational function in the coefficients x_{jm} of our system (5.1). For degree reasons, this rational function is never a non-zero polynomial. It was shown in [6, Theorem 1.4] that the product

$$\mathcal{R}_\Delta(f_0,\ldots,f_r)\cdot\mathrm{Res}^{\Delta}_{f}(t^a)$$

is a polynomial in the variables x_{jm}, where $\mathcal{R}_\Delta(f_0,\ldots,f_r)$ denotes the (unmixed) sparse resultant associated with the polytope Δ; see [10, §8.2].

There is an easy algebraic method [5, Algorithm 2] for computing the rational function $\mathrm{Res}_f^\Delta(t^a)$: translate $f_0, \ldots, f_r, t^a$ into multihomogeneous polynomials $F_0, \ldots, F_r, u^{a'}$ in the homogeneous coordinate ring of X_Δ, compute any Gröbner basis $\mathcal{G}$ for $\langle F_0, \ldots, F_r \rangle$, and finally take the normal form modulo $\mathcal{G}$ of $u^{a'}$ and divide it by the normal form modulo $\mathcal{G}$ of the *toric Jacobian* [7] of $F_0, \ldots, F_r$. This computation yields $\mathrm{Res}_f^\Delta(t^a)$ up to a constant.

Example 5.3. We demonstrate the algorithm of [5] by showing how it computes the rational function (1.10). Here $r = 1$ and Δ is the segment $[0, 2]$ on the line. The system (5.1) consists of two quadratic polynomials $f_0(t) = x_1 + x_2 t + x_3 t^2$ and $f_1(t) = x_4 + x_5 t + x_6 t^2$. The toric variety X_Δ is the projective line $\mathbb{P}^1$. We rewrite our input equations in homogeneous coordinates,

$$F_0(u_1, u_2) \;=\; x_1 u_1^2 + x_2 u_1 u_2 + x_3 u_2^2 \quad \text{and} \quad F_1(u_1, u_2) \;=\; x_4 u_1^2 + x_5 u_1 u_2 + x_6 u_2^2,$$

and we compute any Gröbner basis $\mathcal{G}$ for the ideal $\langle F_0, F_1 \rangle$ in $\mathbf{K}[u_1, u_2]$, where $\mathbf{K} = \mathbb{Q}(x_1, x_2, x_3, x_4, x_5, x_6)$. Here the toric Jacobian equals

$$\begin{aligned} J(u_1, u_2) \quad &= \quad \tfrac{\partial F_0}{\partial u_1}\tfrac{\partial F_1}{\partial u_2} - \tfrac{\partial F_0}{\partial u_2}\tfrac{\partial F_1}{\partial u_1} \\ = \quad 2\,(x_1 x_5 - x_2 x_4)\, u_1^2 \;&+\; 4\,(x_1 x_6 - x_3 x_4)\, u_1 u_2 \;+\; 2\,(x_2 x_6 - x_3 x_5)\, u_2^2. \end{aligned}$$

The residue $\mathrm{Res}_f^\Delta(t^2)$, which appears in (1.10), is computed as 4 times the ratio of the normal form of $u_1 u_2$ over that of $J(u_1, u_2)$, both modulo $\mathcal{G}$.

To establish the connection to hypergeometric functions, we now consider the Cayley configuration of $\Delta, \Delta, \ldots, \Delta$, taken $r + 1$ times:

$$\widehat{\Delta} \quad := \quad \bigcup_{i=0}^{r} (\{e_i\} \times \Delta) \;\subset\; \mathbb{Z}^{r+1} \times \mathbb{Z}^r \;=\; \mathbb{Z}^{2r+1}.$$

The points in $\widehat{\Delta}$ are labeled by the variables x_{im} for $i = 0, \ldots, r$ and $m \in \Delta$.

Lemma 5.4. *The configuration $\widehat{\Delta}$ is gkz-rational.*

Proof. Let $a \in \mathrm{Int}((r{+}1)\cdot\Delta)$ and $f_0, \ldots, f_r$ generic Laurent polynomials as in (5.1). It follows from either the definition or [5, Algorithm 2] that $\mathrm{Res}_f^\Delta(t^a)$ is a homogeneous function with respect to the grading induced by $\widehat{\Delta}$; that is, it satisfies the equations defined by the operators (1.2) for a suitable parameter vector. It follows from [5, Theorem 7] that $\mathrm{Res}_f^\Delta(t^a)$ is also annihilated by the operators (1.1). Hence $\mathrm{Res}_f^\Delta(t^a)$ is a rational $\widehat{\Delta}$-hypergeometric function.

The discriminant associated with $\widehat{\Delta}$ equals the resultant $\mathcal{R}_\Delta = R_{\Delta,\Delta,\ldots,\Delta}$, by Proposition 5.1. This resultant is not a monomial, for instance, by [10, Corollary 8.2.3]. We showed in [6, Theorem 1.4] that $\mathcal{R}_\Delta \cdot \mathrm{Res}_f^\Delta(t^a)$ is a polynomial. It remains to be seen that the toric residue $\mathrm{Res}_f^\Delta(t^a)$ itself is non-zero for at least one lattice point $a \in \mathrm{Int}((r+1)\cdot\Delta)$. Recall from [7, Theorem 5.1] and [6, Proposition 1.2] that the polynomial

$$j(t) \;=\; \det \begin{pmatrix} f_0 & f_1 & \cdots & f_r \\ t_1\frac{\partial f_0}{\partial t_1} & t_1\frac{\partial f_1}{\partial t_1} & \cdots & t_1\frac{\partial f_r}{\partial t_1} \\ \vdots & \vdots & \ddots & \vdots \\ t_r\frac{\partial f_0}{\partial t_r} & t_r\frac{\partial f_1}{\partial t_r} & \cdots & t_r\frac{\partial f_r}{\partial t_r} \end{pmatrix}$$

is supported in $\mathrm{Int}((r+1)\cdot\Delta)$ and $\mathrm{Res}_f^\Delta(j(t)) = n!\cdot\mathrm{vol}(\Delta)$. Here the operator $\mathrm{Res}_f^\Delta(\,\cdot\,)$ is extended from monomials t^a to the polynomial $j(t)$ by linearity. At least one of the residues $\mathrm{Res}_f^\Delta(t^a)$ as a runs over $\mathrm{Int}((r+1)\cdot\Delta)$, does not vanish and hence is a non-Laurent rational $\widehat{\Delta}$-hypergeometric function. ∎

Example 5.5. The reciprocal of the determinant is a hypergeometric function. To see this, take Δ to be the unit simplex, so that, $f_i = x_{i0} + x_{i1}t_1 + \cdots + x_{ir}t_r$ in affine coordinates on $X_\Delta = \mathbb{P}^r$. The scaled simplex $(r+1)\cdot\Delta$ has a unique interior lattice point a, and $\mathrm{Res}_f^\Delta(t^a) = 1/\det(x_{ij})$. Here the Cayley configuration $\widehat{\Delta}$ is the product of two r-simplices $\Delta\times\Delta$. We conclude that $1/\det(x_{ij})$ is a rational $\Delta\times\Delta$-hypergeometric function.

We are now prepared to complete the proof of our main result.

Proof of the if direction in Theorem 1.5. Let A be an essential Cayley configuration with $A_0, A_1, \ldots, A_r$ as in (1.8). Let Δ be the set of all lattice points in a convex polytope containing all the configurations A_i for $i = 0, 1, \ldots, r$. Then, Δ is full-dimensional and $A \subseteq \widehat{\Delta}$.

Consider configurations $B_0, \ldots, B_r$ in $\mathbb{Z}^r$ such that $A_i \subseteq B_i \subseteq \Delta$ for $i = 0, \ldots, r$. The corresponding Cayley configuration B is still essential, since the Minkowski sum $\sum_{i\in I} B_i$ has affine dimension at least $|I|$. This property holds for A and it does for B. We conclude from Proposition 5.1 that the Cayley configuration B is non-degenerate and $D_B = R_{B_0,\ldots,B_r}$.

We would like to show that, in fact, any such configuration B must be gkz-rational. We proceed by induction on the cardinality of $\widehat{\Delta}\backslash B$. The base case is cardinality zero: if $B = \widehat{\Delta}$ then B is gkz-rational by Lemma 5.4.

For the induction step we may suppose that $\tilde{B}$ is obtained from B by removing a point b from $B_0 \backslash A_0$ and assume, inductively, that f is a rational B-hypergeometric function which contains the discriminant D_B in its denominator. Let us denote by t the variable associated with b and by $\tilde{x}$ the variables associated with $\tilde{B}$. Expand as in (3.2):

$$f(\tilde{x}; t) \quad = \quad \sum_{\ell \geq \ell_0} R_\ell(\tilde{x}) \cdot t^\ell \,, \tag{5.3}$$

where each $R_\ell(\tilde{x})$ is a rational $\tilde{B}$-hypergeometric function. We may now argue as in the proof of Theorem 1.2; since B and $\tilde{B}$ have affine dimension $2r$ it follows from Lemma 3.2 that the $\tilde{B}$ discriminant $D_{\tilde{B}}$ divides the specialization $D_B|_{t=0}$. Hence, for some ℓ, the rational function $R_\ell(\tilde{x})$ will lie strictly in the field of fractions of the domain $\mathbb{C}[\tilde{x}, \tilde{x}^{-1}]_{\langle D_{\tilde{B}} \rangle}$ and, consequently, will be a rational $\tilde{B}$-hypergeometric function which contains the discriminant $D_{\tilde{B}}$ in its denominator. In summary, the configuration $\tilde{B} = B \backslash \{b\}$ inherits the property of being gkz-rational from the configuration B. By induction, we conclude that A is gkz-rational. ∎

The results in this paper raise many questions about rational hypergeometric functions. The most obvious one is whether Conjectures 1.3 and 1.4 are true for toric varieties other than hypersurfaces, curves, surfaces and threefolds. Another question which concerns the number of rational solutions is the following: Is the dimension of the vector space of rational function solutions to the hypergeometric system $H_A(\beta)$ always bounded by the normalized volume of A? This volume is the degree of X_A; see (1.3).

It would be nice to extend the observation in Example 5.5 from determinants to hyperdeterminants. Following [10, Chapter 14], the *hyperdeterminant* is the discriminant (= dual hypersurface) associated with any Segre variety $X_A = \mathbb{P}^{k_0} \times \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. Suppose $k_0 \geq k_1 \geq \cdots \geq k_r$. The corresponding configuration is a product of simplices $A = \Delta_{k_0} \times \Delta_{k_1} \times \cdots \times \Delta_{k_r}$. It is known [10, Theorem 14.1.3] that A is non-degenerate if and only if $k_0 \leq k_1 + k_2 + \cdots + k_r$. The case of equality $k_0 = k_1 + k_2 + \cdots + k_r$ is of special interest; it defines the *hyperdeterminants of boundary format*. Since in this case A is an essential Cayley configuration, it follows from Theorem 1.5 that A is gkz-rational. We conjecture the converse of this statement:

Conjecture 5.6. *Let A be the product of simplices $\Delta_{k_0} \times \Delta_{k_1} \times \cdots \times \Delta_{k_r}$ where $k_0 \geq \cdots \geq k_r$. Then A is gkz-rational if and only if $k_0 = k_1 + \cdots + k_r$.*

Finally, we are hoping for a “Universality Theorem for Toric Residues” to the effect that the space of rational hypergeometric functions is spanned by Laurent polynomials and toric residues. This statement is literally false, as the following example shows. Let A be the Cayley configuration of the segments $\{0,1\}$ and $\{0,2\}$. The following rational function is A-hypergeometric:

$$f(x_1,x_2,x_3,x_4) \quad = \quad \frac{x_4\,(-x_1^4x_4^2 - 6x_1^2x_2^2x_3x_4 + 3x_2^4x_3^2)}{x_2^2\,(x_2^2x_3 + x_1^2x_4)^3}.$$

This is not a toric residue because the degree is zero in the variables $\{x_3,x_4\}$. However, an appropriate derivative of f will be a toric residue. For example,

$$\frac{\partial f}{\partial x_4} \quad = \quad 3\,\frac{x_3\,(x_1^4x_4^2 - 6x_1^2x_2^2x_3x_4 + x_2^4x_3^2)}{(x_2^2x_3 + x_1^2x_4)^4}$$

agrees, up to constant, with the toric residue in $\mathbb{P}^1$ associated with the differential form

$$\frac{t^4}{(x_1+x_2t)^4\cdot(x_3+x_4t^2)}\,\frac{dt}{t}.$$

Although this is not explicit in the proof of Theorem 1.5, one can show that every essential Cayley configuration admits rational hypergeometric functions which are toric residues and whose denominators are multiples of the A-discriminant. Families of examples together with extensive computer experiments support the following general conjecture:

Conjecture 5.7. *Every rational A-hypergeometric function f has an iterated derivative*

$$\frac{\partial^{i_1+\cdots+i_s}}{\partial x_1^{i_1}\cdots\partial x_s^{i_s}}f$$

which is a toric residue defined by some facial subset of A.

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