THE QUANTUM GROUPS $\overline{U}_q(sl_2)$ AT THE ROOTS OF UNITY, SELF-DUALITY AND ASCENT

LARS KADISON

Abstract. We compute that $\overline{U}_q(sl_2)$ is self-dual if q is a fourth root of unity. We prove the converse statement by using Radford's formula for the fourth power of the antipode. We end with a discussion of the β-Frobenius extension structure of $\overline{U}_i(sl_2)$ over the Sweedler 4-dimensional Hopf subalgebra, and remark that the triangular structure at $\lambda = 0$ ascends for certain general reasons.

1. Introduction

This paper continues some computations performed in [6]. We will prove that the finite dimensional quantum enveloping algebra $\overline{U}_q(sl_2)$, where q is the fourth root of unity $i = \sqrt{-1}$, is self-dual as a Hopf algebra. More generally, we establish that

Theorem 1.1. *The Hopf algebra $\overline{U}_q(sl_2)$ at the primitive n'th root of unity q is self-dual if and only if $n = 4$.*

We prove this in Sections 2 and 3 below. In Section 2, we make some general observations on the integrals of $\overline{U}_q(sl_2)$, unimodularity, the Nakayama automorphism S^2 where S is the antipode, Radford's formula for S^4 and counimodularity. In Section 3, we continue the computations in [6] by computing the Hopf algebra structure of the dual $\overline{U}_i^*$ of $\overline{U}_i := \overline{U}_i(sl_2)$ and showing it is isomorphic as Hopf algebras to $\overline{U}_i$.

We conclude with Section 4 discussing our results in light of the structure given in [2] for a β-Frobenius algebra extension formed by a Hopf subalgebra pair, such as the e^3-dimensional algebra $\overline{U}_q := \overline{U}_q(sl_2)$ over Taft's e^2-dimensional Hopf algebras [19]. A property P of unital algebras is said to *ascend* along a special type of algebra extension if a subalgebra with P implies the overalgebra has P in such an extension: we have a similar definition for *descent*. Although the Taft algebras are self-dual, non-symmetric and non-unimodular, we see then that none of these properties automatically ascend along β-Frobenius extensions.

1991 *Mathematics Subject Classification.* 16W35,17B37,81R50.

We continue our computations for the fourth root of unity and provide a β-Frobenius structure for the 8-dimensional Hopf algebra $\overline{U}_i$ extending Sweedler's 4-dimensional Hopf algebra. Indepedent of the rest of the section, we observe finally that properties of Hopf algebras like quasi-cocommutativity or being braided or triangular ascend for certain "well-placed" Hopf subalgebras like Sweedler's algebra within $\overline{U}_i$.

2. General Observations on U_q and its quotient

All algebras in this paper are unital and over the complex numbers $\mathcal{C}$. Let q be a complex number different from $0, 1, -1$. The algebra $U_q := U_q(sl_2)$ is the free polynomial algebra in noncommuting variables K, E, F, where K has inverse K^{-1}, and subject to the relations,

$$KE = q^2 EK \tag{1}$$
$$KF = q^{-2} FK \tag{2}$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \tag{3}$$

[7]. The additional structure of a Hopf algebra on U_q is given by the following values of its comultiplication $\Delta : U_q \to U_q \otimes U_q$, counit $\epsilon : U_q \to \mathcal{C}$ and antipode $S : U_q \to U_q$ [16]:

$$\Delta(K) = K \otimes K \tag{4}$$
$$\Delta(E) = E \otimes 1 + K \otimes E \tag{5}$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F, \tag{6}$$

$$\epsilon(K) = 1, \quad \epsilon(E) = 0, \quad \epsilon(F) = 0, \tag{7}$$

$$S(K) = K^{-1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK \tag{8}$$

Recall that Δ and ϵ are algebra homomorphisms, while the antipode S is an algebra anti-homomorphism: $S(ab) = S(b)S(a)$. The antipode S above has an inverse $\overline{S}$ given by

$$\overline{S}(K) = K^{-1}, \quad \overline{S}(E) = -EK^{-1}, \quad \overline{S}(F) = -KF \tag{9}$$

If q is a primitive root of unity of order d, one defines e to be d if odd, or $\frac{d}{2}$ if even. Then K^e, E^e and F^e belong to the center of U_q [5]. One defines the finite dimensional Hopf algebra $\overline{U}_q$ as the quotient of U_q by the ideal I generated by E^e, F^e and $K^e - 1$: I is a Hopf ideal [5]. The algebra $\overline{U}_q$ (called the Frobenius-Lusztig kernel of $U_q(sl_2)$) is e^3-dimensional with basis $\{E^\ell F^j K^k | 0 \leq \ell, j, k \leq e - 1\}$ [5].

We make a series of small observations about the algebras U_q and $\overline{U}_q$ needed later.

The Hopf algebra automorphism S^2 is important here and we record below the well-known fact that S^2 is an inner automorphism:

Lemma 2.1. *$S^2(a) = K^{-1}aK$ for all $a \in U_q$ or $\overline{U}_q$.*

Next we note a fact about the Cartan automorphism α on $H = U_q$ or $\overline{U}_q$, given by $\alpha(K) = K^{-1}, \alpha(E) = F, \alpha(F) = E$, which helps account for the differing Hopf structures on U_q in the literature (cf. [5, 6]).

Lemma 2.2. *The Cartan automorphism α on $H = U_q$ or $\overline{U}_q$ induces a Hopf algebra isomorphism of H with its co-opposite algebra H^{cop}.*

Proof. We note from Equations 4 - 6 that $(\alpha \otimes \alpha)\Delta = \Delta^{\mathrm{op}}\alpha$ where $\Delta^{\mathrm{op}} := \tau \circ \Delta$, and τ is the twist on $H \otimes H$ given by $\tau(a \otimes b) := b \otimes a$. Moreover, $\epsilon \circ \alpha = \epsilon$. It follows that $\alpha : H \stackrel{\cong}{\to} H^{\mathrm{cop}}$ as Hopf algebras and that $\overline{S} \circ \alpha = \alpha \circ S$ [15, Theorem 1.5d], since $\overline{S}$ is the antipode of H^{cop}. □

Since S induces a Hopf algebra isomorphism $H \cong H^{\mathrm{op,cop}}$, it follows from the lemma that H is Hopf isomorphic to its opposite algebra H^{op}.

Recall that a finite dimensional Hopf algebra H has a right (respectively left) integral $t \in H$ with defining property, $ta = \epsilon(a)t$ (respectively $at = \epsilon(a)t$) for all $a \in H$ [9]. Since the space of right integrals is one-dimensional [9], there is an augmentation $m \in H^*$ called the *right modular function or right distinguished group-like element* in the dual Hopf algebra H^* (for dual Hopf algebra: [5]), such that $at = m(a)t$ for all $a \in H$. The left modular function is defined correspondingly and equals $m^{-1} := m \circ S$. If the space of left integrals coincides with the right integrals, H is *unimodular*, i.e. $m = \epsilon$. The integral for $\overline{U}_q$ below resembles somewhat the left and right integrals for Taft's Hopf algebras [15].

Proposition 2.3. *The Hopf algebras $\overline{U}_q$ are unimodular.*

Proof. From relations 1 and 2, we see that for any integer powers ℓ, j we have $K^\ell E^j F^j = E^j F^j K^\ell$. It follows that

$$t := E^{e-1}F^{e-1}(1 + K + \cdots + K^{e-1}) = (1 + K + \cdots + K^{e-1})E^{e-1}F^{e-1}. \tag{10}$$

We compute that $Kt = t = \epsilon(K)t$ and similarly $Et = 0 = Ft$ from $E^e = 0 = F^e$. Thus t is a left integral, and similarly we see that t is a right integral. □

A right integral t provides a nice pair of bases for a finite dimensional Hopf algebra H as follows: compute $\Delta(t) = \sum_{(t)} t_{(1)} \otimes t_{(2)}$ in Sweedler's notation, then $\{\overline{S}(t_{(2)})\}$, $\{t_{(1)}\}$ are *dual bases* for H, with defining property that there is a linear functional $\phi \in H^*$ such that

$$a = \sum_{(t)} \phi(a\overline{S}(t_{(2)}))t_{(1)} \tag{11}$$

$$a = \sum_{(t)} \overline{S}(t_{(2)})\phi(t_{(1)}a) \tag{12}$$

for all $a \in H$. We note that $\langle a, b\rangle := \phi(ab)$ is a nondegenerate associative bilinear form on H. If this bilinear form is symmetric for some ϕ and associated dual bases, H is a *symmetric algebra* with many nice properties from the point of view of representation theory, such as a symmetric Cartan matrix [20]. Equivalently, the algebra H is symmetric if the (Nakayama) algebra automorphism η of H defined by $\phi(xa) = \phi(\eta(a)x)$ for all $a, x \in H$ is an inner automorphism [8].

In terms of the dual bases and the associated $\phi \in H^*$ satisfying $\phi(t) = 1$, the Nakayama automorphism for H is given by

$$\eta(a) = S^2(a) \leftharpoonup m^{-1} \tag{13}$$

[2], where H^* acts on the bimodule H from left and right by $f \rightharpoonup a := \sum_{(a)} a_{(1)}f(a_{(2)})$ and $a \leftharpoonup f := \sum_{(a)} f(a_{(1)})a_{(2)}$ for all $f \in H^*$, $a \in H$. In case $H = \overline{U}_q$, $m = \epsilon$ by Proposition 2.3, whence η equals the *Hopf* algebra automorphism S^2. Putting this together with Lemma 2.1, we arrive at the next observation:

Proposition 2.4. *The $\overline{U}_q$ are symmetric algebras.*

For d odd, the last two propositions also follow from putting together [5, Theorem IX.6.6] with Equation 13 and theorems in [1, 13].

We will need a few more general remarks for a finite dimensional Hopf algebra H. H is said to be *counimodular* if its dual Hopf algebra H^* is unimodular. Of course, a self-dual H is unimodular if and only if it is counimodular. If H is not counimodular, there is a right distinguished group-like element $b \in H$, different from 1, which is the modular function for H^* in $H^{**} \cong H$, as defined above. Radford [12] has the following formula for the fourth power of the antipode: for all $a \in H$

$$S^4(a) = b(m^{-1} \rightharpoonup a \leftharpoonup m)b^{-1}. \tag{14}$$

Thus $S^4 = \mathrm{Id_H}$ if H is unimodular and counimodular.

3. Computations for $U_q(sl_2)^*$ at the fourth root of unity

In this section we compute the structure of the 8-dimensional algebra $\overline{U}_i^* := \mathrm{Hom}_{\mathcal{C}}(\overline{\mathrm{U}}_{\mathrm{i}}, \mathcal{C})$ and show that it is Hopf algebra isomorphic to $\overline{U}_i$.

With $q = i$, the relations 1– 8 and those of the Hopf ideal I become the following defining relations of the Hopf algebra $\overline{U}_i$:

$$\begin{array}{llll}
(15) & EK = -KE & FK = -KF & EF = FE \\
(16) & K^2 = 1 & E^2 = 0 & F^2 = 0 \\
& \Delta(K) = K \otimes K & \Delta(E) = E \otimes 1 + K \otimes E & \Delta(F) = F \otimes K + 1 \otimes F \\
(17) & \epsilon(K) = 1 & \epsilon(E) = 0 & \epsilon(F) = 0 \\
(18) & S(K) = K & S(E) = -KE & S(F) = -FK
\end{array}$$

We base our computations on those in [6][1] for the Hopf algebra $\overline{U}_i$, relative to the basis,

$$\begin{array}{cc}
e_0 := \frac{1+K}{2} & e_1 := \frac{1-K}{2} \\
E_0 := e_0 E & E_1 := e_1 E \\
F_0 := e_0 F & F_1 := e_1 F \\
C_0 := e_0 EF & C_1 := e_1 EF.
\end{array}$$

We note the identities $e_j^2 = e_j$, $e_0 e_1 = 0$, $F_j^2 = E_j^2 = C_j^2 = 0$, $e_0 E = E e_1$, etc. $(j = 0, 1)$.

We denote the dual (ordered) basis in $\overline{U}_i^*$ of the basis above, where $\hat{e_0}(e_0) = 1$, $e_0(x) = 0$ for x another basis element, etc., by

$$\{\hat{e_0}, \hat{e_1}, \hat{E}_0, \hat{E}_1, \hat{F}_0, \hat{F}_1, \hat{C}_0, \hat{C}_1\}.$$

We compute the multiplication table using the formula, $(\chi\rho)(a) = \sum_{(a)} \chi(a_{(1)})\rho(a_{(2)})$ for multiplication in $\overline{U}_i^*$ of $\chi, \rho \in \overline{U}_i$ and the values of Δ on the basis for $\overline{U}_i$ just above [6].

	$\hat{e_0}$	$\hat{e_1}$	$\hat{E}_0$	$\hat{E}_1$	$\hat{F}_0$	$\hat{F}_1$	$\hat{C}_0$	$\hat{C}_1$
$\hat{e_0}$	$\hat{e_0}$	$\hat{e_1}$	$\hat{E}_0$	$\hat{E}_1$	$\hat{F}_0$	$\hat{F}_1$	$\hat{C}_0$	$\hat{C}_1$
$\hat{e_1}$	$\hat{e_1}$	$\hat{e_0}$	$-\hat{E}_1$	$-\hat{E}_0$	$\hat{F}_1$	$\hat{F}_0$	$-\hat{C}_1$	$-\hat{C}_0$
$\hat{E}_0$	$\hat{E}_0$	$\hat{E}_1$	0	0	$\hat{C}_0$	$\hat{C}_1$	0	0
$\hat{E}_1$	$\hat{E}_1$	$\hat{E}_0$	0	0	$\hat{C}_1$	$\hat{C}_0$	0	0
$\hat{F}_0$	$\hat{F}_0$	$-\hat{F}_1$	$-\hat{C}_0$	$\hat{C}_1$	0	0	0	0
$\hat{F}_1$	$\hat{F}_1$	$-\hat{F}_0$	$\hat{C}_1$	$-\hat{C}_0$	0	0	0	0
$\hat{C}_0$	$\hat{C}_0$	$-\hat{C}_1$	0	0	0	0	0	0
$\hat{C}_1$	$\hat{C}_1$	$-\hat{C}_0$	0	0	0	0	0	0

[1]The preprint [6] is available as a postscript file from the internet address www.cpt.univ-mrs.fr

From the multiplication table, we note that the linear transformation $\psi : \overline{U}_i^* \to \overline{U}_i$, given by

$$\psi(\hat{e_0}) = 1 \qquad \psi(\hat{e_1}) = K \qquad \psi(\hat{E}_0) = E \tag{19}$$

$$\psi(\hat{E}_1) = EK \quad \psi(\hat{F}_0) = -FK \quad \psi(\hat{F}_1) = F \tag{20}$$

$$\psi(\hat{C}_0) = -EFK \quad \psi(\hat{C}_1) = EF, \tag{21}$$

is an algebra isomorphism.

We next record the Hopf structure of $\overline{U}_i^*$ below. We first compute its comultiplication Δ^* from the formula $\Delta^*(\rho)(a \otimes b) = \rho(ab)$ for all $\rho \in \overline{U}_i^*$, $a, b \in \overline{U}_i$.

(22)

$$\begin{array}{rr}
\Delta^*(\hat{e_0}) = \hat{e_0} \otimes \hat{e_0} & \Delta^*(\hat{e_1}) = \hat{e_1} \otimes \hat{e_1} \\
\Delta^*(\hat{E}_0) = \hat{E}_0 \otimes \hat{e_1} + \hat{e_0} \otimes \hat{E}_0 & \Delta^*(\hat{E}_1) = \hat{e_1} \otimes \hat{E}_1 + \hat{E}_1 \otimes \hat{e_0} \\
\Delta^*(\hat{F}_0) = \hat{F}_0 \otimes \hat{e_1} + \hat{e_0} \otimes \hat{F}_0 & \Delta^*(\hat{F}_1) = \hat{e_1} \otimes \hat{F}_1 + \hat{F}_1 \otimes \hat{e_0} \\
\Delta^*(\hat{C}_0) = \hat{C}_0 \otimes \hat{e_0} + \hat{e_0} \otimes \hat{C}_0 + \hat{F}_0 \otimes \hat{E}_1 + \hat{E}_0 \otimes \hat{F}_1 & \\
\Delta^*(\hat{C}_1) = \hat{C}_1 \otimes \hat{e_1} + \hat{e_1} \otimes \hat{C}_1 + \hat{F}_1 \otimes \hat{E}_0 + \hat{E}_1 \otimes \hat{F}_0 &
\end{array}$$

We note that $(\psi \otimes \psi) \circ \Delta^* = \Delta^{\mathrm{op}} \circ \psi$. To show that $\psi : \overline{U}_i^* \xrightarrow{\cong} \overline{U}_i^{\mathrm{cop}}$ is a Hopf algebra isomorphism, it suffices to show that $\epsilon \circ \psi = \epsilon^*$ where ϵ^* is defined on $\overline{U}_i^*$ by $\epsilon^*(\rho) := \rho(1)$ [15]. This follows from a comparison with the formulas for ϵ^* below for $j = 0, 1$:

$$\epsilon^*(\hat{e_j}) = 1 \quad \epsilon^*(\hat{E}_j) = 0 = \epsilon^*(\hat{F}_j) = \epsilon^*(\hat{C}_j). \tag{23}$$

Our record of the Hopf structure of $\overline{U}_i^*$ is not complete without the antipode $S^*(\rho) = \rho \circ S$. However, it follows automatically from the fact that ψ is a bialgebra isomorphism that $\psi \circ S^* = \overline{S} \circ \psi$ [15].

$$\begin{array}{llll}
S^*(\hat{e_0}) = \hat{e_0} & S^*(\hat{e_1}) = \hat{e_1} & S^*(\hat{E}_0) = -\hat{E}_1 & S^*(\hat{E}_1) = \hat{E}_0 \\
S^*(\hat{F}_0) = \hat{F}_1 & S^*(\hat{F}_1) = -\hat{F}_0 & S^*(\hat{C}_0) = \hat{C}_0 & S^*(\hat{C}_1) = \hat{C}_1
\end{array} \tag{24}$$

We can now prove:

Theorem 3.1. *The Hopf algebras $\overline{U}_q(sl_2)$ at the primitive n'th root of unity q are self-dual if and only if $n = 4$.*

Proof. If $q = \pm i$, we have shown in this section that $\overline{U}_q^* \cong \overline{U}_q^{\mathrm{cop}}$. But by Lemma 2.2, the Cartan automorphism induces $\overline{U}_q \cong \overline{U}_q^{\mathrm{cop}}$. Whence $\overline{U}_q \cong \overline{U}_q^*$ and $\overline{U}_q$ is self-dual.

Now suppose q is not a fourth root of unity and $\overline{U}_q$ is self-dual. By Proposition 2.3 $\overline{U}_q$ is unimodular, therefore counimodular. Then by Radford's formula 14, $S^4 = \mathrm{Id}$. But from Lemma 2.1

$$F = S^4(F) = K^{-2}FK^2 = q^4F$$

which implies that $q^4 = 1$, a contradiction. □

Sommerhäuser [17] observes another proof for odd n that $\overline{U}_q$ is not self-dual, which is implied by formulas for the dimensions of the radicals of $\overline{U}_q$ and its dual.

4. Observations on Hopf subalgebras of $\overline{U}_q$

In $\overline{U}_q$ is the Hopf algebra T_q generated by F and K. If e is again the order d of the root of unity q if d is odd, $\frac{d}{2}$ if d is even, then T_q is determined by the relations:

$$F^e = 0 \quad K^e = 1 \quad FK^{-1} = q^{-2}K^{-1}F \tag{25}$$

$$\Delta(K) = K \otimes K \qquad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \tag{26}$$

$$\epsilon(K) = 1,\ S(K) = K^{-1} \qquad \epsilon(F) = 0,\ S(F) = -FK \tag{27}$$

On the one hand, this is clearly a Hopf subalgebra of $\overline{U_q}$ (i.e, $\Delta(T_q) \subseteq T_q \otimes T_q$ and $S(T_q) \subseteq T_q$). On the other hand, these are the defining relations and Hopf structure on the Taft algebras (cf. [19, 15]) since q^{-2} is a primitive e'th root of unity.

The Taft algebra T_q is not unimodular, having right integral

$$t = F(1 + K + \cdots + K^{e-1}) \tag{28}$$

and non-proportional left integral $(1 + K + \cdots + K^{e-1})F$. Therefore, T_q is not a symmetric algebra by [4, Corollary 3.3]. But T_q is self-dual with isomorphism $T_q \stackrel{\cong}{\to} T_q^*$ given by $K^{-1} \mapsto G$ where G is the algebra map defined by $G(K^{-1}) = q^2$, $G(F) = 0$, and $F \mapsto X$ where X is the linear map defined by $X(K^{-\ell}F^j) = \delta_{j,1}$ for $0 \leq \ell, j \leq e-1$ [15].

Suppose H is a finite dimensional Hopf algebra with Hopf subalgebra T. Then it turns out that H is free as a right or left T-module [11] with special dual bases and right T-module homomorphism $\Phi : H \to T$: in the special case that T is the unit subalgebra $\mathcal{C}1$, we recover our discussion in Section 2 (cf. Equation 12) [2]. As a left T-module map Φ is twisted by a new algebra automorphism $\beta : T \to T$ which is given in terms of the Nakayama automorphisms of T and H, respectively, by

$$\beta = \eta_T \circ \eta_H^{-1}. \tag{29}$$

There are formulas for the dual bases and homomorphism Φ given in [2] as follows. If t is a right integral for H, n a right integral for T, and $\phi \in H^*$ satisfies Equations 12, then

$$\Phi(a) := \sum_{(a)} \phi(a_{(1)}\overline{S}(n))a_{(2)}, \tag{30}$$

for every $a \in H$. By freeness of H as a right and left T-module, there exists $\Lambda \in H$ such that

$$t = \hat{\Lambda} n. \tag{31}$$

Let

$$\Lambda := \eta_H(\overline{S}(\hat{\Lambda})). \tag{32}$$

Then $\{\overline{S}(\Lambda_{(2)})\}$, $\{\Lambda_{(1)}\}$ are dual bases for the right and left-twisted T-module H as follows: for all $a \in H$,

$$a = \sum_{(\Lambda)} \overline{S}(\Lambda_{(2)})\Phi(\Lambda_{(1)}a) \tag{33}$$

$$a = \sum_{(\Lambda)} \beta^{-1}[\Phi(a\overline{S}(\Lambda_{(2)}))]\Lambda_{(1)}. \tag{34}$$

The advantage of these equations comes about when H extends a well-known algebra T, or *vice versa*. For example, from Equation 31 we deduce that Hopf subalgebras of a semisimple Hopf algebra are themselves semisimple, since $\epsilon(t) = \epsilon(\hat{\Lambda})\epsilon(n) \neq 0 \Rightarrow \epsilon(n) \neq 0$ [10]. However, we add a word of caution next.

For a β-Frobenius extension $T \subseteq H$ of algebras, induction of T-modules, $M \mapsto M \otimes_T H$, is naturally isomorphic to co-induction of T-modules, $M \mapsto \mathrm{Hom_T(H, M)}$, up to a category equivalence determined by the automorphism β of T [3]. Certain properties such as self-injectivity of the subalgebra T will ascend to H [3]. However, we now note a number of properties that will not ascend, including self-duality for Hopf algebras (since T_q is self-dual, but $\overline{U}_q$ is not self-dual for $q \neq \pm i$) and non-symmetry for Hopf algebras (since T_q is not a symmetric algebra, but U_q is).

At the fourth root of unity (cf. Section 3), we have the Hopf subalgebra pair $\overline{U}_i \supset T_i$ where T_i is the well-known 4-dimensional Sweedler algebra with right integral $n = F(1 + K)$ [18]. Since $t = EF(1 + K)$, $\hat{\Lambda} = E$ and $\Lambda = S(E) = -KE$.[2] Then

$$\sum_{(\Lambda)} \overline{S}(\Lambda_{(2)}) \otimes \Lambda_{(1)} = K \otimes EK + E \otimes 1.$$

Furthermore, for all $a \in \overline{U}_i$,

$$\Phi(a) = -\phi(a(1 + K)F)$$

[2]In passing, we should note that t provides an easy counterexample to the third Kaplansky conjecture that there are no nonzero central nilpotent elements in Hopf algebras over a field of characteristic zero, since t is a 2-sided integral and $t^2 = 0$ [17].

where ϕ has dual bases for $\overline{U}_i$, $\{\overline{S}(t_{(2)})\}$, $\{t_{(1)}\}$ as in Equation 12. We compute these ordered bases for $\overline{U}_i$ and then the associated map ϕ:

$$\{\overline{S}(t_{(2)})\} = \{e_0, -e_1, C_0, -C_1, F_1, -F_0, E_1, -E_0\} \tag{35}$$

$$\{t_{(1)}\} = \{C_0, C_1, e_0, e_1, E_0, E_1, F_0, F_1\}. \tag{36}$$

It follows from Equation 12 that $\phi(C_j) = (-1)^j$, $\phi(e_j) = \phi(F_j) = \phi(E_j) = 0$ for $j = 0, 1$. Thus Φ has values

a	1	F	K	KF	E	EF	KEF	EK
$\Phi(a)$	0	0	0	0	1	F	$-KF$	K

Finally, the automorphism β is given by

x	1	F	K	KF
$\beta(x)$	1	F	$-K$	$-KF$

We note that $\beta(x)$ is not of the form uxu^{-1} for any $u \in \overline{U}_i$ by applying ϵ to this equation and setting $x = K$. If it were of this form, one could choose new Φ and dual bases (by multiplying appropriately by u and its inverse) and get rid of the twist β in Equation 34.

Finally we note that $\overline{U}_i$ is a ribbon algebra like T_i (with braided category of modules [5]) for reasons that generalize. For T_i has a one-parameter family of universal R-matrices R_λ [13], which at $\lambda = 0$ is

$$R := R_0 = \frac{1}{2}(1 \otimes 1 + 1 \otimes K + K \otimes 1 - K \otimes K). \tag{37}$$

Then $\Delta^{\mathrm{op}}(a)R = R\Delta(a)$ for $a = K, F$. The Cartan automorphism α fixes K, transposes E and F and satisfies $\Delta^{\mathrm{op}} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta$, equivalently $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta^{\mathrm{op}}$. Since $R^{-1} = R = \tau(R)$, it follows that

$$\begin{aligned} \Delta^{\mathrm{op}}(E)R &= \Delta^{\mathrm{op}}(\alpha(F))(\alpha \otimes \alpha)(R) = \\ (\alpha \otimes \alpha)(\Delta(F)R) &= (\alpha \otimes \alpha)(R\Delta^{\mathrm{op}}(F) \\ &= R\Delta(E) \end{aligned}$$

It follows by multiplicativity of Δ and Δ^{op} that $\overline{U}_i$ is quasi-cocommutative (almost cocommutative) with same universal R-matrix R. That $\overline{U}_i$ is braided (indeed, triangular) follows at once from R satisfying $(\Delta \otimes 1)(R) = R_{13}R_{23}$ and $(1 \otimes \Delta)(R) = R_{13}R_{12}$ [13].

Recall that a braided Hopf algebra H is a ribbon algebra if there is a central element $\theta \in H$ such that $\Delta(\theta) = (\tau(R)R)^{-1}(\theta \otimes \theta)$, $\epsilon(\theta) = 1$, and $S(\theta) = \theta$ [14]. It follows from taking $\theta = 1$ that $\overline{U}_i$ is a ribbon algebra.

The preceding generalizes to theorems of the following kind, providing properties that ascend for certain "well-placed" Hopf subalgebras:

Proposition 4.1. *Suppose H is a Hopf algebra with braided Hopf subalgebra T and Hopf algebra automorphism $\alpha : H \to H$ fixing the universal R-matrix R. If H is generated as an algebra by T and $\alpha(T)$, then H is braided.*

Proof. The conclusion follows, as a variation on the theme above, by noting

$$\Delta^{\mathrm{op}}(\alpha(x))R = (\alpha \otimes \alpha)(\Delta^{\mathrm{op}}(x)R) = (\alpha \otimes \alpha)(R\Delta(x)) = R\Delta(\alpha(x)),$$

for every $x \in T$, since $(\alpha \otimes \alpha)(R) = R$. □

Finally it should be noted that $\overline{U}_i$ is the Radford biproduct of the Hopf subalgebra generated by K and a zero-dimensional local Gorenstein algebra $G := \mathcal{C}\langle E, F \,|\, EF = FE, E^2 = 0 = F^2 \rangle$. G is the algebra of the isolated singularity at $(0,0)$ of the variety defined by $x^3 + y^3$. G is equivalent to $\overline{U}_i$ as algebras in a slightly weaker sense than Morita equivalence [3], for the bimodule projection $G \oplus GK \to G$ has very pleasant dual bases given by $1 \otimes_G 1 + K \otimes_G K$, which has etale-like properties and no twisting by an automorphism.

References

[1] V.G. Drinfeld, On almost commutative Hopf algebras, *Leningrad Math. J.* **1** (1990), 321–342 (Russian original in Algebra and Analysis, 1989).

[2] D. Fischman, S. Montgomery, and H.-J. Schneider, Frobenius extensions of subalgebras of Hopf algebras, *Trans. Amer. Math. Soc.* **349** (1997), 4857–4895.

[3] L. Kadison, *New Examples of Frobenius Extensions*, Univ. Lect. Series **14**, A.M.S., Providence, 1999.

[4] L. Kadison and A.A. Stolin, Frobenius systems and Hopf algebras over rings, Chalmers preprint 20:1999.

[5] C. Kassel, *Quantum Groups*, Grad. Texts Math. **155**, Springer, New York, 1995.

[6] D. Kastler, T. Krajewski, P. Seibt, and K. Valavane, $U_q(sl_2)$ at fourth root of unity, Centre de Physique Theorique, Luminy preprint, 1998.

[7] P.P. Kulish and N.Yu. Reshetikhin, Quantum linear problems for the sine-Gordon equation and higher representations. *Zap. Nauchn. Sem. L.O.M.I.* **101** (1981), 101–110. English transl.: J. Soviet Math. **23** (1983), 2435–2441.

[8] T.Y. Lam, *Lectures on Modules and Rings*, Grad. Texts Math. **189**, Springer-Verlag, Heidelberg-Berlin-New York, 1999.

[9] R.G. Larson and M. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75–93.

[10] S. Montgomery, Hopf algebras and their actions on rings, *CBMS Regional Conf. Series in Math.* **82**, A.M.S., 1993.
[11] W.D. Nichols and M.B. Zoeller, A Hopf algebra freeness theorem, *Amer. J. Math.* **111** (1989), 381–385.
[12] D. Radford, The order of the antipode of a finite dimensional Hopf algebra is finite, Amer. J. Math. **98** (1976), 333–355.
[13] D. Radford, Minimal quasi-triangular Hopf algebras, *J. Algebra* **157** (1993), 285–315.
[14] N.Yu. Reshetikhin and V.G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.* **127** (1990), 1-26.
[15] H.-J. Schneider, Lectures on Hopf Algebras, *Preprint Ciudad University,* Cordoba, Argentina, Trabajos de Matematica, serie B, no. **31**, 1995.
[16] E.K. Sklyanin, On an algebra generated by quadratic relations. *Uspekhi Mat. Nauk* **40:2 (242)** (1985), 214. (In Russian.)
[17] Y. Sommerhäuser, On Kaplansky's conjectures, preprint, Munich, 1998.
[18] M.E. Sweedler, *Hopf Algebras*, W.A. Benjamin, New York, 1969.
[19] E.J. Taft, The order of the antipode of finite-dimensional Hopf algebras. *Proc. Nat. Acad. Sci. U.S.A.* **68** (1971), 2631–2633.
[20] K. Yamagata, Frobenius Algebras, in: *Handbook of Algebra, Vol. 1*, ed. M. Hazewinkel, Elsevier, Amsterdam, 1996, 841–887.

Chalmers University of Technology/Göteborg University, S-412 96 Göteborg, Sweden

E-mail address: kadison@math.ntnu.no